The Dual Diameter of Triangulations*

Matias Korman[a,**], Stefan Langerman[b], Wolfgang Mulzer[c], Alexander Pilz[d], Maria Saumell[e], Birgit Vogtenhuber[d]

[a] *Tohoku University, Sendai, Japan*
[b] *Université Libre de Bruxelles (ULB), Brussels, Belgium*
[c] *Institut für Informatik, Freie Universität Berlin, Germany*
[d] *Institute for Software Technology, Graz University of Technology, Graz, Austria*
[e] *Department of Mathematics and European Centre of Excellence NTIS (New Technologies for the Information Society), University of West Bohemia, Czech Republic*

Abstract

Let $\mathcal{P}$ be a simple polygon with n vertices. The *dual graph* T^* of a triangulation T of $\mathcal{P}$ is the graph whose vertices correspond to the bounded faces of T and whose edges connect those faces of T that share an edge. We consider triangulations of $\mathcal{P}$ that minimize or maximize the diameter of their dual graph. We show that both triangulations can be constructed in $O(n^3 \log n)$ time using dynamic programming. If $\mathcal{P}$ is convex, we show that any minimizing triangulation has dual diameter exactly $2 \cdot \lceil \log_2(n/3) \rceil$ or $2 \cdot \lceil \log_2(n/3) \rceil - 1$, depending on n. Trivially, in this case any maximizing triangulation has dual diameter $n - 2$. Furthermore, we investigate the relationship between the dual diameter and the number of *ears* (triangles with exactly two edges incident to the boundary of $\mathcal{P}$) in a triangulation. For convex $\mathcal{P}$, we show that there is always a triangulation that simultaneously minimizes the dual diameter and maximizes the number of ears. In contrast, we give examples of general simple polygons where every triangulation that maximizes the number of ears has dual diameter that is quadratic in the minimum possible value. We also consider the case of point sets in general position in the plane. We show that for any such set of n points there are triangulations with dual diameter in $O(\log n)$ and in $\Omega(\sqrt{n})$.

Keywords: Triangulation, Dual Graph, Diameter, Optimization, Simple Polygon

In memoriam: Ferran Hurtado (1951–2014)

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1. Introduction

Let $\mathcal{P}$ be a simple polygon with $n > 3$ vertices. We regard $\mathcal{P}$ as a closed two-dimensional subset of the plane, containing its boundary. A *triangulation* T of $\mathcal{P}$ is a maximal crossing-free geometric (i.e., straight-line) graph whose vertices are the vertices of $\mathcal{P}$ and whose edges lie inside $\mathcal{P}$. Hence, T is an outerplanar graph. Similarly, for a set S of n points in the plane, a *triangulation* T of S is a maximal crossing-free geometric graph whose vertices are exactly the points of S. It is well known that in both cases all bounded faces of T are triangles. The *dual graph* T^* of T is the graph with a vertex for each bounded face of T and an edge between two vertices if and only if the corresponding triangles

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**Corresponding author

Email addresses: mati@dais.is.tohoku.ac.jp (Matias Korman), stefan.langerman@ulb.ac.be (Stefan Langerman), mulzer@inf.fu-berlin.de (Wolfgang Mulzer), apilz@ist.tugraz.at (Alexander Pilz), saumell@kma.zcu.cz (Maria Saumell), bvogt@ist.tugraz.at (Birgit Vogtenhuber)

share an edge in T. If all vertices of T are incident to the unbounded face, then T^* is a tree. An *ear* in a triangulation of a simple polygon is a triangle whose vertex in the dual graph is a leaf (equivalently, two out of its three edges are edges of $\mathcal{P}$). We call the diameter of the dual graph T^* the *dual diameter (of the triangulation T)*. In the following, we will study combinatorial and algorithmic properties of *minimum* and *maximum dual diameter triangulations* for simple polygons and for planar point sets (minDTs and maxDTs for short). Note that both triangulations need not to be unique.

Previous Work. Shermer [11] considers *thin* and *bushy* triangulations of simple polygons, i.e., triangulations that minimize or maximize the number of ears. He presents algorithms for computing a thin triangulation in time $O(n^3)$ and a bushy triangulation in time $O(n)$. Shermer also claims that bushy triangulations are useful for finding paths in the dual graph, as is needed, e.g., in geodesic algorithms. In that setting, however, the running time is not actually determined by the number of ears, but by the dual diameter of the triangulation. Thus, bushy triangulations are only useful for geodesic problems if there is a connection between maximizing the number of ears and minimizing the dual diameter. While this holds for convex polygons, we show that, in general, there exist polygons for which no minDT maximizes the number of ears. Moreover, we give examples where forcing a single ear into a triangulation may almost double the dual diameter, and the dual diameter of any bushy triangulation may be quadratic in the dual diameter of a minDT.

The dual diameter also plays a role in the study of edge flips: given a triangulation T, an *edge flip* is the operation of replacing a single edge of T with another one so that the resulting graph is again a valid triangulation. In the case of convex polygons, edge flips correspond to rotations in the dual binary tree [12]. For this case, Hurtado, Noy, and Urrutia [4, 13] show that a triangulation with dual diameter k can be transformed into a fan triangulation by a sequence of most k parallel flips (i.e., two edges not incident to a common triangle may be flipped simultaneously). They also obtain a triangulation with logarithmic dual diameter by recursively cutting off a linear number of ears.

While we focus on the dual graph of a triangulation, distance problems in the primal graph have also been considered. For example, Kozma [8] addresses the problem of finding a triangulation that minimizes the total link distance over all vertex pairs. For simple polygons, he gives a sophisticated $O(n^{11})$ time dynamic programming algorithm. Moreover, he shows that the problem is strongly NP-complete for general point sets when arbitrary edge weights are allowed and the length of a path is measured as the sum of the weights of its edges.

Our Results. In Section 2, we present several properties of the dual diameter for triangulations of simple polygons. Among other results, we calculate the exact dual diameter of minDTs and maxDTs of convex polygons, which can be obtained by maximizing and minimizing the number of ears of the triangulation, respectively. On the other hand, we show that there exist simple polygons where the dual diameter of any minDT is $O(\sqrt{n})$, while that of any triangulation that maximizes the number of ears is in $\Omega(n)$. Likewise, there exist simple polygons where the dual diameter of any triangulation that minimizes the number of ears is in $O(\sqrt{n})$, while the maximum dual diameter is linear. In Section 3, we present efficient algorithms to construct a minDT and a maxDT for any given simple polygon.

Finally, in Section 4 we consider the case of planar point sets, showing that for any point set in the plane in general position there are triangulations with dual diameter in $O(\log n)$ and in $\Omega(\sqrt{n})$, respectively.

2. The Number of Ears and the Diameter

The dual graph of any triangulation T has maximum degree 3. In this case, the so-called *Moore bound* implies that the dual diameter of T is at least $\log_2(\frac{t+2}{3})$, where t is the number of triangles in T (see, e.g., [9]). For convex polygons, we can compute the minimum dual diameter exactly.

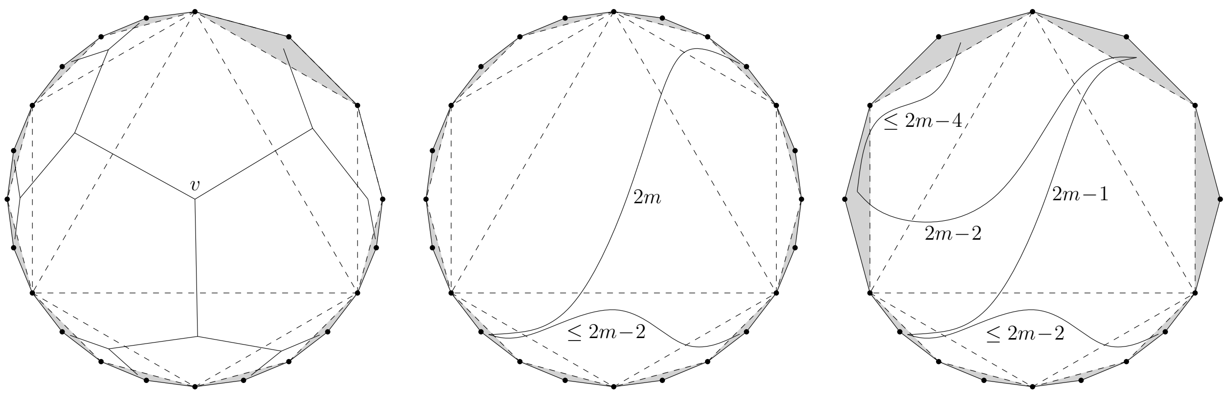
Proposition 2.1. *Let $\mathcal{P}$ be a convex polygon with $n \geq 3$ vertices, and let $m \geq 1$ such that $n \in \{3 \cdot 2^{m-1} + 1, \ldots, 3 \cdot 2^m\}$. Then any* minDT *of $\mathcal{P}$ has dual diameter $2 \cdot \lceil \log_2(n/3) \rceil - 1$ if $n \in \{3 \cdot 2^{m-1} + 1, \ldots, 4 \cdot 2^{m-1}\}$, and $2 \cdot \lceil \log_2(n/3) \rceil$ if $n \in \{4 \cdot 2^{m-1} + 1, \ldots, 3 \cdot 2^m\}$, for some $m \geq 1$.*

Proof. The dual graph of any triangulation of $\mathcal{P}$ is a tree with $n-2$ vertices and maximum degree 3; see Figure 1(a) for an example. Furthermore, every tree with $n-2$ vertices and maximum degree 3 is dual to some triangulation of $\mathcal{P}$.

For the upper bound, suppose first that $n = 3 \cdot 2^m$, for some $m \geq 1$. We define a triangulation T_1 as follows. It has a central triangle that splits $\mathcal{P}$ into three sub-polygons, each with 2^m edges on the boundary. For each sub-polygon, the dual tree for T_1 is a full binary tree of height $m-1$ with 2^{m-1} leaves; see Figure 1(b). The leaves of T_1^* correspond to the ears of T_1. The shortest path between any two ears in two different sub-polygons has length exactly $2(m-1) + 2 = 2\log_2(n/3)$. The shortest path between any two ears in the same sub-polygon has length at most $2(m-1)$. Thus, the dual diameter of T_1 is $2\log_2(n/3)$.

Now let $n \in \{3 \cdot \frac{4}{3} \cdot 2^{m-1} + 1, \ldots, 3 \cdot 2^m - 1\}$, and consider the triangulation T_2 of $\mathcal{P}$ obtained by cutting off $3 \cdot 2^m - n \leq 2 \cdot 2^{m-1} - 1$ ears that are consecutive in the convex hull from T_1. Then T_2^* is a subtree of T_1^*. Since T_1 has $3 \cdot 2^{m-1}$ ears, T_2 has at least $2^{m-1} + 1$ ears in common with T_1. Two of them lie in different sub-polygons, so the dual diameter remains $2m = 2 \cdot \lceil \log_2(n/3) \rceil$.

Finally, for $n \in \{3 \cdot 2^{m-1} + 1, \ldots, 3 \cdot \frac{4}{3} \cdot 2^{m-1}\}$, if we remove $3 \cdot 2^m - n \leq 3 \cdot 2^{m-1} - 1$ ears from T_1 such that all ears in two of the sub-polygons are removed, we get a triangulation with dual diameter $2m - 1 = 2 \cdot \lceil \log_2(n/3) \rceil - 1$; see Figure 1(c).



(a) (b) (c)

Figure 1: The convex case. (a) A triangulation and its dual tree. The ears are gray. (b) The triangulation T_1 for $m = 3$. The central triangle creates sub-polygons with 2^m edges of $\mathcal{P}$ each. Any path between ears in different sub-polygons has length $2m$. Other paths are shorter. (c) The triangulation T_2 for $4 \cdot 2^{m-1}$ vertices ($m = 3$). The central triangle creates three sub-polygons, one with 2^m edges of $\mathcal{P}$ and two with 2^{m-2} edges of $\mathcal{P}$.

For the lower bound, assume there is a tree T^* with $n-2$ vertices, maximum degree 3, and diameter k strictly smaller than in the proposition. Consider a longest path π in T^* and a vertex v on π for which the distances to the endpoints of π differ by at most one. By adding vertices, we can turn T^* into a tree with $n' - 2 > n - 2$ vertices, diameter k, and the same structure as T_1^* or T_2^* for a convex polygon with n' vertices (with v as central vertex). Since the upper bound on the dual diameter grows monotonically, this means that the triangulation T_1 or T_2 for a convex polygon with n vertices has diameter k, a contradiction. $\square$

As the dual graph of a triangulation of *any* simple polygon has maximum degree 3, the proof of Proposition 2.1 yields the following corollary.

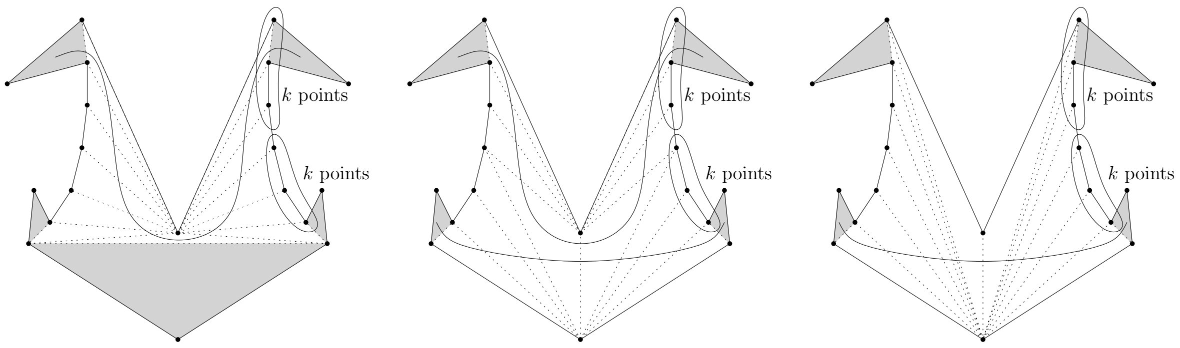
Corollary 2.2. *Let $\mathcal{P}$ be a simple polygon with $n \geq 3$ vertices, and let $m \geq 1$ such that $n \in \{3 \cdot 2^{m-1} + 1, \ldots, 3 \cdot 2^m\}$. The dual diameter of any triangulation of $\mathcal{P}$ is at least $2 \cdot \lceil \log_2(n/3) \rceil - 1$ if $n \in \{3 \cdot 2^{m-1} + 1, \ldots, 4 \cdot 2^{m-1}\}$, and $2 \cdot \lceil \log_2(n/3) \rceil$ if $n \in \{4 \cdot 2^{m-1} + 1, \ldots, 3 \cdot 2^m\}$.*

Proposition 2.1 also shows that if $\mathcal{P}$ is convex, there exists a minDT with a maximum number of ears. Next, we show that this does not hold for general simple polygons. Hence, any approach that tries to construct minDTs by maximizing the number of ears is doomed to fail.

Proposition 2.3. *For arbitrarily large n, there exist simple polygons with n vertices in which any* minDT *minimizes the number of ears.*

Proof. Let $k \geq 1$ and consider the polygon $\mathcal{P}$ with $n = 4k + 8$ vertices sketched in Figure 2. Any triangulation of $\mathcal{P}$ has either 4 or 5 ears. The triangulation in Figure 2(a) is the only triangulation with 5 ears, and it has dual diameter $4k+2$. However, as depicted in Figure 2(b), omitting the large ear at the bottom allows a triangulation with 4 ears and dual diameter $2k + 3$. Thus, forcing even one additional ear may nearly double the dual diameter. □

Figure 2(c) shows a triangulation of $\mathcal{P}$ with 4 ears and almost twice the diameter as in Figure 2(b). Thus, neither for minimizing the diameter nor for maximizing the number of ears this triangulation is desirable. However, it has the nice property that the two top ears are connected by a dual path with four interior vertices. Below, this property will be useful when making a larger construction.

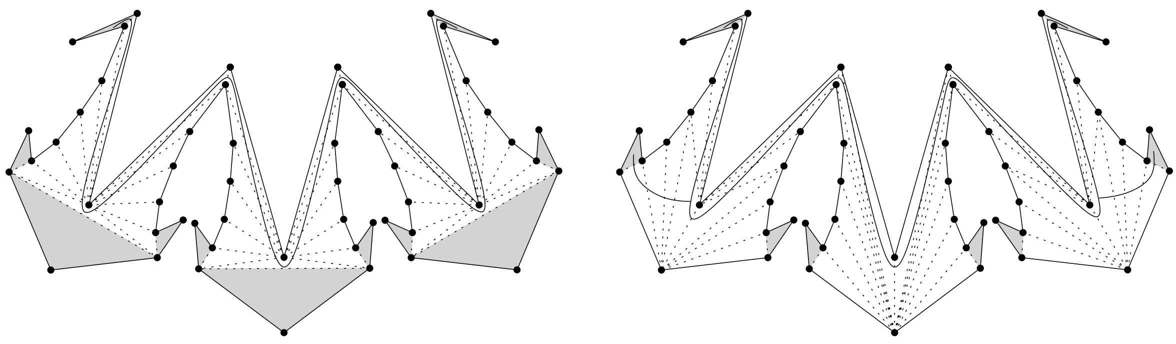


(a) 5 ears, dual diameter $4k + 2$. (b) 4 ears, dual diameter $2k + 3$ (c) 4 ears, dual diameter $4k + 3$

Figure 2: Three triangulations of a polygon with $n = 4k + 8$ vertices ($k = 3$) and paths that define their dual diameters. The ears are shaded.

Theorem 2.4. *For arbitrarily large n, there is a simple polygon with n vertices that has minimum dual diameter $O(\sqrt{n})$ while any triangulation that maximizes the number of ears has dual diameter $\Omega(n)$.*

Proof. Let c be a parameter to be determined later, and let $\mathcal{P}'$ be the polygon constructed in Proposition 2.3. We construct a polygon $\mathcal{P}$ by concatenating c copies of $\mathcal{P}'$ as in Figure 3. $\mathcal{P}$ has $n = c(4k + 4) + 4$ vertices. Using the triangulation from Figure 2(a) for each copy, we obtain a triangulation with the maximum number $3c + 2$ of ears and dual diameter $c(4k + 1) + 1$ (the curved line in Figure 3 indicates a longest path). On the other hand, using the triangulation from Figure 2(b) for the leftmost and rightmost part of the polygon and the one from Figure 2(c) for all intermediate parts yields a triangulation with dual diameter $4c + 4k - 3$ that has only $2c + 2$ ears. For $c = k$, we obtain $c, k = \Theta(\sqrt{n})$. Thus, the dual diameter for the triangulation with maximum number of ears is $\Theta(n)$, while the optimal dual diameter is $O(\sqrt{n})$. □



(a) $3c + 2$ ears, dual diameter $c(4k + 1) + 1$. (b) $2c + 2$ ears, dual diameter $4c + 4k - 3$.

Figure 3: Two triangulations of a polygon with $n = c(4k + 4) + 4$ vertices ($c = k = 3$) and corresponding longest paths. The ears are shaded.

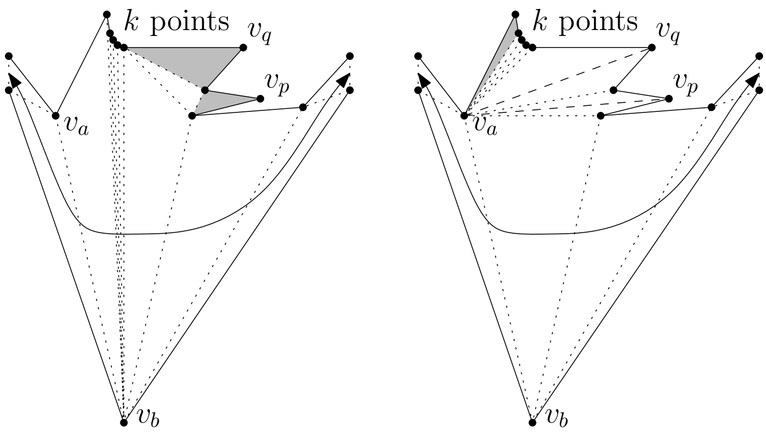


Figure 4: Two triangulations of a part of a polygon where the dual diameter is locally decreased by k when minimizing the number of ears.

Similarly, for maximizing the dual diameter, we can give examples where the dual diameter is suboptimal when the number of ears is minimized.

Theorem 2.5. *For arbitrarily large n, there is a simple polygon with n vertices that has maximum dual diameter $\Omega(n)$ while any triangulation that minimizes the number of ears has dual diameter $O(\sqrt{n})$.*

Proof. Figure 4 shows a triangulation of a part of a simple polygon. We suppose that the indicated dual path π is the only one of maximum length. In addition to the ears at the endpoints of π, there are two ears at the vertices v_p and v_q. If we want to have at most one ear in this part of the polygon, at least one of v_p and v_q must be connected to a non-neighboring vertex by a triangulation edge. For this, the only possibilities are $v_p v_a$ and $v_q v_a$. But then there cannot be any edge between the bottommost vertex v_b and the k vertices between v_q and v_a. In particular, that part must be triangulated as shown to the right of Figure 4. Here, there is only one ear, but the dual diameter is reduced by k (assuming the remainder of the polygon is large enough). As in the proof of Theorem 2.4, we concatenate $\Theta(\sqrt{n})$ copies of this construction and choose $k = \Theta(\sqrt{n})$. The parts are independent in the sense that they are separated by *unavoidable* edges (i.e., edges that are present in any triangulation of the resulting polygon).[1] Hence, while the dual diameter of a maxDT is linear in n, it is in $O(\sqrt{n})$ for any triangulation that minimizes the number of ears. □

It is easy to construct polygons for which the dual graph of any triangulation is a path, forcing minimum dual diameter $\Omega(n)$. The other direction is slightly less obvious.

Proposition 2.6. *For any n, there exists a simple polygon $\mathcal{P}$ with n vertices such that the dual diameter of any* maxDT *of $\mathcal{P}$ is in $\Theta(\log n)$.*

Proof. We incrementally construct $\mathcal{P}$ by starting with an arbitrary triangle t. See Figure 5 for an accompanying illustration. We replace every corner of t by four new vertices so that two of them can see only these four new vertices. This means that the edge between the other two newly added vertices is unavoidable. We repeat this construction recursively in a balanced way. If necessary, we add dummy vertices to obtain exactly n vertices. The unavoidable edges partition $\mathcal{P}$ into convex regions, either hexagons or quadrilaterals. The dual tree of this partition is balanced with diameter $\Theta(\log n)$. Since every triangulation of $\mathcal{P}$ contains all unavoidable edges, the maximum possible dual diameter is $O(\log n)$. □

[1] Unavoidable edges are defined by segments between two vertices s.t. no other edge crosses them. Hence, they have to be present in every triangulation. Unavoidable edges of point sets have been investigated by Karoly and Welzl [5] (as "crossing-free segments"), and Xu [14] (as "stable segments").

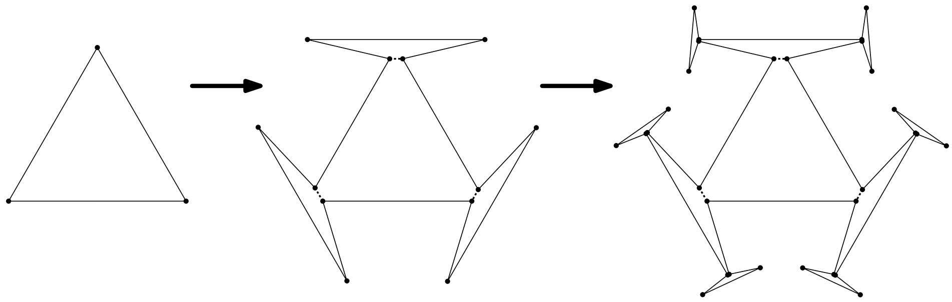


Figure 5: The convex vertices of a polygon are incrementally replaced by four new vertices, resulting in unavoidable edges (dotted).

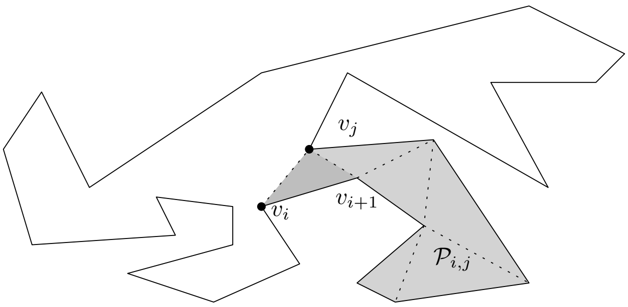


Figure 6: Any triangulation of $\mathcal{P}_{i,j}$ (gray) has exactly one triangle adjacent to v_iv_j (dark gray).

3. Optimally Triangulating a Simple Polygon

We now consider the algorithmic question of constructing a minDT and a maxDT of a simple polygon $\mathcal{P}$ with n vertices. Let $v_1, \ldots, v_n$ be the vertices of $\mathcal{P}$ in counterclockwise order. The segment v_iv_j is a *diagonal* of $\mathcal{P}$ if it lies completely in $\mathcal{P}$ but is not part of the boundary of $\mathcal{P}$. For a diagonal v_iv_j, $i < j$, we define $\mathcal{P}_{i,j}$ as the polygon with vertices $v_i, v_{i+1}, \ldots, v_{j-1}, v_j$; see Figure 6. Observe that $\mathcal{P}_{i,j}$ is a simple polygon contained in $\mathcal{P}$. If v_iv_j is not a diagonal, we set $\mathcal{P}_{i,j} = \emptyset$.

Theorem 3.1. *For any simple polygon $\mathcal{P}$ with n vertices, we can compute a* minDT *in $O(n^3 \log n)$ time.*

Proof. We use the classic dynamic programming approach [6], with an additional twist to account for the non-local nature of the objective function. Let v_iv_j be a diagonal. Any triangulation T of $\mathcal{P}_{i,j}$ has exactly one triangle t incident to v_iv_j; see Figure 6. Let $f(T)$ be the maximum length of a path in T^* that has t as an endpoint.

For $d > 0$ and $i, j = 1, \ldots, n$, with $i < j$, let $\mathcal{T}_d(i, j)$ be the set of all triangulations of $\mathcal{P}_{i,j}$ with dual diameter at most d (we set $\mathcal{T}_d(i, j) = \emptyset$ if v_iv_j is not a diagonal of $\mathcal{P}$). We define $M_d[i, j] = \min_{T \in \mathcal{T}_d(i,j)} f(T) + 1$, if $\mathcal{T}_d(i, j) \neq \emptyset$, or $M_d[i, j] = \infty$, otherwise. Intuitively, we aim for a triangulation that minimizes the distance from v_iv_j to all other triangles of $\mathcal{P}_{i,j}$ while keeping the dual diameter below d (the value of $M_d[i, j]$ is the smallest possible distance that can be obtained). Let $\mathcal{V}(i, j)$ be all vertices v_l of $\mathcal{P}_{i,j}$ such that the triangle $v_iv_jv_l$ lies inside $\mathcal{P}_{i,j}$. We claim that $M_d[i, j]$ obeys the following recursion:

$$M_d[i, j] = \begin{cases} 0, & \text{if } i + 1 = j, \\ \infty, & \text{if } v_iv_j \text{ is not a diagonal}, \\ \min_{v_l \in \mathcal{V}(i,j)} I_d[i, j, l], & \text{otherwise}, \end{cases}$$

where

$$I_d[i, j, l] = \begin{cases} \infty, & \text{if } M_d[i, l] + M_d[l, j] > d, \\ \max\{M_d[i, l], M_d[l, j]\} + 1, & \text{otherwise}. \end{cases}$$

We minimize over all possible triangles t in $\mathcal{P}_{i,j}$ incident to v_iv_j. For each t, the longest path to v_iv_j is the longer of the paths to the other edges of t plus t itself. If t joins two longest paths of total length more than d, there is no valid solution with t. Thus, we can decide in $O(n^3)$ time whether there is a triangulation with dual diameter at most d, i.e., if $M_d[1, n] \neq \infty$. Since the dual diameter is at most $n-3$, a binary search gives an $O(n^3 \log n)$ time algorithm. □

We can use a very similar approach to obtain some maxDT.

Theorem 3.2. *For any simple polygon $\mathcal{P}$ with n vertices, we can compute a* maxDT *in $O(n^3 \log n)$ time.*

Proof. The proof is similar to the one of Theorem 3.1. This time, we are looking for triangulations that have dual diameter at least d. Let $f(T)$ be defined as before, and let $\mathcal{T}(i, j)$ be the set of all triangulations of $\mathcal{P}_{i,j}$ (this time, we do not need the third parameter). We define $M_d[i, j]$ in the following way. If $\mathcal{T}(i, j) = \emptyset$, then $M_d[i, j] = -\infty$. If $\mathcal{T}(i, j)$ contains a triangulation with diameter at least d, $M_d[i, j] = \infty$. Otherwise, let $M_d[i, j] = \max_{T \in \mathcal{T}(i,j)} f(T) + 1$. Clearly, there is a triangulation with diameter at least d if and only if $M_d[1, n] = \infty$. With $\mathcal{V}(i, j)$ defined as before, the recursion for $M_d[i, j]$ is

$$M_d[i, j] = \begin{cases} 0, & \text{if } i + 1 = j, \\ -\infty, & \text{if } v_i v_j \text{ is not a diagonal,} \\ \max_{v_l \in \mathcal{V}(i,j)} I_d[i, j, l], & \text{otherwise,} \end{cases}$$

where

$$I_d[i, j, l] = \begin{cases} -\infty, & \text{if } M_d[i, l] \text{ or } M_d[l, j] \text{ is } -\infty, \\ \infty, & \text{if } M_d[i, l] + M_d[l, j] \geq d, \\ \max\{M_d[i, l], M_d[l, j]\} + 1, & \text{otherwise.} \end{cases}$$

For the given diagonal $v_i v_j$, we maximize over all possible triangles. If at some point the triangle t at $v_i v_j$ closes a path of length at least d, we are basically done, as any triangulation of the remainder of the polygon results in a triangulation with dual diameter at least d. If the triangulation of $\mathcal{P}_{i,j}$ does not contain such a long path, we store the longer one to $v_i v_j$, as before. Again, we can find the optimal dual diameter via a binary search, giving an $O(n^3 \log n)$ time algorithm. □

4. Bounds for Point Sets

We are now given a set S of n points in the plane in general position, and we need to find a triangulation of S whose dual graph optimizes the diameter. Since the dual graph has maximum degree 3, it is easy to see that the $\Omega(\log n)$ lower bound for simple polygons extends for this case. It turns out that this bound can always be achieved, as we show in Section 4.1. In Section 4.2, we find a triangulation of S that has dual diameter in $\Omega(\sqrt{n})$.

4.1. Minimizing the Dual Diameter

Theorem 4.1. *Given a set S of n points in the plane in general position, we can compute in $O(n \log n)$ time a triangulation of S with dual diameter $\Theta(\log n)$.*

Proof. Let $\mathcal{P}$ be a convex polygon with n vertices and T' a triangulation of $\mathcal{P}$ with dual diameter $\Theta(\log n)$ (e.g., the triangulation from Proposition 2.1). The triangulation T' is an outerplanar graph. Any outerplanar graph of n vertices has a plane straight-line embedding on any given n-point set [10]. Furthermore, such an embedding can be found in $O(nd)$ time and $O(n)$ space, where d is the dual diameter of the graph [1].

Let T_S be the embedding of T' on S. In general, T_S does not triangulate S; see Figure 7. Consider the convex hull of T_S (which equals the convex hull of S). The untriangulated *pockets* are simple polygons. We triangulate each pocket arbitrarily to obtain a triangulation T of S. We claim that the dual diameter of T is $O(\log n)$.

Lemma 4.2. *The dual distance from any triangle in a pocket to any triangle in T_S is $O(\log n)$.*

Proof. Let Q be a pocket, and T_Q a triangulation of Q. Since Q is a simple polygon, the dual T_Q^* is a tree with maximum degree 3. A triangle t of T_Q not incident to the boundary of T_S either has degree 3 in T_Q^*, or it is the unique triangle in T_Q that shares an edge with the convex hull of S. We perform a breadth-first-search in T_Q^* starting from t, and let k be the maximum number of consecutive layers from the root of the BFS-tree that do not contain a triangle incident to the boundary. By the above observation, all vertices in the first $k-1$ levels have degree three in T_Q^*. Thus, each vertex of level $k-1$ or lower has two children. In particular, at each level the number of vertices must double (except at the topmost level where the number of vertices is tripled), hence $k = O(\log n)$. □

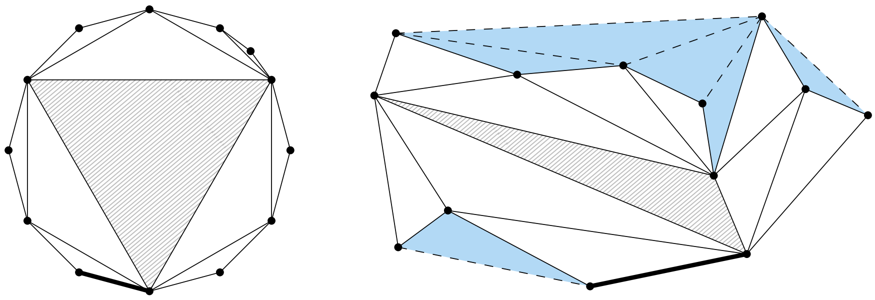


Figure 7: When computing a minDT of a point set S, we first view it as if it were in convex position and construct a minDT (left image). Then, we embed T_S into the actual point set (solid edges in the right image). (The correspondence is marked by the central triangle and the thick boundary edge.) All remaining untriangulated pockets (highlighted region in the figure) are triangulated arbitrarily (dashed edges).

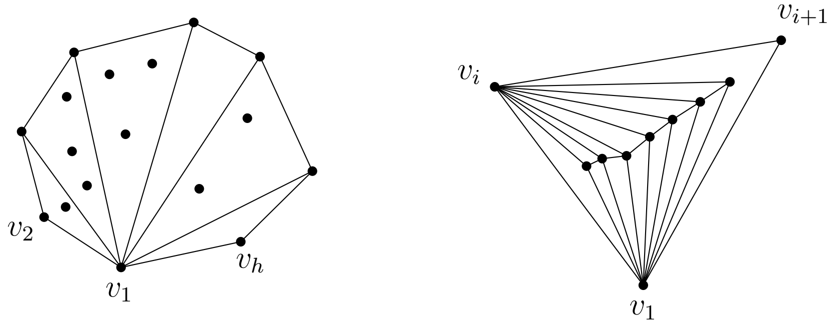


Figure 8: Left: v_1 is connected to all remaining vertices in the convex hull. Right: additional edges added inside Δ_i. We connect the points of an increasing subsequence of $v_{j_1}, v_{j_2}, \ldots, v_{j_{n_i}}$ to both v_1, v_i as well as the predecessor and successor in the subsequence.

Given Lemma 4.2 and the fact that T_S has dual diameter $O(\log n)$, Theorem 4.1 is now immediate. □

4.2. Obtaining a Large Dual Diameter

We now focus our attention on the problem of triangulating S so that the dual diameter is maximized.

Theorem 4.3. *Given a set S of $n \geq 3$ points in the plane in general position, we can compute in $O(n \log n)$ time a triangulation of S with dual diameter at least $\sqrt{n-3}$.*

Proof. Naturally, the triangulation T must contain the edges of the convex hull of S. Let $v_1, v_2, \ldots, v_h$ be the vertices of the convex hull of S in clockwise order. We connect v_1 to the vertices $v_3, v_4, \ldots, v_{h-1}$; see Figure 8 (left). In order to complete this set of edges to a triangulation, it suffices to consider the triangular regions $v_1 v_i v_{i+1}$ (for $2 \leq i \leq h-1$) with at least one point of S in their interior.

Let $\Delta_i = v_1 v_i v_{i+1}$ be such a triangular region, $S_i \subset S$ the points in the interior of Δ_i, and $n_i = |S_i|$. Let $w_1, w_2, \ldots, w_{n_i}$ denote the points in S_i sorted in clockwise order with respect to v_1, and $w_{j_1}, w_{j_2}, \ldots, w_{j_{n_i}}$ denote the same points sorted in counterclockwise order with respect to v_i. By the Erdős-Szekeres theorem [2], the index sequence j_k contains an increasing or decreasing subsequence σ_i of length at least $\sqrt{n_i}$.

If σ_i is increasing, we connect all points of σ_i to both v_1 and v_i. In addition, we connect each point of σ_i to its predecessor and successor in σ_i; see Figure 8 (right). Since the clockwise order with respect to v_1 coincides with the counterclockwise order with respect to v_i, the new edges do not create any crossing.

If σ_i is decreasing, we claim that the corresponding point sequence is in counterclockwise order around v_{i+1}. Indeed, let w and w' be two vertices of S_i whose indices appear consecutively in σ_i (with w before w'). By definition, the segment $v_1 w'$ crosses the segment $v_i w$. Moreover, points v_1 and v_i are on the same side of the line through w and w'; see Figure 9 (left). Since w and w' are contained in Δ_i, we conclude that v_{i+1} must lie on the opposite side of the line. Thus, v_i, w, w', v_1 form a counterclockwise sequence around v_{i+1}, and we can connect each point of σ_i to

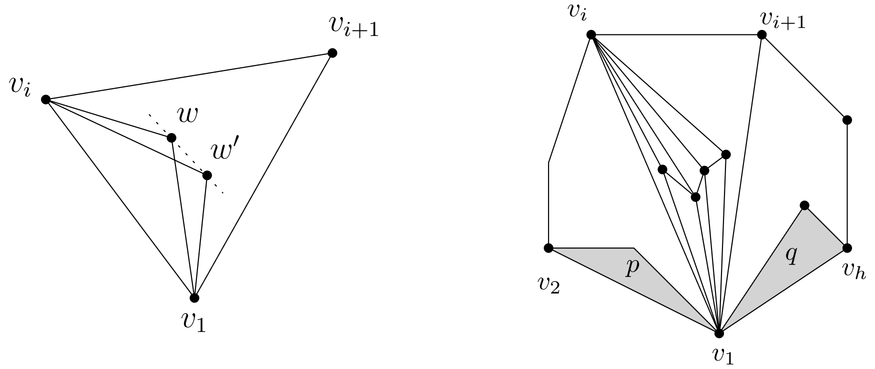


Figure 9: Left: if σ_i generates a decreasing sequence, the same sequence must be increasing when we view the angles with v_{i+1} instead. Right: any path between p and q in the dual graph must visit all triangles Δ_i and at least $\sqrt{n_i}$ additional triangles between the crossing of segments v_1v_i and v_1v_{i+1}.

v_1, v_{i+1}, and its predecessor and successor in σ_i without crossings. Finally, we add arbitrary edges to complete the resulting graph inside Δ_i into a triangulation T_i.

We claim that, regardless of how we complete the triangulation, there are two triangles whose distance in the dual graph is at least $\sqrt{n-3}$. Let p and q be the triangles of T incident to edges v_1v_2 and v_1v_h, respectively (since both segments are on the convex hull, p and q are uniquely defined). Let π be the shortest path from p to q. Clearly, π must cross each segment v_1v_i, for $i \in \{3, \ldots h-1\}$, exactly once and in increasing order. This gives $h-3$ steps (one step for each triangle incident in clockwise order around v_1 on an edge v_1v_i, $i \in \{3, \ldots, h-1\}$). In addition, at least $\sqrt{n_i}$ additional triangles must be traversed between the segments v_1v_i and v_1v_{i+1} (for all $i \in \{2, \ldots, h-1\}$): indeed, for each vertex $w \in \sigma_i$, the edges v_1w and either wv_i or wv_{i+1} (depending on whether σ_i was increasing or decreasing) disconnect p and q., Hence at least one of the two must be crossed by π, and the triangles following these edges are pairwise distinct and distinct from the triangles following the segments v_1v_i. Summing over i, we get

$$|\pi| \geq h - 3 + \sum_{i=2}^{h-1} \sqrt{n_i} \geq h + \sqrt{n-h} - 3 \geq 3 + \sqrt{n-3} - 3 = \sqrt{n-3}.$$

In the second inequality we used the fact that $\sum_{i=2}^{h-1} \sqrt{n_i} \geq \sqrt{\sum_{i=2}^{h-1} n_i} = \sqrt{n-h}$. (since a point is either on the convex hull or in its interior), the third inequality follows from $h \geq 3$ (and the fact that the expression is minimized when h is as small as possible).

Finding a longest increasing (or decreasing) subsequence of n_i numbers takes $O(n_i \log n_i)$ time, which is optimal in the comparison model [3]. Hence, all subsequences, as well as the whole triangulation can be computed in $O(n \log n)$ time, where the last part also uses the fact that point location in a triangulation on $\leq n$ vertices can be done in $O(\log n)$ time after $O(n \log n)$ preprocessing. □

Proposition 4.4. *Any set of n points in the plane in general position with k points in convex position has a triangulation with dual diameter in* $\Omega(k)$.

Proof. See Figure 10 for an accompanying illustration. Let S be such a point set with $C \subseteq S$ being the convex subset of size k. First, triangulate only C by a zig-zag chain of edges: for the convex hull of C being defined by the sequence $(c_1, \ldots, c_k)$, add the edges c_ic_{k-i} and c_ic_{k-i-1} for $1 \leq i < \lfloor k/2 \rfloor$, as well as the boundary of the convex hull of C. Then, add the extreme points of S and triangulate the convex hull of S without C such that each added edge is incident to a point of C (this is not necessary to obtain the result, but it makes our arguments simpler). The resulting triangulation has dual diameter $\Omega(k)$, as is witnessed by the triangles at c_k and $c_{\lfloor k/2 \rfloor}$ inside C: if we label each triangle with the index of the incident point in C that is closest to $c_{\lfloor k/2 \rfloor}$, then this index can change by at most 1 along a step in any dual path between the triangles inside C incident to c_k and $c_{\lfloor k/2 \rfloor}$. The dual diameter does not decrease when adding the remaining points of S and completing the triangulation arbitrarily. □

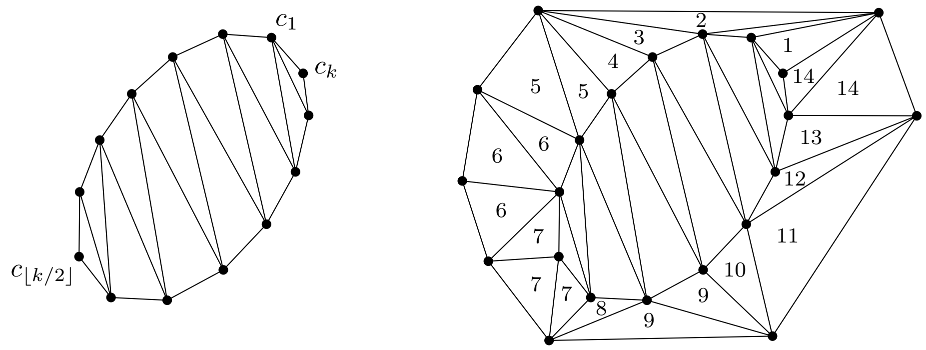


Figure 10: Left: zig-zag triangulation of a convex subset of size k. Right: adding the edges to the extreme points implies a labeling of the triangles that relate to their distance to $c_{\lfloor k/2 \rfloor}$.

5. Conclusions

The proof of Corollary 2.2 (lower bound for simple polygons) is essentially based on fundamental properties of graphs (i.e., bounded degree) rather than geometric properties. Since the bound is tight even for the convex case, it cannot be tightened in general. However, we wonder if, by using geometric tools, one can construct a bound that depends on the number of reflex vertices of the polygon (or interior points for the case of sets of points). Another natural open problem is to extend our dynamic programming approach for simple polygons to general polygonal domains (or even sets of points).

It is open whether Theorem 4.3 is tight. That is: does there exist a point set S such that the diameter of the dual graph of any triangulation of S is in $O(\sqrt{n})$? From Proposition 4.4, we see that any such point set can contain at most $O(\sqrt{n})$ points in convex position. Thus, the point set must have $\Theta(\sqrt{n})$ convex hull layers, each with $\Theta(\sqrt{n})$ points. We suspect that some smart perturbation of the grid may be an example, but we have been unable to prove so.

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