DIPLOMARBEIT

Minimum Dilation Triangulations for the Regular n-gon

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Hiermit versichere ich, dass ich die vorliegende Diplomarbeit selbständig verfasst und keine anderen als die von mir angegebenen Quellen und Hilfsmittel verwendet habe.

Berlin, Oktober 2004

Und die Fische, sie verschwinden!
Und man kann nicht an ihn ran
Denn ein Haifisch ist kein Haifisch
Wenn man's nicht beweisen kann.
(Bert Brecht — Die Moritat von Mackie Messer)

Danksagung

Zuallererst möchte ich mich natürlich bei meinem Betreuer Prof. Dr. Christian Knauer bedanken, der mir das Thema zur Verfügung gestellt hat und mir immer engagiert mit Rat und Tat zur Seite gestanden ist. Herrn Prof. Dr. Günther Rote möchte ich für die Anfertigung des Zweitgutachtens danken.

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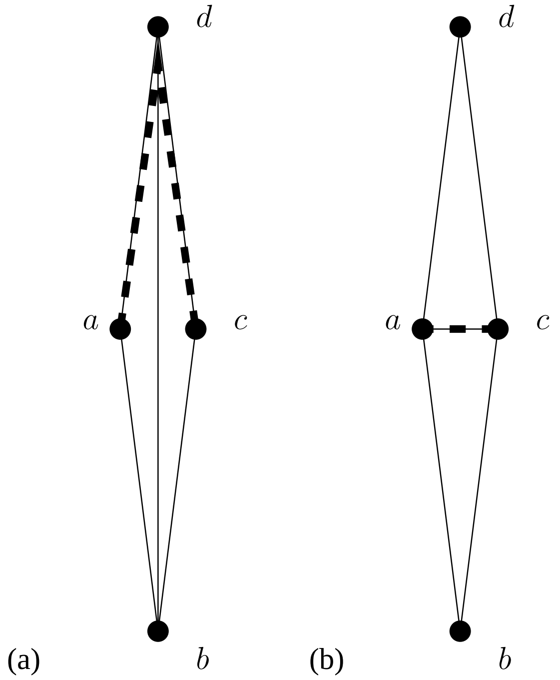
Chapter 1

Introduction

In this thesis we are going to deal with the graph theoretic dilation of planar point sets. The problem is as follows: Given a set S of points in the Euclidean plane, find a triangulation T of S such that the maximum detour between any pair of these points in T is minimal, where the *detour* between a pair of points (u, v) in S^2 is defined as the ratio between the shortest path distance of (u, v) in T and the Euclidean distance $|uv|$ (see chapter 2 for formal definitions of these terms). In figure 1.1 we can see an example of a planar point set and two triangulations, one of which achieves a very low maximum detour, while the other triangulation has a very high maximum detour. The maximum detour between any pair of points in S^2 with respect to a triangulation T of S is called the *graph theoretic dilation* of T, and the minimum graph theoretic dilation that any triangulation of S can achieve is called the graph theoretic dilation of S.

Figure 1.1: Two triangulations of point set $\{a, b, c, d\}$. In triangulation (a) the detour between points a and c is very high, whereas triangulation (b) achieves a very low detour. The bold dashed lines represent a shortest path between a and c in the respective triangulation.



The problem of computing the minimum dilation triangulation of a given planar point set belongs to a larger class of problems that constitute the main topic of *geometric network design theory* (see [Epp96] for a survey on which most of this introduction is based). The general problem in geometric network design theory is to connect a given set of points, or *sites,* by a network that has certain "good" properties. It is clear that geometric network design theory has applications in numerous fields, such as telecommunications, road design, and medical imaging, to cite some examples from [Epp96]. Naturally, there are many ways in which the general problem from network design theory can be specialized. For example, we could require that the network optimize a certain quantity. Some natural quantities to look at are

1. the *weight, i.e.,* the total length of all edges in the network,
2. the *diameter, i.e.,* the longest network distance between two sites, or finally
3. the *dilation,* which we are going to consider in this thesis.

Furthermore, it is possible to impose certain restrictions on the network by which the sites can be connected. For instance, the desired network could be

1. a tree,
2. a planar graph, or
3. a general graph.

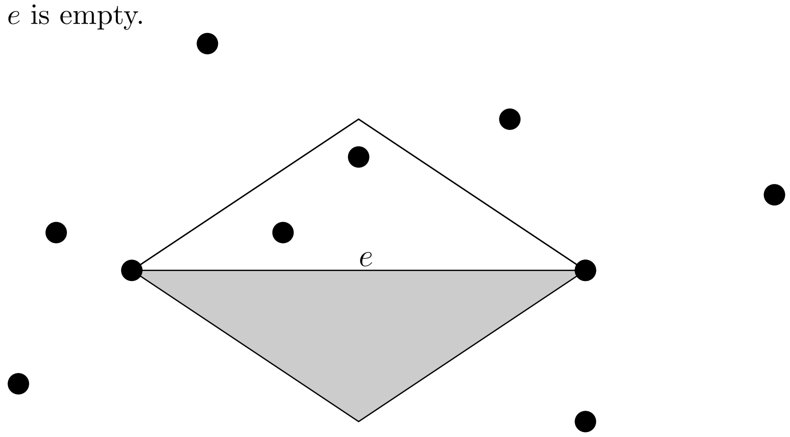
Some problems in geometric design theory, like for example the minimum weight spanning tree problem, are very well studied, and many algorithms exist that solve these problems efficiently (see *e.g.,* [Tar83] for classical results and [Cha00] for the state of the art). Other problems, like the minimum weight triangulation problem, seem to be very hard, and it is not known whether it is possible to solve them in polynomial time. Actually, the minimum weight triangulation problem is one of the last unsolved problems from Garey and Johnson's book on NP-completeness [GJ79].

Let us now look at the problems which are concerned with dilation. Naturally, if we ask for a general graph that achieves dilation 1, the complete graph is an obvious choice. Unfortunately, the complete graph has a quadratic number of edges. However, for any fixed $\varepsilon > 0$ it is possible to compute graphs with $O(n)$ edges whose dilation is at most $1 + \varepsilon$ in $O(n \log n)$ time, where n is the number of sites. Such graphs are usually called *spanners.*

Much less is known about minimum dilation spanning trees. It is clear that the minimum weight spanning tree has dilation $O(n)$, since the minimum weight spanning tree has the property that the shortest path between any two sites in the minimum weight spanning tree has the smallest possible maximum edge length (this property is called the *bottleneck shortest path property*). Indeed, the bottleneck shortest path property implies that the length of each edge along a shortest path between two sites is at most the Euclidean distance of the two sites, and there are at most $n - 1$ edges along any shortest path in a tree (since this is the number of edges in a tree with n vertices). Nonetheless, it is possible to construct point sets such that the minimum weight spanning tree yields a dilation that is $\Omega(n)$, even when a dilation of $O(\sqrt{n})$ can be achieved. There are also some lower bounds known for minimum dilation spanning trees. For example, it can be shown that any tree that connects the vertices of a regular n-gon has dilation $\Omega(\sqrt{n})$.

Little research has been done on minimum dilation triangulations, even though there has been some work on estimating the dilation of certain types of triangulations that had already been studied in other contexts. In [Che86] and [Che89], Chew shows that the rectilinear Delaunay triangulation has dilation at most $\sqrt{10}$. A similar result for the Euclidean Delaunay

Figure 1.2: An illustration of the Diamond property: One of the two isosceles triangles on edge e is empty.



triangulation is given by Dobkin *et al.* in [DFS90] where they show that the dilation of the Euclidean Delaunay triangulation is bounded above by $\left(\left(1+\sqrt{5}\right)/2\right)\pi \approx 5.08$. This bound was further improved to $\frac{2\pi}{3\cos(\pi/6)} \approx 2.42$ by Keil and Gutwin [KG89].

Das and Joseph generalize these results by identifying two properties of planar graphs such that if A is an algorithm that computes a planar graph from a given set of sites and if all the graphs constructed by A meet these properties, then the dilation of all the graphs constructed by A is bounded by a constant [DJ89]. The properties are

- **Diamond property.** There is some angle $\alpha < \pi$ such that for any edge e in a graph constructed by the algorithm, one of the isosceles triangles with e as a base and with apex angle α contains no other site (see figure 1.2). Intuitively, this property means that it cannot happen that a shortest path between two sites is obstructed by an edge of the graph.

- **Good polygon property.** There is some constant d such that for each face f of a graph constructed by the algorithm and any two sites u, v that are visible to each other across the face, one of the two paths around f from u to v has detour at most d. This constraint means that a shortest path between two sites does not incur a large detour by going around a large oval face. Note that this property is met by any triangulation, since all bounded faces of a triangulation are triangles and obviously the graph theoretic dilation of a triangle is 1.

Surprisingly, there are almost no results known about the actual minimum dilation triangulation, which constitutes the main topic of this thesis.

Before we proceed by giving a brief outline of the contents of this thesis, let us quickly mention a notion that is very similar to the graph theoretic dilation. This concept is called the *geometric dilation* of a planar point set. The difference between geometric and graph theoretic dilation is that the geometric dilation is defined as the supremum of the detour between any pair of points on a graph, not just between the vertices of the graph as it is the case for the graph theoretic dilation.

Even though at the first glance the geometric dilation appears to be very similar to the graph theoretic dilation, there are major structural differences. Perhaps the largest difference between the two concepts lies in the fact that it can be shown that if there is a pair of points that achieves maximum geometric dilation, then there is one that is co-visible, *i.e.,* the two points can be connected by a straight line that does not intersect any other line of the graph. Using this property, it is possible to devise several efficient algorithms that approximate or compute exactly the geometric dilation of polygonal curves and polygons [AKKS02, EBKLL01, Gru02, LMS02]. The author is not aware of any algorithms for computing a planar graph that achieves the optimum geometric dilation for a given point set. However, there is an upper bound of 1.678 and a lower bound of $\left(1+10^{-11}\right)\pi/2$ for the geometric dilation of any finite planar point set [EBGR03, DGR04].

This thesis, however, deals with the graph theoretic dilation of finite planar point sets. In particular, we will consider the graph theoretic dilation of the set of nodes of a regular n-gon. Even though this seems to be a very special case, it turns out that it is even nontrivial to find an algorithm that approximates the graph theoretic dilation of the regular n-gon and to prove its correctness. Furthermore, it seems that some of our results should be generalizable to *fat point sets*, *i.e.,* planar point sets that can be sandwiched between two circles whose radii have a constant ratio.

In addition to this introduction, this thesis has seven more chapters and two appendices.

In chapter 2, we are going to define all the terms that are central to the ensuing discussion and mention some simple consequences of these definitions. These definitions will be filled with life in chapter 3, which contains the results of some experiments that were conducted with a small *Java*-program that computes minimum dilation triangulations for small point sets. We begin our theoretical analysis of minimum dilation triangulations in chapter 4, where we give a general upper bound for the graph theoretic dilation of any triangulation of the regular n-gon, estimate the value of the dilation of a special triangulation of the regular n-gon, the canonical triangulation, and derive an upper bound for the case that $n = 3 \cdot 2^i$. These upper bounds are complemented by a lower bound that we shall derive in chapter 5, where we will carry out a detailed analysis in order to obtain a lower bound on the graph theoretic dilation of any triangulation of the regular n-gon. Some interesting consequences of this lower bound will be exposed in chapter 6, in which we show that the Euclidean distance between the two vertices of any maximum detour pair is bounded from below by a large constant and that in any triangulation there are three distinguished vertices such that any shortest path between the two vertices of a maximum detour pair must include at least one of them. These consequences will be put to use in chapter 7, whose purpose is to present a polynomial time approximation algorithm that approximates the graph theoretic dilation of the regular n-gon within a factor of $1 + 1/\sqrt{\log n}$. Finally, we conclude in chapter 8 with some final remarks and some possible directions for further work.

In appendix A, we present two heuristics that can be used to approximate the graph theoretic dilation of the regular n-gon, and appendix B contains a description of an efficient method to enumerate all the triangulations of a convex planar point set.

Chapter 2

Definitions

In this chapter we are going to define all the central notions we are going to use throughout this thesis.

Let S be a finite set of points in the Euclidean plane, and let $G = (V, E)$ be a planar graph, *i.e.,* a graph which has a planar embedding. By a *planar embedding* of a graph H we mean a drawing of H in the Euclidean plane such that all the edges of H are represented by smooth, simple curves which do not meet except in the points that represent the vertices of H. In the following, let us assume that $V(G) = S$ and that G is embedded in the plane such the vertices of G are represented by the corresponding points in S.

Let u and v be two points in S. We can think of many different ways to define a notion of distance between u and v, but there are two distance metrics that are of particular practical interest: On the one hand, there is the *Euclidean distance* between u and v, which we will denote by $|uv|$ and which represents the length of the direct connection between u and v, as the bird flies. On the other hand, we can also look at the *shortest path distance* between u and v with respect to G, which we are going to call $\pi_G(u, v)$. For the shortest path distance, we assign each edge $e \in E(G)$ the length of the curve which represents e in the planar embedding of G. The shortest path distance represents the minimum distance we need to cover in order to travel from u to v when we are only allowed to use the edges in G.

The ratio between the shortest path distance and the Euclidean distance is called the *(relative) detour* between u and v with respect to G, which we shall denote by $\delta_G(u, v)$. Formally, the detour is defined as follows:

$$\delta_G(u, v) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } u = v, \\ \frac{\pi_G(u,v)}{|uv|}, & \text{if } u \neq v. \end{cases}$$

The convention to define $\delta_G(v, v) = 1$ for any $v \in S$ is very natural, since from the definition it is immediate that $\delta_G(u, v) \geq 1$ for every $u, v \in S$, as clearly we have $\pi_G(u, v) \geq |uv|$ for all $u, v \in S$. Further properties that can be derived directly from the definition are $\delta_G(u, v) = \delta(v, u)$ (*i.e.,* the detour is symmetric) and $\delta_G(u, v) = 1 \Leftrightarrow \{u, v\} \in E(G)$ for $u \neq v \in S$, where the second property holds only if S is in *general position*, *i.e.,* no three points in S lie on a common line.

Intuitively, the detour is a measure for the quality of the connection between u and v in G. If the detour is large, this means that we have to travel a long way along the edges in G in order to reach v from u even though the direct route would be much shorter.

In order to get a measure for the quality of the connection between any two vertices of G, it is natural to take the maximum over all the detours between pairs of vertices in G. This quantity is called the *graph theoretic dilation* of G. We will denote it by $\delta(G)$. The formal definition is this:

$$\delta(G) \stackrel{\text{def}}{=} \max_{(u,v)\in V(G)^2} \delta_G(u,v).$$

Now, given a finite set $S \subseteq \mathbb{E}^2$ of points in the Euclidean plane, the question arises how to connect these points in such a way that the graph theoretic dilation of the resulting graph is as small as possible. This quantity is called the *graph theoretic dilation* of S and will be denoted by $\delta(S)$. Formally, we write:

$$\delta(S) \stackrel{\text{def}}{=} \min_{\substack{G=(V,E) \text{ planar} \\ V(G)=S}} \delta(G).$$

Let G^* be a planar graph that achieves the optimal graph theoretic dilation for point set S. Without loss of generality, we may assume that G^* is a *triangulation* of S, since the graph theoretic dilation of G^* can only decrease when further edges are added. By a *triangulation* of point set S we mean a maximal planar subdivision H whose vertex set is S, where maximal means that $H \cup \{u,v\}$ is not planar any more for any $\{u,v\} \in \binom{S}{2} \backslash E(H)$. Here $\binom{S}{2}$ denotes the set of all unordered pairs of distinct elements in S. All the faces of a triangulation except for the unbounded face are triangles, since every simple polygon can be triangulated. For more information on triangulations of planar point sets, refer to any textbook on computational geometry, *e.g.,* [dBKOS00].

At this point, one might ask why we restrict our attention to planar graphs, and why we do not allow arbitrary graphs or *Steiner points.* A Steiner point is an additional point that is not part of the input and that is added in order to improve the dilation of the point set. However, if we allow Steiner points or edge crossings, a new problem arises: Do we still consider only the detours between the vertices of the graph, or do we also include the Steiner points and the points in which the edges cross into our consideration? If we do so, the problem becomes much more complicated, and if we don't, the problem seems somewhat implausible. Therefore – for the time being – we are going to focus our attention on planar graphs.

Chapter 3

Experimental Results

In the course of this thesis a small *Java*-program was implemented that computes the optimal graph theoretic dilation of the set of nodes of the regular n-gon, which we denote by S_n. The program works by exhaustively enumerating all possible triangulations of S_n and choosing those which achieve the minimum graph theoretic dilation. Naturally, this approach is only feasible for very small values for n, and indeed, for $n = 21$ it took about seven days on a current PC to get the desired answer.

The results which we obtained are shown in table 3.1 on page 14. The column labeled *# Minima* contains the number of triangulations that achieve the minimum graph theoretic dilation for a given value of n. A glance at this column reveals that the minimum dilation triangulation is not unique. Actually, this is not a big surprise since it is possible that there are some edges in a minimum dilation triangulation that do not participate in the shortest path between the two vertices of any maximum detour pair. Sometimes, these edges can be flipped without affecting the maximum detour, and hence the minimum dilation triangulation is not unique. It is interesting to note that for $n = 7$ all triangulations of S_7 are optimal.

The column labeled *Total #* contains the total number of possible triangulations of the regular n-gon. One can show that this number, and in fact the number of triangulations of any planar convex set of n points, equals C_{n-2}, where C_n denotes the n-th *Catalan-number* for $n \in N_0$, which is defined as $C_n \stackrel{\text{def}}{=} \frac{1}{n+1}\binom{2n}{n}$ (see *e.g.,* [Aig01] for a proof of this statement). It is well known that the Catalan numbers grow exponentially with n.

% Minima shows what percentage of the triangulations achieve the optimum graph theoretic dilation. As expected, this number becomes negligible very soon. *Ticks* denotes the number of clock ticks the program needed in order to compute the desired information for a given n. It is a measure for the running time. This quantity was added in order to demonstrate the dramatic effect of the combinatorial explosion on the performance of the program.

Finally, the field *Minimum Dilation* shows the minimum graph theoretic dilation of the regular n-gon $\delta(S_n)$ as it was computed by our program. Please be aware that the results that are shown in this table should be taken with a certain caution. Since our program relies on double precision floating point arithmetic, the results are not entirely accurate. However, table 3.1 is very useful as a guide to give some intuition on the behavior of the graph theoretic dilation.

Perhaps the most astonishing feature of the results shown in table 3.1 is that the graph theoretic dilation does not show any clear monotonicity properties. One might have expected that $\delta(S_n)$ grows monotonically with n, but – at least for small values of n – this is not the case. One might also assume that the graph theoretic dilation increases as n is multiplied with an integral factor, since this would mean that points are added outside the convex hull of S_n. However, this conjecture also turns out to be false, as the values of $\delta(S_4)$, $\delta(S_8)$, $\delta(S_{12})$, and

$\delta(S_{16})$ demonstrate. Nonetheless, the values in table 3.1 suggest that $\delta(S_n)$ "tends to grow" as n increases, and it should be expected that $\delta(S_n)$ does not change too much if n is large. We will use this intuition in chapter 7 where we devise an algorithm that quickly approximates the minimum detour triangulation.

Figures 3.1–3.4 show some of the minimum dilation triangulations that were computed by our *Java*-program. Note that these triangulations tend to be very regular and symmetric, a fact which we will exploit in section 4.3 where we consider a regular triangulation of S_n in order to devise an upper bound for all n which are of the form $n = 3 \cdot 2^i$ and in appendix A.2 where we present a heuristic that tries to approximate the minimum detour triangulation by a symmetric triangulation.

Table 3.1: Experimental Results for small values of n.

n	**# Minima**	**Total #**	**% Minima**	**Ticks**	**Minimum Dilation**
4	2	2	100	0	1.41422
5	5	5	100	0	1.23607
6	8	14	57	0	1.36603
7	42	42	100	0	1.33513
8	16	132	12	0	1.41422
9	87	429	20	10	1.34730
10	20	1,430	1	60	1.39681
11	44	4,862	≤ 1	240	1.37704
12	500	16,796	≤ 1	1,031	1.38367
13	1,248	58,786	≤ 1	4,406	1.39122
14	784	208,012	≤ 1	19,488	1.40533
15	660	742,900	≤ 1	81,478	1.40898
16	2,000	2,674,440	≤ 1	355,762	1.40925
17	6,732	9,694,845	≤ 1	1,499,897	1.40845
18	6	35,357,670	≤ 1	6,418,740	1.38170
19	4,560	129,644,790	≤ 1	26,149,270	1.40989
20	5,040	477,638,700	≤ 1	112,332,436	1.41422
21	18,816	1,767,263,190	≤ 1	466,895,338	1.41611

Figure 3.1: Examples of minimum dilation triangulations for S_{14} and S_{15}. We have $\delta(S_{14}) \approx 1.40533$ and $\delta(S_{15}) \approx 1.40898$. A maximum detour pair and a shortest path between its two vertices are shown in gray.

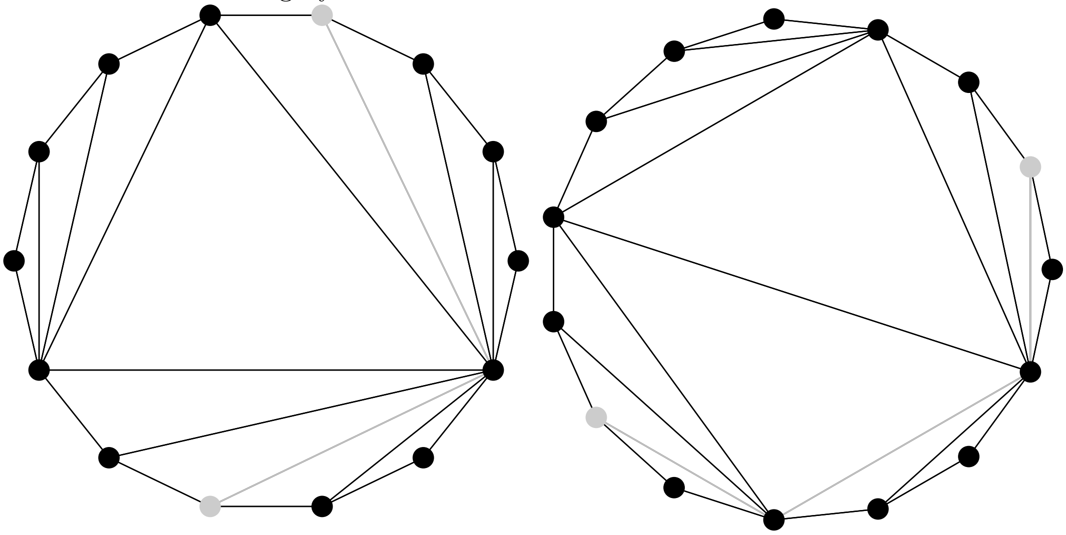


Figure 3.2: Examples of minimum dilation triangulations for S_{16} and S_{17}. We have $\delta(S_{16}) \approx 1.40925$ and $\delta(S_{17}) \approx 1.40845$. A maximum detour pair and a shortest path between its two vertices are shown in gray.

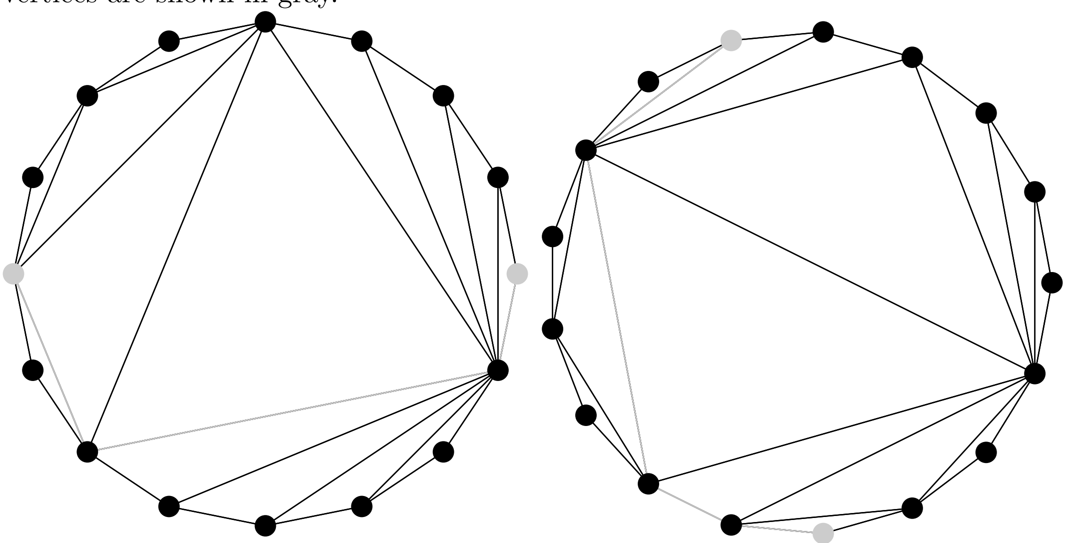


Figure 3.3: Examples of minimum dilation triangulations for S_{18} and S_{19}. We have $\delta(S_{18}) \approx 1.38170$ and $\delta(S_{19}) \approx 1.40989$. A maximum detour pair and a shortest path between its two vertices are shown in gray.

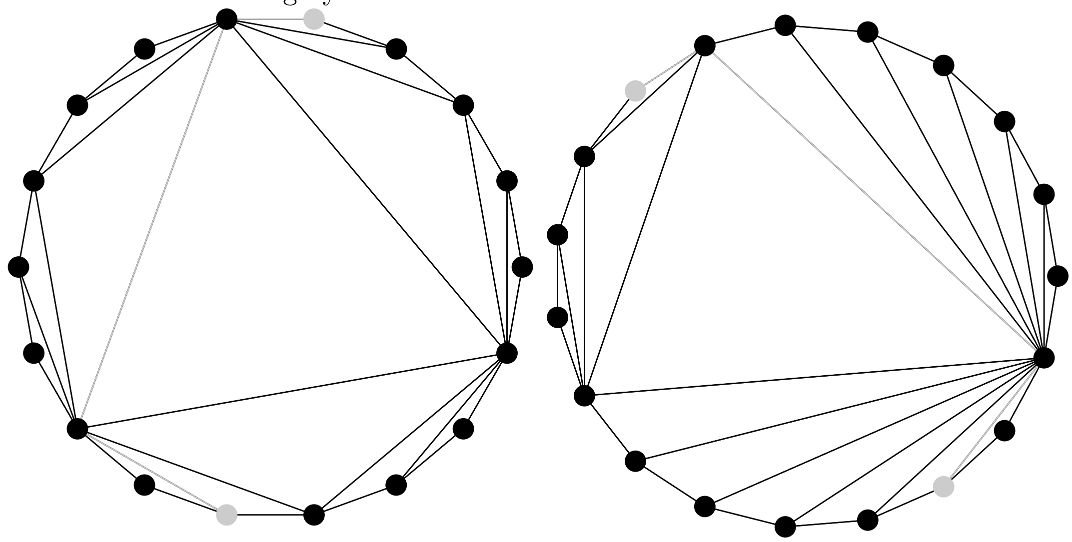
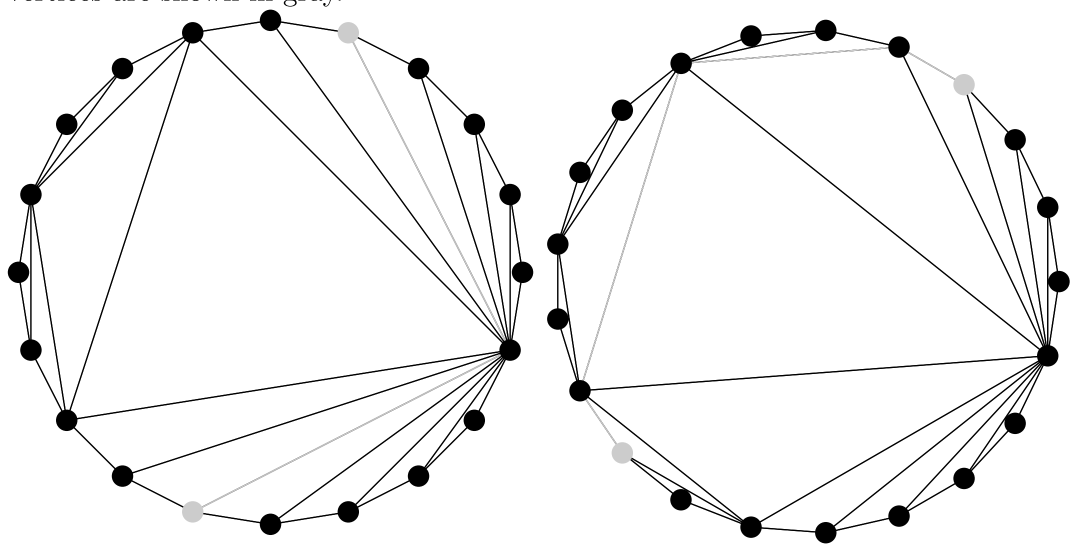


Figure 3.4: Examples of minimum dilation triangulations for S_{20} and S_{21}. We have $\delta(S_{20}) \approx 1.41422$ and $\delta(S_{21}) \approx 1.41611$. A maximum detour pair and a shortest path between its two vertices are shown in gray.



Chapter 4

Upper Bounds for the Regular n-Gon

Let $S_n = \{s_0, s_1, \ldots, s_{n-1}\}$ be the set of nodes of a regular n-gon in counter-clockwise order. Without loss of generality, we may assume that the radius of S_n is 1, because in a regular n-gon of radius r the distance between any two nodes is r times the corresponding distance in a regular n-gon of radius 1, and this factor cancels when we consider the detour.

In this chapter we shall try to develop some intuition for the graph theoretic dilation of S_n. In order to do this, we will derive a simple upper bound on the graph theoretic dilation of any triangulation of S_n and afterwards look at the graph theoretic dilation of two very simple triangulations of S_n, namely the *canonical triangulation* K_n and the *star triangulation* R_n.

More precisely, in section 4.1 we will show that *any* triangulation of S_n has graph theoretic dilation at most $\frac{\pi}{2}$. This follows easily from the definition. Furthermore, we will also show that this bound is asymptotically tight, which means that there exist triangulations of S_n that actually achieve this upper bound asymptotically.

In section 4.2 we will give an estimate for the graph theoretic dilation of the canonical triangulation of S_n, *i.e.,* the triangulation of S_n in which every vertex is connected to s_0. Even though this is a very simple triangulation, it turns out to be nontrivial to compute its graph theoretic dilation, and we will not succeed in obtaining a precise value (and it seems that the precise value cannot be represented in closed form). The value we obtain can be considered as an upper bound on the graph theoretic dilation of S_n.

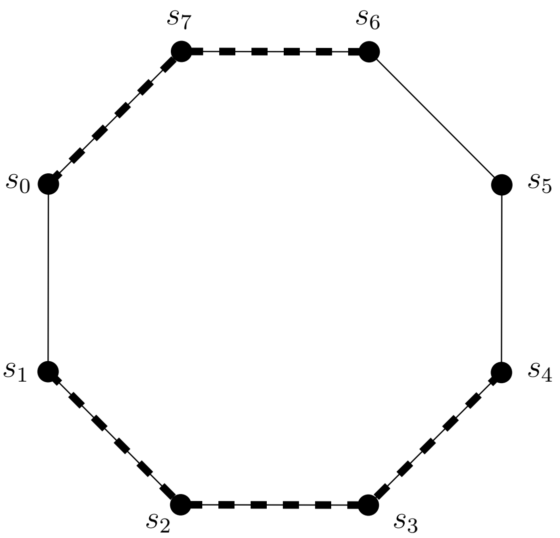
Finally, in section 4.3 we will consider the special case that n is of the form $n = 3 \cdot 2^i$ for an $i \in \mathbb{N}_0$. In this case we can give a good upper bound on the graph theoretic dilation of S_n by considering the so called regular triangulation of S_n, *i.e.,* the triangulation of S_n that is "as symmetric as possible".

In the course of the following discussion we will often need to refer to the number of vertices that lie between a given pair of nodes on the convex hull of S_n. Let $s_a, s_b \in S_n$. Then we define the *convex hull distance* between s_a and s_b, which we will denote by $\Delta_{S_n}(s_a, s_b)$, as

$$\Delta_{S_n}(s_a, s_b) \stackrel{\text{def}}{=} \min\{|b-a|, n-|b-a|\}.$$

Intuitively, $\Delta_{S_n}(s_a, s_b)$ counts the minimum number of "hops" we need to make when traveling from s_a to s_b along the convex hull of S_n (see figure 4.1).

Figure 4.1: An example of convex hull distance for S_8. The convex hull distance between s_1 and s_4 is 3, the convex hull distance between s_0 and s_6 is 2.



4.1 An Upper Bound on the Graph Theoretic Dilation of Any Triangulation of the Regular n-Gon

Let T be any triangulation of S_n. In this section we shall show that $\delta(T) \leq \frac{\pi}{2}$. The main idea behind this upper bound is that between any two points of S_n there always exists a path that goes along the convex hull of S_n, as the edges that constitute the border of the convex hull are contained in any triangulation. More precisely, any two points in S_n can be connected through a path that uses at most $\lceil \frac{n}{2} \rceil$ nodes (the start and end node included), since the points in S_n are uniformly distributed on the unit circle.

Hence, let s_a and s_b be two distinct points in S_n, and let $\Delta = \Delta_{S_n}(s_a, s_b)$ be the convex hull distance between s_a and s_b. Then the Euclidean distance between s_a and s_b is $2\sin\left(\frac{\Delta\pi}{n}\right)$, while the length of the path along the convex hull is $2\Delta\sin\left(\frac{\pi}{n}\right)$ (see figure 4.2).

Therefore, we can bound the detour between s_a and s_b as follows:

$$\delta_T(s_a, s_b) \leq \frac{\Delta \sin\left(\frac{\pi}{n}\right)}{\sin\left(\Delta\frac{\pi}{n}\right)}.$$

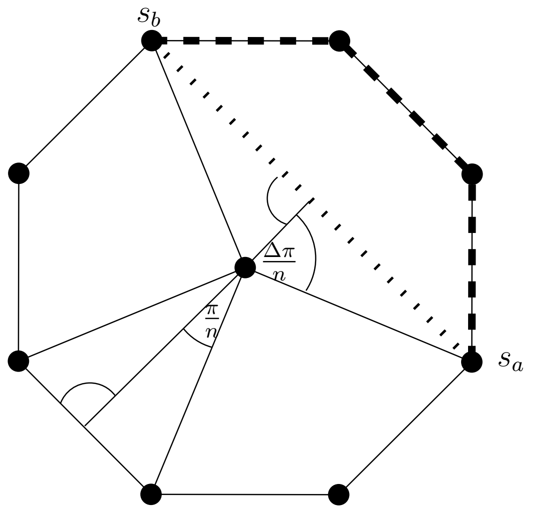
Since we are looking for an upper bound for the graph theoretic dilation of T, we need to find the value of Δ for which the function $\Delta \mapsto \Delta\sin\left(\frac{\pi}{n}\right) / \sin\left(\Delta\frac{\pi}{n}\right)$ takes on its maximum value, where Δ ranges from 1 to $\lfloor \frac{n}{2} \rfloor$ as we noted above. Unfortunately, it is not immediately obvious where this maximum lies. Therefore, now, as well as in the future, we shall need to employ some calculus in order to obtain the desired results. The behavior of the upper bound function is settled by the following claim:

Claim 4.1 *For $n \in \mathbb{N}$, let $\alpha = \frac{\pi}{n}$ and $f : \{1, \ldots, \lfloor n/2 \rfloor\} \to \mathbb{R}$ be given by*

$$f(\Delta) \stackrel{\text{def}}{=} \frac{\Delta}{\sin(\Delta\alpha)}.$$

Then f is monotonically increasing.

Figure 4.2: The Euclidean distance between s_a and s_b is $2\sin\left(\frac{\Delta\pi}{n}\right)$ (bold dotted line). The length of the path along the convex hull (bold dashed line) is $2\Delta\sin\left(\frac{\pi}{n}\right)$, because it consists of Δ line segments each of which has length $2\sin\left(\frac{\pi}{n}\right)$.



Proof: For the proof, we shall extend f to the domain $[1, n/2]$ and use calculus to show that f is monotonically increasing on this domain. Then the result follows immediately.

The derivative of f can be computed as follows:

$$f'(\Delta) = \frac{\sin(\Delta\alpha) - \Delta\alpha\cos(\Delta\alpha)}{\sin^2(\Delta\alpha)}.$$

In order to show that f is monotonically increasing we need to verify that the numerator of the derivative is positive. If we let $\mu = \Delta\alpha$, we can deduce that $\frac{\pi}{n} \leq \mu \leq \frac{\pi}{2} < 2$. From the power series expansion of the sine and the cosine function it now follows that

$$\sin(\mu) \geq \mu - \frac{\mu^3}{3!} \overset{(1)}{>} \mu\left(1 - \frac{\mu^2}{2!} + \frac{\mu^4}{4!}\right) \geq \mu\cos(\mu),$$

where inequality (1) is due to the fact that $0 < \mu < 2$.

Thus, $f'(\Delta) > 0$, and the claim follows. ■

Consequently, the upper bound on the detour is maximal for $\Delta = \frac{n}{2}$ (note that we do not need to worry about integrality here, as we are merely interested in an upper bound). Hence, all that remains to be done is to compute the value of the upper bound for $\Delta = \frac{n}{2}$. This is done as follows:

$$\begin{aligned} \frac{\frac{n}{2}\sin\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{2}\right)} &= \frac{\pi}{2}\frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} \\ &\nearrow \frac{\pi}{2}, \qquad \text{as } n \to \infty, \end{aligned}$$

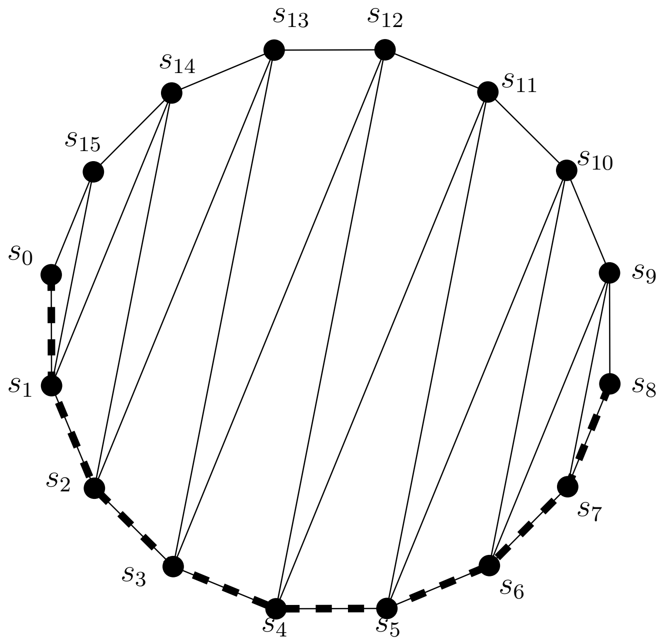
since $\sin x/x \to 1$ as $x \to 0$ and by an argument that is almost identical to the proof of claim 4.1 we can show that $\sin x/x$ is monotonically increasing as x approaches 0.

Therefore, we find that for any $s_a, s_b \in_n S$ we have

$$\delta_T(s_a, s_b) \leq \frac{\pi}{2}.$$

Let us summarize this result in our first

Figure 4.3: Worst case triangulation W_{16} for S_{16}. The detour between s_0 and s_8 is about 1.561.



Theorem 4.2 *Let S_n be the set of nodes of a regular n-gon. Then any triangulation of S_n has graph theoretic dilation at most $\frac{\pi}{2}$.*

It is not hard to show that this upper bound is asymptotically tight, *i.e.,* there are triangulations of S_n whose graph theoretic dilation approaches $\frac{\pi}{2}$ as n goes to infinity. We shall describe such a *worst case triangulation* for even n. The construction can easily be adapted for the case that n is odd.

Let W_n be the triangulation of S_n whose edge set $E(W_n)$ is given by

$$\begin{aligned} E(W_n) &= \{\{s_i, s_{i+1}\} \mid 0 \leq i \leq n-2\} \cup \{\{s_0, s_{n-1}\}\} \cup \\ &\quad \left\{\{s_i, s_{n-i}\} \middle| 1 \leq i < \frac{n}{2}\right\} \cup \left\{\{s_i, s_{n-i-1}\} \middle| 1 \leq i < \frac{n}{2}\right\}. \end{aligned}$$

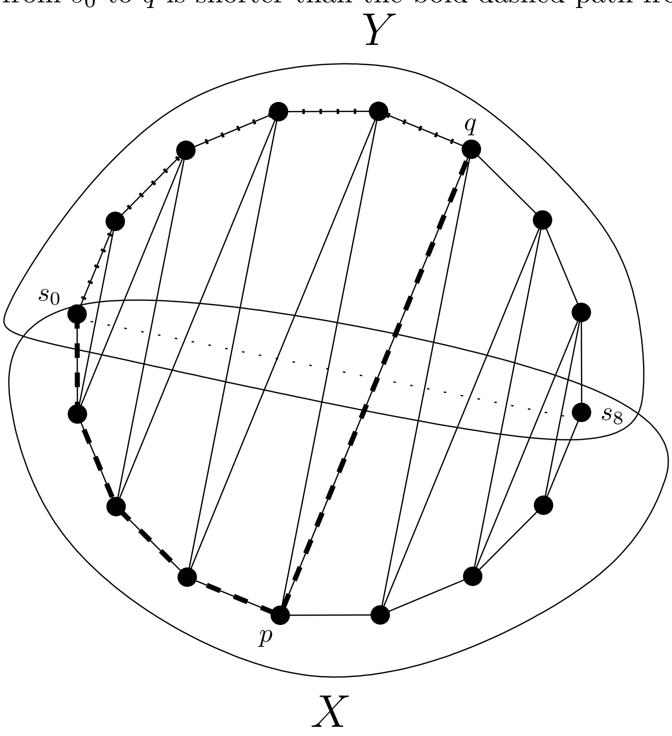
An example for $n = 16$ is shown in figure 4.3.

The diagonal between s_0 and $s_{\frac{n}{2}}$ partitions S_n into two parts, say X and Y. Then it is clear that a shortest path Π between s_0 and $s_{\frac{n}{2}}$ cannot change between X and Y. [Suppose it does, and let $p \to q$ be the first edge on the path such that $p \in X$ and $q \notin X$ or vice versa. We shall consider the case that $p \in X$ and $q \notin X$, the other case is analogous. In this case it follows that up to vertex p, Π goes along the convex hull in X. However, it is easy to see that line segment $\overline{pq}$ is longer than the line segments on the convex hull, since all of the diagonals between X and Y are longer than line segment $\overline{s_1 s_{n-1}}$ which is the base of an isosceles triangle whose other two sides are line segments on the convex hull and whose apex angle is obtuse. Thus, it would have been shorter to follow the convex hull in Y up to q, contradicting the fact that that Π is a shortest path (see figure 4.4).]

Thus, any shortest path between s_0 and $s_{\frac{n}{2}}$ follows the convex hull of S_n and hence has length $\frac{n}{2} \cdot 2\sin\left(\frac{\pi}{n}\right)$. Since the Euclidean distance between s_0 and $s_{\frac{n}{2}}$ is 2, it follows that

$$\lim_{n\to\infty} \delta_{W_n}\left(s_0, s_{\frac{n}{2}}\right) = \lim_{n\to\infty} \frac{n}{2} \sin\left(\frac{\pi}{n}\right),$$

Figure 4.4: Since $\overline{pq}$ is longer than any line segment on the convex hull, the bold dotted path from s_0 to q is shorter than the bold dashed path from s_0 over p to q.



and we have already seen that this limit equals $\frac{\pi}{2}$. Thus, we have shown that W_n is asymptotically a worst case triangulation of S_n.

4.2 The Graph Theoretic Dilation of the Canonical Triangulation

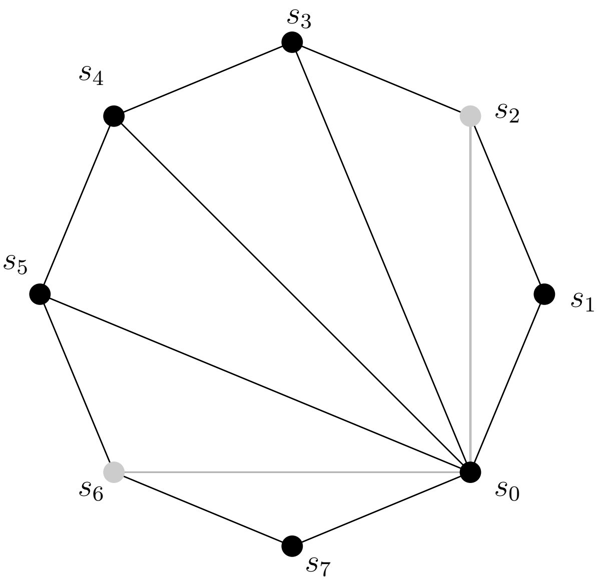
In this section we are going to estimate the graph theoretic dilation of the canonical triangulation, which will give us an upper bound on the graph theoretic dilation of S_n. The canonical triangulation K_n of S_n is defined as the triangulation of S_n in which s_0 is connected to every other node by a line segment (see figure 4.5). Elementary trigonometry tells us that for $0 \leq a \leq n-1$ the length of line segment $\overline{s_0 s_a}$ equals $2\sin\left(\frac{\pi a}{n}\right)$.

Now let $(s_a, s_b) \in S_n^2$ be a pair of nodes such that $a < b$. As we have already seen in the previous section, the Euclidean distance between s_a and s_b is $2\sin\left(\frac{(b-a)\pi}{n}\right)$. In order to compute the detour between s_a and s_b we also need to know the length of the shortest path between them. There are exactly two possibilities: Either the shortest path between s_a and s_b in K_n follows the convex hull of the triangulation, or it has the form $s_a \to s_0 \to s_b$. This is true, since if s_0 is a node on the shortest path, the triangle inequality tells us that the shortest path must be $s_a \to s_0 \to s_b$, and if s_0 is not on the shortest path, then the path must be $s_a \to s_{a+1} \to \ldots \to s_{b-1} \to s_b$, since no vertex appears twice on a shortest path.

In the former case the length of the shortest path is $2\sin\left(\frac{\pi a}{n}\right) + 2\sin\left(\frac{\pi b}{n}\right)$, as this is the combined length of line segments $\overline{s_0 s_a}$ and $\overline{s_0 s_b}$. In the latter case it is $2(b-a)\sin\left(\frac{\pi}{n}\right)$, since the length of a single line segment on the convex hull equals $2\sin\left(\frac{\pi}{n}\right)$, as we have seen already in the previous section.

Hence, the detour between s_a and s_b is given by

Figure 4.5: The canonical triangulation K_8 for S_8. Its graph theoretic dilation is about 1.4142. The maximum detour occurs between s_2 and s_6.



$$\delta_{K_n}(s_a, s_b) = \frac{\min\left\{2\sin\left(\frac{\pi a}{n}\right) + 2\sin\left(\frac{\pi b}{n}\right), 2(b-a)\sin\left(\frac{\pi}{n}\right)\right\}}{2\sin\left(\frac{(b-a)\pi}{n}\right)}.$$

As we are interested in the graph theoretic dilation of K_n, we need to find the values of a and b for which $\delta_{K_n}(s_a, s_b)$ achieves its maximum value. Note that the expressions $2(b-a)\sin(\pi/n)$ and $2\sin((b-a)\pi/n)$ do not depend on the actual values of a and b, but only on their difference. Hence, it seems promising to substitute $\Delta = b - a$. For the time being, let us fix the value of Δ. This means that we are looking for

$$\begin{aligned}
&\max_{\substack{1\leq a<b\leq n-1 \\ b-a=\Delta}} \frac{1}{\sin\left(\frac{\Delta\pi}{n}\right)} \min\left\{\sin\left(\frac{\pi a}{n}\right) + \sin\left(\frac{\pi b}{n}\right), \Delta\sin\left(\frac{\pi}{n}\right)\right\} \\
= \; & \frac{1}{\sin\left(\frac{\Delta\pi}{n}\right)} \min\left\{\max_{\substack{1\leq a<b\leq n-1 \\ b-a=\Delta}} \left\{\sin\left(\frac{\pi a}{n}\right) + \sin\left(\frac{\pi(a+\Delta)}{n}\right)\right\}, \Delta\sin\left(\frac{\pi}{n}\right)\right\}.
\end{aligned}$$

It follows that we need to compute for which values of a and b the maximum is achieved. We will do this in the next

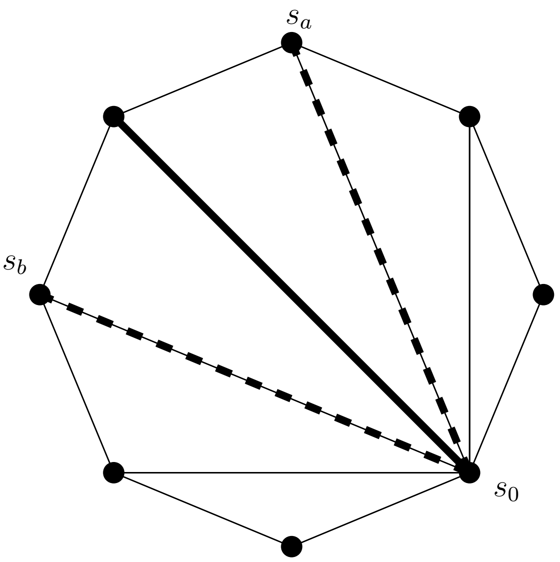
Claim 4.3 *Let $\lambda \in (0, \pi)$ and $f : (0, \pi - \lambda) \to \mathbb{R}$ be defined as*

$$f(x) \stackrel{\text{def}}{=} \sin x + \sin(x + \lambda).$$

Then f is maximal for

$$x = \frac{\pi - \lambda}{2}.$$

Figure 4.6: The length of the path $s_a \to s_0 \to s_b$ is maximal if (s_a, s_b) is symmetric with respect to the diagonal through s_0.



Proof: We have that $f'(x) = \cos x + \cos(x+\lambda)$, and therefore

$$\begin{aligned} f'(x) = 0 \quad &\Leftrightarrow \quad \cos x = -\cos(x+\lambda) \\ &\Leftrightarrow \quad \frac{\pi}{2} - x = x + \lambda - \frac{\pi}{2} \\ &\Leftrightarrow \quad x = \frac{\pi - \lambda}{2}, \end{aligned}$$

because $\cos\left(x - \frac{\pi}{2}\right) = \sin x$ is an odd function and all the values we consider lie in the interval $(0,\pi)$. This tells us that $x_0 \stackrel{\text{def}}{=} \frac{\pi-\lambda}{2}$ is a maximum, since $f'(x) > 0$ for $x \in (0, x_0)$ and $f'(x) < 0$ for $x \in (x_0, \pi)$. ∎

Consequently, since

$$\frac{\pi a}{n} = \frac{\pi}{2} - \frac{\pi\Delta}{2n} \Leftrightarrow a = \frac{n}{2} - \frac{\Delta}{2},$$

we can conclude that the length of the path $s_a \to s_0 \to s_b$ where s_a and s_b have a fixed convex hull distance Δ is maximal if (s_a, s_b) is symmetric with respect to the diagonal through s_0 (see figure 4.6). Note that no problem arises if $\frac{n}{2} - \frac{\Delta}{2}$ is not integral, since we are merely interested in an upper bound.

Now we can eliminate a and b completely from our consideration and obtain an upper bound that depends solely on Δ:

$$\begin{aligned} \max_{\substack{1 \le a < b \le n-1 \\ b-a=\Delta}} \delta_K(s_a, s_b) \quad &\le \quad \frac{1}{\sin\left(\frac{\pi\Delta}{n}\right)} \min\left\{\sin\left(\frac{\pi}{2} - \frac{\pi\Delta}{2n}\right) + \sin\left(\frac{\pi}{2} + \frac{\pi\Delta}{2n}\right), \Delta \sin\left(\frac{\pi}{n}\right)\right\} \\ &= \quad \frac{1}{\sin\left(\frac{\pi\Delta}{n}\right)} \min\left\{2\cos\left(\frac{\pi\Delta}{2n}\right), \Delta \sin\left(\frac{\pi}{n}\right)\right\}, \end{aligned}$$

since $\sin\left(\frac{\pi}{2} \pm x\right) = \sin\left(\frac{\pi}{2}\right)\cos x \pm \cos\left(\frac{\pi}{2}\right)\sin x = \cos x$ for all $x \in \mathbb{R}$.

The only thing that is left to do now is to determine the value of Δ for which the above expression achieves its maximum value. The strategy will be to show that one of the two functions in the above expression is increasing, while the other function decreases. Then the y-coordinate of the point in which the graphs of these functions intersect will be the value of the maximum.

In the last section, we have already seen that the function $\Delta \mapsto \Delta \sin\left(\frac{\pi}{n}\right) / \sin\left(\frac{\pi\Delta}{n}\right)$ grows with Δ (claim 4.1 on page 18).

Thus, let us focus our attention on the behavior of the other function, namely

$$f(\Delta) \stackrel{\text{def}}{=} \frac{2\cos\left(\frac{\pi\Delta}{2n}\right)}{\sin\left(\frac{\pi\Delta}{n}\right)}, \qquad \text{where } 1 \leq \Delta \leq n-1.$$

We shall now verify that this function decreases with Δ. Elementary calculus yields

$$f'(\Delta) = \frac{-\sin\left(\frac{\pi\Delta}{n}\right)\frac{\pi}{n}\sin\left(\frac{\pi\Delta}{2n}\right) - 2\cos\left(\frac{\pi\Delta}{2n}\right)\frac{\pi}{n}\cos\left(\frac{\pi\Delta}{n}\right)}{\sin^2\left(\frac{\pi\Delta}{n}\right)}.$$

For the sake of readability, let us substitute $\mu \stackrel{\text{def}}{=} \frac{\pi\Delta}{n}$. We find that $0 < \mu < \pi$, and hence

$$\begin{aligned}
& -\sin\left(\frac{\mu}{2}\right)\sin\mu - 2\cos\left(\frac{\mu}{2}\right)\cos\mu \leq 0 \\
\Leftrightarrow \quad & -\sin\left(\frac{\mu}{2}\right)\sin\mu \leq 2\cos\left(\frac{\mu}{2}\right)\cos\mu \\
\Leftrightarrow \quad & -\sqrt{\frac{1-\cos\mu}{2}}\sin\mu \leq 2\sqrt{\frac{1+\cos\mu}{2}}\cos\mu \\
\Leftrightarrow \quad & -\frac{1-\cos\mu}{2}\sin\mu \ \leq 2\frac{\sqrt{\sin^2\mu}}{2}\cos\mu \\
\Leftrightarrow \quad & \frac{\cos\mu - 1}{2} \leq \cos\mu \\
\Leftrightarrow \quad & -1 \leq \cos\mu,
\end{aligned}$$

which is definitely a true statement. In the above calculation we used the well known facts that $|\sin(x/2)| = \sqrt{(1-\cos x)/2}$, $|\cos(x/2)| = \sqrt{(1+\cos x)/2}$, and the trigonometric Pythagorean theorem $(1+\cos x)(1-\cos x) = \sin^2 x$ for all $x \in \mathbb{R}$. Furthermore, for the values of μ we are interested in we have that $\sin\mu, \sin(\mu/2), \cos(x/2) > 0$, as $0 < \mu < \pi$.

Hence, we can conclude that the function $\Delta \mapsto 2\cos\left(\frac{\pi\Delta}{2n}\right)/\sin\left(\frac{\pi\Delta}{n}\right)$ decreases monotonically for $1 \leq \Delta \leq n-1$.

In order to get the desired upper bound, we would now like to compute the value of Δ for which we have

$$2\cos\left(\frac{\pi\Delta}{2n}\right) = \Delta\sin\left(\frac{\pi}{n}\right).$$

Unfortunately, we do not know how to do this. However, by dint of numerical methods, we can arrive at a quite accurate estimation. Let $n \geq 31$. We claim that in this case it is true that $2\cos\left(\frac{\pi\Delta}{2n}\right) \leq \Delta\sin\left(\frac{\pi}{n}\right)$ for $\Delta \geq 0.471n$, *i.e.,* $0.471n$ is an estimate of the value of Δ for which the two functions meet.

To see why the claim holds, note that it follows from the power series expansion of the sine and cosine function that

$$\begin{aligned}
0.471n\sin\left(\frac{\pi}{n}\right) &\geq 0.471\left(\pi-\frac{\pi^3}{6n^2}+\frac{\pi^5}{120n^4}-\frac{\pi^7}{5040n^6}\right)\\
&\geq 0.471\left(\pi-\frac{\pi^3}{6\cdot 31^2}+\frac{\pi^5}{120\cdot 31^4}-\frac{\pi^7}{5040\cdot 31^6}\right)\\
&\geq 1.477158\\
&\geq 2-0.471^2\frac{\pi^2}{2^2}+0.471^4\frac{\pi^4}{12\cdot 2^4}-0.471^6\frac{\pi^6}{360\cdot 2^6}+0.471^8\frac{\pi^8}{20160\cdot 2^8}\\
&\geq 2\cos\left(0.471\frac{\pi}{2}\right).
\end{aligned}$$

Now, since the function $\Delta \mapsto \Delta\sin\left(\frac{\pi}{n}\right)/\sin\left(\frac{\Delta\pi}{n}\right)$ is monotonically increasing and the function $\Delta \mapsto 2\cos\left(\frac{\pi\Delta}{2n}\right)/\sin\left(\frac{\Delta\pi}{n}\right)$ is monotonically decreasing, it follows that an upper bound for the graph theoretic dilation of K_n is given by $0.471n\sin\left(\frac{\pi}{n}\right)/\sin\left(0.471\frac{\pi}{n}\right)$.

Consequently, since we have already seen in the previous section that $\frac{n}{2}\sin\left(\frac{\pi}{n}\right) \nearrow \frac{\pi}{2}$ – and hence $n\sin\left(\frac{\pi}{n}\right) \nearrow \pi$ – we can upperbound the graph theoretic dilation of the canonical triangulation by $0.471\pi/\sin(0.471\pi)$, which is approximately 1.48586. (This bound has been proven for $n \geq 31$, but by a straightforward yet tedious calculation it can also be verified for smaller values of n.)

We note that already for $\Delta = 0.47n$ we have $2\cos\left(0.47\frac{\pi}{2}\right) \geq 0.47n\sin(\frac{\pi}{n})$ for n large enough, as can be shown in the same way as above. Furthermore, we have $0.47\pi/\sin(0.47\pi) \approx 1.48313$. Thus, asymptotically, our estimate is off by at most $3\cdot 10^{-3}$.

In conclusion, we have just proved

Theorem 4.4 *Let S_n be the set of nodes of a regular n-gon, and let K_n be the canonical triangulation of S_n. Then the graph theoretic dilation of K_n can be estimated by*

$$\frac{0.471\pi}{\sin(0.471\pi)} \approx 1.48586.$$

In particular, this implies an upper bound of 1.48586 on the graph theoretic dilation of the minimum dilation triangulation of S_n.

4.3 An Upper Bound for $n = 3\cdot 2^i$

Intuitively, it seems clear that the minimum dilation triangulation should be very regular, and this expectation is also corroborated by the structure of the minimum dilation triangulations for small values of n which we saw in chapter 3. In appendix A.2 we will use this idea in order to design a simple heuristic. In this section we will give a simple symmetric triangulation for those S_n where $n = 3\cdot 2^i$ with $i \in \mathbb{N}_0$. The triangulation is as follows: we start with an equilateral triangle and continue by adding line segments to the points that lie exactly halfway between the endpoints of the existing line segments until the triangulation is complete. We will call this triangulation the *star triangulation* of S_n and denote it by R_n. In figure 4.7 we show star triangulations R_{12} and R_{24} as an example, and table 4.1 shows the values of the graph theoretic dilation of the star triangulations for some small values of n. The figure also shows the two vertices of a maximum detour pair. We see that these two vertices are nearly diametrically opposite, and that one vertex is close to a vertex of the central triangle, while the other vertex is close to a vertex of the opposite second largest triangle (see also figure A.3 on page 70).

Unfortunately, it is not clear how to compute the graph theoretic dilation of the star triangulation of S_n for $n = 2\cdot 3^i$ explicitly. However, with a simple trick it is possible to bound

Figure 4.7: Star triangulations R_{12} and R_{24} for S_{12} and S_{24}. We have $\delta(R_{12}) \approx 1.38367$ and $\delta(R_{24}) \approx 1.40134$

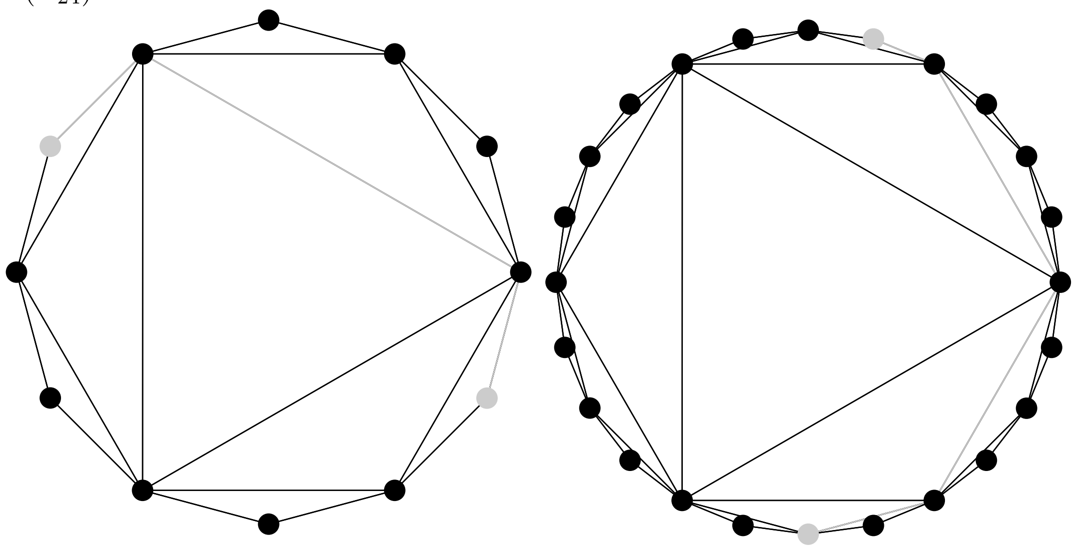


Table 4.1: Graph theoretic dilation of the star triangulation for some values of n

n	6	12	24	48	96
$\delta(T)$	1.36603	1.38367	1.40134	1.45442	1.45474
n	192	384	768	1536	3072
$\delta(T)$	1.45474	1.45474	1.45672	1.45812	1.45815

the dilation for any star triangulation: Observe that the value of the graph theoretic dilation of the star triangulation is monotonically increasing. This is clear, because as n increases, we add nodes outside the existing triangulation and connect them to the existing nodes. Thus, the detour between the existing nodes cannot change, and hence the graph theoretic dilation can only increase.

Let R_n be the star triangulation for an $n > 3072$. We have

$$\delta(R_n) = \max_{a',b' \in V(S_n)} \frac{\pi(a',b')}{2\sin(\angle(a',b')/2)}, \tag{4.1}$$

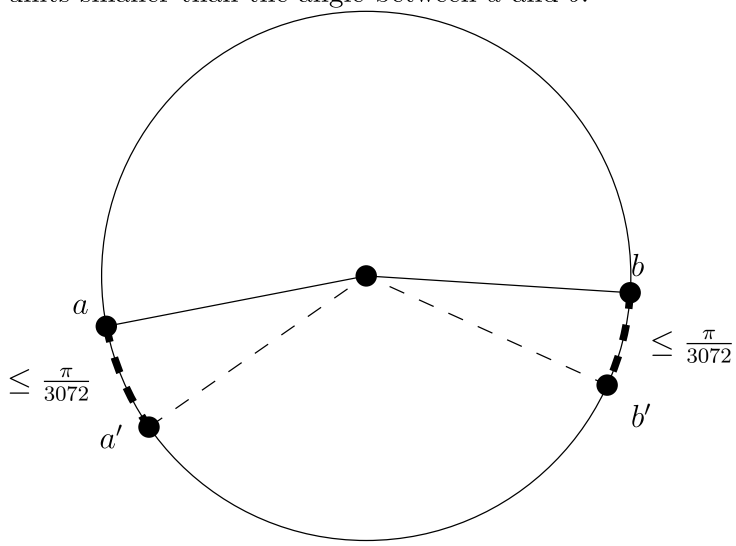
where $\angle(a',b')$ denotes the angle between a' and b'.

It now follows that

$$\max_{a',b' \in V(S_n)} \frac{\pi(a',b')}{2\sin(\angle(a',b')/2)} \leq \max_{a,b \in V(S_{3072})} \frac{\pi(a,b) + 2\frac{2\pi}{6144}}{2\sin\left(\angle(a,b)/2 - \frac{2\pi}{6144}\right)}.$$

To see why this inequality holds, consider a pair of vertices $a', b' \in S_n$, and let $a, b \in S_{3072}$ be the vertices in S_{3072} that are closest to a' and b', respectively. We will assume that a' and b' are far enough apart such that $a \neq b$, an assumption that is justified later. It follows that the shortest path distance between a' and b' is at most $2\frac{2\pi}{6144}$ units longer than the shortest path between a and b, since a' can be reached from a by a path of length at most $\frac{2\pi}{6144}$, which is half the length of an arc of radius 1 between two successive points in S_{3072}, and the same holds for b. Similarly, the Euclidean distance between a and b is at least $2\sin\left(\angle(a,b)/2 - \frac{2\pi}{6144}\right)$, since $\angle(a',b') \geq \angle(a,b) - 2\frac{2\pi}{6144}$ (see figure 4.8).

Figure 4.8: The shortest path distance between a' and b' is at most $2\pi/3072$ units larger than the shortest path distance between a and b and the angle between a' and b' is at most $2\pi/3072$ units smaller than the angle between a and b.



By means of a computer program we can calculate that $\delta(R_{3072}) < 1.45815$, and consequently for any $a, b \in S_{3072}$, we have $\pi(a,b) < 1.45815\,|ab| = 1.45815 \cdot 2\sin(\angle(a,b)/2)$.

Thus, we have that

$$\max_{a,b\in V(S_{3072})} \frac{\pi(a,b) + 2\frac{2\pi}{6144}}{2\sin\left(\angle(a,b)/2 - \frac{2\pi}{6044}\right)} \leq \max_{a,b\in V(S_{3072})} \frac{1.45815\sin\left(\angle(a,b)/2\right) + \frac{\pi}{3072}}{\sin\left(\angle(a,b)/2 - \frac{\pi}{3072}\right)}.$$

We will show that this function of $\angle(a,b)$ is monotonically decreasing. To do this, consider the function $f : \mathbb{R} \to \mathbb{R}$ which is given by

$$f(\gamma) = \frac{a\sin\gamma + b}{\sin(\gamma - c)},$$

for certain $a, b, c > 0$.

The derivative of f can be computed as follows:

$$\begin{aligned} f'(\gamma) &= \frac{\sin(\gamma-c)\,a\cos\gamma - \cos(\gamma-c)\,(a\sin\gamma + b)}{\sin^2(\gamma-c)} \\ &= a\frac{\left(\sin\gamma\cos\gamma\cos c - \cos^2\gamma\sin c\right)}{\sin^2(\gamma-c)} \\ &\quad + a\frac{\left(-\sin\gamma\cos\gamma\cos c - \sin^2\gamma\sin c\right)}{\sin^2(\gamma-c)} \\ &\quad - \frac{b\cos(\gamma-c)}{\sin^2(\gamma-c)} \\ &= \frac{-a\sin c - b\cos(\gamma-c)}{\sin^2(\gamma-c)}, \end{aligned}$$

and for the values of a, b, c, and γ that we are interested in (*i.e.*, $a = 1.45815$, $b = \pi/3072$, $c = \pi/3072$, $0 < \gamma < \pi/2$), this derivative is negative.

In chapter 6 we will show that for any maximum detour pair $a, b \in S_n$ we have $\angle(a,b)/2 > \frac{5}{12}\pi$. This justifies our assumption that $a \neq b$ and implies that equation (4.1) can be strengthened to

$$\delta(R_n) = \max_{\substack{a',b' \in V(S_n) \\ \angle(a',b')/2 > \frac{5}{12}\pi}} \frac{\pi(a',b')}{2\sin(\angle(a',b')/2)},$$

and hence

$$\delta(R_n) \leq \max_{\substack{a,b \in V(S_{3072}) \\ \angle(a,b)/2 > \frac{5}{12}\pi - \frac{\pi}{1536}}} \frac{1.45815 \sin(\angle(a,b)/2) + \frac{\pi}{3072}}{\sin\left(\angle(a,b)/2 - \frac{\pi}{3072}\right)}.$$

Since the function on the right hand side is monotonically decreasing as we have seen, it follows that

$$\delta(R_n) \leq \frac{1.45815 \sin\left(\frac{5}{12}\pi - \frac{\pi}{1536}\right) + \frac{\pi}{3072}}{\sin\left(\frac{5}{12}\pi - \frac{\pi}{1536} - \frac{\pi}{3072}\right)} < 1.4597.$$

In conclusion, we have shown the following

Theorem 4.5 *Let $n = 3 \cdot 2^i$ for $i \in \mathbb{N}_0$. Then the graph theoretic dilation of S_n is at most 1.4597.*

In chapter 7 we will use a similar analysis when we compute the approximation factor of our approximation algorithm.

Chapter 5

Lower Bounds for the Regular n-Gon

Once again, let $S_n = \{s_0, s_1, \ldots, s_{n-1}\}$ be the set of nodes of a regular n-gon in counter-clockwise order. Let T be any triangulation of S_n. In this section we are going to determine lower bounds on the graph theoretic dilation $\delta(T)$ of T. The main strategy will be to look at a distinguished pair of vertices and to determine the minimum detour between this pair of vertices that any triangulation can achieve.

How can we find such a distinguished pair? The starting point for us will be the following lemma which reveals a simple property of any triangulation of S_n.

Lemma 5.1 *Let S_n be the set of nodes of a regular n-gon with radius 1, and let T be any triangulation of S_n. Then the longest line segment in T has length at least $\sqrt{3}$.*

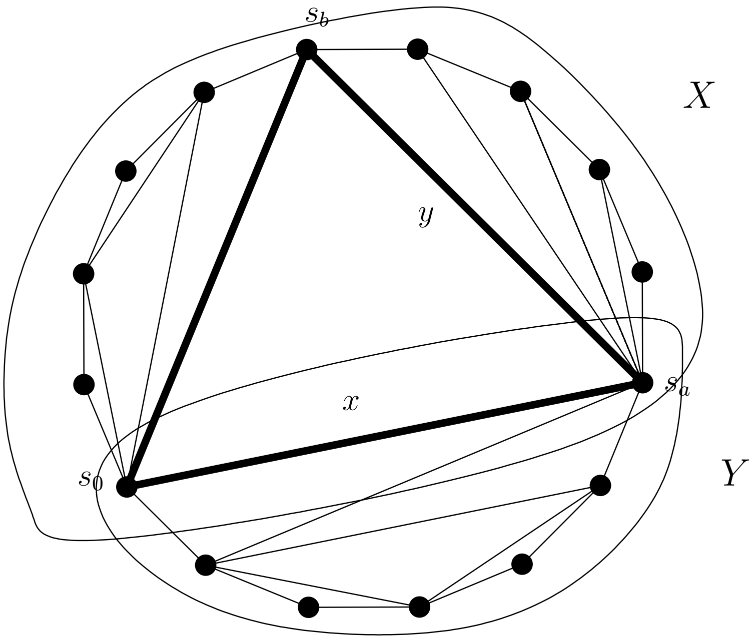
Proof: Let $x = \overline{s_\alpha s_a}$ be a longest line segment in T. Without loss of generality, we may assume that $\alpha = 0$ and $a \leq \frac{n}{2}$. There has to be another line segment $y \neq x$ that has a common endpoint with x and whose other endpoint s_b has the property that $b > a$. We may assume that $y = \overline{s_a s_b}$. Furthermore, let y be chosen in such a way that there are no further line segments $\overline{s_a s_c}$ with $c > b$. Since T is a triangulation, it follows that $\overline{s_0 s_b}$ also is a line segment in T (see figure 5.1).

Therefore, as x is a longest line segment and since the triangle defined by s_0, s_a and s_b contains the center, we can conclude that $a \geq \frac{n}{3}$ and that the length of x is $|x| = 2\sin\left(\frac{\pi a}{n}\right) \geq 2\sin\left(\frac{\pi}{3}\right) = \sqrt{3}$. ■

Now, let x be a longest line segment in T. x divides S_n into two parts X and Y such that all the points to the left of x lie in X and all the points to the right of x lie in Y and the endpoints of x are contained in both X and Y. Without loss of generality, let us assume that that $|X| \geq |Y|$. It is clear that x participates in exactly two different triangles in T, one of which lies in X and one of which lies in Y. Let D be the triangle whose vertices lie in X. In the following, we will refer to this triangle quite frequently. Thus, we will call it the *central triangle* of triangulation T (see figure 5.1).

There are two things to remark about this definition. First, the definition is ambiguous for the case that $|X| = |Y|$. If this happens, we break the tie by defining the central triangle as the one of the two triangles whose third vertex has the smaller index in S_n. Second, we need to convince ourselves that the notion of a central triangle is well-defined, *i.e.,* that the central triangle is still unique in the case that there is more than one longest line segment in T. However, it is easy to see that all the longest line segments in T have to participate in a common triangle, since if x is a longest line segment, then the edges of the two triangles D_1 and D_2 in which x participates cannot be longer than x. Therefore, any edge that lies inside the circular segments that are defined by the edges $\neq x$ of D_1 and D_2 must be shorter than x.

Figure 5.1: The central triangle in a triangulation of S_{16}. The central triangle has vertices s_0, s_a, s_b, the longest line segment is $\overline{s_0 s_a}$



Thus, any longest line segment must lie either in D_1 or in D_2. However, it cannot happen that there are two longest line segments $y, z \neq x$ such that y is in D_1 and z is in D_2, because then y and z could not lie on a common triangle, while the argument we just gave shows that any *two* longest line segments must lie on a common triangle. Hence, it follows that *all* longest line segments lie on a common triangle. In particular, we can conclude that there are at most three of them.

Now, obviously, the central triangle constitutes an obstacle that a shortest path between any two vertices that lie in different circular segments of the central triangle must pass. Therefore, in a first attempt, we will look at the middle points of the circular segments that are defined by the central triangle and compute a lower bound on the detour between them. We will follow this approach in section 5.1. However, it turns out that this approach only yields a relatively weak lower bound. The problem is that the points we consider vary as the central triangle varies. A more successful approach will be to consider points that are at a fixed distance of a fixed vertex of the central triangle. We will do this in section 5.2 and achieve a pretty strong lower bound, from which we will be able to deduce some interesting properties of the graph theoretic dilation of S_n.

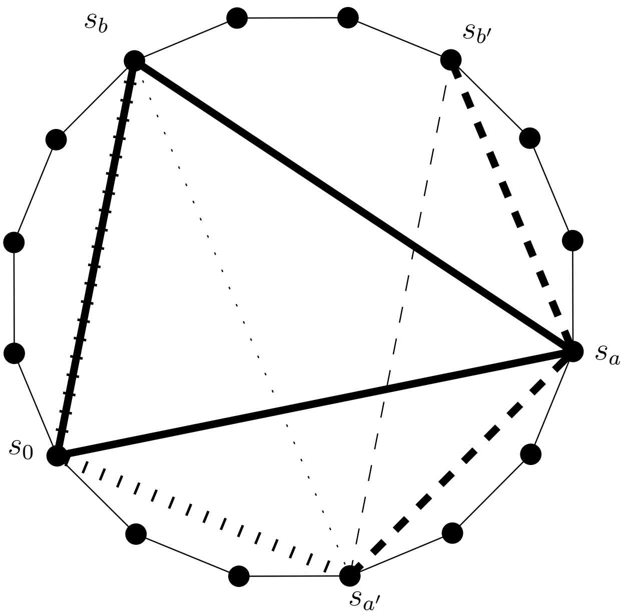
5.1 A Weak Lower Bound

Let T be any triangulation of S_n, and let s_0, s_a, and s_b be the vertices of the central triangle of T such that $a < b$, $a \geq \frac{n}{3}$ and $|\overline{s_a s_b}| \geq |\overline{s_b s_0}|$. Note that in this section we *do not* require that $|\overline{s_0 s_a}| \geq |\overline{s_a s_b}|$. This makes the calculations more convenient.

Hence, a and b must fulfill the constraints

$$\frac{n}{3} \leq a \leq \frac{n}{2}$$

Figure 5.2: We consider the detour between $s_{a'}$ and $s_{b'}$ as well as the detour between $s_{a'}$ and s_b. The thin dashed line shows the Euclidean distance between $s_{a'}$ and $s_{b'}$, and the bold dashed line shows the path between $s_{a'}$ and $s_{b'}$, which we use in order to bound $\pi_T(s_{a'}, s_{b'})$. Similarly, the thin and the bold dotted lines visualize the Euclidean distance and the lower bound on the shortest path length for $s_{a'}$ and s_b.



and

$$a + \frac{n-a}{2} = \frac{n+a}{2} \leq b \leq n-1.$$

In this section we shall consider the middle points of the circular segment between s_0 and s_a and the circular segment between s_a and s_b. Thus, let $a' \stackrel{\text{def}}{=} \frac{a}{2}$ and $b' \stackrel{\text{def}}{=} a + \frac{b-a}{2} = \frac{a+b}{2}$ (see figure 5.2). In the following we will give a lower bound on the maximum of the detour between $s_{a'}$ and $s_{b'}$ and the detour between $s_{a'}$ and s_b. It is necessary to look at these two detour values simultaneously, since otherwise we would always end up with a degenerate configuration of the central triangle that does not give any meaningful lower bound at all.

Now, let us set up the equations for the two detours we are interested in. Of course, we do not know anything about T except for the position of the central triangle. Thus, we cannot exactly say how long the shortest paths between $s_{a'}$ and $s_{b'}$ and the vertices of the central triangle actually are. However, we can lower-bound these lengths by the Euclidean distance between the respective points. Therefore, basic trigonometry yields the following estimates

$$\delta_T\left(s_{a'}, s_{b'}\right) \geq \frac{2\sin\left(\frac{a'}{n}\pi\right) + 2\sin\left(\frac{b'-a}{n}\pi\right)}{2\sin\left(\frac{a'+b'-a}{n}\pi\right)}, \tag{5.1}$$

and

$$\delta_T\left(s_{a'}, s_b\right) \geq \frac{2\sin\left(\frac{a'}{n}\pi\right) + 2\sin\left(\frac{n-b}{n}\pi\right)}{2\sin\left(\frac{n-b+a'}{n}\pi\right)}. \tag{5.2}$$

To make equations (5.2) and (5.1) more legible let us substitute $\lambda \stackrel{\text{def}}{=} \frac{a'\pi}{n} = \frac{a\pi}{2n}$ and $\beta \stackrel{\text{def}}{=} \frac{b'-a}{n}\pi$, *i.e.,* α is half the angle between s_0 and $s_{a'}$ and β is half the angle between s_a and s_b'. The constraints on a and b tell us that

$$\frac{\pi}{6} \leq \lambda \leq \frac{\pi}{4}, \tag{5.3}$$

as well as

$$\frac{\pi}{n} \leq \frac{n-b}{n}\pi \leq \frac{n-\frac{n+a}{2}}{n}\pi = \frac{n-a}{2n}\pi = \frac{\pi}{2} - \lambda,$$

and

$$\frac{\pi}{4} - \frac{\lambda}{2} = \frac{n-a}{4n}\pi \overset{(1)}{\leq} \frac{b'-a}{n}\pi = \beta = \frac{b-a}{2n}\pi \leq \frac{n-1-a}{2n}\pi \leq \frac{\pi}{2} - \lambda, \tag{5.4}$$

where (1) holds because $b' - a = (b-a)/2$ and $b - a \geq (n-a)/2$.

The substitution has the effect that our detours take a much more handy form, namely

$$\delta_T\left(s_{a'}, s_{b'}\right) \geq \frac{\sin\lambda + \sin\beta}{\sin\left(\lambda+\beta\right)}, \tag{5.5}$$

and

$$\delta_T\left(s_{a'}, s_b\right) \geq \frac{\sin\lambda + \sin\left(\pi - 2\lambda - 2\beta\right)}{\sin\left(\pi - \lambda - 2\beta\right)}, \tag{5.6}$$

because

$$\frac{n-b}{n}\pi = \pi\left(1 - \frac{2b'-a}{n}\right) = \pi\left(1 - \frac{a}{n} - \frac{2b'-2a}{n}\right) = \pi - 2\lambda - 2\beta.$$

Given the constraints on λ and β, we can now immediately see that the two lower bounds (5.5) and (5.6) are equal if $\beta = \pi - 2\lambda - 2\beta$, *i.e.*, $\beta = \frac{1}{3}\pi - \frac{2}{3}\lambda$. We shall denote this latter quantity by $\beta^*(\lambda)$. As a sanity check, let us briefly verify that

$$\frac{\pi}{4} - \frac{\lambda}{2} \leq \beta^*(\lambda) \Leftrightarrow \frac{1}{6}\lambda \leq \frac{1}{12}\pi \Leftrightarrow \lambda \leq \frac{\pi}{2}$$

and that

$$\beta^*(\lambda) \leq \frac{\pi}{2} - \lambda \Leftrightarrow \frac{1}{3}\lambda \leq \frac{1}{6}\pi \Leftrightarrow \lambda \leq \frac{\pi}{2}.$$

Hence, $\beta^*(\lambda)$ fulfills the constraints (5.4) imposed on β.

Of course, it does not help us much to know a point in which the two lower bounds intersect without any further information about the behavior of these functions. Ultimately, we are going to follow the same general strategy which we used in section 4.2 in order to compute the graph theoretic dilation of the canonical triangulation. That is, we are going to show that one of the two functions is monotonically increasing, while the other function monotonically decreases, and hence the y-coordinate of the point in which their graphs intersect (of which we then know that it is unique) will be our lower bound on the graph theoretic dilation of S_n for a fixed value of λ. In a second step we will need to determine the value of λ for which this lower bound is minimal. This eventually yields our desired lower bound.

In order to obtain the required monotonicity properties we need to explore the behavior of the lower bounds (5.5) and (5.6) as β varies. To that end we will consider a slightly more general function whose properties will also be useful later on. This function is given by

$$f\left(\alpha, \beta\right) \overset{\text{def}}{=} \frac{\sin\alpha + \sin\beta}{\sin\left(\alpha+\beta\right)},$$

and its properties are explored in the following

Claim 5.2 *The function* $f : \left\{(\alpha,\beta) \in (0,\pi)^2 \,|0 < \alpha+\beta < \pi\right\} \to \mathbb{R}$ *given by*

$$f(\alpha,\beta) = \frac{\sin\alpha + \sin\beta}{\sin(\alpha+\beta)}$$

has derivative

$$f'(\alpha,\beta) = \frac{1-\cos(\alpha+\beta)}{\sin^2(\alpha+\beta)}(\sin\beta, \sin\alpha) \tag{5.7}$$

and grows as one argument increases and the other argument remains fixed.

Proof: We can compute the partial derivative of f as follows:

$$\begin{aligned}
\frac{\partial f(\alpha,\beta)}{\partial\beta} &= \frac{\sin(\alpha+\beta)\cos\beta - (\sin\alpha+\sin\beta)\cos(\alpha+\beta)}{\sin^2(\alpha+\beta)} \\
&= \frac{(\sin\alpha\cos\beta + \cos\alpha\sin\beta)\cos\beta}{\sin^2(\alpha+\beta)} \\
&\quad + \frac{-(\sin\alpha+\sin\beta)(\cos\alpha\cos\beta - \sin\alpha\sin\beta)}{\sin^2(\alpha+\beta)} \\
&= \frac{\sin\alpha\cos^2\beta + \cos\alpha\sin\beta\cos\beta}{\sin^2(\alpha+\beta)} \\
&\quad + \frac{-\sin\alpha\cos\alpha\cos\beta + \sin^2\alpha\sin\beta}{\sin^2(\alpha+\beta)} \\
&\quad + \frac{-\cos\alpha\sin\beta\cos\beta + \sin\alpha\sin^2\beta}{\sin^2(\alpha+\beta)} \\
&= \frac{\sin\alpha(1-\cos\alpha\cos\beta + \sin\alpha\sin\beta)}{\sin^2(\alpha+\beta)} \\
&= \frac{\sin\alpha(1-\cos(\alpha+\beta))}{\sin^2(\alpha+\beta)}.
\end{aligned}$$

$\frac{\partial f(\alpha,\beta)}{\partial\alpha}$ can be computed in a similar fashion. Now the rest of the claim follows immediately, since $|\cos(\alpha+\beta)| \le 1$ and $\sin\alpha \ge 0$ for the values of α at consideration. ■

In particular, the lower bound for $\delta_T(s_{a'}, s_{b'})$ (5.1) is monotonically increasing as b' grows and a' remains fixed, while the lower bound for $\delta_T(s_{a'}, s_b)$ (5.2) decreases as b grows and a' remains fixed. Hence, the maximum of the two lower bounds is minimal for fixed a if $\frac{\pi}{n}b = 2\lambda + 2\beta^*(\lambda) = \frac{2}{3}\lambda + \frac{2}{3}\pi$, *i.e.*, if $b = \frac{a}{3} + \frac{2}{3}n$. Thus, we have successfully eliminated β from consideration.

In order to get the desired lower bound we now need to find out for which value of λ (or equivalently for which value of a) this lower bound reaches its minimum.

Thus, let us substitute the value of β^* into the lower bounds for $\delta_T(s_{a'}, s_{b'})$ (5.5) and $\delta_T(s_{a'}, s_b)$ (5.6). We thus obtain a function that solely depends on λ, which we will denote by $\delta(\lambda)$:

$$\delta(\lambda) \stackrel{\text{def}}{=} \frac{\sin\lambda + \sin\left(\frac{\pi}{3} - \frac{2}{3}\lambda\right)}{\sin\left(\frac{\pi}{3} + \frac{\lambda}{3}\right)}$$

Since $\delta(\lambda) = f\left(\lambda, \frac{\pi}{3} - \frac{2}{3}\lambda\right)$, the chain rule and equation (5.7) in claim 5.2 help us compute the derivative of $\delta(\lambda)$ as follows:

$$\begin{aligned}\delta'(\lambda) &= f'\left(\lambda, \frac{\pi}{3} - \frac{2}{3}\lambda\right)\begin{pmatrix}1 \\ -\frac{2}{3}\end{pmatrix} \\ &= \frac{1-\cos\left(\frac{\pi}{3}+\frac{\lambda}{3}\right)}{\left(\sin\left(\frac{\pi}{3}+\frac{\lambda}{3}\right)\right)^2}\left(\sin\left(\frac{\pi}{3}-\frac{2}{3}\lambda\right)-\frac{2}{3}\sin\lambda\right) \\ &\geq 0,\end{aligned}$$

since

$$\sin\left(\frac{\pi}{3}-\frac{2}{3}\lambda\right) \geq \sin\left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} > \frac{\sqrt{2}}{3} = \frac{2}{3}\sin\left(\frac{\pi}{4}\right) \geq \frac{2}{3}\sin\lambda.$$

Therefore, $\delta(\lambda)$ increases as λ grows, and thus constraint (5.3) on λ tells us that $\delta(\lambda)$ takes on its minimum value for $\lambda = \frac{\pi}{6}$. Hence, the maximum of the detours between $(s_{a'}, s_b)$ and $(s_{a'}, s_{b'})$ in any triangulation of S_n is at least

$$\delta\left(\frac{\pi}{6}\right) = \frac{\sin\left(\frac{\pi}{6}\right)+\sin\left(\frac{2}{9}\pi\right)}{\sin\left(\frac{7}{18}\pi\right)} \approx 1.2161.$$

Thus, we have obtained a lower bound on the graph theoretic dilation of S_n. However, throughout our calculations we have always assumed that a' and b' are integral, which does not always have to be the case. Therefore, a rigorous treatment would now necessitate a detailed analysis of the error we have introduced through this assumption. Nonetheless, we will omit these calculations, as a slightly different approach yields a much more powerful lower bound, which we will derive in the next section.

5.2 A Better Lower Bound

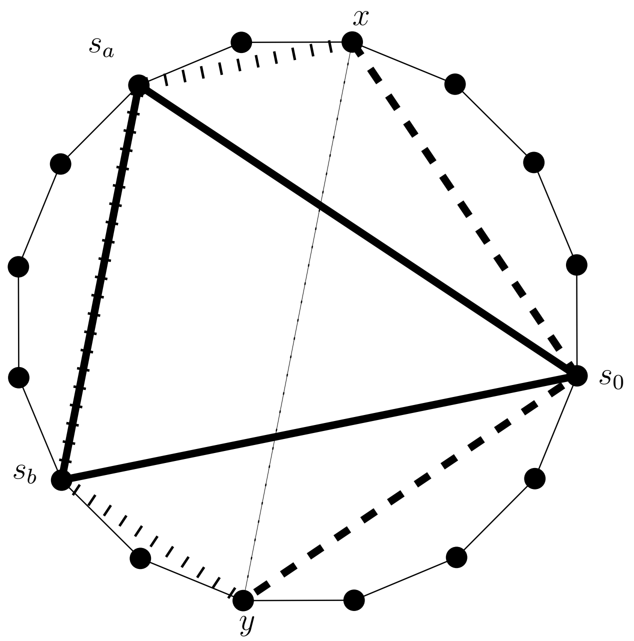
As the experimental results in table 3.1 on page 14 suggest, the lower bound that we obtained in the previous section is pretty weak. In this section we will derive a better bound by using a slightly modified approach. The basic idea remains the same: we are going to look at a pair of distinguished vertices in S_n. The problem in the previous section seems to be that the vertices which we considered there could vary as the central triangle varies. In this section we shall consider a pair of vertices that lie diametrically opposed to each other such that each vertex is one quarter of a circle away from a fixed endpoint of the central triangle. If you think about it, this approach is very natural and intuitively promising, since the configuration we consider is very similar to a square, which is the simplest example that is used to show that in general the graph theoretic dilation of a point set cannot be arbitrarily close to 1. Indeed, it will turn out that this approach yields a quite powerful lower bound.

In subsection 5.2.1 we will first derive the bound for the case that n is divisible by 4. Since it is clear that the lower bound cannot be valid for all possible values of n (obviously, the graph theoretic dilation of S_3 is 1), we will analyze for which n the lower bound holds in subsection 5.2.2, where we show that the lower bound is valid if n is large enough.

5.2.1 A Lower Bound for $n \equiv 0 \pmod 4$

Let T be an arbitrary triangulation of S_n. We already know that in T there is a longest line segment $\ell = \overline{s_\gamma s_a}$ such that the convex hull distance $\Delta_{S_n}(s_\gamma, s_a)$ between s_γ and s_s is at least $\frac{n}{3}$. Furthermore, the proof of lemma 5.1 on page 29 about the central triangle implies that ℓ is adjacent to another line segment $\ell' = \overline{s_\gamma s_b}$ such that the convex hull distance $\Delta_{S_n}(s_\gamma, s_b)$

Figure 5.3: We are looking at the detour between x and y. The shortest path between x and y either includes s_0 (dashed line) or it uses line segment $\overline{s_a s_b}$ (bold dotted line).



between s_γ and s_b is at least half of $n - \Delta_{S_n}(s_\gamma, s_a)$. For the sake of simplicity we will assume that $\gamma = 0$. Furthermore, throughout this subsection we will assume that $n \equiv 0 \pmod 4$, *i.e.,* that n is a multiple of 4. We will drop this assumption in the next subsection. Now let $x = s_{\frac{n}{4}}$ and $y = s_{\frac{3n}{4}}$. In the following we shall compute a lower bound the detour between x and y (see figure 5.3).

As before, let us introduce some shorthand notation in order to simplify our formulae. Thus, let $\alpha \stackrel{\text{def}}{=} \frac{a\pi}{n}$ and $\beta \stackrel{\text{def}}{=} \frac{(n-b)\pi}{n}$. This means that α denotes half the angle between s_0 and s_a, while β represents half the angle between s_0 and s_b. By our assumptions it follows that $\lceil \frac{n}{3} \rceil \leq a \leq \lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n-a}{2} \rceil \leq n - b \leq a$, since a is a longest line segment. This implies the following bounds on α and β:

$$\frac{\pi}{3} \leq \alpha \leq \frac{\pi}{2}, \tag{5.8}$$

and

$$\frac{\pi - \alpha}{2} \leq \beta \leq \alpha.$$

Note that these bounds imply in particular that x always lies between s_0 and s_a and that y always lies between s_0 and s_b, as shown in figure 5.3. This is so because from the bounds it follows that $\beta \geq \frac{\pi}{4}$.

Now let us compute the detour between x and y. Clearly, the Euclidean distance between x and y is 2. The shortest path between the two points either passes s_0 or it uses line segment $\overline{s_a s_b}$. In the former case the length of the shortest path has to be at least $2\sin\left(\frac{\pi}{4}\right) + 2\sin\left(\frac{\pi}{4}\right)$, in the latter case the length is bounded from below by $2\sin\left(\alpha - \frac{\pi}{4}\right) + 2\sin\left(\beta - \frac{\pi}{4}\right) + 2\sin\left(\pi - (\alpha + \beta)\right)$, since the shortest path length can never be less than the Euclidean distance (see figure 5.3). Thus, we have

$$\begin{aligned} \delta_T(x, y) &\geq \frac{\min\left\{2\sin\left(\frac{\pi}{4}\right), \sin\left(\alpha - \frac{\pi}{4}\right) + \sin\left(\beta - \frac{\pi}{4}\right) + \sin\left(\pi - (\alpha + \beta)\right)\right\}}{\sin\left(\frac{\pi}{2}\right)} \\ &= \min\left\{\sqrt{2}, \sin\left(\alpha - \frac{\pi}{4}\right) + \sin\left(\beta - \frac{\pi}{4}\right) + \sin\left(\alpha + \beta\right)\right\}. \end{aligned} \tag{5.9}$$

In order to compute the minimum (5.9) we need to examine the function

$$f(\alpha,\beta) \stackrel{\text{def}}{=} \sin\left(\alpha-\frac{\pi}{4}\right)+\sin\left(\beta-\frac{\pi}{4}\right)+\sin(\alpha+\beta). \tag{5.10}$$

We will first figure out for which value of β $f(\alpha,\beta)$ is minimal when α is fixed. Therefore, we compute the partial derivative $\frac{\partial f(\alpha,\beta)}{\partial\beta}$ as follows:

$$\frac{\partial f(\alpha,\beta)}{\partial\beta} = \cos\left(\beta-\frac{\pi}{4}\right)+\cos(\alpha+\beta). \tag{5.11}$$

Now let us compute the zeros of (5.11). We know that $\cos\xi=-\cos\phi \Leftrightarrow \phi=(2k+1)\pi-\xi \vee \phi=(2k+1)\pi+\xi, k\in\mathbb{Z}$, and since we are only interested in the case that $\phi,\xi\in[0,\pi]$, the only values of interest in which the derivative vanishes are

$$\beta-\frac{\pi}{4}=\alpha+\beta-\pi \Leftrightarrow \alpha=\frac{3}{4}\pi \tag{5.12}$$

and

$$\beta-\frac{\pi}{4}=\pi-\alpha-\beta \Leftrightarrow \beta=\frac{5}{8}\pi-\frac{\alpha}{2}. \tag{5.13}$$

The first case (5.12) cannot occur due to the constraints on α (5.8). The second case (5.13) occurs if and only if $\alpha\geq\frac{5}{12}\pi$, since

$$\frac{5}{8}\pi-\frac{\alpha}{2}\leq\alpha \Leftrightarrow \alpha\geq\frac{5}{12}\pi.$$

Let us first assume that $\alpha<\frac{5}{12}\pi$. In this case $\frac{\partial f(\alpha,\beta)}{\partial\beta}$ does not vanish for $\beta\in\left[\frac{\pi-\alpha}{2},\alpha\right]$, and we have

$$\begin{aligned}\frac{\partial f}{\partial\beta}\left(\alpha,\frac{\pi-\alpha}{2}\right) &= \cos\left(\frac{\pi-\alpha}{2}-\frac{\pi}{4}\right)+\cos\left(\frac{\pi-\alpha}{2}+\alpha\right)\\ &= \cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)+\cos\left(\frac{\pi}{2}+\frac{\alpha}{2}\right)\\ &= \cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)-\sin\left(\frac{\alpha}{2}\right)\\ &> 0,\end{aligned} \tag{5.14}$$

since $\alpha/2$ ranges from $\frac{\pi}{6}$ to $\frac{\pi}{4}$ and hence we have $\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)>\sqrt{2}/2$ and $\sin\left(\frac{\alpha}{2}\right)<\sqrt{2}/2$.

Consequently, $f(\alpha,\beta)$ is monotonically increasing if $\alpha<\frac{5}{12}\pi$ remains fixed and $\beta\in\left[\frac{\pi-\alpha}{2},\alpha\right]$ and hence $f(\alpha,\beta)$ takes on its minimum value for $\beta=\frac{\pi-\alpha}{2}$.

Now let $\alpha\geq\frac{5}{12}\pi$. We will now show that in this case it is still true that $f(\alpha,\beta)$ takes on its minimum value for $\beta=\frac{\pi-\alpha}{2}$. To that end let us consider the function $g(x)\stackrel{\text{def}}{=} f\left(\alpha,\frac{5}{8}\pi-\frac{\alpha}{2}+x\right)$. We get

$$\begin{aligned} g(x) &= \sin\left(\alpha-\frac{\pi}{4}\right)+\sin\left(\frac{3}{8}\pi-\frac{\alpha}{2}+x\right)+\sin\left(\frac{5}{8}\pi+\frac{\alpha}{2}+x\right)\\ &= \sin\left(\alpha-\frac{\pi}{4}\right)+\sin\left(\frac{3}{8}\pi-\frac{\alpha}{2}+x\right)-\sin\left(-\left(\frac{3}{8}\pi-\frac{\alpha}{2}\right)+x\right),\end{aligned}$$

since $\sin(\lambda+\pi)=-\sin(\lambda)$.

It is now easy to see that g is an even function, because

$$\begin{aligned}
g(-x) &= \sin\left(\alpha-\frac{\pi}{4}\right)+\sin\left(\frac{3}{8}\pi-\frac{\alpha}{2}-x\right)-\sin\left(-\left(\frac{3}{8}\pi-\frac{\alpha}{2}\right)-x\right)\\
&= \sin\left(\alpha-\frac{\pi}{4}\right)-\sin\left(-\left(\frac{3}{8}\pi-\frac{\alpha}{2}\right)+x\right)+\sin\left(\frac{3}{8}\pi-\frac{\alpha}{2}+x\right)\\
&= g(x).
\end{aligned}$$

Therefore, it follows that $f(\alpha,\beta)$ is symmetric with respect to the line $\beta=\frac{5}{8}\pi-\frac{\alpha}{2}$, and furthermore, since $\frac{5}{8}\pi-\frac{\alpha}{2}\geq\frac{1}{2}\left(\alpha+\frac{\pi-\alpha}{2}\right)\Leftrightarrow\alpha\leq\frac{\pi}{2}$, we know that this line lies to the right of the center of the interval at consideration. Consequently, since we saw in (5.14) that

$$\frac{\partial f}{\partial\beta}\left(\alpha,\frac{\pi-\alpha}{2}\right)\geq 0$$

and since the partial derivative only vanishes once in the critical interval, we can conclude that if α is fixed, then $f(\alpha,\beta)$ is minimal for $\beta=\frac{\pi-\alpha}{2}$.

Thus, we have successfully eliminated β from consideration. By substituting $\beta=\frac{\pi-\alpha}{2}$ into the lower bound (5.10) we obtain a function that solely depends on α, which we shall denote by $h(\alpha)$. We have

$$\begin{aligned}
h(\alpha) &= f\left(\alpha,\frac{\pi-\alpha}{2}\right)\\
&= \sin\left(\alpha-\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}-\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}+\alpha\right)\\
&= \sin\left(\alpha-\frac{\pi}{4}\right)-\sin\left(\frac{\alpha}{2}-\frac{\pi}{4}\right)+\cos\left(\frac{\alpha}{2}\right),
\end{aligned}$$

for $\alpha\in\left[\frac{\pi}{3},\frac{\pi}{2}\right]$.

In order to obtain the desired lower bound we need to find the value of α for which $h(\alpha)$ is minimal. Thus, let us compute the first and the second derivative of $h(\alpha)$. We get

$$h'(\alpha)=\cos\left(\alpha-\frac{\pi}{4}\right)-\frac{1}{2}\cos\left(\frac{\alpha}{2}-\frac{\pi}{4}\right)-\frac{1}{2}\sin\left(\frac{\alpha}{2}\right),$$

and

$$h''(\alpha)=-\sin\left(\alpha-\frac{\pi}{4}\right)+\frac{1}{4}\sin\left(\frac{\alpha}{2}-\frac{\pi}{4}\right)-\frac{1}{4}\cos\left(\frac{\alpha}{2}\right).$$

Elementary properties of the sine and cosine function imply that $h''(\alpha)<0$ for $\alpha\in\left[\frac{\pi}{3},\frac{\pi}{2}\right]$, because $\alpha-\frac{\pi}{4}\in\left[\frac{\pi}{12},\frac{\pi}{4}\right]$ and $\frac{\alpha}{2}-\frac{\pi}{4}\in\left[-\frac{1}{12}\pi,0\right]$, hence $-\sin\left(\alpha-\frac{\pi}{4}\right)<0$ and $\frac{1}{4}\sin\left(\frac{\alpha}{2}-\frac{\pi}{4}\right)\leq 0$. It follows that $h'(\alpha)$ strictly decreases for $\alpha\in\left[\frac{\pi}{3},\frac{\pi}{2}\right]$, and as

$$h'\left(\frac{\pi}{3}\right)=\frac{1}{2}\cos(\frac{\pi}{12})-\frac{1}{4}=\frac{\sqrt{2+\sqrt{3}}-1}{4}>0$$

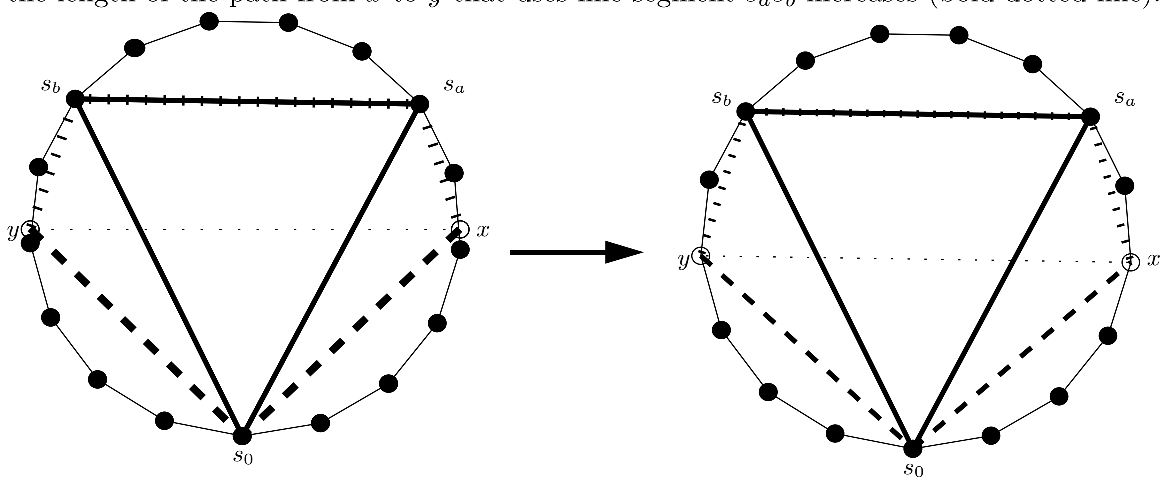
and

$$h'\left(\frac{\pi}{2}\right)=\frac{\sqrt{2}}{4}-\frac{1}{2}<0,$$

we can conclude that $h(\alpha)$ first grows monotonically and then decreases monotonically. This implies that the only candidate points for a minimum of $h(\alpha)$ are the endpoints of the interval $\left[\frac{\pi}{3},\frac{\pi}{2}\right]$. By plugging those candidate points into h we obtain

$$h\left(\frac{\pi}{3}\right)=2\sin\left(\frac{\pi}{12}\right)+\frac{\sqrt{3}}{2}=\sqrt{2-\sqrt{3}}+\frac{\sqrt{3}}{2}\approx 1.3836$$

Figure 5.4: x and y are rounded towards s_0. The Euclidean distance $|xy|$ (length of the thin dotted line) and the length of the path from x to y via s_0 (bold dashed line) decrease, whereas the length of the path from x to y that uses line segment $s_a s_b$ increases (bold dotted line).



and

$$h\left(\frac{\pi}{2}\right) = \sqrt{2} \approx 1.4142,$$

and hence we have proved the following

Theorem 5.3 *Let S_n be the set of vertices of a regular n-gon, and let $n \equiv 0 \pmod 4$. Then any triangulation of S_n has graph theoretic dilation at least $\sqrt{2-\sqrt{3}} + \frac{\sqrt{3}}{2} \approx 1.3836$.*

5.2.2 A Lower Bound for Arbitrary n

Naturally, we would like to generalize the result of the previous section for arbitrary n. Unfortunately, a glance at table 3.1 on page 14 reveals that the lower bound does not hold for any possible value of n (for instance, it does not hold for $n = 5$). However, it is obvious that if n is large enough, there should not be a large difference between the cases $n \equiv 0 \pmod 4$ and $n \not\equiv 0 \pmod 4$. Thus, our strategy will be to look at a pair of points that is very close to the pair of points that we considered in the previous subsection. We will obtain this pair of points by rounding the indices we used in the previous subsection, $\frac{n}{4}$ and $\frac{3}{4}n$. We then consider the detour between the points we have thus obtained, and in particular we will compute a value n_0 such that the lower bound from the previous section holds for all $n \geq n_0$.

It is clear that there are several ways to perform this rounding. To be precise, there are exactly four different ways, since we can round each index either up or down. The simplest method, which we shall explore first, is to move both points towards the distinguished vertex s_0 of the central triangle. This approach yields a relatively low value for n_0. Of the other three methods only one can be used to improve the bound on n_0 a little bit, which we will do in the second paragraph of this subsection. The reader is invited to verify (or to take it on faith) that the other rounding methods do not yield any substantial improvements.

A Simple Rounding Method

In this paragraph we are going to find out what happens if we move both distinguished points towards the distinguished vertex of the central triangle s_0 (see figure 5.4).

Thus, let $x = s_{\lfloor \frac{n}{4} \rfloor}$ and $y = s_{n-\lfloor \frac{n}{4} \rfloor}$. As in the last subsection, we would like to bound the detour between x and y from below. Just as before in any triangulation T of S_n, the shortest path between x and y either passes s_0 or it uses line segment $\overline{s_a s_b}$, where s_a and s_b denote the two other vertices of the central triangle. In the latter case the lower bound on the detour between x and y can only increase compared to the lower bound we computed in the previous subsection. This is so because x and y are now further away from s_a and s_b, respectively, and thus the lower bound on the shortest path length can only become larger, whereas the Euclidean distance between x and y decreases (see figure 5.4).

Consequently, we only need to consider the case that the shortest path from x to y includes s_0. By the standard argument which relates the shortest path length to the Euclidean distance and since the Euclidean distance between y and s_0 and the Euclidean distance between x and s_0 both equal $2\sin\left(\lfloor \frac{n}{4} \rfloor \frac{\pi}{n}\right)$, we can conclude that the length of such a shortest path is at least $4\sin\left(\lfloor \frac{n}{4} \rfloor \frac{\pi}{n}\right)$, which we write as $4\sin\left(\frac{\pi}{4} - \left(\frac{n}{4} - \lfloor \frac{n}{4} \rfloor\right)\frac{\pi}{n}\right)$, since that form will be more convenient for our analysis. Similarly, the Euclidean distance between x and y equals $2\sin\left(2\lfloor \frac{n}{4} \rfloor \frac{\pi}{n}\right)$, which we will write as $2\sin\left(\frac{\pi}{2} - \left(\frac{n}{2} - 2\lfloor \frac{n}{4} \rfloor\right)\frac{\pi}{n}\right)$. Hence, for the case that the shortest path between x and y passes s_0 the detour between x and y can be bounded from below by the following function which we will denote by $f(n)$:

$$f(n) \stackrel{\text{def}}{=} \frac{4\sin\left(\frac{\pi}{4} - \left(\frac{n}{4} - \lfloor \frac{n}{4} \rfloor\right)\frac{\pi}{n}\right)}{2\sin\left(\frac{\pi}{2} - \left(\frac{n}{2} - 2\lfloor \frac{n}{4} \rfloor\right)\frac{\pi}{n}\right)}.$$

Thus, we can obtain a lower bound on the graph theoretic dilation $\delta(T)$ of triangulation T by taking the minimum of $f(n)$ and $\sqrt{2-\sqrt{3}} + \frac{\sqrt{3}}{2} \approx 1.3836$, *i.e.,* the lower bound on the detour between x and y if the shortest path between x and y uses line segment $\overline{s_a s_b}$. We will now show that in a certain sense $f(n)$ is monotonically increasing. As soon as this claim is established, it will be easy to compute a n_0 such that the lower bound from the previous section holds for all $n \geq n_0$.

In claim 5.2 on page 33 we already showed that the function

$$z(\alpha, \beta) = \frac{\sin\alpha + \sin\beta}{\sin(\alpha+\beta)}$$

is monotonically increasing in each of its arguments as long as the other parameter remains fixed for $\alpha, \beta \in (0, \pi)$ and $\alpha + \beta \in (0, \pi)$. Thus, if we take into consideration that $\frac{n}{4} - \lfloor \frac{n}{4} \rfloor = \frac{n+4}{4} - \lfloor \frac{n+4}{4} \rfloor$, it follows that

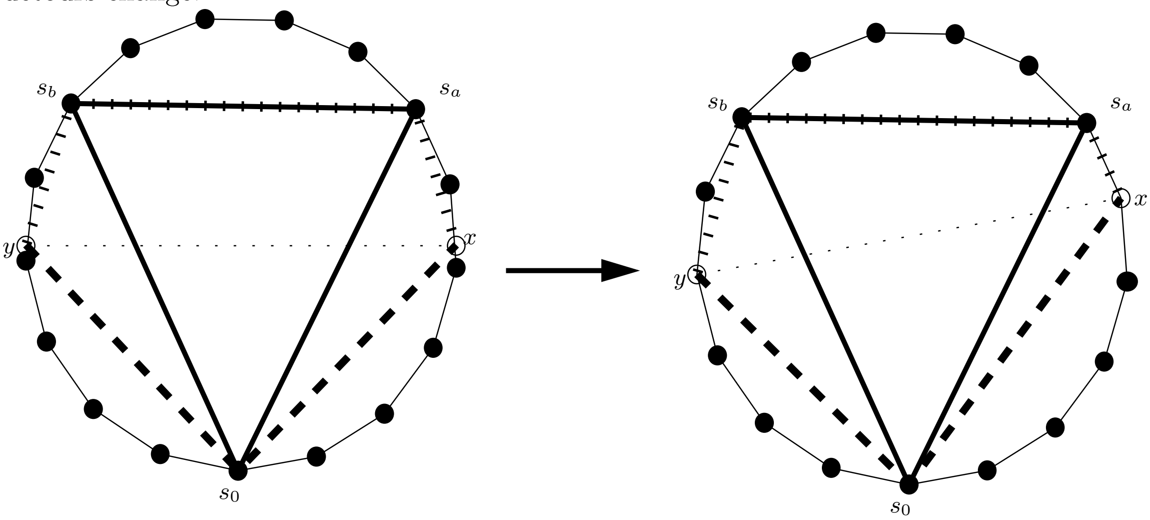
$$\begin{aligned}
f(n) &= \frac{\sin\left(\frac{\pi}{4} - \left(\frac{n}{4} - \lfloor \frac{n}{4} \rfloor\right)\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{4} - \left(\frac{n}{4} - \lfloor \frac{n}{4} \rfloor\right)\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{2} - \left(\frac{n}{2} - 2\lfloor \frac{n}{4} \rfloor\right)\frac{\pi}{n}\right)} \\
&< \frac{\sin\left(\frac{\pi}{4} - \left(\frac{n+4}{4} - \lfloor \frac{n+4}{4} \rfloor\right)\frac{\pi}{n+4}\right) + \sin\left(\frac{\pi}{4} - \left(\frac{n}{4} - \lfloor \frac{n}{4} \rfloor\right)\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{2} - \left(\frac{n}{4} - \lfloor \frac{n}{4} \rfloor\right)\frac{\pi}{n} - \left(\frac{n}{4} - \lfloor \frac{n}{4} \rfloor\right)\frac{\pi}{n+4}\right)} \\
&< \frac{2\sin\left(\frac{\pi}{4} - \left(\frac{n}{4} - \lfloor \frac{n}{4} \rfloor\right)\frac{\pi}{n+4}\right)}{\sin\left(\frac{\pi}{2} - \left(\frac{n}{2} - 2\lfloor \frac{n}{4} \rfloor\right)\frac{\pi}{n+4}\right)} \\
&= f(n+4).
\end{aligned}$$

Thus, f grows as n increases in steps of 4, and all that remains to be done is to determine for each congruence class $\pmod 4$ the value of n for which $f(n) \geq \sqrt{2-\sqrt{3}} + \frac{\sqrt{3}}{2} \approx 1.3836$. This is easily done by dint of a pocket calculator which readily yields the results that are shown in table 5.1.

Table 5.1: Validity of the Lower Bound

$n \pmod 4$	**Lower bound holds for** $n \geq$
0	4
1	37
2	74
3	107, can be improved to 39

Figure 5.5: x is moved towards s_a, y is moved towards s_0. *A priori*, it is not clear how the detours change.



A More Complicated Rounding Method

A natural question that arises at this point is whether the threshold values for n which are shown in table 5.1 can be improved if we choose a different rounding method. In general, the answer to this question is that different rounding methods do not yield any significant improvements. Nonetheless, there is one exception which we will describe in this paragraph.

Thus, let us consider the detour between x and y where x is given by $s_{\lceil \frac{n}{4} \rceil}$ and y is given by $s_{n-\lfloor \frac{n}{4} \rfloor}$. This means that we move x closer to s_a and y closer to s_0 (see figure 5.5). For this pair of points (x, y) we will show that the lower bound holds for $n \geq 39$ given that $n \equiv 3 \pmod 4$. The analysis is very similar to the previous one. However, it is rendered more cumbersome by the fact that we now also need to consider the case that the shortest path between x and y uses line segment $\overline{s_a s_b}$.

Recall our previous definition of $\alpha \overset{\text{def}}{=} \frac{a\pi}{n}$ and $\beta \overset{\text{def}}{=} \frac{(n-b)\pi}{n}$, where α is half the angle between s_0 and s_a, while β is half the angle between s_0 and s_b. In the previous subsection we found that α and β fulfill the constraints

$$\frac{\pi}{3} \leq \alpha \leq \frac{\pi}{2}$$

and

$$\frac{\pi - \alpha}{2} \leq \beta \leq \alpha.$$

Please keep in mind that we are only interested in the case $n \equiv 3 \pmod 4$. By the standard argument the length of any shortest path between x and y that employs line segment $\overline{s_a s_b}$ is at least $2\sin(\alpha+\beta) + 2\sin\left(\alpha - \frac{\pi}{4} - \frac{\pi}{4n}\right) + 2\sin\left(\beta - \frac{\pi}{4} + \frac{3\pi}{4n}\right)$, and the Euclidean distance between x and y equals $2\sin\left(\frac{\pi}{2} - \frac{\pi}{2n}\right)$ (these are the same expressions we used in subsection 5.2.1 with the only difference that we added some "rounding terms"). Thus, the function $g(\alpha,\beta,n)$ given by

$$g(\alpha,\beta,n) \stackrel{\text{def}}{=} \frac{\sin(\alpha+\beta)+\sin\left(\alpha-\frac{\pi}{4}-\frac{\pi}{4n}\right)+\sin\left(\beta-\frac{\pi}{4}+\frac{3\pi}{4n}\right)}{\sin\left(\frac{\pi}{2}-\frac{\pi}{2n}\right)}$$

bounds the ratio between the length of any x-y-path that includes line segment $\overline{s_a, s_b}$ and the Euclidean distance between x and y from below. By considering the partial derivative $\frac{\partial g(\alpha,\beta,n)}{\partial n}$, we can see that g decreases as n grows. Indeed, elementary calculus yields

$$\begin{aligned}\frac{\partial g(\alpha,\beta,n)}{\partial n} &= \frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{2n}\right)\left(\frac{\pi}{4n^2}\cos\left(\alpha-\frac{\pi}{4}-\frac{\pi}{4n}\right)-\frac{3\pi}{4n^2}\cos\left(\beta-\frac{\pi}{4n}+\frac{3\pi}{4n}\right)\right)}{\sin^2\left(\frac{\pi}{2}-\frac{\pi}{2n}\right)}\\ &- \frac{\frac{\pi}{2n^2}\cos\left(\frac{\pi}{2}-\frac{\pi}{2n}\right)\left(\sin(\alpha+\beta)+\sin\left(\alpha-\frac{\pi}{4}-\frac{\pi}{4n}\right)+\sin\left(\beta-\frac{\pi}{4}+\frac{3\pi}{4n}\right)\right)}{\sin^2\left(\frac{\pi}{2}-\frac{\pi}{2n}\right)}\end{aligned}$$

Now obviously the term $\frac{\frac{\pi}{2n^2}\cos\left(\frac{\pi}{2}-\frac{\pi}{2n}\right)\left(\sin(\alpha+\beta)+\sin\left(\alpha-\frac{\pi}{4}-\frac{\pi}{4n}\right)+\sin\left(\beta-\frac{\pi}{4}+\frac{3\pi}{4n}\right)\right)}{\sin^2\left(\frac{\pi}{2}-\frac{\pi}{2n}\right)}$ is positive for the values of α and β we are interested in, since $\alpha \geq \frac{\pi}{3}, \beta \geq \frac{\pi}{4}$, and $n \geq 3$. Furthermore, since $\alpha \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ and $\alpha > \beta$, we can choose $n \geq 7$ in order to get

$$\frac{1}{4}\cos\left(\alpha-\frac{\pi}{4}-\frac{\pi}{4n}\right) < \frac{3}{4}\cos\left(\beta-\frac{\pi}{4}+\frac{3\pi}{4n}\right),$$

because the left hand side decreases as n grows, while the right hand side becomes larger, and $\cos\left(\sigma-\frac{\pi}{28}\right) < 3\cos\left(\sigma+\frac{3\pi}{28}\right)$ for $\sigma \in \left[\frac{\pi}{12}, \frac{\pi}{4}\right]$. Hence, for $n \geq 4$, the first term of $\frac{\partial g(\alpha,\beta,n)}{\partial n}$ is negative.

In conclusion, we have just convinced ourselves that $\frac{\partial g(\alpha,\beta,n)}{\partial n}$ is negative for $n \geq 7$, and therefore $g(\alpha,\beta,n)$ falls as n approaches infinity. However, $\lim_{n\to\infty} g(\alpha,\beta,n)$ is nothing but the function $f(\alpha,\beta)$ we already defined in equation (5.10) on page 36 in section 5.2.1. There, we found out that this function is bounded from below by $\sqrt{2-\sqrt{3}}+\frac{\sqrt{3}}{2} \approx 1.3836$. Thus, if the shortest path from x to y uses line segment $\overline{s_a s_b}$, the lower bound remains valid after rounding as long as $n \geq 7$.

But what happens if the shortest path uses s_0? The Euclidean distance between x and s_0 is $2\sin\left(\frac{\pi}{4}+\frac{\pi}{4n}\right)$, while the Euclidean distance between y and s_0 equals $2\sin\left(\frac{\pi}{4}-\frac{3\pi}{4n}\right)$. Thus, any shortest path from x to y has length at least $2\sin\left(\frac{\pi}{4}-\frac{3\pi}{4n}\right)+2\sin\left(\frac{\pi}{4}+\frac{\pi}{4n}\right)$. Therefore, the detour quotient is at least as large as

$$f(n) = \frac{\sin\left(\frac{\pi}{4}-\frac{3\pi}{4n}\right)+\sin\left(\frac{\pi}{4}+\frac{\pi}{4n}\right)}{\sin\left(\frac{\pi}{2}-\frac{\pi}{2n}\right)} \tag{5.15}$$

For the sake of readability let us substitute $\tau = \frac{\pi}{n}$ into equation (5.15). This leads us to the function $h(\tau)$ which looks like this:

$$h(\tau) \stackrel{\text{def}}{=} \frac{\sin\left(\frac{\pi}{4}-\frac{3}{4}\tau\right)+\sin\left(\frac{\pi}{4}+\frac{\tau}{4}\right)}{\sin\left(\frac{\pi}{2}-\frac{\tau}{2}\right)}.$$

Using claim 5.2 (page 33) and the chain rule to compute the derivative of $h(\tau)$, we will now show that $h(\tau)$ decreases for $\tau \in \left[0, \frac{\pi}{4}\right]$ and hence $f(n)$ grows as n approaches infinity. Indeed, we get

$$\begin{aligned} h'(\tau) &= \frac{1-\cos\left(\frac{\pi}{2}-\frac{\tau}{2}\right)}{\left(\sin\left(\frac{\pi}{2}-\frac{\tau}{2}\right)\right)^2}\left(\sin\left(\frac{\pi}{4}+\frac{\tau}{4}\right), \sin\left(\frac{\pi}{4}-\frac{3}{4}\tau\right)\right)\begin{pmatrix}-\frac{3}{4}\\ \frac{1}{4}\end{pmatrix}\\ &= \frac{1-\cos\left(\frac{\pi}{2}-\frac{x}{2}\right)}{\left(\sin\left(\frac{\pi}{2}-\frac{x}{2}\right)\right)^2}\left(-\frac{3}{4}\sin\left(\frac{\pi}{4}+\frac{\tau}{4}\right)+\frac{1}{4}\sin\left(\frac{\pi}{4}-\frac{3}{4}\tau\right)\right)\\ &< 0,\end{aligned}$$

since $\sin\left(\frac{\pi}{4}+\frac{\tau}{4}\right) > \sin\left(\frac{\pi}{4}-\frac{3}{4}\tau\right)$ for $0 \leq \tau \leq \frac{\pi}{4}$.

Consequently, just like in the previous paragraph all that is left to do now is to determine the value of n for which $f(n) > \sqrt{2-\sqrt{3}}+\frac{\sqrt{3}}{2} \approx 1.3836$. A pocket calculator shows that this happens for $n = 39$.

Hence, all in all, we have eventually proved the following

Theorem 5.4 *Let S_n be the set of vertices of a regular n-gon, and let $n \geq 74$. Then any triangulation of S_n has graph theoretic dilation at least $\sqrt{2-\sqrt{3}}+\frac{\sqrt{3}}{2} \approx 1.3836$.*

5.3 More on Lower Bounds

It is clear that we cannot expect to get any nontrivial lower bounds (*i.e.,* any lower bound > 1) for the graph theoretic dilation of an arbitrary point set. For example, if we arrange n points on a line, then the graph theoretic dilation of the resulting point set is 1. Naturally, this example is not very convincing since such a point set is not in general position. In this section, however, we will give a simple example which shows that even for n points that lie on the unit circle we cannot hope for any nontrivial lower bounds on the graph theoretic dilation.

Let $\varepsilon > 0$ and $n \geq 3 \in \mathbb{N}$ be given. Without loss of generality, we may assume that $\varepsilon < 1$. Now consider the set $X = \{x_0, \ldots, x_{n-1}\}$ of n points which are given by

$$x_0 = (1,0)\,, x_1 = \left(\cos\left(\frac{2}{3}\pi\right), \sin\left(\frac{2}{3}\pi\right)\right), x_2 = \left(\cos\left(\frac{4}{3}\pi\right), \sin\left(\frac{4}{3}\pi\right)\right),$$

and

$$x_i = \left(\cos\left(i\frac{\varepsilon}{n}\right), \sin\left(i\frac{\varepsilon}{n}\right)\right), \qquad \text{for } 3 < i \leq n-1.$$

Obviously, all the points in X lie on a unit circle whose center is the origin. We will show that the graph theoretic dilation of X is at most $1+O(\varepsilon)$. To that end we consider the triangulation T of X in which all the x_i are adjacent to x_1 (see figure 5.6).

Let us now compute the graph theoretic dilation of T. It is clear that the maximum detour can only occur between two nodes x_i, x_j with $i,j > 2$ or between an x_i and x_2 for $i > 2$.

First, let us look at a pair of nodes (x_i, x_j) with $i,j > 2$. Let μ be the angle $\angle(x_i \mathrm{O} x_j)$, where O denotes the origin. Naturally, the length of a shortest path between x_i and x_j is at most μ. Furthermore, the Euclidean distance between x_i and x_j is $2\sin(\mu/2)$. Hence, we have

$$\delta_T(x_i, x_j) \leq \frac{\mu}{2\sin\left(\frac{\mu}{2}\right)} \leq 1+\mu^2,$$

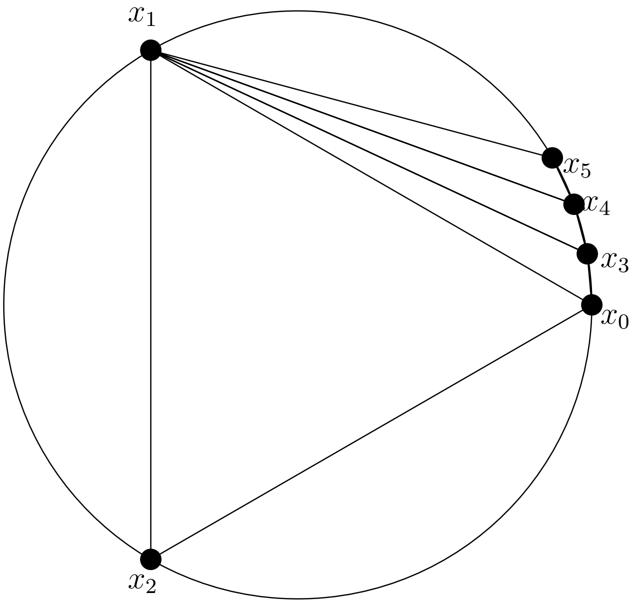
since $\mu/\sin(\mu/2) \leq \mu/\left(\mu-\frac{\mu^3}{24}\right)$ and

$$\frac{\mu}{\mu-\frac{\mu^3}{24}}-1 = \frac{\mu^3}{24\mu-\mu^3} \leq \mu^2,$$

because of our choice of μ. Hence, it follows that for any $i,j > 2$ we have $\delta_T(i,j) \leq 1+\varepsilon^2$, since $\mu \leq \varepsilon$.

Now let us look at $\delta_T(x_2, x_i)$ for $i > 2$. Let μ be the angle $\angle(x_0 \mathrm{O} x_i)$. Then the shortest path between x_i and x_2 has length at most $\sqrt{3}+\mu$, since the length of line segment $\overline{x_0 x_2}$ is $\sqrt{3}$.

Figure 5.6: A triangulation of X that has graph theoretic dilation $1 + O(\varepsilon)$. All vertices are adjacent to x_1.



The Euclidean distance between x_i and x_2 is $2\sin\left(\left(\frac{2}{3}\pi + \mu\right)/2\right)$. Consequently,

$$\begin{aligned}
\delta_T(x_2, x_i) \le \frac{\sqrt{3}+\mu}{2\sin\left(\frac{\pi}{3}+\frac{\mu}{2}\right)} &= \frac{\sqrt{3}+\mu}{2\left(\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\mu}{2}\right)+\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\mu}{2}\right)\right)} \\
&= \frac{\sqrt{3}+\mu}{\sqrt{3}\cos\left(\frac{\mu}{2}\right)+\sin\left(\frac{\mu}{2}\right)} \\
&\le \frac{\sqrt{3}+\mu}{\sqrt{3}\left(1-\frac{\mu^2}{4}\right)+\frac{\mu}{2}-\frac{\mu^3}{12}} \\
&\le 1+4\mu,
\end{aligned}$$

because of our choice of μ. Once again, since $\mu \le \varepsilon$, it follows that for any $i > 2$ we have $\delta_T(x_2, x_i) \le 1 + O(\varepsilon)$, and hence $\delta(T) \le 1 + O(\varepsilon)$. Hence, we have just proved

Theorem 5.5 *Let $n \in \mathbb{N}$ and $\varepsilon > 0$. It is possible to arrange n points on the unit circle in such a way that the graph theoretic dilation of the resulting point set is at most $1 + O(\varepsilon)$.*

Chapter 6

Implications of the Lower Bound

As usual, let $S_n = \{s_0, s_1, \ldots, s_{n-1}\}$ be the set of nodes of a regular n-gon in counter-clockwise order, and let T be any triangulation of S_n. In the previous chapters we have gone through some long and cumbersome calculations in order to prove some bounds on the graph theoretic dilation of S_n. In this chapter we are going to use these bounds in order to establish some structural properties of T. In particular, we are interested in the properties of any pair of nodes $(s_a, s_b) \in S_n^2$ such that $\delta(T) = \delta_T(s_a, s_b)$. We shall call such a pair a *maximum detour pair*, since the detour between s_a and s_b is maximal. It is trivial that such a pair always exists, but it does not have to be unique.

In section 6.1 we will prove that the Euclidean distance between the vertices of a maximum detour pair has to be at least 1.93185 if the radius of the regular n-gon is 1. If the radius of the n-gon is not 1, it is still true that the convex hull distance between the two vertices of a maximum detour pair is at least $\frac{5}{12}n$.

Another interesting property that we will derive in section 6.2 is that the shortest path between the two nodes of a maximum detour pair always includes at least one vertex of the central triangle, *i.e.,* in any triangulation T of S_n there are three distinguished vertices such that the shortest path between the vertices of any maximum detour pair for T has to include at least one of them.

The proofs of these two properties rely heavily on the lower bound which we have proved in the previous chapter. Therefore, we can only be sure that they are valid if we can be sure that the lower bound is valid, *i.e.,* for $n \geq 74$ or $n \equiv 0 \pmod 4$.

Finally, in section 6.3 we will outline a possible algorithmic use of the results that we obtain in this chapter.

6.1 A Bound on the Euclidean Distance of a Maximum Dilation Pair

In this section we shall prove that the Euclidean distance between the two vertices of a maximum detour pair is bounded from below by a pretty large constant. The main idea is that the fact that between any two vertices in S_n there is a path that goes along the convex hull of S_n gives us an upper bound on the detour between these two vertices. This upper bound can then be compared with the lower bound on $\delta(T)$ to obtain the desired property of a maximum detour pair.

More precisely, in section 4.1 we saw that an upper bound for the detour between any two distinct points s_a and s_b in S_n that have convex hull distance $\Delta = \Delta_{S_n}(s_a, s_b)$ is given by

$$\delta_T(s_a, s_b) \leq \frac{\Delta \sin\left(\frac{\pi}{n}\right)}{\sin\left(\Delta \frac{\pi}{n}\right)}. \tag{6.1}$$

On the other hand, theorem 5.4 from the previous chapter tells us that

$$\delta(T) \geq \sqrt{2-\sqrt{3}} + \frac{\sqrt{3}}{2}.$$

This means in particular that for any two points $s_x, s_y \in S_n$ with $\delta(T) = \delta_T(s_x, s_y)$, *i.e.*, for any maximum dilation pair (s_x, s_y), we have

$$\delta_T(s_x, s_y) \geq \sqrt{2-\sqrt{3}} + \frac{\sqrt{3}}{2}.$$

From this we can show that the convex hull distance Δ between s_x and s_y has to be more than $\frac{5}{12}n$. Indeed, in section 4.1, claim 4.1 on page 18 we have already seen that the upper bound defined in equation (6.1) grows monotonically with Δ and hence for $1 \leq \Delta \leq \frac{5}{12}n$ we have

$$\begin{aligned}
\frac{\Delta \sin\left(\frac{\pi}{n}\right)}{\sin\left(\Delta \frac{\pi}{n}\right)} &\leq \frac{\frac{5}{12}n \sin\left(\frac{\pi}{n}\right)}{\sin\left(\frac{5}{12}n\frac{\pi}{n}\right)} \\
&\overset{(1)}{\leq} \frac{\frac{5}{12}\pi}{\sin\left(\frac{5}{12}\pi\right)} \\
&< \sqrt{2-\sqrt{3}} + \frac{\sqrt{3}}{2},
\end{aligned}$$

where (1) is due to the fact that $\frac{5}{12}n \sin\left(\frac{\pi}{n}\right) \nearrow \frac{5}{12}\pi$, a variation of the limit $\frac{n}{2}\sin\left(\frac{\pi}{n}\right) \nearrow \frac{\pi}{2}$ which we computed in section 4.1.

Consequently, the Euclidean distance between s_x and s_y is larger than $2\sin\left(\frac{5}{12}\pi\right)$, which can be computed as

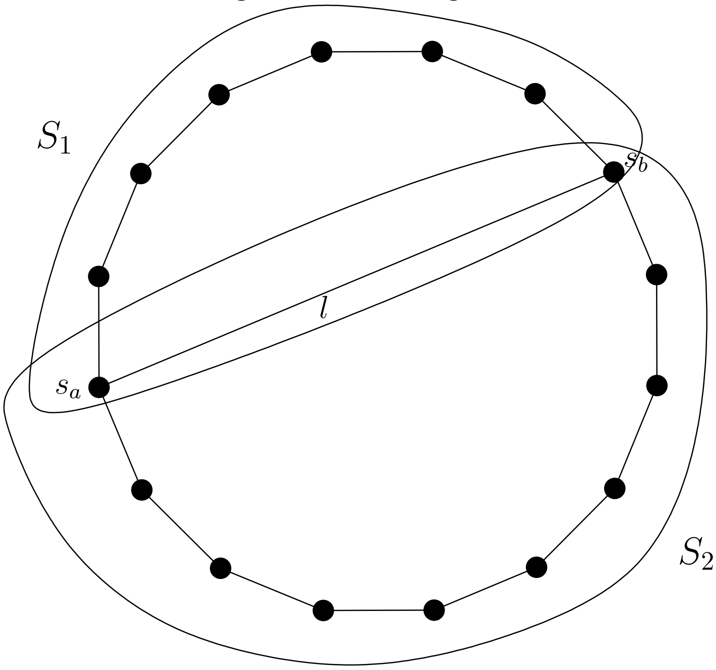
$$2\sin\left(\frac{5}{12}\pi\right) = \frac{1}{2}\left(\sqrt{6+3\sqrt{3}} + \sqrt{2-\sqrt{3}}\right) \approx 1.93185.$$

Let us state this result as a corollary:

Corollary 6.1 *Let $n \geq 74$ and S_n be the set of vertices of a regular n-gon. For any triangulation T of S_n and any $s_x, s_y \in S_n$ such that $\delta_T(s_x, s_y) = \delta(T)$, the convex hull distance $\Delta_{S_n}(s_a, s_b)$ between s_a and s_b is larger than $\frac{5}{12}n$. Furthermore, if the radius of S_n is 1, the Euclidean distance $|s_x s_y|$ is more than $\frac{1}{2}\left(\sqrt{6+3\sqrt{3}} + \sqrt{2-\sqrt{3}}\right) \approx 1.93185$.*

There are two things to remark about this corollary. First, it is interesting to note that this corollary implies that the vertices of maximum detour pairs are nearly diametrically opposite, which is also what one might expect intuitively. Furthermore, let us mention that obviously these estimates are not tight. The bound of $\frac{5}{12}n$ on the convex hull distance can definitely be strengthened. However, we have chosen to state the corollary as above, because this value gives a clean statement of the corollary above and turns out to be sufficient to prove the central triangle property of the next subsection.

Figure 6.1: Line segment l divides S_n into two sets S_1 and S_2.



6.2 The Shortest Path Crosses the Central Triangle

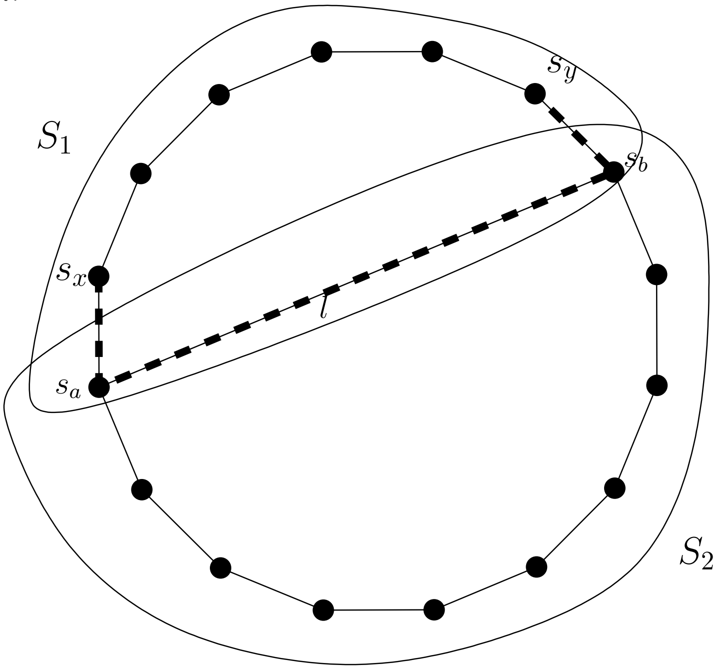
Let $s_x, s_y \in S_n$ such that $\delta_T(s_x, s_y) = \delta_T(S_n)$. Furthermore, let $l = \overline{s_a s_b}$ be a line segment of T. l divides S_n into two parts which we will call S_1 and S_2, where S_1 contains all the points in S_n that lie to the left of l and the endpoints of l and S_2 contains all the points that lie to the right of l and the endpoints of l (see figure 6.1). Without loss of generality, we may assume that $|S_1| \leq |S_2|$. The goal of this section is to show that it is not possible that both s_x and s_y lie in the smaller part S_1. The main idea behind the proof is as follows: Due to the results of the previous chapter it is not possible that the convex hull distance between s_x and s_y is too small. Therefore, s_x and s_y have to be pretty close to l. However, this gives us a path between s_x and s_y that achieves a detour that is below the detour that is necessitated by the lower bound of the previous chapter.

Therefore, let $\Delta = \Delta_{S_n}(s_a, s_b)$ be the convex hull distance between s_a and s_b. If $\Delta \leq \frac{5}{12}n$, it is obvious that the claim follows immediately from corollary 6.1. Otherwise, some more work is required.

Thus, let us assume that $\Delta > \frac{5}{12}n$, and for the sake of contradiction, we will suppose that both s_x and s_y lie in S_1. We may assume that s_a is closer to s_x than s_b and that s_b is closer to s_y than s_a (see figure 6.2). By corollary 6.1 the convex hull distance between s_x and s_y is larger than $\frac{5}{12}n$. We will now derive a contradiction to theorem 5.4 by showing that the detour between s_x and s_y is less than $\sqrt{2-\sqrt{3}} + \frac{\sqrt{3}}{2} \approx 1.3836$.

First, let us define $\lambda \stackrel{\text{def}}{=} \Delta_{S_n}(s_a, s_x) + \Delta_{S_n}(s_b, s_y)$ and $\mu \stackrel{\text{def}}{=} \Delta_{S_n}(s_a, s_x)$. With this notation, the Euclidean distance between s_x and s_y is $2\sin\left((\Delta - \lambda)\frac{\pi}{n}\right)$. Furthermore, there is a path between s_x and s_y that follows the convex hull from s_x to s_a, then uses line segment $\overline{s_a s_b}$ and finally goes along the convex hull from s_b to s_y (see figure 6.2). The length of this path is $2\sin\left(\Delta\frac{\pi}{n}\right) + 2\mu\sin\left(\frac{\pi}{n}\right) + 2(\lambda - \mu)\sin\left(\frac{\pi}{n}\right) = 2\sin\left(\Delta\frac{\pi}{n}\right) + 2\lambda\sin\left(\frac{\pi}{n}\right)$. Consequently, we can

Figure 6.2: We assume that both s_x and s_y lie in S_1, where s_x is closer to s_a and s_y is closer to s_b. The bold dashed line shows a path from s_x to s_y that uses the convex hull and line segment l.



bound the detour between s_x and s_y from above as follows:

$$\delta_T(s_x, s_y) \leq \frac{\sin\left(\Delta\frac{\pi}{n}\right) + \lambda\sin\left(\frac{\pi}{n}\right)}{\sin\left((\Delta - \lambda)\frac{\pi}{n}\right)},$$

where $\frac{5}{12}n \leq \Delta \leq \lfloor\frac{n}{2}\rfloor$ and $0 \leq \lambda \leq \Delta - \frac{5}{12}n$. Clearly, this upper bound grows as λ increases. As $\lambda \leq \Delta - \frac{5}{12}n$, we can conclude that

$$\delta_T(s_x, s_y) \leq \frac{\sin\left(\Delta\frac{\pi}{n}\right) + \left(\Delta - \frac{5}{12}n\right)\sin\left(\frac{\pi}{n}\right)}{\sin\left(\frac{5}{12}\pi\right)},$$

and this obviously grows with Δ. Thus, by substituting the upper bound for Δ we get

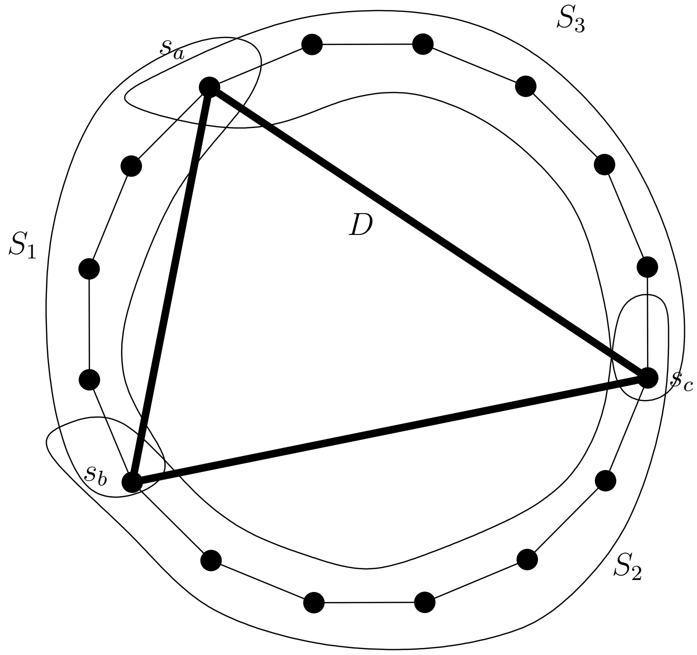
$$\begin{aligned} \delta_T(s_x, s_y) &\leq \frac{\sin\left(\frac{\pi}{2}\right) + \frac{n}{12}\sin\left(\frac{\pi}{n}\right)}{\sin\left(\frac{5}{12}\pi\right)} \\ &\leq \frac{1 + \frac{\pi}{12}}{\sin\left(\frac{5}{12}\pi\right)} \approx 1.306 \\ &< \sqrt{2 - \sqrt{3}} + \frac{\sqrt{3}}{2}. \end{aligned}$$

Since we assumed that the detour between s_x and s_y is maximal, we have thus reached a contradiction to theorem 5.4.

We can recapitulate the result of this section in the following

Corollary 6.2 *Let $n \geq 74$ and let T be an arbitrary triangulation of S_n, the set of vertices of a regular n-gon. Then for any $s_x, s_y \in S$ such that $\delta_T(s_x, s_y) = \delta_T(S)$ and any line segment $\overline{s_a s_b}$ in $E(T)$ that divides S into two subsets S_1 and S_2 with $|S_1| \leq |S_2|$ and $S_1 \cap S_2 = \{s_a, s_b\}$, it cannot be the case that both $s_x \in S_1$ and $s_y \in S_1$.*

Figure 6.3: The central triangle s_a, s_b, s_c divides S_n into three circular segments S_1, S_2, and S_3.



In particular, this corollary implies a very interesting property of maximum detour pairs which we state in the following corollary:

Corollary 6.3 *Let $n \geq 74$, let T be an arbitrary triangulation of S_n, the set of vertices of a regular n-gon, and let s_a, s_b, and s_c be the vertices of the central triangle of T. Then, for any $s_x, s_y \in S$ such that $\delta_T(s_x, s_y) = \delta_T(S)$, the shortest path in T between s_x and s_y contains at least one vertex in $\{s_a, s_b, s_c\}$.*

6.3 Possible Algorithmic Use

The results of this chapter suggest the following strategy for approximating the detour of a regular n-gon: Corollary 6.1 tells us that if we disregard the Euclidean distance in the definition of detour and merely optimize shortest path distances in a triangulation, we will still get a triangulation whose dilation is a constant factor approximation of the minimum graph theoretic dilation. Corollary 6.3 even tells us that we do not need to minimize all shortest path distances, but only the maximum length of any shortest path that crosses the central triangle.

In the following we will make this intuition more precise. Let $\mathcal{T}_n$ be the set of all triangulations of S_n and let $T \in \mathcal{T}_n$ be any triangulation of S_n with central triangle D. Furthermore, let us call the vertices of D s_a, s_b, and s_c. Obviously, D divides S_n into three circular segments which we will denote by S_1, S_2, and S_3 (see figure 6.3). Let

$$\mathcal{N}_T \stackrel{\text{def}}{=} \left\{(s_x, s_y) \in S_n^2 \,|\, s_x \in S_i, s_y \in S_j \text{ such that } i \neq j\right\}$$

denote the set of all pairs of nodes that lie in different circular segments. The shortest path between any pair of nodes in $\mathcal{N}_T$ has to pass a vertex of the central triangle.

Now assume that we have an algorithm that computes a triangulation T^* of S_n which minimizes the function $f : \mathcal{T}_n \to \mathbb{R}$ given by

$$f(T) = \max_{(s_a, s_b) \in \mathcal{N}_T} \pi_T(s_a, s_b).$$

Then it follows that T^* is an $\frac{25}{24}$ approximation for $\delta(S_n)$. To see why this is true let T_{OPT} be a minimum dilation triangulation for S_n. Then by corollary 6.3 it follows that

$$\begin{aligned}
\delta(T_{\mathrm{OPT}}) &= \max_{(s_a,s_b)\in\mathcal{N}_{T_{\mathrm{OPT}}}} \frac{\pi_{T_{\mathrm{OPT}}}(s_a,s_b)}{|s_a s_b|} \\
&\overset{(1)}{\geq} \frac{1}{2}\max_{(s_a,s_b)\in\mathcal{N}_{T_{\mathrm{OPT}}}} \pi_{T_{\mathrm{OPT}}}(s_a,s_b) \\
&\overset{(2)}{\geq} \frac{1}{2}\max_{(s_a,s_b)\in\mathcal{N}_{T^*}} \pi_{T^*}(s_a,s_b) \\
&\overset{(3)}{\geq} \frac{24}{25}\delta(T^*),
\end{aligned}$$

where inequality (1) follows from the fact that the Euclidean distance between any two points in S_n is at most 2 and inequality (2) follows from the fact that T^* minimizes $f(T)$. Finally, inequality (3) follows from the fact that $\frac{1.93185}{2} \geq \frac{24}{25}$ and corollary 6.1, which implies that in any triangulation T of S_n a maximum detour pair (s_a, s_b) fulfills

$$\delta_T(s_a,s_b) = \frac{\pi_T(s_a,s_b)}{|s_a s_b|} \leq \frac{\pi_T(s_a,s_b)}{1.93185}.$$

Thus, we have

$$\delta(T^*) \leq \frac{25}{24}\delta(T_{\mathrm{OPT}}).$$

It seems plausible that a technique that uses dynamic programming could be used in order to compute a triangulation which minimizes $f(T)$. Unfortunately, we could not figure out how to do this. We will report the results of some of our attempts in appendix A.1. Based on the ideas of this section, we have implemented a very crude heuristic which we also present in appendix A.1. The description of the algorithm is accompanied by some experimental results on the performance of this heuristics which suggest that the approximation obtained is not too bad.

Chapter 7

A $1+\frac{1}{\sqrt{\log n}}$ Approximation Algorithm

Let $S_n = \{s_0, s_1, \ldots, s_{n-1}\}$ be a planar point set in convex position. This chapter is devoted to the description of a fast approximation algorithm for the graph theoretic dilation of S_n. The underlying idea is very simple: If there are a lot of points on the convex hull of S_n, then it is likely that the detour will not change too much if we just throw away some of these points. Taking this idea to an extreme, if we could throw away all but logarithmically many points of S_n without affecting the graph theoretic dilation of S_n too much, we could use exhaustive search in order to find a minimum dilation triangulation for the logarithmic sample. Then we add the vertices we discarded before and add edges in an arbitrary manner until we obtain a triangulation T^* of S_n. This triangulation T^* will be our approximation of a triangulation that achieves the optimal graph theoretic dilation. In this chapter we are going to show that the graph theoretic dilation $\delta(T^*)$ of triangulation T^* approximates $\delta(S_n)$ up to a factor of $1+O\left(\frac{1}{\sqrt{\log n}}\right)$ if S_n is the set of vertices of a regular n-gon. This means that the approximation is arbitrarily close as n approaches infinity.

We will give a detailed description of the algorithm and an analysis of its running time in section 7.1.

The proof of correctness has two parts. The first part, which is presented in section 7.2, examines how the graph theoretic dilation of the sample changes when the remaining vertices of S_n are added. This step is necessary in order to relate the graph theoretic dilation of the sample to the graph theoretic dilation of S_n. In the second part of the proof in section 7.3 we compute how much larger the graph theoretic dilation of T^* can be compared to the graph theoretic dilation of the sample. Eventually, we will sandwich the graph theoretic dilation of S_n between two terms that approach the graph theoretic dilation of the logarithmic sample as n approaches infinity, which then proves the desired approximation.

7.1 Description of the Algorithm

In this section we will give a detailed description of the algorithm. As mentioned above, the algorithm takes a logarithmic sample A of S_n for which it computes a minimum dilation triangulation T_A and then extends this triangulation to a triangulation T^* of S_n.

The first step of the algorithm is to compute the total length of the convex hull of S_n, which we will denote by l. It is clear that l can be computed in linear time.

Using l, the algorithm then computes a distance d that is given by

$$d = \frac{2l}{\log n}.$$

The distance d is used in order to compute the logarithmic sample A of S_n. This is done as follows: The algorithm picks an arbitrary start vertex, say s_0, and then proceeds counter-clockwise along the convex hull of S_n. During this process it picks the first vertex that has distance at least d from s_0 along the convex hull, say s_α, and includes it in the sample. Then it picks the first vertex that has distance at least d from s_α, say s_β, and includes it in the sample, and so on. This process continues until the whole convex hull of S_n has been processed.

To formalize the process we just described, let us define inductively

$$\begin{aligned} a_0 &= s_0, \\ A_0 &= \left\{ s_j \,\middle|\, \sum_{k=0}^{j-1} |s_k s_{k+1}| < d \right\}. \end{aligned} \tag{7.1}$$

and

$$\begin{aligned} a_{i+1} &= s_{\lambda_{i+1}}, \text{where } \lambda_{i+1} = \min \left\{ \alpha \geq 0 \,\middle|\, s_\alpha \notin \bigcup_{k=0}^{i} A_k \right\}, \\ A_{i+1} &= \left\{ s_j \,\middle|\, j \geq \lambda_{i+1}, \sum_{k=\lambda_{i+1}}^{j-1} |s_k s_{k+1}| < d \right\}, \end{aligned} \tag{7.2}$$

as long as $\lambda_{i+1} \neq -\infty$. Then our sample is given by

$$A = \{a_0, a_1, \ldots, a_k\}, \tag{7.3}$$

where k is the number of sets that have been defined by equations (7.1) and (7.2) (see figure 7.1 for an illustration). Note that this process can be performed on any arbitrary convex set.

It is clear that this sampling process can be carried out in linear time by a single walk along the convex hull of S_n and that the size of the sample A is at most $\frac{\log n}{2}$.

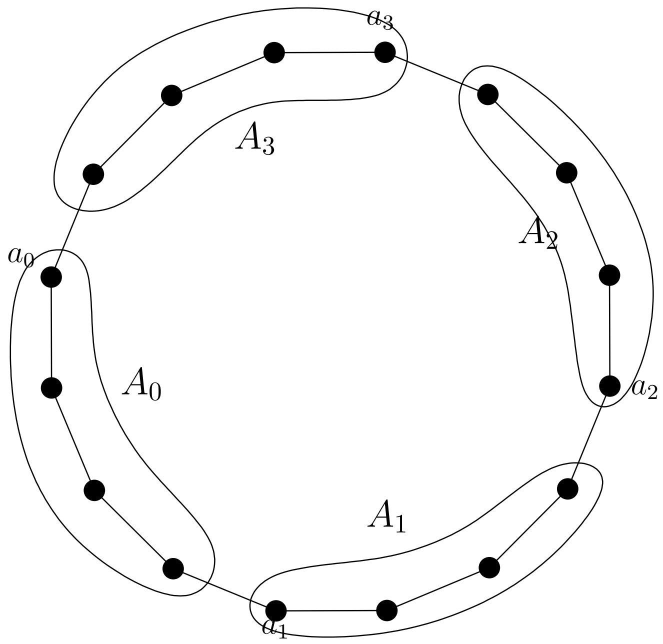
Now the algorithm determines a triangulation T_A of A such that $\delta(A) = \delta(T_A)$. This is done by brute force by enumerating all possible triangulations of A and choosing one that achieves the minimum graph theoretic dilation. As we mentioned already in chapter 3, for a convex set X with m points there are exactly $C_{m-2} = \frac{1}{m-1}\binom{2m-4}{m-2}$ different triangulations of X, where C_m is called the m-th Catalan number. From *Stirling's Formula* $m! \sim \left(\frac{m}{e}\right)^m \sqrt{2\pi m}$ (see, *e.g.*, [AE99]), it follows that

$$\begin{aligned} C_m &= \frac{1}{m+1} \frac{(2m)!}{(m!)^2} \\ &\sim \frac{1}{m+1} \frac{\left(\frac{2m}{e}\right)^{2m} \sqrt{4\pi m}}{\left(\frac{m}{e}\right)^{2m} 2\pi m} \\ &= \frac{2^{2m}}{(m+1)\sqrt{\pi m}}. \end{aligned}$$

Hence, the number of different triangulations of A is at most

$$C_{(\log n)/2} = \frac{2^{\log n}}{((\log n)/2 + 1)\sqrt{\pi (\log n)/2}} = O\left(\frac{n}{\log^{\frac{3}{2}} n}\right).$$

Figure 7.1: Computing the logarithmic sample: Each set A_i is represented by the respective vertex a_i. In this example, we have $k = 3$.



It is possible to enumerate all these triangulations with very small overhead (see appendix B), and the graph theoretic dilation of a given triangulation of A can be computed in time $O\left(|A|^2\right)$ [RKRS97]. Thus, it is possible to compute T_A in time $O\left(n\sqrt{\log n}\right)$.

Finally, the algorithm proceeds to extend T_A to a triangulation T^* of S_n. To do this we connect any point in $S_n \backslash A$ with the vertex in A that is closest to it and add edges in an arbitrary manner as long as the graph remains planar. It is clear that this step can be carried out in $O(n)$ time, since the maximum number of edges in a planar graph is linear in the number of vertices. The resulting triangulation T^* is our approximation of a minimum dilation triangulation. We shall examine the quality of this approximation in the ensuing two sections.

Table 7.1 summarizes the main steps of the algorithm. Note that the algorithm is stated in such a way that S_n can be an arbitrary set of n points in convex position. In our analysis, however, we will require that S_n be the vertices of a regular n-gon.

Before we proceed, let us state the running time of the algorithm in the following:

Theorem 7.1 *The number of steps the algorithm described in table 7.1 needs in order to compute its result is at most* $O\left(n\sqrt{\log n}\right)$*, where* $n = |S_n|$ *denotes the number of points in the input.*

7.2 A Bound on the Dilation of the Sample

In this section we will relate the graph theoretic dilation of the sample A to the detour between any pair of points in A when the points from $S_n \backslash A$ are added. The experimental results from chapter 3 show that it is possible that the graph theoretic dilation of A decreases when points are added outside the convex hull of A. We are going to show that this decrease cannot be arbitrarily large.

Table 7.1: Pseudo-code for the Approximation Algorithm

Input: A convex, finite planar point set $S_n = \{s_0, \ldots, s_{n-1}\}$ in counter-clockwise order.
Output: A triangulation T of S_n that approximates $\delta(S_n)$.

1. COMPUTE THE LENGTH l OF THE CONVEX HULL OF S_n.
2. $d \leftarrow \frac{2l}{\log n}$
3. INDUCTIVELY COMPUTE A SAMPLE SET $A \subseteq S_n$ AS GIVEN BY EQUATION (7.3)
4. COMPUTE THE MINIMUM DILATION TRIANGULATION T_A FOR A.
5. ADD THE POINTS OF $S_n \backslash A$ TO T_A AS DESCRIBED ABOVE IN ORDER TO OBTAIN A TRIANGULATION T^* OF S_n.
6. OUTPUT T^*.

We start by considering a shortest path P between two points a and b in an arbitrary triangulation T of a finite, and planar point set X in convex position. The line segment $\overline{ab}$ divides X into two sets $Y, Z \subseteq X$ such that $Y \cap Z = \{a, b\}$ and Y is the set of points to the left of $\overline{ab}$ and Z is the set of points to the right of $\overline{ab}$. Let us order the vertices in $Y = \{y_1, y_2, \ldots, y_\sigma\}$ and $Z = \{z_1, z_2, \ldots, z_\tau\}$ in increasing convex hull distance from a, *i.e.,* $y_i \leq y_j \Leftrightarrow \Delta_X(a, y_i) \leq \Delta_X(a, y_j)$ and $z_i \leq z_j \Leftrightarrow \Delta_X(a, z_i) \leq \Delta_X(a, z_j)$. Furthermore, assume that Y and Z are numbered in that order. Now, let

$$a = p_1 \rightarrow p_2 \rightarrow \cdots \rightarrow p_{m-1} \rightarrow p_m = b$$

be the sequence of nodes along P. Then the two sub-sequences

$$P \cap Y = \left(a = y_{i_0}, y_{i_1}, y_{i_2}, \ldots, y_{i_{\alpha-1}}, y_{i_\alpha} = b\right)$$

and

$$P \cap Z = \left(a = z_{j_0}, z_{j_1}, z_{j_2}, \ldots, z_{i_{\beta-1}}, z_{i_\beta} = b\right)$$

are strictly monotonically increasing (see figure 7.2).

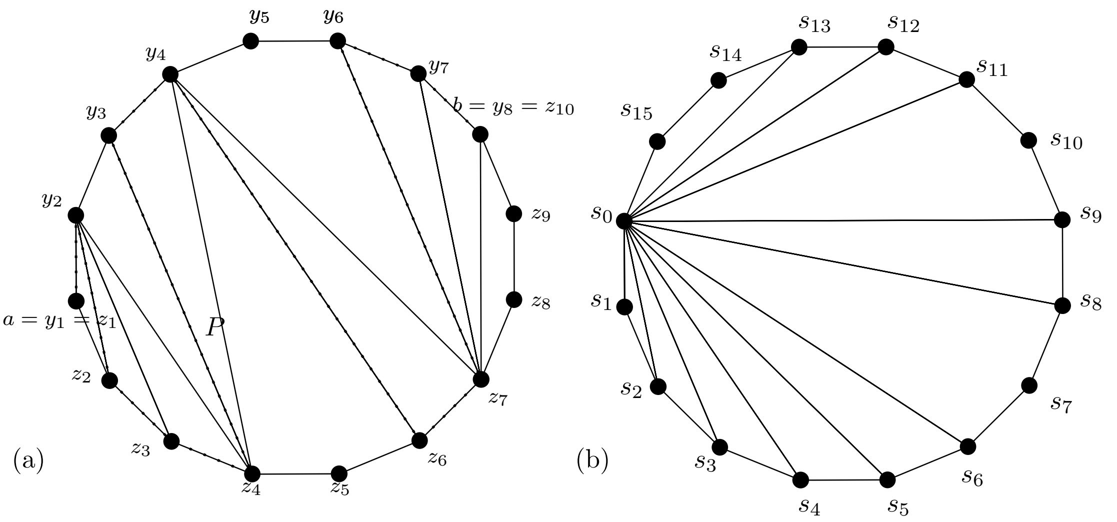
To see why this claim holds observe that obviously no vertex in X can appear twice on a shortest path from a to b. Now suppose that there is a pair of consecutive vertices $(y_{i_\lambda}, y_{i_{\lambda+1}})$ in, say, $P \cap Y$ such that $y_{i_{\lambda+1}} < y_{i_\lambda}$ and that λ is the smallest index with that property (obviously, we have $y_{i_{\lambda+1}} \neq a$). This means that up to λ the sequence $P \cap Y$ is strictly monotonically increasing, and therefore there is a $\mu < \lambda + 1$ such that $y_{i_{\mu-1}} < y_{i_{\lambda+1}} < y_{i_\mu}$. Again, we have $y_{i_\mu} \neq a$. Now, let p be the vertex on P that immediately precedes y_{i_μ}. It follows that $y_{i_{\lambda+1}}$ lies to the left of line segment $\overline{py_{i_\mu}}$, while b lies to the right of line segment $\overline{py_{i_\mu}}$. Since T is planar, every path from $y_{i_{\lambda+1}}$ to b must pass p or y_{i_μ}, which contradicts the fact that P is a shortest path from a to b and that $y_{i_{\lambda+1}}$ appears on P after p and y_{i_μ}. $P \cap Z$ is handled in an analogous manner.

From now on, we will consider the case that $X = S_n$, the set of nodes of a regular n-gon with radius 1. Let us look at the following sequence

$$d_i = |p_\alpha p_\beta|\,, \text{ where } \alpha = \max\{k \leq i \,|p_k \in P \cap Y\} \text{ and } \beta = \max\{k \leq i \,|p_k \in P \cap Z\}\,,$$

for $1 \leq i \leq m$.

Figure 7.2: In (a), we see a shortest path from a to b (bold dotted line). The sequences $P \cap Y$ and $P \cap Z$ are strictly monotonically increasing. The thin chords are the lines that define the values of the d_i. Figure (b) shows the equivalent arrangements of these chords that is used in the proof that $(d_i)_{1 \le i \le m}$ is bitonic (claim 7.2).



Claim 7.2 *The sequence $(d_i)_{1 \le i \le m}$ is* bitonic, *i.e., it is first monotonically increasing and then monotonically decreasing.*

Proof: For each d_i, let $(p_{\alpha_i}, p_{\beta_i})$ be the corresponding pair of nodes on P. It is clear that the sequence $h_i = \Delta_{S_n}(p_{\alpha_i}, a) + \Delta_{S_n}(a, p_{\beta_i})$, which counts the "hops" along the convex hull from p_{α_i} to p_{β_i} over a, is strictly increasing and that d_i is the same as $|s_0 s_{h_i}|$. Now the claim follows immediately, since every circle is unimodal (see figure 7.2). ■

We will try to describe geometrically what is going in the above proof. It is clear that d_i is bitonic as long as P does not change sides, *i.e.*, as long as P does not change from Y to Z or vice versa. Thus, the only critical situation arises when P changes sides. However, what we do in the proof is not to let P change sides but to flip S_n. From this point of view, it is obvious that claim 7.2 holds.

Now let us consider the following set of of indices:

$$M \stackrel{\text{def}}{=} \left\{ 1 \le i \le m \,\middle|\, d_i \ge 2 \sin\left(\frac{l}{2\sqrt{\log n}}\right) \right\}, \tag{7.4}$$

where l denotes the length of the convex hull of S_n. Observe that M is an interval since $(d_i)_{1 \le i \le m}$ is a bitonic sequence. Furthermore, let us call a node p_i on P a *jump node*, if either $p_i \in P \cap Y$ and $p_{i+1} \in P \cap Z$ or $p_i \in P \cap Z$ and $p_{i+1} \in P \cap Y$, *i.e.*, P changes sides between p_i and p_{i+1}. In the following claim we shall bound the number of jump nodes with indices in M.

Claim 7.3 *The number of jump nodes with indices in M is $O\left(\sqrt{\log n}\right)$, or, more formally:*

$$|\{p_i \in P \,|\, i \in M \text{ and } p_i \text{ is jump node }\}| = O\left(\sqrt{\log n}\right).$$

Proof: If p_i is a jump node, this means that the edge $\overline{p_i p_{i+1}}$ increases the length of P by d_i. However, the length of P can be at most $\frac{l}{2}$, while each jump node with index in M increases

Figure 7.3: An example of a contraction step. x and y are contracted into y, *i.e.,* the line segment connecting x and y becomes a loop and all the line segments that ended in x now end in y.

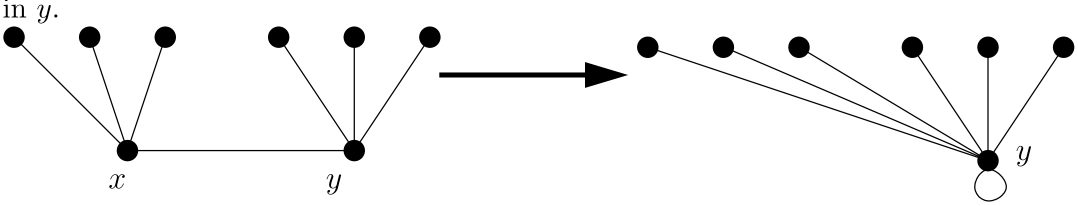
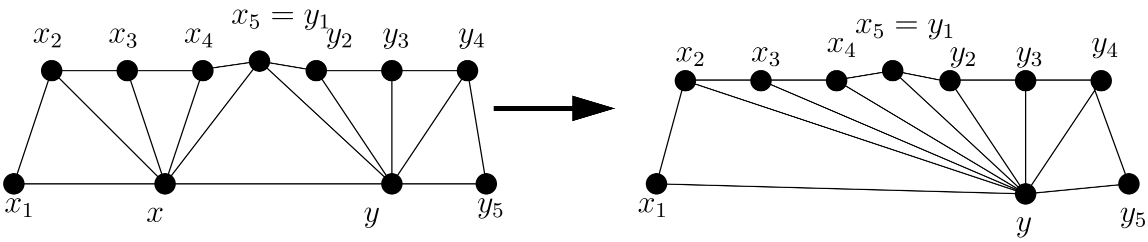


Figure 7.4: A typical step during the contraction that produces $\hat{T}$. The x_i are all the neighbors of x, the y_i are all the neighbors of y. x and y have exactly one common neighbor.



the length of P by at least $2\sin\left(\frac{l}{2\sqrt{\log n}}\right)$. Furthermore, we have that $2\sin\left(\frac{l}{2\sqrt{\log n}}\right) \geq \frac{l}{2\sqrt{\log n}}$ if n is large enough, because $\sin(x) \geq \frac{x}{2}$ for $0 \leq x \leq \sqrt{3}$. It follows that the number of jump nodes with index in M is at most

$$\frac{l}{2} \cdot \frac{2\sqrt{\log n}}{l} = \sqrt{\log n},$$

which proves our claim. ■

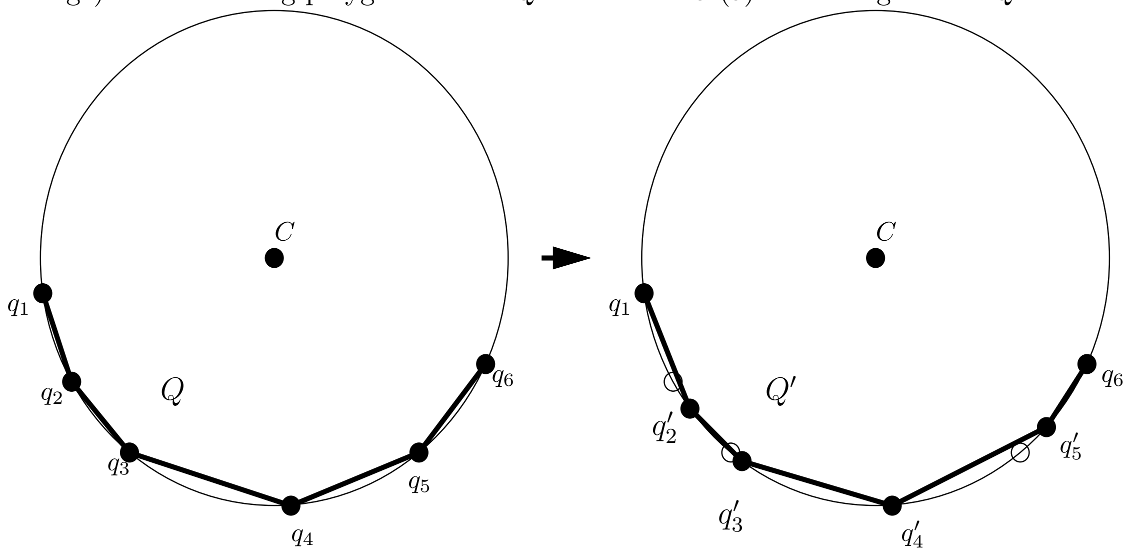
After these preparations, let us now come to the heart of the argument. Let T be an arbitrary triangulation of S_n, and let A_i and a_i be the sets and points defined by equations (7.1) and (7.2). Now let us consider the graph $\hat{T}$ which we get when we contract all the vertices in A_i into vertex a_i for each i and then delete all the loops and multiple edges. This contraction process is performed as a series of single *contraction steps*, in which two vertices of T are contracted into one vertex. An example of one such contraction step is shown in figure 7.3.

Claim 7.4 *$\hat{T}$ is a planar subdivision with vertex set A.*

Proof: What we need to prove is that $\hat{T}$ is planar, *i.e.,* we need to show that it cannot happen that two edges cross after all the contraction steps have taken place. Obviously, it suffices to verify this claim for a single contraction step. Thus, assume that vertices x and y are contracted into y. Since T is a triangulation and S_n is convex, the situation looks exactly as in figure 7.4, where the $x_1, \ldots, x_\sigma$ are the neighbors of x and $y_1, \ldots, y_\tau$ are the neighbors of y.

The only line segments that change throughout the contraction step are those incident with x. The polygon P defined by the vertices $x, x_1, \ldots, x_\sigma = y_1, \ldots, y_\tau, y$ (in that order) is convex, and no line segment outside this polygon is affected by the contraction step. After the contraction step P is subdivided such that all vertices are adjacent to y, and since P is convex this subdivision of P is planar. ■

Figure 7.5: Points q_2, q_3, q_4, and q_5 are perturbed to q_2', q_3', q_4', and q_5' (note that q_4 does not change). The resulting polygonal chain Q' is at most $O(\varepsilon)$ units longer than Q.



Let $a, b \in A$ be two vertices in A. Now that we have established that $\hat{T}$ is a planar subdivision with vertex set A, we would like to know how the length of the shortest path $\pi_T(a, b)$ between a and b in T relates to the length of the shortest path $\pi_{\hat{T}}(a, b)$ between a and b in $\hat{T}$.

Let us first consider a very special case.

Claim 7.5 *Let $q_1, q_2, \ldots, q_s$ be a set of points on a semicircle of radius 1 with center C in counter-clockwise order and let $Q = q_1 \to q_2 \to \cdots \to q_s$ be the polygonal chain along these points. If we perturb each of the points $q_2, \ldots, q_{s-1}$ by an angle of at most ε along the semicircle in counter-clockwise direction and call the new points $q_2', \cdots, q_{s-1}'$, then the length of polygonal chain $Q' = q_1 \to q_2' \to \ldots \to q_{s-1}' \to q_s$ can be bounded by $|Q'| \le |Q| + O(\varepsilon)$, where $|Q|$ and $|Q'|$ denote the length of polygonal chains Q and Q', respectively.*

See figure 7.5 for an illustration of this claim.

Proof: Let $l_i = \overline{q_i q_{i+1}}$ and $l_i' = \overline{q_i' q_{i+1}'}$ be the line segments of polygonal chains Q and Q' for $1 \le i < s$ (naturally, we set $q_1' = q_1$ and $q_s' = q_s$). Furthermore, for each line segment $l_i = \overline{q_i q_{i+1}}$ let $\alpha(l_i)$ denote the angle $\angle(q_i C q_{i+1})$. We will prove the following stronger claim:

$$\left|Q'\right| - |Q| \le \left(\alpha\left(q_1, q_s\right)\right)^2 \sin\left(\frac{\varepsilon}{2}\right). \tag{7.5}$$

Let us first consider the case that $s = 3$, and let $\sigma = \alpha(q_1, q_2)$ and $\tau = \alpha(q_2, q_3)$. Obviously, σ and τ fulfill the constraints $0 \le \sigma, \tau \le \alpha(q_1, q_3) \le \pi$. Furthermore, we have $|l_1| = 2\sin\left(\frac{\sigma}{2}\right)$ and $|l_2| = 2\sin\left(\frac{\tau}{2}\right)$, and if we perturb q_2 by an angle μ, we get that

$$f(\sigma, \mu, \tau) = \left|Q'\right| - |Q| = 2\sin\left(\frac{\sigma}{2} + \frac{\mu}{2}\right) + 2\sin\left(\frac{\tau}{2} - \frac{\mu}{2}\right) - 2\sin\left(\frac{\sigma}{2}\right) - 2\sin\left(\frac{\tau}{2}\right).$$

Elementary calculus shows that if we fix τ and μ this expression is maximal if $\sigma = 0$, since $\frac{\partial f(\sigma,\mu,\tau)}{\partial \sigma} = \cos\left(\frac{\sigma}{2} + \frac{\mu}{2}\right) - \cos\left(\frac{\sigma}{2}\right) \le 0$.

Therefore, we have that

$$f(\sigma, \mu, \tau) \le 2\sin\left(\frac{\mu}{2}\right) + 2\sin\left(\frac{\tau}{2} - \frac{\mu}{2}\right) - 2\sin\left(\frac{\tau}{2}\right),$$

and again simple calculus shows that this upper bound is maximal if $\tau = \alpha(q_1, q_3)$, since $\frac{\partial f(\sigma,\mu,\tau)}{\partial \tau} = \cos\left(\frac{\tau}{2} - \frac{\mu}{2}\right) - \cos\left(\frac{\tau}{2}\right) \geq 0$.

Consequently, we can conclude that

$$\begin{aligned}
|Q'| - |Q| &\leq 2\sin\left(\frac{\mu}{2}\right) + 2\sin\left(\frac{\alpha(p_1,p_3)}{2} - \frac{\mu}{2}\right) - 2\sin\left(\frac{\alpha(p_1,p_3)}{2}\right) \\
&= 2\sin\left(\frac{\mu}{2}\right) + 2\sin\left(\frac{\alpha(p_1,p_3)}{2}\right)\cos\left(\frac{\mu}{2}\right) \\
&\quad -2\cos\left(\frac{\alpha(p_1,p_3)}{2}\right)\sin\left(\frac{\mu}{2}\right) - 2\sin\left(\frac{\alpha(p_1,p_3)}{2}\right) \\
&\leq 2\left(1 - \cos\left(\frac{\alpha(p_1,p_3)}{2}\right)\right)\sin\left(\frac{\mu}{2}\right) \\
&\leq \frac{(\alpha(p_1,p_3))^2}{4}\sin\left(\frac{\mu}{2}\right), \qquad (7.6)
\end{aligned}$$

as $\cos(x) \geq 1 - \frac{x^2}{2}$. Since $\mu \leq \varepsilon$, claim (7.5) follows.

Now let us look at the case $s > 3$. Let β_i denote the angle $\angle(q_i C q_{i+2})$ for $1 \leq i \leq s-2$. We perturb the q_i as follows: in the first step, we fix $q_1, q_3, \ldots, q_s$ and perturb q_2 to get q_2'. In the second step, we fix $q_1, q_2', q_4, \ldots, q_s$ and perturb q_3 to get q_3'. We continue in this fashion, until all the q_i have been moved. From (7.6) it follows that in the i-th step the length of the polygonal chain increases by at most $\beta_i^2 \sin\left(\frac{\varepsilon}{2}\right)/4$, since all the points are perturbed in counter-clockwise direction and hence we have $\angle(q_i' C q_{i+2}) \leq \beta_i$. Thus, the total increase in length is at most

$$\frac{1}{4}\sin\left(\frac{\varepsilon}{2}\right)\sum_{i=1}^{s-2}\beta_i^2.$$

Now claim (7.5) follows from the fact that

$$\sum_{i=1}^{s-2}\beta_i \leq 2\alpha(q_1, q_s),$$

and hence

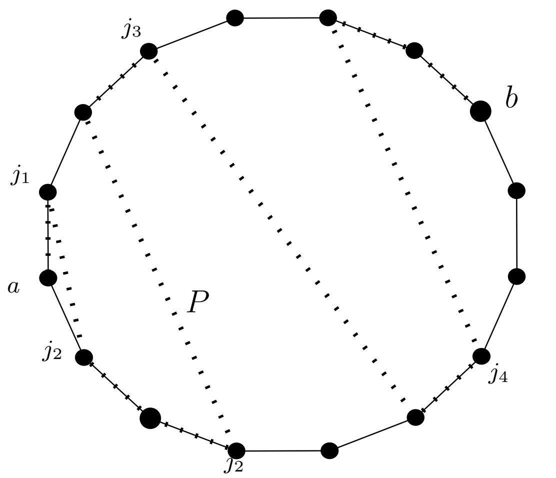
$$\sum_{i=1}^{s-2}\beta_i^2 \leq 4\alpha(q_1, q_s)^2,$$

since all the β_i are positive.

Thus, claim 7.5 follows, since $\sin\left(\frac{\varepsilon}{2}\right) \leq \varepsilon$. ■

Now let us consider a shortest path P between two points a and b in S_n. First, we look at $P \cap M$, the sub-path of P whose vertices are those whose indices lie in M, where M is the index set defined in equation (7.4). We have already seen that the shortest path $P \cap M$ changes sides at most $O\left(\sqrt{\log n}\right)$ times. The perturbation of each of these jump nodes can increase the length of P by at most $O(l/\log n)$, and between the jump nodes we have the situation we examined in the last claim, where we saw that the length of the shortest path between the jump nodes can increase by at most $O(l/\log n)$, since the perturbation angle is larger than the perturbation distance (see figure 7.6). Thus, the total length of the sub-path $P \cap M$ increases by at most $O\left(l\sqrt{\log n}/\log n\right) = O\left(l/\sqrt{\log n}\right)$. We do not know what happens to the shortest path outside M, but by our choice of M we do know that the increase in length is $O\left(l/\sqrt{\log n}\right)$, since the shortest path outside M consists of two paths. each of which is restricted to a circular segment

Figure 7.6: A typical shortest path from a to b. Between any two sequential jump nodes, the length of the shortest path increases by at most $O\left(\frac{l}{\log n}\right)$.



that is defined by a chord of length at most $2\sin\left(l/2\sqrt{\log n}\right)$, and the length of such a path can be at most $l/\sqrt{\log n}$, which bounds the length of the convex hull.

Therefore, we have proved that the for any $a, b \in A^2$ we have $\pi_{\hat{T}}(a,b) \leq \pi_T(a,b) + O\left(l/\sqrt{\log n}\right)$. Furthermore, by corollary 6.1 on page 46 we know that any pair of vertices in A that achieves maximum detour has a Euclidean distance that is $\Omega(l)$. Thus, we have shown that for any triangulation T of S_n in which the maximum detour between any pair of points in A is δ, there is a triangulation $\hat{T}$ of A such that $\delta\left(\hat{T}\right) \leq \delta + O\left(1/\sqrt{\log n}\right)$.

Consequently, by taking the minimum on both sides we can conclude with the following

Lemma 7.6 *Let S_n be the set of vertices of a regular n-gon, and let $A \subseteq S_n$ be the subset of S_n that is defined by equation (7.3). Then we have*

$$\delta(S_n) \geq \delta(A) - O\left(\frac{1}{\sqrt{\log n}}\right).$$

7.3 Computing the Approximation Factor

In this section we will relate the graph theoretic dilation $\delta(T^*)$ of the triangulation T^* of the whole set S_n that is computed by our algorithm to the graph theoretic dilation $\delta(T_A)$ of the minimum dilation triangulation T_A of the sample A. The argument we use is very similar to the analysis of the graph theoretic dilation of the regular triangulation which we gave in section 4.3.

In order to get the desired relationship let us compute an upper bound on the detour between the vertices of a maximum detour pair $x, y \in S_n \backslash A$. Let $a, b \in A$ be the points in A that are closest to x and y, respectively. Since we know that the Euclidean distance between the vertices of a maximum detour pair is $\Omega(l)$, we can assume that $a \neq b$ and $|ab| > \frac{4l}{\log n}$. By our definition of A it follows that the distance between x and a as well as y and b along the convex hull (and hence the Euclidean distance) is at most $d = \frac{2l}{\log n}$. Thus, we can upperbound the shortest path length $\pi_{T^*}(x,y)$ between x and y by $\pi_{T^*}(x,y) \leq \pi_{T_A}(a,b) + 2d$. Furthermore, we can lower-bound the Euclidean distance $|xy|$ between x and y by

$$|xy| \geq |ab| - 2d. \tag{7.7}$$

Thus, the detour $\delta_{T^*}(x,y)$ between x and y in T^* is at most

$$\begin{aligned}
\delta_{T^*}(x,y) &\leq \frac{\pi_{T_A}(a,b)+2d}{|xy|} \\
&= \frac{\pi_{T_A}(a,b)}{|ab|} + \frac{(\pi_{T_A}(a,b)+2d)\,|ab| - \pi_{T_A}(a,b)\,|xy|}{|ab|\,|xy|} \\
&\overset{(1)}{\leq} \frac{\pi_{T_A}(a,b)}{|ab|} + \frac{(\pi_{T_A}(a,b)+2d)\,|ab| - \pi_{T_A}(a,b)\,(|ab|-2d)}{|ab|\,|xy|} \\
&= \frac{\pi_{T_A}(a,b)}{|ab|}\left(1+\frac{2d\,|ab| + 2d\pi_{T_A}(a,b)}{\pi_{T_A}(a,b)\,|xy|}\right) \\
&\overset{(2)}{\leq} \frac{\pi_{T_A}(a,b)}{|ab|}\left(1+\frac{4d}{|xy|}\right) \\
&\overset{(3)}{=} \delta(T_A)\left(1+O\left(\frac{1}{\log n}\right)\right).
\end{aligned}$$

In this chain of inequalities, (1) is due to equation (7.7), (2) is due to the fact that $\frac{|ab|}{\pi_{T_A}(a,b)} = \frac{1}{\delta_{T_A}(a,b)} \leq 1$, and (3) holds because a maximum detour pair in S_n has detour $\Omega(l)$ and $d = 2l/\log n$. Since x and y were arbitrary, we can now conclude the following fundamental inequality

$$\delta(T^*) \leq \delta(T_A)\left(1+O\left(\frac{1}{\log n}\right)\right). \tag{7.8}$$

Together with lemma 7.6 this inequality finally yields the desired theorem that proves the correctness of our algorithm:

Theorem 7.7 *Let S_n be the vertex set of a regular n-gon. Then the triangulation T^* of S_n that is computed by the algorithm described in section 7.1 has the property that $\delta(T^*)$ approximates $\delta(S_n)$ up to a factor of $1+O\left(1/\sqrt{\log n}\right)$, i.e.,*

$$\delta(T^*) \leq \left(1+O\left(\frac{1}{\sqrt{\log n}}\right)\right)\delta(S_n).$$

Proof: Obviously, we have $\delta(T^*) \geq \delta(T_A)$ and thus inequality (7.8) yields

$$\delta(T_A) \leq \delta(T^*) \leq \delta(T_A)\left(1+O\left(\frac{1}{\log n}\right)\right).$$

Furthermore, by definition we have $\delta(S_n) \leq \delta(T^*)$ and thus by lemma 7.6 it follows that

$$\delta(T_A) - O\left(\frac{1}{\sqrt{\log n}}\right) \leq \delta(S_n) \leq \delta(T^*).$$

Therefore, we get

$$\delta(T_A)\left(1-O\left(\frac{1}{\sqrt{\log n}}\right)\right) \overset{(1)}{\leq} \delta(T_A) - O\left(\frac{1}{\sqrt{\log n}}\right) \leq \delta(S_n) \overset{(2)}{\leq} \delta(T_A)\left(1+O\left(\frac{1}{\sqrt{\log n}}\right)\right),$$

where (1) follows from the fact that $\delta(T_A) \geq 1$ and (2) is due to the fact that $1/\log n \leq 1/\sqrt{\log n}$. Hence, the approximation factor follows, because

$$\frac{1+\frac{1}{\sqrt{\log n}}}{1-\frac{1}{\sqrt{\log n}}} = \frac{1+\frac{2}{\sqrt{\log n}}+\frac{1}{\log n}}{1-\frac{1}{\log n}} = 1+O\left(\frac{1}{\sqrt{\log n}}\right).$$

■

Chapter 8

Concluding Remarks

Let us conclude this thesis with some reflections on the results we obtained and with some possible directions for further work.

In the course of this thesis, we have made some progress in the field of minimum dilation triangulations, and we have identified some useful properties of minimum dilation triangulations of the regular n-gon which we used in order to obtain an efficient approximation algorithm. In particular, the property that any maximum detour pair must have a large Euclidean distance has proved very useful. However, the main question how to compute a minimum detour triangulation for an *arbitrary* planar point set remains wide open.

A promising direction seems to try to generalize the results of this thesis to arbitrary fat point sets, *i.e.,* point sets that can bounded by two circles whose radii have a constant ratio. The approximation algorithm we devised can be applied to arbitrary convex sets, but in order to prove the approximation factor, we need a lower bound on the minimum Euclidean distance between the two vertices of a maximum detour pair. If such a bound can be shown for certain types of convex planar point sets, the approximation algorithm will yield a triangulation whose graph theoretic dilation approximates the graph theoretic dilation for these point sets.

Another direction might be to look for other local properties that characterize the minimum dilation triangulation, for example something that has the flavor of the diamond property mentioned in the introduction. Such properties will extend our understanding of minimum dilation triangulations and could be useful in devising new efficient approximation or exact algorithms for the problem.

Appendix A

Some Heuristics

In this appendix we present two heuristic approaches to the minimum dilation triangulation problem. One of these heuristics is based on the observation that the Euclidean distance between the vertices of a maximum detour pair must be large and therefore it suffices to optimize the shortest path distances in order to get a close approximation. However, we do not know how to compute explicitly a triangulation which optimizes the shortest path distances. Therefore, the heuristic only approximates such a triangulation. It is described in section A.1.

The other heuristic uses the intuition that the minimum detour triangulation should be symmetric. Thus, the heuristic proceeds by recursively constructing a triangulation that is as symmetric as possible. The details of this algorithm are presented in section A.2.

A.1 A Shortest Path Heuristic

As always, let $S_n = \{s_0, s_1, \dots, s_{n-1}\}$ be the nodes of a regular n-gon in counter-clockwise order. In this section we will describe a crude heuristic that is based on the general idea that is outlined in section 6.3.

We will describe the basic idea and the detailed steps of the heuristic in subsection A.1.1.

In subsection A.1.2 we will describe some experimental results that show how well the algorithm performs in practice.

Unfortunately, we cannot prove anything about the quality of approximation that the algorithm in this section achieves. However, the intuition from section 6.3 and the experimental results suggest that the heuristic will yield triangulations that are not too bad.

A.1.1 Description of the Algorithm

Let us first describe the basic idea of the algorithm. Corollary 6.1 shows that any pair of nodes which achieves the maximum graph theoretic dilation has a Euclidean distance that is bounded from below by a constant. Therefore, if we merely optimize the maximum shortest path distance between any two nodes while we disregard their Euclidean distance, it is still possible to achieve a triangulation whose graph theoretic dilation is within a constant factor of the optimum. This in itself is not very spectacular, since we have already seen that *any* triangulation of the nodes of a regular n-gon has a graph theoretic dilation that is within a constant factor of the optimum, but we expect the approximation factor of our heuristic to be better than that (and indeed in section 6.3 we show that an algorithm that actually computes a triangulation that minimizes the maximum length of any shortest path that crosses the central triangle can achieve an approximation factor of $\frac{25}{24}$).

The heuristic consists of two main steps. In the first step the algorithm computes for every circular segment that consists of at most $\lfloor \frac{n}{2} \rfloor$ nodes a triangulation that approximates a triangulation that minimizes the maximum shortest path distance from any node inside the circular segment to any of the two endpoints of the circular segment. The second step pieces the circular segments together in order to obtain a triangulation of S_n that minimizes the maximum length of a shortest path between any point in S_n and any of the vertices of the central triangle.

Now we describe the details of the heuristic. For $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ let $S_k \stackrel{\text{def}}{=} \{s_0, \ldots, s_k\}$ and let $\mathcal{T}_k$ be the set of all triangulations of S_k. In the first step the algorithm approximates a triangulation that minimizes the functions $f_k : \mathcal{T}_k \to \mathbb{R}$ for $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ which are given by

$$f_k(T) \stackrel{\text{def}}{=} \max_{0 \leq i \leq k} \max \{\pi_T(s_0, s_i), \pi_T(s_i, s_k)\},$$

where $\pi_T(x, y)$ denotes the shortest path distance between x and y in T. This means that $f_k(T)$ denotes the maximum length of a shortest path from any vertex of the circular segment S_k to one of its endpoints. The approximation is done using a dynamic programming approach. For any $1 \leq \mathtt{k} \leq \frac{n}{2}$ we maintain a structure called `segment[k]` which stores information about the triangulation of S_k which is supposed to approximate a triangulation that minimizes $f_k(T)$. The fields of `segment[k]` consist of two real arrays π_a, π_b and an integer `vertex`. The arrays `segment[k].`π_a`[i]` and `segment[k].`π_b`[i]` store the length of the shortest path from s_i to s_0 and s_k, respectively, where $0 \leq \mathtt{i} \leq \mathtt{k}$. The field `segment[k].vertex` stores the index of the third vertex of the triangle in which line segment $\overline{s_0 s_{\mathtt{k}}}$ participates.

The contents of `segment[k]` are computed recursively. If $\mathtt{k} = 1$, we obviously have

$$\begin{aligned} \mathtt{segment[1].vertex} &= 0, \\ \mathtt{segment[1].}\pi_a\mathtt{[0]} &= 0, \\ \mathtt{segment[1].}\pi_a\mathtt{[1]} &= 2\sin\left(\frac{\pi}{n}\right), \\ \mathtt{segment[1].}\pi_b\mathtt{[0]} &= 2\sin\left(\frac{\pi}{n}\right), \\ \mathtt{segment[1].}\pi_b\mathtt{[1]} &= 0. \end{aligned} \tag{A.1}$$

For $2 \leq \mathtt{k} \leq \lfloor \frac{n}{2} \rfloor$ we proceed as follows: Let $0 \leq \mathtt{vertex} \leq \mathtt{k}$ and $0 \leq \mathtt{i} \leq \mathtt{k}$. Furthermore, let $\pi_a^{\mathtt{vertex}}\mathtt{[i]}$ denote the length of the shortest path between s_0 and $s_{\mathtt{i}}$ in the triangulation $T_{\mathtt{k}}^{\mathtt{vertex}}$ of $S_{\mathtt{k}}$ that is obtained by using the triangulation that is stored in `segment[vertex]` for $\{s_0, \ldots, s_{\mathtt{vertex}}\}$ and the triangulation that is stored in `segment[k - vertex]` for point set $\{s_{\mathtt{vertex}}, \ldots, s_{\mathtt{k}}\}$ and by adding line segment $\overline{s_0 s_{\mathtt{k}}}$. It is clear that

$$\pi_a^{\mathtt{vertex}}\mathtt{[i]} = \begin{cases} \mathtt{segment[vertex].}\pi_a\mathtt{[i]}, & \text{if } \mathtt{i} < \mathtt{vertex}, \\ |s_0 s_{\mathtt{vertex}}|, & \text{if } \mathtt{i} = \mathtt{vertex}, \\ \min\{\mathtt{segment[k - vertex].}\pi_a\mathtt{[i - k]} + |s_0 s_{\mathtt{vertex}}|, & \\ \quad \mathtt{segment[k - vertex].}\pi_b\mathtt{[i - k]} + |s_0 s_{\mathtt{k}}|\}, & \text{if } \mathtt{i} > \mathtt{vertex}. \end{cases} \tag{A.2}$$

Similarly, the length of the shortest path between s_i and s_k in $T_k^{\mathtt{vertex}}$ which we call $\pi_b^{\mathtt{vertex}}\mathtt{[i]}$ can be computed by

$$\pi_b^{\mathtt{vertex}}\mathtt{[i]} = \begin{cases} \min\{\mathtt{segment[vertex].}\pi_a\mathtt{[i]} + |s_0 s_{\mathtt{k}}|, & \\ \quad \mathtt{segment[vertex].}\pi_b\mathtt{[i]} + |s_{\mathtt{vertex}} s_{\mathtt{k}}|\}, & \text{if } \mathtt{i} < \mathtt{vertex}, \\ |s_{\mathtt{vertex}} s_{\mathtt{k}}|, & \text{if } \mathtt{i} = \mathtt{vertex}, \\ \mathtt{segment[k - vertex].}\pi_b\mathtt{[i - k]}, & \text{if } \mathtt{i} > \mathtt{vertex}. \end{cases} \tag{A.3}$$

Thus, in order to compute `segment[k]`, the algorithm can go through all the possible values for $0 \le \texttt{vertex} \le k$ and choose the value which minimizes

$$\max_{0 \le i \le k} \max \left\{ \pi_a^{\texttt{vertex}}\texttt{[i]}, \pi_b^{\texttt{vertex}}\texttt{[i]} \right\}.$$

This value is stored in `segment[k].vertex`, and the corresponding arrays $\pi_a^{\texttt{vertex}}$ and $\pi_b^{\texttt{vertex}}$ are stored in $\texttt{segment[k].}\pi_a$ and $\texttt{segment[k].}\pi_b$, respectively. This concludes the description of the first step of the algorithm. The reason why this step only provides a heuristic and does not actually compute triangulations that minimize the functions f_k is that f_k does not exhibit the optimal substructure property which would be necessary for the algorithm to work, *i.e.,* it is not necessarily true that there exists an optimal triangulation for f_k that contains two optimal sub-triangulations for f_s and f_t for some $1 \le s, t < k, s + t = k$.

The purpose of the second step of the algorithm is to piece together the triangulations from the first step in order to obtain a triangulation that approximates a triangulation which minimizes the maximum length of a shortest path that crosses the central triangle. This is done as follows: Let s_0, s_a and s_b be the the vertices of the central triangle, where $a \le b$. S_n is divided into three subsets $X = \{s_0, \ldots, s_a\}$, $Y = \{s_a, \ldots, s_b\}$, and $Z = \{s_b, \ldots, s_{n-1}, s_0\}$. For each of these sets, we use the appropriate triangulation which we computed in the first step, and compute the following quantity

$$\max \left\{ \max_{\substack{x \in X \\ y \in Y}} \{\pi(x, s_a) + \pi(y, s_a)\}, \max_{\substack{x \in X \\ z \in z}} \{\pi(x, s_0) + \pi(z, s_0)\}, \max_{\substack{y \in Y \\ z \in Z}} \{\pi(y, s_b) + \pi(z, s_b)\} \right\}. \tag{A.4}$$

This quantity can be computed quickly by using the values stored in `segment[k]`. By trying all possible values for s_a and s_b we obtain a triangulation T that minimizes (A.4) among all the triangulations at consideration. Triangulation T is the output of the algorithm. A summary of the steps can be found in table A.1.1. It is clear that the heuristic can be implemented to run in time $O\left(n^3\right)$.

A.1.2 Experimental Results

Even though the heuristic described in the previous section is very crude and nothing can be shown about its quality of approximation, it does not perform too bad in practice. Table A.2 shows the dilation of the triangulations that were computed by the algorithm for some n between 100 and 1000, and figure A.1 shows two of these triangulations for $n = 100$ and $n = 500$. The structure of these triangulations is quite interesting. The requirement that it must be possible to reach any of the vertices of the central triangle quickly leads to "highways" in the circular segments, *i.e.,* diagonals which make it possible to get from one end of a circular segment to the other end.

The results in table A.2 show that the heuristic yields triangulations that have lower graph theoretic dilations than the canonical triangulation. However, the results are not as good as the results that can be obtained by the heuristic we present in the next section.

A.1.3 A Possible Dynamic Program

In the first step the algorithm from this section tries to find a triangulation T^* of $S_k = \{s_0, \ldots, s_k\}$ that minimizes the function $f_k : \mathcal{T}_k \to \mathbb{R}$ which is given by

$$f_k(T) \stackrel{\text{def}}{=} \max_{0 \le i \le k} \max \{\pi_T(s_0, s_i), \pi_T(s_i, s_k)\},$$

Table A.1: Pseudo-code for the Shortest Path Heuristic

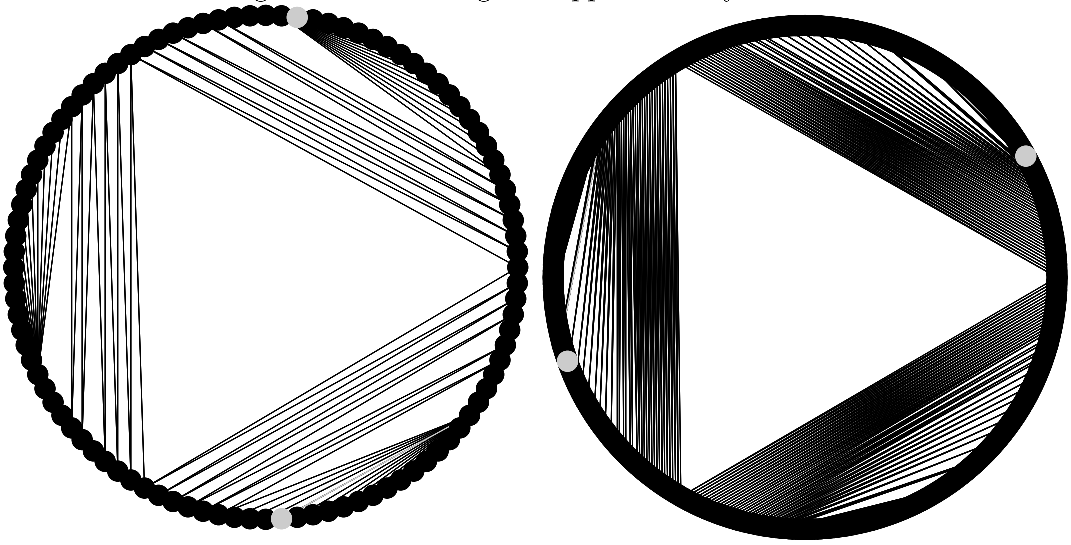
Input: The set $S_n = \{s_0, \ldots, s_{n-1}\}$ of vertices of a regular n-gon in counter-clockwise order.
Output: A triangulation T of S_n that approximates $\delta(S_n)$.

1. INITIALIZE `segment[1]` AS GIVEN BY EQUATION (A.1).
2. FOR $\texttt{k} \leftarrow 2$ `TO` $\lfloor \frac{n}{2} \rfloor$ DO
3. $\quad$ $\texttt{minimum} \leftarrow \infty$; $\quad \texttt{minVertex} \leftarrow 0$
4. $\quad$ FOR $\texttt{vertex} \leftarrow 0$ `TO` `k` DO
5. $\quad\quad$ COMPUTE $\pi_a^{\texttt{vertex}}, \pi_b^{\texttt{vertex}}$ AS GIVEN BY EQUATIONS (A.2) AND (A.3).
6. $\quad\quad$ $\texttt{maxPath} \leftarrow \max_{0 \leq i \leq \texttt{k}} \max \{\pi_a^{\texttt{vertex}}\texttt{[i]}, \pi_b^{\texttt{vertex}}\texttt{[i]}\}$
7. $\quad\quad$ IF $\texttt{maxPath} < \texttt{minimum}$ THEN
8. $\quad\quad\quad$ $\texttt{minimum} \leftarrow \texttt{maxPath}$; $\quad \texttt{minVertex} \leftarrow \texttt{vertex}$
9. $\quad$ $\texttt{segment[k].vertex} \leftarrow \texttt{minVertex}$
10. $\quad$ $\texttt{segment[k].}\pi_a \leftarrow \pi_a^{\texttt{minVertex}}$; $\quad \texttt{segment[k].}\pi_b \leftarrow \pi_b^{\texttt{minVertex}}$
11. DETERMINE THE TRIANGULATION `T` AS DESCRIBED.
12. OUTPUT `T`.

Table A.2: Graph theoretic dilation of the triangulations computed by the two heuristics for some values of n. T_{sp} denotes the triangulations produced by the shortest path heuristic. T_{sym} denotes the triangulations computed by the symmetry heuristic.

n	$\delta(T_{sp})$	$\delta(T_{sym})$
100	1.46852	1.45737
192	1.47101	1.45474
200	1.47411	1.45743
300	1.47410	1.45416
384	1.47633	1.45474
400	1.47508	1.45981
500	1.47528	1.45823
600	1.47225	1.45720
700	1.47560	1.45769
768	1.47249	1.45672
800	1.47401	1.45930
900	1.47466	1.45806
1000	1.47323	1.45833

Figure A.1: The triangulations computed by the shortest path heuristic for $n = 100$ and $n = 500$. The graph theoretic dilation of triangulation on the left is about 1.46852, the graph theoretic dilation of the triangulation on the right is approximately 1.47528.



where $\pi_T(x, y)$ denotes the shortest path distance between x and y in T. However, the algorithm does not really compute a triangulation that minimizes f_k, but merely an approximation for such a triangulation, since it is not clear how to compute such a triangulation in polynomial time. In this subsection we shall describe a way to solve this problem exactly by dint of a dynamic programming approach. Unfortunately, this approach takes exponential time.

The approach is to solve a more general problem. Instead of minimizing f_k, we minimize a function $g_k : \mathcal{T}_k \times \mathbb{R}^4 \to \mathbb{R}$ which is given by

$$\begin{aligned} g_k(T, \alpha, \beta, \xi, \zeta) &= \max_{0 \le i \le k} \min\bigl\{\pi_T(s_0, s_i) + \xi, \pi_T(s_i, s_k) + \zeta, \\ &\qquad \max\{\pi_T(s_0, s_i) + \alpha, \pi_T(s_i, s_k) + \beta\}\bigr\}. \end{aligned}$$

Let T^* be a triangulation of S_k that minimizes g_k for fixed values of α, β, ξ, and ζ. Line segment $\overline{s_0 s_k}$ participates in exactly one triangle in T^*. Let the third vertex of this triangle be s_u. We have

$$\begin{aligned} g_k(T^*, \alpha, \beta, \xi, \zeta) &= \max_{0 \le i \le k} \min\bigl\{\pi_{T^*}(s_0, s_i) + \xi, \pi_{T^*}(s_i, s_k) + \zeta, \\ &\qquad \max\{\pi_{T^*}(s_0, s_i) + \alpha, \pi_{T^*}(s_i, s_k) + \beta\}\bigr\}, \end{aligned}$$

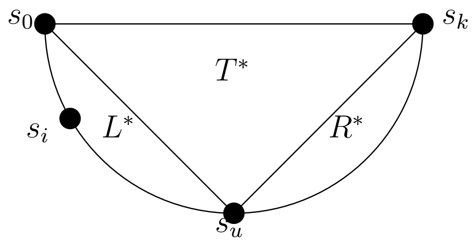
and we can split up the maximum term according to whether $i \le u$ or $i \ge u$. Thus, we get

$$g_k(T^*, \alpha, \beta, \xi, \zeta) = \max\Bigl\{g_k^l(L^*, \alpha, \beta, \xi, \zeta), g_k^r(R^*, \alpha, \beta, \xi, \zeta)\Bigr\},$$

where L^* is the sub-triangulation of T^* on the vertex set $\{s_0, \ldots, s_u\}$, R^* is the sub-triangulation of T^* on the vertex set $\{s_u, \ldots, s_k\}$, and

$$\begin{aligned} g_k^l(L^*, \alpha, \beta, \xi, \zeta) &= \max_{0 \le i \le u} \min\bigl\{\pi_{T^*}(s_0, s_i) + \xi, \pi_{T^*}(s_i, s_k) + \zeta, \\ &\qquad \max\{\pi_{T^*}(s_0, s_i) + \alpha, \pi_{T^*}(s_i, s_k) + \beta\}\bigr\}, \end{aligned}$$

Figure A.2: The shortest path from s_i to s_0 lies completely in L^*. The shortest path from s_i to s_k either passes s_0 or s_u. In the former case, is has length $\pi_{L^*}(s_0, s_i) + |\overline{s_0 s_k}|$, in the latter case it has length $\pi_{L^*}(s_i, s_u) + |\overline{s_u s_k}|$.



and

$$g_k^r(R^*, \alpha, \beta, \xi, \zeta) = \max_{u \le i \le k} \min\Big\{\pi_{T^*}(s_0, s_i) + \xi, \pi_{T^*}(s_i, s_k) + \zeta, \max\{\pi_{T^*}(s_0, s_i) + \alpha, \pi_{T^*}(s_i, s_k) + \beta\}\Big\}.$$

Let us now consider the sub-triangulation L^*. If s_i lies in L^*, *i.e.,* $0 \le i \le u$, then it is clear that the shortest path between s_0 and s_i lies completely in L^*, and thus we have $\pi_{T^*}(s_0, s_i) = \pi_{L^*}(s_0, s_i)$. The shortest path from s_i to s_k has to pass either s_0 or s_u. In the former case, it uses line segment $\overline{s_0 s_k}$, in the latter case it uses line segment $\overline{s_u s_k}$. Thus, we get $\pi_{T*}(s_0, s_i) = \min\{\pi_{L^*}(s_0, s_i) + |\overline{s_0 s_k}|, \pi_{L^*}(s_i, s_u) + |\overline{s_u s_k}|\}$ (see figure A.2). Hence, it follows that

$$\begin{aligned} g_k^l(L^*, \alpha, \beta, \xi, \zeta) = \max_{0 \le i \le u} \min\Big\{ & \pi_{L^*}(s_0, s_i) + \xi, \pi_{L^*}(s_0, s_i) + \zeta + |\overline{s_0 s_k}|, \\ & \pi_{L^*}(s_i, s_u) + \zeta + |\overline{s_u s_k}|, \\ & \max\big\{\pi_{L^*}(s_0, s_i) + \alpha, \min\{\pi_{L^*}(s_0, s_i) + \beta + |\overline{s_0 s_k}|, \\ & \qquad \pi_{L^*}(s_i, s_u) + \beta + |\overline{s_u s_k}|\}\big\}\Big\}. \end{aligned}$$

Now the lattice structure of $(\mathbb{R}, \min, \max)$ implies that

$$\max\{x, \min\{y, z\}\} = \min\{\max\{x, y\}, \max\{x, z\}\}.$$

Thus, we can conclude that

$$\begin{aligned} g_k^l(L^*, \alpha, \beta, \xi, \zeta) &= \max_{0 \le i \le u} \min\Big\{\pi_{L^*}(s_0, s_i) + \min\{\xi, \zeta + |\overline{s_0 s_k}|, \max\{\alpha, \beta + |\overline{s_0 s_k}|\}\}, \\ & \qquad \pi_{T^*}(s_i, s_u) + \zeta + |\overline{s_u s_k}|, \\ & \qquad \max\{\pi_{L^*}(s_0, s_i) + \alpha, \pi_{L^*}(s_0, s_i) + \beta + |\overline{s_u s_k}|\}\Big\} \\ &= g_u(L^*, \alpha, \beta + |\overline{s_u s_k}|, \min\{\xi, \zeta + |\overline{s_0 s_k}|, \max\{\alpha, \beta + |\overline{s_0 s_k}|\}\}, \zeta + |\overline{s_u s_k}|). \end{aligned}$$

By an analogous argument we get

$$g_k^r\left(R^*,\alpha,\beta,\xi,\zeta\right) = g_{k-u}\big(R^*,\alpha+\left|\overline{s_0s_u}\right|,\beta,\xi+\left|\overline{s_0s_u}\right|, \min\left\{\zeta,\xi+\left|\overline{s_0s_k}\right|,\max\left\{\alpha+\left|\overline{s_0s_k}\right|,\beta\right\}\right\}\big).$$

Since $g_k\left(T,\alpha,\beta,\xi,\zeta\right)$ is minimal if both $g_k^l\left(L,\alpha,\beta,\xi,\zeta\right)$ and $g_k^r\left(R,\alpha,\beta,\xi,\zeta\right)$ are minimal for fixed α, β, ξ, and ζ, it now follows that

$$\begin{aligned}\mathrm{OPT}_k\left(\alpha,\beta,\xi,\zeta\right) = \min_{1\le u<k}\max\Big\{&\mathrm{OPT}_u\big(\alpha,\beta+\left|\overline{s_us_k}\right|,\min\big\{\xi,\zeta+\left|\overline{s_0s_k}\right|, \\ &\qquad\max\left\{\alpha,\beta+\left|\overline{s_0s_k}\right|\right\}\big\},\zeta+\left|\overline{s_us_k}\right|\big), \\ &\mathrm{OPT}_{k-u}\big(\alpha+\left|\overline{s_0s_u}\right|,\beta,\xi+\left|\overline{s_0s_u}\right|, \\ &\qquad\min\big\{\zeta,\xi+\left|\overline{s_0s_k}\right|,\max\left\{\alpha+\left|\overline{s_0s_k}\right|,\beta\right\}\big\}\big)\Big\},\end{aligned}$$

where $\mathrm{OPT}_k\left(\alpha,\beta,\xi,\zeta\right)$ denotes the minimum value of $g_k\left(T,\alpha,\beta,\xi,\zeta\right)$ for α, β, ξ, and ζ fixed. This is the recursion that can be used for dynamic programming. The optimum of f_k is given by $\mathrm{OPT}_k\left(0,0,\infty,\infty\right)$. Unfortunately, this dynamic program needs exponentially many steps, since exponentially many values for α,β,ξ, and ζ need to be considered.

A.2 A Symmetry Heuristic

Intuitively, we expect that the minimum dilation triangulation for S_n is very regular and symmetric. Therefore, it seems very plausible that we get a very good approximation for the minimum dilation triangulation when we construct a triangulation of S_n that is "as symmetric as possible". Indeed, the heuristic we are going to describe yields the best results of all the algorithms we tried, where our experiments were performed for values of n below 1000.

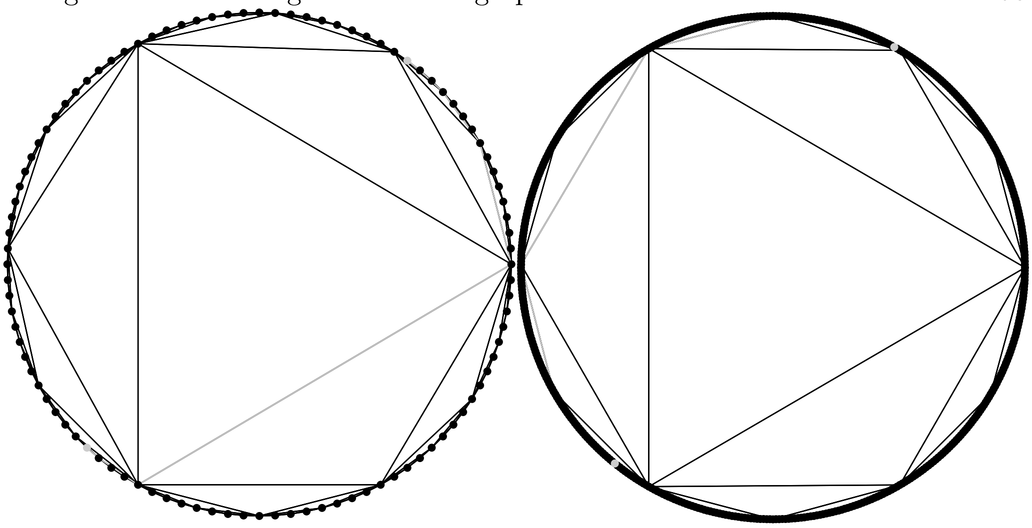
In section A.2.1 we shall describe the details of the heuristic. In section A.2.2 we present some experimental results that show that the heuristic performs very well in practice (at least for small values of n).

A.2.1 Description of the Algorithm

As mentioned above, the goal of the heuristic is to produce a triangulation that is as symmetric as possible. Thus, we proceed as follows: We pick an arbitrary start vertex, say s_0. Then we take the two vertices s_a and s_b that have convex hull distance $\lfloor\frac{n}{3}\rfloor$ from s_0, *i.e.,* $\Delta_{S_n}(s_0,s_a) = \Delta_{S_n}(s_0,s_b) = \lfloor\frac{n}{3}\rfloor$. The triangle s_0,s_a,s_b divides S_n into three subsets $X=\{s_0,\dots,s_a\}$, $Y=\{s_a,\dots,s_b\}$ and $Z=\{s_b,\dots,s_{n-1},s_0\}$. Now each of these subsets is triangulated separately. These triangulations for X,Y, and Z will be called T_X,T_Y, and T_Z, respectively. In order to obtain a triangulation T of S_n, triangulations T_X,T_Y and T_Z are simply glued together, *i.e.,* $T=T_X\cup T_Y\cup T_Z$.

It remains to describe how T_X,T_Y, and T_Z are computed. We just describe T_X, since T_Y and T_Z are produced in the same manner. T_X is computed recursively. If X consists of three or fewer points, then T_X is the unique triangulation of X, *i.e.,* a triangle or a line segment (or just a single point, if X is a singleton). Otherwise, let $X=\{x_0,x_1,\dots,x_q\}$ with $q\ge 4$, and let $a=x_{\lfloor\frac{q}{2}\rfloor}$. Furthermore, let $X_l=\{x_0,x_1,\dots,a\}$, and let $X_r=\left\{a,x_{\lfloor\frac{q}{2}\rfloor+1},\dots,x_q\right\}$. Recursively we triangulate X_l and X_r in order to get triangulations T_{X_l} and T_{X_r}. From these

Figure A.3: The triangulations computed by the symmetry heuristic for $n = 100$ and $n = 500$. The graph theoretic dilation of the triangulation on the left is approximately 1.45737. The triangulation on the right achieves a graph theoretic dilation that is about 1.45823.



triangulations we construct T_X by putting T_{X_l} and T_{X_r} together and adding line segment $\overline{x_0 x_q}$, *i.e.,* $T_x = T_{X_l} \cup T_{X_r} \cup \{\overline{x_0 x_q}\}$. This concludes the construction of T_X. It is clear that the algorithm needs $O(n)$ steps.

The triangulation produced by the heuristic is very regular. It contains a large regular triangle in the middle, and the sub-triangulations of subsets X, Y, and Z are chosen such that the paths to the vertices of the central triangle are relatively short. The minimum dilation triangulations shown in chapter 3 suggest that this is not too far from the structure of the actual minimum dilation triangulation.

A.2.2 Experimental Results

The symmetry heuristic yields the best results of all the heuristics we tried. This is also what we expect intuitively, since regularity seems to favor a small detour. Table A.2 shows the dilation of the triangulation that was computed by the algorithm for some n between 100 and 1000, and figure A.3 shows two of these triangulations for $n = 100$ and $n = 500$.

Appendix B

Enumerating Triangulations

Let $X = \{x_0, \ldots, x_{n-1}\}$ be a planar convex set of points in counter-clockwise order. In this appendix we are going to describe an efficient way to enumerate all the triangulations of X. The algorithm we present is fairly standard, but for the sake of completeness we decided to include it in this appendix.

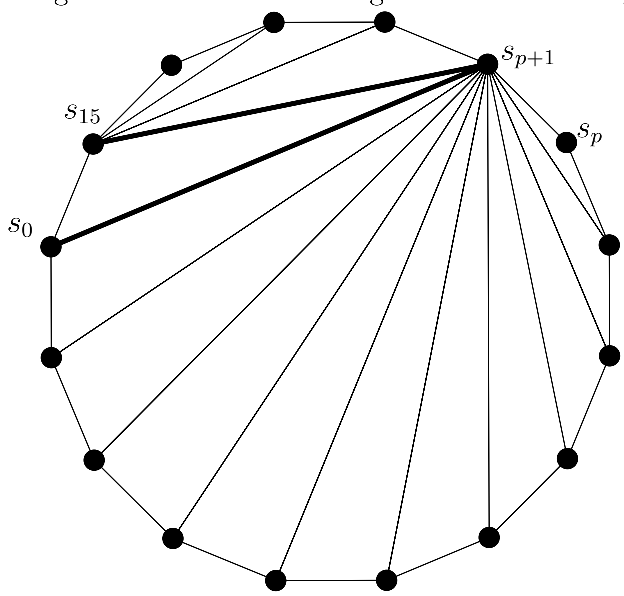
Before we describe the algorithm, let us mention how a triangulation is represented: We use an adjacency matrix representation, *i.e.,* a triangulation T is represented by an $n \times n$-matrix $A = (a_{ij})_{0 \le i,j < n}$ such that $a_{ij} \neq 0$ if and only if there is an edge between x_i and x_j in T. Since T is undirected, it follows that A is symmetric.

The idea of the enumeration algorithm is very simple. It is clear that line segment $\overline{x_0 x_{n-1}}$ participates in exactly one triangle in a given triangulation T. Let the third vertex of this triangle be x_p. Triangle x_0, x_p, x_{n-1} divides X into two sets $L = \{x_0, \ldots, x_p\}$ and $R = \{x_p, \ldots, x_{n-1}\}$. And T is divided into two sub-triangulations $T_L = T \cap L$ and $T_R = T \cap R$. In order to find the successor T' of T, we first try to find the successor T'_L of T_L recursively. If T'_L exists, we take $T' = T'_L \cup T_R \cup \{\overline{x_0, x_{n-1}}\}$. Otherwise, we try to find the successor T'_R of T_R. If T'_R exists, we take $T' = K_L \cup T'_R \cup \{\overline{x_0, x_{n-1}}\}$, where K_L denotes the canonical triangulation of L in which all nodes of L are adjacent to x_p (see section 4.2). Finally, when neither T_L nor T_R has a successor, we need to advance x_p by one node. That is, we take T' to be the triangulation in which the third vertex of the triangle in which line segment $\overline{x_0 x_{n-1}}$ participates is x_{p+1} and in which the set $\{x_0, \ldots, x_{p+1}\}$ is triangulated by a canonical triangulation such that all nodes are adjacent to x_{p+1} and the set $\{x_{p+1}, \ldots, x_{n-1}\}$ is triangulated in such a way that all nodes are adjacent to x_{n-1} (see figure B.1). The base of the recursion occurs when T is a triangle. In this case T does not have a successor. We can continue this enumeration process until $x_p = x_{n-1}$. In this case there does not exist any successor for T.

In order to implement the algorithm efficiently it is necessary to be able to retrieve the third vertex of the triangle in which line segment $x_0 x_{p-1}$ participates efficiently. The obvious way to do this is to store the index of this vertex in the entries of the adjacency matrix $a_{0(n-1)}$ and $a_{(n-1)0}$ that correspond to line segment $x_0 x_{p-1}$. Care has to be taken to initialize these values properly, but this is easily done.

This concludes our description of the enumeration algorithm. Obviously, the worst case running time of the algorithm in order to determine a successor triangulation is linear in the number of nodes in X. This is sufficient for our purposes, since this linear term is dominated by the time we need in order to compute the graph theoretic dilation of the successor triangulation. However, the algorithm is very similar to increasing a binary counter, and hence it is very likely that a more detailed analysis will yield better bounds on the amortized running time of the enumeration algorithm.

Figure B.1: The next triangulation immediately after the third vertex was moved to x_{p+1}.



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