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Diplomarbeit

The Mahler Conjecture

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Contents

Preface

Kurt Mahler was born in 1903 into a Jewish middle-class family in Krefeld, Germany. In his youth, he early began to read mathematical textbooks and soon wrote small articles about the content and formulated implications of the material. Without knowledge of his son, Kurt Mahler's father sent these notes to a mathematician at the local school. After a short while, the material got into the office of Carl Siegel, who later invited him to study in Frankfurt am Main. Mahler attended his first lectures there at the age of 20 and his mathematical career started with his doctoral thesis, that was approved in 1927. Due to the political situation in Germany, he had to change his working place several times, spending years in the United Kingdom, the Netherlands and in Canberra, Australia, where he died in 1988 at the age of 85. Mahler mainly worked in transcendental number theory, Diophantine approximation, p-adic analysis and geometry of numbers. A detailed description of his contributions to mathematics and more biographical background is presented in an obituary by Cassels [5].

During his time in Manchester, Mahler submitted his famous work "Ein Übertragunsprinzip für konvexe Körper" [15]. In this article, Mahler raised the problem, that the present work is devoted to. On pages 96 and 97 he wrote

> "Wir werden jetzt zeigen, daß Δ für jedes konvexe Körperpaar zwischen zwei positiven, allein von der Dimension n abhängigen Schranken bleibt. Wahrscheinlich lauten die exacten Ungleichungen
>
> $$\frac{4^n}{n!} \leqq JJ' \leqq \frac{\pi^n}{\Gamma\left(\frac{n}{2}+1\right)^2},$$
>
> wobei die linke Seite zutrifft für das polare Körperpaar von Parallelepiped und Oktaeder [..] und die rechte Seite für die polaren Ellipsoide."

The quantity Δ denotes the product of the volume J of an arbitrary centrally symmetric convex body and the volume J' of its polar. Mahler's Conjecture turned out to be a persistent problem, yet no general proof is known for dimensions greater than two. Mahler [14] himself showed the validity in the plane and applied his arguments also to the non-symmetric case, where the conjectured lower bound is given by simplices. But also in this case, the problem remains open for bodies of more than two dimensions.

However, the present work only covers the centrally symmetric case of Mahler's Conjecture and is organised as follows. In Chapter 1 we introduce the reader to the notations and conventions from Convex Geometry that we use throughout. It is followed by a survey of the current state of the knowledge. Mahler's original proof in the plane and an investigation on the upper bound are also discussed in Chapter 2. The third chapter deals with an estimation to the lower bound that is the best known result, which can be obtained without use of deep theory. Since Hanner polytopes are conjectured to characterise the lower bound, Chapter 4 takes a detailed look at these bodies. Additionally, we give an approach to combine the known results by proving a transference property of the p-product relation that is introduced in Chapter 3. The second part of this work, that consists of the Chapters 5, 6 and 7, discusses the validity of the Mahler Conjecture for the classes of zonoids, polytopes with at most $2n + 2$ facets and 1-unconditional bodies.

We are grateful to Prof. Dr. Martin Henk for proposing that interesting problem as the topic of this work, for valuable suggestions and indispensable discussions. We also thank Dr. Gennadiy Averkov for useful hints concerning the POV-Ray programme and his remarks on the style of writing, and furthermore Prof. Dr. Shlomo Reisner, who provided the main literature for Chapter 7.

1 Introduction

The purpose of this introductory chapter is to provide definitions from Convex Geometry that will be used throughout the present work without further reference. Additionally, basic properties of some of these concepts are stated.

Convex bodies

A *convex body* in n-dimensional Euclidean space $\mathbb{R}^n$ is a compact convex set with nonempty interior. We denote with $\mathcal{K}^n$ the set of all convex bodies in $\mathbb{R}^n$ and with $\mathcal{K}_0^n$ all those being additionally centrally symmetric, i.e., for $x \in K$ it also holds $-x \in K$. A centrally symmetric convex body K is in correspondence to a norm on $\mathbb{R}^n$ which is given by

$$||x||_K = \min\{t \geq 0 \mid x \in tK\},\ x \in \mathbb{R}^n.$$

It holds $K = \{x \in \mathbb{R}^n \mid ||x||_K \leq 1\}$. We omit the subscript, when we consider the *Euclidean unit ball* B^n, whose norm is given by $||x|| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$, $x \in \mathbb{R}^n$. Other frequently used examples of centrally symmetric convex bodies are the unit *cube* $C_n = [-1, 1]^n$ and the *crosspolytope* $C_n^\star = \mathrm{conv}\{\pm e_1, \ldots, \pm e_n\}$, where $e_i, 1 \leq i \leq n$, denote the i-th unit normal vectors in $\mathbb{R}^n$. The boundary of some convex body K will be denoted with ∂K and the special case $S^{n-1} = \partial B^n = \{x \in \mathbb{R}^n \mid ||x|| = 1\}$ is called the *unit n-sphere*.

Polarity of convex bodies is of major interest in this work. Thus, we define the *polar body* of $K \in \mathcal{K}^n$ as

$$K^\star = \{y \in \mathbb{R}^n \mid x^\intercal y \leq 1 \text{ for all } x \in K\}.$$

The polar operation is inclusion inversive, since $A \subset B$ implies $B^\star \subset A^\star$. If K is centrally symmetric, we have $(K^\star)^\star = K$ and furthermore, for any invertible linear transformation T on $\mathbb{R}^n$ it holds $(TK)^\star = (T^\intercal)^{-1}K^\star$.

The family $\mathcal{K}^n$ is equipped with a metric. The *Hausdorff distance* of two convex bodies K_1 and K_2 is defined by

$$\delta(K_1, K_2) = \min\{\varepsilon \geq 0 \mid K_1 \subset K_2 + \varepsilon B^n \text{ and } K_2 \subset K_1 + \varepsilon B^n\}.$$

Two common geometric quantities of a convex body K are *circumradius* and *inradius*, that are given by $R(K) = \inf\{R > 0 \mid K \subset RB^n\}$ and $r(K) = \sup\{r > 0 \mid rB^n \subset K\}$, respectively. Additionally, the *eccentricity* of K is said to be the quotient $e(K) = \frac{R(K)}{r(K)}$ of these numbers.

Let χ_A denote the characteristic function of a subset A of R^n. If A is measurable, its *volume* $\mathrm{vol}(A) = \int_{R^n} \chi_A(x)\ dx = \int_A dx$ is declared by the usual Lebesgue measure on $\mathbb{R}^n$. The volume of the n-dimensional Euclidean unit ball is abbreviated by $\kappa_n = \mathrm{vol}(B^n)$.

The *orthogonal complement* of a linear subspace S is defined by $S^\perp = \{x \in \mathbb{R}^n \mid x^\intercal y = 0 \text{ for all } y \in S\}$. For fixed $x \in \mathbb{R}^n$ there exist unique $y \in S$ and $z \in S^\perp$ such that $x = y + z$. The operator that maps x to $y \in S$ is called the *orthogonal projection* of x onto S. The image of $A \subset \mathbb{R}^n$ under this mapping is called the projection of A onto S and is denoted with $A|S$. For a centrally symmetric convex body K, we have $(K|S)^\star = K^\star \cap S$.

Support and radial function

A useful concept in Convex Geometry is the *support function* $h_K : \mathbb{R}^n \to \mathbb{R}$ of a convex body K which is given by

$$h_K(x) = \max_{y \in K} x^\intercal y,\ x \in \mathbb{R}^n.$$

It is continuous and positive homogeneous of degree one, i.e., for $c > 0$ we have $h_K(cx) = c \cdot h_K(x)$. The *radial function* $\rho_K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ of K is the dual concept and defined by

$$\rho_K(x) = \max\{c \in \mathbb{R} \mid cx \in K\},\ x \in \mathbb{R}^n \setminus \{0\}.$$

Support and radial function determine the convex body K and moreover, K is centrally symmetric if and only if $h_K(-u) = h_K(u)$ $[\rho_K(-u) = \rho_K(u)]$, for all unit vectors $u \in S^{n-1}$. Let the origin be an interior point of K. Then, for all $u \in S^{n-1}$ we have $\rho_{K^\star}(u) = \frac{1}{h_K(u)}$, which affirms the duality of both concepts. Furthermore, if K is centrally symmetric, the following relations hold

$$||x||_K = \frac{1}{\rho_K(x)},\ x \in \mathbb{R}^n \setminus \{0\}, \quad \text{and} \quad ||y||_{K^\star} = h_K(y),\ y \in \mathbb{R}^n.$$

These two functionals enable us to construct new bodies from a given centrally symmetric $K \in \mathcal{K}_0^n$. Let $n \geq 2$ and define the *projection body* of K as the convex body ΠK such that

$$h_{\Pi K}(u) = \mathrm{vol}_{n-1}(K|u^\perp),\ \text{for all } u \in S^{n-1},$$

and the *intersection body* of K as the convex body IK such that

$$\rho_{IK}(u) = \mathrm{vol}_{n-1}(K \cap u^\perp),\ \text{for all } u \in S^{n-1}.$$

Both constructions ΠK and IK yield centrally symmetric convex bodies, and for a detailed investigation we refer to Gardner [6].

2 An overview of the current state

Let K be a convex body in $\mathbb{R}^n$ that contains the origin in its interior. The functional $\mathbf{M}$, that maps K to its volume product $\mathbf{M}(K) = \mathrm{vol}(K)\mathrm{vol}(K^\star)$, is called the *Mahler volume* of K. This mapping $\mathbf{M}$ is affine invariant, i.e., for any invertible linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, we have $\mathbf{M}(TK) = \mathbf{M}(K)$. This can be seen by $(TK)^\star = (T^\intercal)^{-1}K^\star$ and $\mathrm{vol}(TK) = |\det(T)|\mathrm{vol}(K)$. Moreover, for centrally symmetric K, it holds $\mathbf{M}(K) = \mathbf{M}(K^\star)$.

In 1939, Mahler [15] proved, that if we restrict the functional $\mathbf{M}$ to centrally symmetric convex bodies K, it is bounded from below as well as from above by constants which only depend on the dimension n. Precisely, he showed

$$\frac{4^n}{(n!)^2} \leq \mathbf{M}(K) \leq 4^n, \ K \in \mathcal{K}_0^n. \tag{2.1}$$

Furthermore, he conjectured that the sharp inequalities are given by

$$\frac{4^n}{n!} \leq \mathbf{M}(K) \leq \frac{\pi^n}{\Gamma(\frac{n}{2}+1)^2}, \ K \in \mathcal{K}_0^n, \tag{2.2}$$

where the lower bound is attained for the cube C_n and its polar, the crosspolytope $C_n^\star$, and the upper estimate holds for the Euclidean unit ball B^n. On the right hand side, it appears the well-known Gamma function $\Gamma(x) = \int_0^\infty s^{x-1}e^{-s}\,ds, x > 0$.

The upper bound in (2.2) was shown by Blaschke [3] and Santaló [24], and equality holds only for ellipsoids. Details are discussed in Section 2.1 below. However, the lower bound remains to be an open problem and is called the *Mahler Conjecture*. There is an analogue conjecture for the general case of convex bodies, which are not necessary centrally symmetric. It states

$$\mathbf{M}(K) \geq \frac{(n+1)^{n+1}}{(n!)^2}, \ K \in \mathcal{K}^n. \tag{2.3}$$

It is also conjectured that the case of equality is characterised by simplices in $\mathbb{R}^n$. The current state of the art is given in [17], where (2.3) is shown to be true for polytopes with at most $n+3$ vertices. Mahler [14] verified this inequality in dimension $n = 2$.

This work exclusively deals with the centrally symmetric case (2.2), and before we work through any details, we survey the current knowledge on

Mahler's Conjecture. Mahler himself gave a proof in dimension two, that is discussed in Section 2.2, but for dimensions $n \geq 3$ no general argument was found yet. In the literature there are two different approaches to deal with the problem. On the one hand, special classes of bodies in $\mathcal{K}_0^n$ are investigated. Contributions in this direction were made by Saint-Raymond [23] for 1-unconditional bodies, by Reisner [21] for zonoids and by Lopez and Reisner [13] for polytopes with at most $2n+2$ facets. On the other hand, one is interested in a general estimation of $\mathbf{M}(K)$ from below up to a certain factor. Those estimates are either given in terms of $\mathbf{M}(C_n)$ or $\mathbf{M}(B^n)$. Mahler's Conjecture says

$$\mathbf{M}(K) \geq \mathbf{M}(C_n) \quad \text{and} \quad \mathbf{M}(K) \geq m^n \mathbf{M}(B^n), \;\; K \in \mathcal{K}_0^n, \tag{2.4}$$

where $m = \frac{4}{\pi}\left(\frac{\Gamma(\frac{n}{2}+1)^2}{n!}\right)^{\frac{1}{n}}$ tends to $\frac{2}{\pi}$ from above, when n goes to infinity. Mahler initiated this approach, since (2.1) yields $\mathbf{M}(K) \geq \frac{1}{n!}\mathbf{M}(C_n), K \in \mathcal{K}_0^n$. In 1987, Bourgain and Milman [4] obtained

2.1 Theorem (Bourgain, Milman [4]).
There is a universally constant $c > 0$ such that for every body $K \in \mathcal{K}_0^n$

$$\mathbf{M}(K) \geq c^n \mathbf{M}(B^n).$$

The original arguments heavily rely on the local theory of Banach spaces. A simplified proof can be found in Chapter 7 of Pisier's book [18]. Kuperberg [12] used more elementary geometric reasoning to get a weaker result than the Bourgain-Milman Theorem, which states that the constant m in (2.4) can be chosen as $\frac{1}{\log_2 n}$, for dimensions greater than or equal to four. The best known lower bound on $\mathbf{M}(K)$ is also due to Kuperberg [11]. In 2006, he published a paper with the following outcome

2.2 Theorem (Kuperberg [11]).
For any centrally symmetric convex body K it holds

$$\mathbf{M}(K) \geq \left(\frac{\pi}{4}\right)^n \gamma_n \mathbf{M}(C_n),$$

where $\gamma_n \geq \frac{4}{\pi}$ only depends on the dimension and tends to $\sqrt{2}$, when n goes to infinity.

2.1 The Blaschke-Santaló inequality

This section deals with the upper bound in (2.2). In 1923, it was shown by Blaschke [3], that it is true in dimensions $n = 2, 3$, and 26 years later Santaló

[24] found a proof for arbitrary dimensions. The literature contains several different approaches on the inequality (see [16], [23]). These two papers also treat the case of equality, which is characterised by ellipsoids.

What follows, is a sketch of the proof by Meyer and Pajor [16], who utilise Steiner symmetrisations.

2.3 Definition (Steiner symmetrisation).
Let $K \in \mathcal{K}^n$ and let $H \subset \mathbb{R}^n$ be a hyperplane. The set

$$st_H(K) = \left\{ x + \frac{1}{2}(v - w) \;\middle|\; x \in K|H, x + v, x + w \in K \right\}$$

is called the Steiner symmetrisation *of K with respect to H.*

Figure 1 illustrates the Steiner symmetrisation of a polygon with respect to $H = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$.

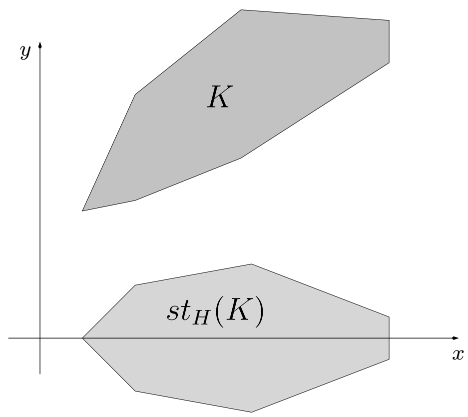

Figure 1: Steiner symmetrisation of K

Two essential properties of $st_H(K)$ are given in the following proposition.

2.4 Proposition. *Let $K \in \mathcal{K}^n$ and let $H \subset \mathbb{R}^n$ be a hyperplane. Steiner symmetrisation does not change the volume, i.e., it holds*

$$\mathrm{vol}(st_H(K)) = \mathrm{vol}(K).$$

Furthermore, $st_H(\cdot)$ is a continuous function on $\{K \in \mathcal{K}^n \mid \dim K = n\}$.

2.5 Lemma. *Let $K \in \mathcal{K}^n$ with $\dim K = n$. Furthermore, write S_K for the family of all convex bodies, that can be derived of K by finitely many Steiner symmetrisations with respect to hyperplanes, that contain the origin. Then* *there is a sequence* $(K_i)_{i\in\mathbb{N}} \subset S_K$ *that converges to* $\left(\frac{\mathrm{vol}(K)}{\mathrm{vol}(B^n)}\right)^{\frac{1}{n}} B^n$, *with respect to the Hausdorff metric.*

Proof. For $L \in \mathcal{K}^n$, define $\rho(L) = \min\{\gamma \geq 0 \mid L \subset \gamma B^n\}$ and let ρ be the infimum of $\rho(S)$ taken over all $S \in S_K$. Thus, there exists a sequence $(S_i)_{i\in\mathbb{N}} \subset S_K$, such that $\rho(S_i)$ tends to ρ, when i tends to infinity. Due to the fact that we only consider Steiner symmetrisations of K with respect to hyperplanes that contain the origin, and since, taking the Steiner symmetral does not increase the diameter of K, we have $\rho(S) \leq \rho(K)$ for all $S \in S_K$. Therefore, all the bodies $S \in S_K$ are contained in $\rho(K)B^n$. Under these conditions, Blaschke's selection theorem (see [25], Theorem 1.8.6) yields a convergent subsequence of $(S_i)_{i\in\mathbb{N}}$. Without loss of generality, let S_i tend to M, for $i \to \infty$. Note, that $\rho(\cdot)$ is continuous on S_K and so we have $\rho(M) = \rho$. This means, that $M \subset \rho B^n$.

Suppose, that we have $M \neq \rho B^n$. Then there is an $x_0 \in \rho S^{n-1}\backslash M$. Since, M is compact, we can find by the separation theorem (see [10], Chapter 3) a strict separating hyperplane $H(a, \alpha) = \{x \in \mathbb{R}^n \mid a^\intercal x = \alpha\}$ of M and x_0 with $a^\intercal x_0 > \alpha$ and $a^\intercal x \leq \alpha$, for all $x \in M$. Next, we consider for $v \in \rho S^{n-1}$ the spherical cap around v with "radius" α, which is given by $C(v, \alpha) = \{x \in \rho S^{n-1} \mid v^\intercal x > \alpha\}$. Since, S^{n-1} is compact, we find points $x_1, \ldots, x_m \in \rho S^{n-1}$ such that

$$\rho S^{n-1} = \bigcup_{i=0}^{m} C(x_i, \alpha).$$

Now, write $H_i = H(x_i - x_0, 0)$, for $1 \leq i \leq m$, and $C_j = C(x_j, \alpha)$, for $0 \leq j \leq m$. An important observation is, that the cap C_i is the reflection of C_0 at H_i. To see this, it is sufficient to show, that x_i is mapped to x_0 by that reflection R_i. We have

$$R_i(x_i) = x_i - \frac{2(x_i - x_0)^\intercal x_i}{(x_i - x_0)^\intercal(x_i - x_0)}(x_i - x_0) = x_i - (x_i - x_0) = x_0,$$

because $\frac{2(x_i-x_0)^\intercal x_i}{(x_i-x_0)^\intercal(x_i-x_0)} = \frac{2x_i^\intercal x_i - 2x_0^\intercal x_i}{x_i^\intercal x_i - 2x_0^\intercal x_i + x_0^\intercal x_0} = 1$, since x_0 and x_i are of the same length. By definition of C_0, we get $C_0 \cap M = \emptyset$ and therefore, $st_{H_1}(M) \cap (C_0 \cup C_1) = \emptyset$. Similarly, we obtain $st_{H_2}(st_{H_1}(M)) \cap (C_0 \cup C_1 \cup C_2) = \emptyset$ and inductively

$$\bar{M} = st_{H_m}(st_{H_{m-1}}(\ldots(st_{H_1}(M))\ldots)) \cap \rho S^{n-1} = \emptyset.$$

This means that $\bar{M} \subset \mathrm{int}(\rho B^n)$. Let us now denote

$$\bar{S}_i = st_{H_m}(st_{H_{m-1}}(\ldots(st_{H_1}(S_i))\ldots)).$$

Then, clearly $\bar{S}_i \in S_K$ and hence $\rho(\bar{S}_i) \geq \rho$. The continuity of the Steiner symmetrisation (see Proposition 2.4) yields, that $\bar{S}_i$ converges to $\bar{M}$, since S_i converges to M, for i tending to infinity. But this creates a contradiction, because we have $\rho(\bar{S}_i) \to \rho(\bar{M}) < \rho$. Thus, we must drop the assumption and get $M = \rho B^n$.

All bodies in S_K have the same volume, as Steiner symmetrisation does not change it. So, we are done, because $\mathrm{vol}(K) = \mathrm{vol}(M) = \rho^n \mathrm{vol}(B^n)$, or equivalently,

$$\rho = \left(\frac{\mathrm{vol}(K)}{\mathrm{vol}(B^n)} \right)^{\frac{1}{n}}.$$

□

2.6 Lemma. *Let $K \in \mathcal{K}_0^n$ with $\dim K = n$ and let $H \subset \mathbb{R}^n$ be a hyperplane, that contains the origin. Then*

$$\mathrm{vol}((st_H(K))^\star) \geq \mathrm{vol}(K^\star).$$

Proof. First of all, for all affine transformations $A : \mathbb{R}^n \to \mathbb{R}^n$ we have

$$\mathrm{vol}((AK)^\star) = \mathrm{vol}(A^{-\intercal} K^\star) = |\det A|^{-1} \mathrm{vol}(K^\star).$$

Since, H contains the origin, we can therefore find a suitable rotation A ($|\det A| = 1$) of K, to assume that $H = \{x \in \mathbb{R}^n \mid x_n = 0\}$. For this special hyperplane H we get

$$st_H(K) = \left\{ (x, s)^\intercal \;\middle|\; x \in K|H, s = \frac{1}{2}(a - b), (x, a)^\intercal, (x, b)^\intercal \in K \right\}$$

and

$$K^\star = \{(y, t)^\intercal \mid x^\intercal y + st \leq 1, x \in K|H, (x, s)^\intercal \in K\}.$$

Combining these two identities yields

$$(st_H(K))^\star = \left\{ (y, t)^\intercal \;\middle|\; x^\intercal y + \frac{1}{2}(a - b)t \leq 1, x \in K|H, (x, a)^\intercal, (x, b)^\intercal \in K \right\}.$$

Next, for $A \subset \mathbb{R}^n$ and $t \in \mathbb{R}$, let $A(t) = \{x \in \mathbb{R}^{n-1} \mid (x, t)^\intercal \in A\}$.

Claim: For $t \in \mathbb{R}$ it holds $\frac{1}{2}(K^\star(t) + K^\star(-t)) \subset (st_H(K))^\star(t)$.

To see this, pick some fixed $y_1 \in K^\star(t)$ and $y_2 \in K^\star(-t)$. Then, for all $(x, a)^\intercal, (x, b)^\intercal \in K$ we have

$$\begin{aligned} \left(x, \frac{1}{2}(a - b) \right)^\intercal \left(\frac{1}{2}y_1 + \frac{1}{2}y_2, t \right) &= \frac{1}{2}x^\intercal y_1 + \frac{1}{2}x^\intercal y_2 + \frac{1}{2}at - \frac{1}{2}bt \\ &= \frac{1}{2}(\underbrace{x^\intercal y_1 + at}_{\leq 1}) + \frac{1}{2}(\underbrace{x^\intercal y_2 + b(-t)}_{\leq 1}) \\ &\leq 1. \end{aligned}$$

Therefore, by the above identity, this implies $\frac{1}{2}(y_1 + y_2) \in (st_H(K))^\star(t)$.

Since, $K \in \mathcal{K}_0^n$, the polar $K^\star$ is also centrally symmetric, which leads to $K^\star(t) = -K^\star(-t)$. Thus, we obtain $\mathrm{vol}_{n-1}(K^\star(t)) = \mathrm{vol}_{n-1}(K^\star(-t))$ and an application of the famous Brunn-Minkowski inequality (see [25], Theorem 6.1.1), yields, for all real t,

$$\begin{aligned}
\mathrm{vol}_{n-1}((st_H(K))^\star(t)) &\geq \mathrm{vol}_{n-1}\left(\frac{1}{2}K^\star(t) + \frac{1}{2}K^\star(-t)\right) \\
&\geq \left(\frac{1}{2}\mathrm{vol}_{n-1}(K^\star(t))^{\frac{1}{n-1}} + \frac{1}{2}\mathrm{vol}_{n-1}(K^\star(-t))^{\frac{1}{n-1}}\right)^{n-1} \\
&= \mathrm{vol}_{n-1}(K^\star(t)).
\end{aligned}$$

Eventually, we use Cavalieri's principle and derive

$$\begin{aligned}
\mathrm{vol}((st_H(K))^\star) &= \int_{-\infty}^{\infty} \mathrm{vol}_{n-1}((st_H(K))^\star(t))\ dt \\
&\geq \int_{-\infty}^{\infty} \mathrm{vol}_{n-1}(K^\star(t))\ dt \ = \ \mathrm{vol}(K^\star),
\end{aligned}$$

and the lemma is shown. □

We are now well-prepared to prove the main theorem.

2.7 Theorem (Blaschke [3], Santaló [24]).
Let K be a centrally symmetric convex body in $\mathbb{R}^n$. Then $\mathbf{M}(K) \leq \mathbf{M}(B^n)$.

Proof. As a consequence of Lemma 2.5, we can derive, that there is a sequence of hyperplanes $H_i, i \in \mathbb{N}$, such that

$$K_i := st_{H_i} \dots st_{H_1}(K) \to \left(\frac{\mathrm{vol}(K)}{\mathrm{vol}(B^n)}\right)^{\frac{1}{n}} B^n.$$

This yields that the "polar sequence" $(K_i^\star)_{i\in\mathbb{N}}$ converges to $\left(\frac{\mathrm{vol}(K)}{\mathrm{vol}(B^n)}\right)^{-\frac{1}{n}} B^n$. By Proposition 2.4 and Lemma 2.6, we have

$$\mathbf{M}(K) \leq \mathbf{M}(K_1) \leq \ldots \leq \mathbf{M}(K_i),$$

for all $i \in \mathbb{N}$. The continuity of the Mahler volume, together with the above statements, imply that this sequence of inequalities converges to $\mathbf{M}(B^n)$. □

2.2 Mahler's proof in dimension 2

In a previous work to his famous paper "Ein Übertragungsprinzip für konvexe Körper" [15], Kurt Mahler showed in [14], that in dimension two the functional $\mathbf{M}(\cdot)$ on $\mathcal{K}_0^2$ is bounded from below by the constant $8 = \frac{4^2}{2!}$. Probably, this result was the main aspect that lead him to state his conjecture afterwards.

Preparatory, we have to introduce a definition for polarity, that Mahler used in his proof and which differs from the one we denote with * in this work. A vector $p \in \mathbb{R}^2 \setminus \{0\}$ is said to be polar to the line $l_p = \{x \in \mathbb{R}^2 \mid p_1x_1 + p_2x_2 = 1\}$ and vice versa. Consider a convex polygon $P = \mathrm{conv}\{x_1, \ldots, x_m\}$ and collect for all edges $[x_i, x_j]$ of P the polar point v_{ij} to the line through x_i and x_j. The polar set P' of P is given by the convex hull of $\{v_{ij} \mid [x_i, x_j] \text{ is an edge of } P\}$.

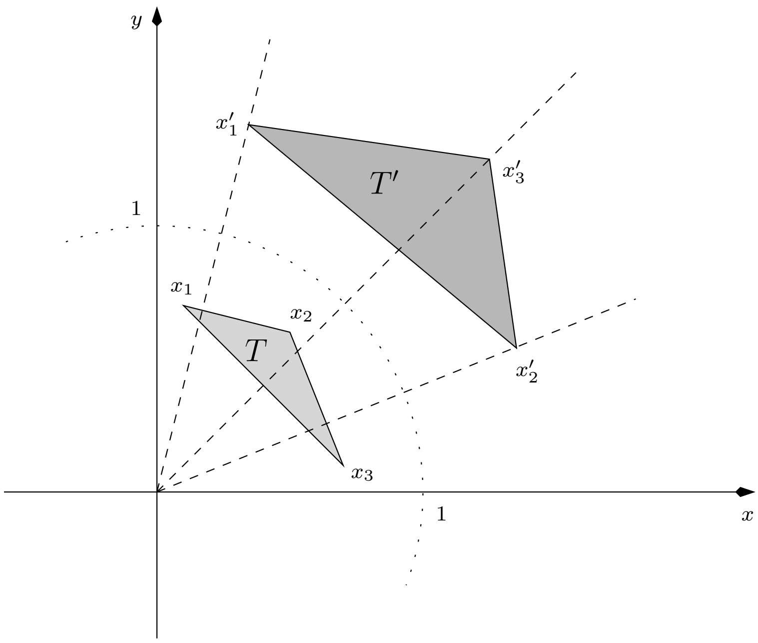

Figure 2: Triangle and its polar

If P contains the origin in its interior, it holds $P' = P^\star$. But if $0 \notin P$, then $P^\star$ is unbounded, while P' is surely not, being the convex hull of finitely many points. Figure 2 stresses that difference by illustrating a triangle $T = \mathrm{conv}\{x_1, x_2, x_3\}$, which does not contain the origin, and its polar $T' = \mathrm{conv}\{x'_1, x'_2, x'_3\}$. The dashed lines enlighten the fact that say x'_1 lies on the line through the origin perpendicular to the edge $[x_1, x_2]$ and the section of the unit circle shows that the length of x_1' is given by the reciprocal of the distance from $[x_1, x_2]$ to 0.

Next, we write for $x, y \in \mathbb{R}^2$

$$d_{xy} = \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = x_1 y_2 - x_2 y_1.$$

The following proposition relates the area of a triangle with that of its polar.

2.8 Proposition. *Let $T = \mathrm{conv}\{x, y, z\}$ be a triangle in $\mathbb{R}^2$ and let $T' = \mathrm{conv}\{x', y', z'\}$ be its polar. Furthermore, assume that no two of x, y and z are linearly dependent. Then, it holds*

$$\mathrm{vol}(T') = \frac{2\mathrm{vol}(T)^2}{d_{xy} d_{yz} d_{zx}}. \tag{2.5}$$

Proof. First of all, note that the assumption on say x and y not to be linearly dependent prevents d_{xy} from being zero. Now, x' is given as the intersection point of the lines l_x and l_y polar to x and y. Therefore, by solving a system of two linear equations we obtain $x' = \frac{1}{d_{xy}} \begin{pmatrix} y_2 - x_2 \\ x_1 - y_1 \end{pmatrix}$. Similarly, we have $y' = \frac{1}{d_{yz}} \begin{pmatrix} z_2 - y_2 \\ y_1 - z_1 \end{pmatrix}$ and $z' = \frac{1}{d_{zx}} \begin{pmatrix} x_2 - z_2 \\ z_1 - x_1 \end{pmatrix}$. Elementary calculations yield, that the area of T' is given by

$$\begin{aligned}
\mathrm{vol}(T') &= \frac{1}{2}\det(y' - x', z' - x') = \frac{1}{2}(d_{x'y'} + d_{y'z'} + d_{z'x'}) \\
&= \frac{1}{2}\left(\frac{1}{d_{xy}d_{yz}} \begin{vmatrix} y_2 - x_2 & z_2 - y_2 \\ x_1 - y_1 & y_1 - z_1 \end{vmatrix} + \frac{1}{d_{yz}d_{zx}} \begin{vmatrix} z_2 - y_2 & x_2 - z_2 \\ y_1 - z_1 & z_1 - x_1 \end{vmatrix} \right. \\
&\quad \left. + \frac{1}{d_{zx}d_{xy}} \begin{vmatrix} x_2 - z_2 & y_2 - x_2 \\ z_1 - x_1 & x_1 - y_1 \end{vmatrix} \right) \\
&= \frac{1}{2 d_{xy} d_{yz} d_{zx}} \left(d_{xy}^2 + d_{yz}^2 + d_{zx}^2 + 2d_{xy}d_{yz} + 2d_{yz}d_{zx} + 2d_{zx}d_{xy}\right) \\
&= \frac{(d_{xy} + d_{yz} + d_{zx})^2}{2 d_{xy} d_{yz} d_{zx}} = \frac{2\mathrm{vol}(T)^2}{d_{xy} d_{yz} d_{zx}}.
\end{aligned}$$

□

Let no two of x, y and z be linearly dependent. Suppose, that $0 \notin T = \mathrm{conv}\{x, y, z\}$. Then, d_{xy}, d_{yz}, d_{zx} are not all of the same sign, so we can assume without loss of generality, that $d_{xy}, d_{yz} > 0$ and $d_{zx} < 0$. Next, fix x, z and $\mathrm{vol}(T)$. The equation

$$d_{xy} + d_{yz} - |d_{zx}| = 2\mathrm{vol}(T)$$

defines a line g parallel to $[x,z]$ from which we pick y arbitrarily. Due to the positivity of d_{xy}, we get all points on g, when $0 < d_{xy} < 2\mathrm{vol}(T) + |d_{zx}|$. Bounding d_{xy} by $0 < \alpha \le d_{xy} \le \beta < 2\mathrm{vol}(T) + |d_{zx}|$ yields a stretch S of g. By equation (2.5) we get

$$\mathrm{vol}(T') = \frac{2\mathrm{vol}(T)^2}{d_{xy}(2\mathrm{vol}(T) + |d_{zx}| - d_{xy})|d_{zx}|}.$$

Thus, $\mathrm{vol}(T')$ depends continuously on $y \in S$ and attains a maximum on that stretch. Furthermore, we have

$$\begin{aligned}\left(\mathrm{vol}(T) + \frac{|d_{zx}|}{2}\right)^2 - \left(\mathrm{vol}(T) + \frac{|d_{zx}|}{2} - d_{xy}\right)^2 &= \left(\mathrm{vol}(T) + \frac{|d_{zx}|}{2}\right)^2 \\ &\quad - \left(\left(\mathrm{vol}(T) + \frac{|d_{zx}|}{2}\right)^2 - 2d_{xy}\left(\mathrm{vol}(T) + \frac{|d_{zx}|}{2}\right) + d_{xy}^2\right) \\ &= d_{xy}(2\mathrm{vol}(T) + |d_{zx}| - d_{xy}) = \frac{2\mathrm{vol}(T)^2}{|d_{zx}|\mathrm{vol}(T')}.\end{aligned}$$

The left hand side attains a maximum at $d_{xy} = \mathrm{vol}(T) + \frac{|d_{zx}|}{2}$ and decreases thereof strongly monotonically in both directions. So, $\mathrm{vol}(T')$ is maximal, when y is a boundary point of S. Precisely,

- if $\mathrm{vol}(T) + \frac{|d_{zx}|}{2} < \alpha$, the maximum is attained at $d_{xy} = \beta$,
- if $\mathrm{vol}(T) + \frac{|d_{zx}|}{2} > \beta$, the maximum is attained at $d_{xy} = \alpha$,
- if $\alpha \le \mathrm{vol}(T) + \frac{|d_{zx}|}{2} \le \beta$, the maximum is attained at $d_{xy} \in \{\alpha, \beta\}$, depending on which point the area of T' is bigger.

The subsequent lemma contains the crucial construction, from that the proof will be obtained.

2.9 Lemma. *Let $P \in \mathcal{K}_0^2$ be a polygon with $2m \ge 6$ vertices. Then, there exists a polygon $Q \in \mathcal{K}_0^2$ with $2(m-1)$ vertices, such that*

$$\mathrm{vol}(Q)\mathrm{vol}(Q') < \mathrm{vol}(P)\mathrm{vol}(P').$$

Proof. Let $v_1, \dots, v_m, -v_1, \dots, -v_m$ be the clockwise ordered vertices of P and let $P' = \mathrm{conv}\{\pm v_1', \dots, \pm v_m'\}$, where $\pm v_i'$ corresponds to the edge $[\pm v_i, \pm v_{i+1}]$, for $1 \le i \le m-1$, and $\pm v_m'$ corresponds to $[\pm v_m, \mp v_1]$.

Due to the assumption, that $2m \ge 6$, we can find a denotation that the triangle $T = \mathrm{conv}\{v_1, v_2, v_3\}$ does not contain the origin. Additionally, let $R = \mathrm{conv}\{\pm v_1, \pm v_3, \dots, \pm v_m\}$, so we have $P = R \cup T \cup -T$.

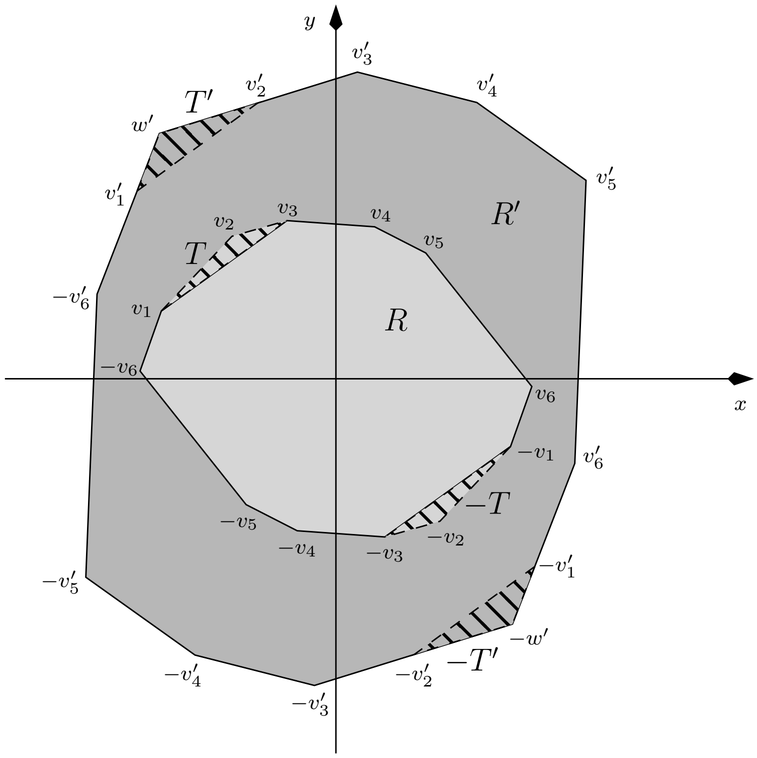

Figure 3: Cutting the vertices $\pm v_2$

The polar triangles $T' = \text{conv}\{v_1', w', v_2'\}$ and $-T'$ are contained in $R' = \text{conv}\{\pm w', \pm v_3', \dots, \pm v_m'\}$ and it holds $P' = R' \setminus (T' \cup -T')$. The just described situation is depicted in Figure 3.

Consider the line g through v_2 parallel to the edge $[v_1, v_3]$. By extending the edges $[-v_m, v_1]$ and $[v_3, v_4]$ towards g we resect a stretch S of that line. As shown in Figure 4, we can move v_2 arbitrarily on S and will always maintain the convexity of P and its volume, since the volume of T would not change. Let us call $T(v) = \text{conv}\{v_1, v, v_3\}$, for $v \in S$. By the considerations after Proposition 2.8, we choose v as the boundary point of S such that the area of $T(v)'$ becomes maximal over all $v \in S$.

Now, write $Q = \text{conv}\{\pm v_1, \pm v, \pm v_3, \dots, \pm v_m\}$ and we get $\text{vol}(Q) = \text{vol}(P)$ and

$$\text{vol}(Q') = \text{vol}(P') + \text{vol}(T') - \text{vol}(T(v)') < \text{vol}(P').$$

Notice, that Q indeed has $2(m-1)$ vertices, since the choice of v ensures that it lays on the extension of either $[-v_m, v_1]$ or $[v_3, v_4]$. In conclusion, we thus have found the desired polygon fulfilling $\text{vol}(Q)\text{vol}(Q') < \text{vol}(P)\text{vol}(P')$. $\square$

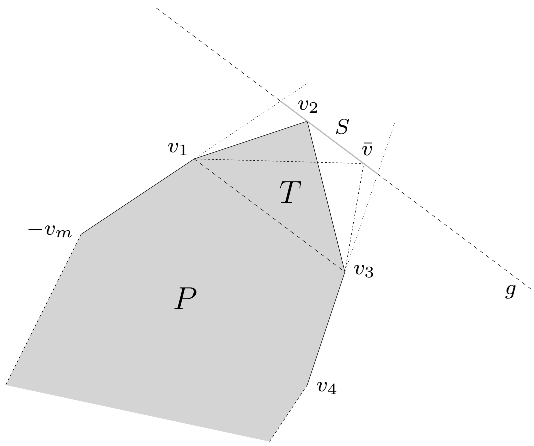

Figure 4: Varying v_2 on S

2.10 Theorem (Mahler [14]).
Let $P \in \mathcal{K}_0^2$ be a polygon. Then we have $\mathbf{M}(P) \geq 8$ and equality is attained if and only if P is a parallelogram.

Proof. We proceed by induction on the number of vertices of P. Let's make a start with $P = \operatorname{conv}\{v, w, -v, -w\}$, i.e., P is a parallelogram. Subdividing P into four triangles $\Delta(0, v, w), \Delta(0, w, -v), \Delta(0, -v, -w)$ and $\Delta(0, -w, v)$, which are all of the same area, yields

$$\operatorname{vol}(P) = \frac{1}{2}d_{vw} + \frac{1}{2}d_{w,-v} + \frac{1}{2}d_{-v,-w} + \frac{1}{2}d_{-w,v} = 2d_{vw}.$$

Note, that we do not have to distinguish P' from $P^\star$ here, since $0 \in \operatorname{int}(P)$. Similarly to the reasoning in the proof of Proposition 2.8, we obtain $P^\star = \operatorname{conv}\{\pm v', \pm w'\}$, where $v' = \frac{1}{d_{vw}} \begin{pmatrix} w_2 - v_2 \\ v_1 - w_1 \end{pmatrix}$ and $w' = \frac{1}{d_{vw}} \begin{pmatrix} -v_2 - w_2 \\ v_1 + w_1 \end{pmatrix}$, which implies $\operatorname{vol}(P^\star) = \frac{4}{d_{vw}}$ and therefore $\mathbf{M}(P) = 8$.

For the induction step let $P = \operatorname{conv}\{\pm v_1, \ldots, \pm v_m\}$ with $m \geq 3$. Lemma 2.9 states the existence of a polygon Q with $2(m-1)$ vertices and

$$\mathbf{M}(P) > \mathbf{M}(Q) \geq 8.$$

The latter comes from the induction hypothesis. And the strict inequality also yields the characterisation of the case of equality. □

Mahler also remarked, that for an arbitrary centrally symmetric convex body K in $\mathbb{R}^2$ we can find a sequence of polygons in $\mathcal{K}_0^2$ that converges to K. Since, the Mahler volume is a continuous functional on $\mathcal{K}_0^2$, we also have $\mathbf{M}(K) \geq 8$, but the characterisation of the equality case is lost.

3 Kuperberg's "low-technology" approach

In this chapter we deal with a work by Kuperberg [12] from 1992. He proves a weaker result than Theorem 2.1, but the arguments are more elementary.

3.1 Theorem (Kuperberg [12]).
Let $n \geq 4$ and $K \in \mathcal{K}_0^n$. Then

$$\mathbf{M}(K) \geq (\log_2 n)^{-n}\mathbf{M}(B^n).$$

In order to get this result, Kuperberg considers special constructions of convex bodies that turn out to be very useful. The definitions and important properties of those convex sets are given in the subsequent part. Afterwards, the details of the arguments are discussed.

3.1 p-sums, p-intersections and p-products

In the following, we declare that $\frac{1}{\infty} = 0$.

3.2 Definition (p-sum, p-intersection, p-product).
Let $K, L \in \mathcal{K}_0^n$, $K' \in \mathcal{K}_0^m$ and $1 \leq p \leq \infty$. We define the p-sum set *of K and L as*

$$K +_p L := \bigcup_{0 \leq t \leq 1} \left((1-t)^{\frac{1}{p}} K + t^{\frac{1}{p}} L \right),$$

the p-intersection set *of K and L as*

$$K \cap_p L := \bigcup_{0 \leq t \leq 1} \left((1-t)^{\frac{1}{p}} K \cap t^{\frac{1}{p}} L \right),$$

and the p-product set *of K and K' as*

$$K \times_p K' := \bigcup_{0 \leq t \leq 1} \left((1-t)^{\frac{1}{p}} K \times t^{\frac{1}{p}} K' \right).$$

The above constructions are generalisations of very common operations, it holds $K +_\infty L = K + L$, $K \cap_\infty L = K \cap L$, $K \times_\infty K' = K \times K'$ and $K +_1 L = \mathrm{conv}\{K \cup L\}$.

3.3 Proposition. *The operations $+_p$, $\cap_p$ and $\times_p$ are associative and the result is always centrally symmetric.*

Proof. The argument to prove the associativity is independent on the operation $+_p$, $\cap_p$ and $\times_p$ and it therefore suffices to show for $p \geq 1$ and $K, L, M \in \mathcal{K}_0^n$ that $(K +_p L) +_p M = K +_p (L +_p M)$. We have

$$\begin{aligned}(K +_p L) +_p M &= \bigcup_{0 \leq t \leq 1} \left((1-t)^{\frac{1}{p}} \bigcup_{0 \leq s \leq 1} \left((1-s)^{\frac{1}{p}} K + s^{\frac{1}{p}} L \right) + t^{\frac{1}{p}} M \right) \\ &= \bigcup_{0 \leq t \leq 1} \bigcup_{0 \leq s \leq 1} \left((1-t)^{\frac{1}{p}} (1-s)^{\frac{1}{p}} K + (1-t)^{\frac{1}{p}} s^{\frac{1}{p}} L + t^{\frac{1}{p}} M \right).\end{aligned}$$

Now, we consider the following substitutions:

$$\tilde{t} = s + t - st \text{ and } \tilde{s} = \frac{t}{s + t - st} \text{ for } s + t \neq 0 \text{ and } \tilde{s} = 1 \text{ if } s = t = 0.$$

The maps $(t, s) \mapsto \tilde{t}$ and $(t, s) \mapsto \tilde{s}$ are both surjective on $[0, 1] \times [0, 1]$ and it holds $1 - \tilde{t} = (1 - t)(1 - s)$, $\tilde{t}(1 - \tilde{s}) = (1 - t)s$ and $\tilde{t}\tilde{s} = t$. This leads to

$$\begin{aligned}(K +_p L) +_p M &= \bigcup_{0 \leq \tilde{t} \leq 1} \bigcup_{0 \leq \tilde{s} \leq 1} \left((1-\tilde{t})^{\frac{1}{p}} K + \tilde{t}^{\frac{1}{p}} (1-\tilde{s})^{\frac{1}{p}} L + \tilde{t}^{\frac{1}{p}} \tilde{s}^{\frac{1}{p}} M \right) \\ &= \bigcup_{0 \leq \tilde{t} \leq 1} \left((1-\tilde{t})^{\frac{1}{p}} K + \tilde{t}^{\frac{1}{p}} \bigcup_{0 \leq \tilde{s} \leq 1} \left((1-\tilde{s})^{\frac{1}{p}} L + \tilde{s}^{\frac{1}{p}} M \right) \right) \\ &= K +_p (L +_p M).\end{aligned}$$

The property that the three described constructions always lead to centrally symmetric sets simply inherits from the assumption on the original bodies to be centrally symmetric. □

3.4 Proposition. *Let $K, L \in \mathcal{K}_0^n, K' \in \mathcal{K}_0^m$ and $1 \leq p < \infty$. Then*

i) $K +_p L = \{x + y \mid x, y \in \mathbb{R}^n$ *and* $||x||_K^p + ||y||_L^p \leq 1\}$.

ii) For $x \in \mathbb{R}^n$ we have $||x||_{K \cap_p L}^p = ||x||_K^p + ||x||_L^p$.

iii) For $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ we have $||(x, y)||_{K \times_p K'}^p = ||x||_K^p + ||y||_{K'}^p$.

iv) By identifying K with $K \times \{0\}^m \in \mathcal{K}_0^{n+m}$ and K' with $\{0\}^n \times K' \in \mathcal{K}_0^{n+m}$, we have $K \times_p K' = K +_p K' \subset \mathbb{R}^n \times \mathbb{R}^m$.

Proof. i) To see the inclusion from left to right, let $w = (1 - t)^{\frac{1}{p}} x + t^{\frac{1}{p}} y \in K +_p L$, where $t \in [0, 1], x \in K$ and $y \in L$. We get

$$||(1-t)^{\frac{1}{p}} x||_K^p + ||t^{\frac{1}{p}} y||_L^p = (1-t)||x||_K^p + t||y||_L^p \leq 1.$$

Conversely, let $x, y \in \mathbb{R}^n, s = ||x||_K$ and $t = ||y||_L$ such that $s^p + t^p \leq 1$. Therefore, since $s, t \leq 1$, there are $\bar{x} \in K$ and $\bar{y} \in L$ with $x = s\bar{x}$ and $y = t\bar{y}$. Without loss of generality we assume, that s is not equal to 1, since otherwise t would be zero and $x + y = s\bar{x} \in K \subset K +_p L$. We obtain

$$x + y = s\bar{x} + t\bar{y} = s\bar{x} + (1 - s^p)^{\frac{1}{p}} \underbrace{(t(1 - s^p)^{-\frac{1}{p}}\bar{y})}_{=:\tilde{y} \in L} = r^{\frac{1}{p}}\bar{x} + (1 - r)^{\frac{1}{p}}\tilde{y},$$

where $r = s^p \in [0, 1]$. This means, that $x + y \in K +_p L$.

ii) We have to show that $K \cap_p L = \{x \in \mathbb{R}^n \mid ||x||_K^p + ||x||_L^p \leq 1\} =: D$. On the one hand, for $x = (1-t)^{\frac{1}{p}}y = t^{\frac{1}{p}}z \in K \cap_p L$, it holds $||x||_K \leq (1-t)^{\frac{1}{p}}$ and $||x||_L \leq t^{\frac{1}{p}}$. Thus, $||x||_K^p + ||x||_L^p \leq 1 - t + t = 1$ and $x \in D$. On the other hand, let $x \in D$ and $s = ||x||_K,\ t = ||x||_L$, so $s^p + t^p \leq 1$. Therefore, there exist $y \in K$ and $z \in L$ such that $x = sy = tz$. Analoguesly to i) we define $r = s^p$ and $\tilde{z} = t(1 - s^p)^{\frac{1}{p}}z \in L$ and get $x = r^{\frac{1}{p}}y = (1 - r)^{\frac{1}{p}}\tilde{z} \in K \cap_p L$.

The statement iii) can be proven similarly to the second one and iv) directly follows from the definitions. □

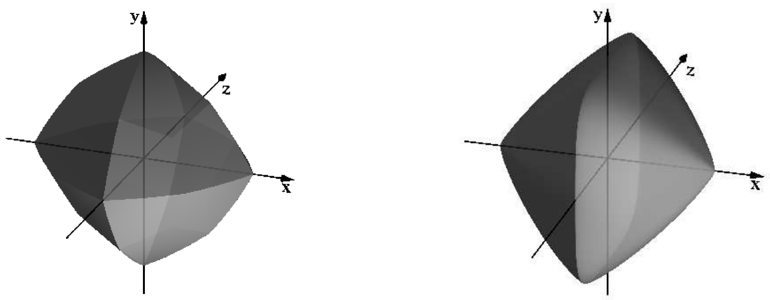

Figure 5: $C_3^\star \cap_3 C_3$ and $C_1 \times_{1.2} C_1 \times_4 C_1$

Since, we wish to investigate the Mahler volume of sets coming from p-constructions, it is inevitable to study the relation to the polar operation, which is stated without proof.

3.5 Lemma. *Let $K, L \in \mathcal{K}_0^n$, $K' \in \mathcal{K}_0^m$ and $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then the following relations hold.*

i) $(K +_p L)^\star = K^\star \cap_q L^\star$, *ii)* $(K \cap_p L)^\star = K^\star +_q L^\star$,

iii) $(K \times_p K')^\star = K^\star \times_q (K')^\star$.

3.6 Remark. *Proposition 3.4 and the preceding lemma show that p-sums, p-intersections and p-products of centrally symmetric convex bodies always correspond to a norm, which means that the outcome of such constructions is always in $\mathcal{K}_0^n$.*

We utilise the properties above to compute some examples for bodies that come from p-constructions. The pictures in Figure 5 were rendered with POV-Ray for Windows v3.5 [19].

The following formula enables us to derive the volume of a p-product set from that of the involved bodies.

3.7 Lemma. *For $K \in \mathcal{K}_0^n$ and $1 \leq p < \infty$ we have*

$$\int_{\mathbb{R}^n} e^{-||x||_K^p} dx = \Gamma(\tfrac{n}{p}+1)\mathrm{vol}(K).$$

Proof. The statement is shown by the following computation.

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-||x||_K^p} dx &= \int_{\mathbb{R}^n} \left(-e^{-s}\Big|_{||x||_K^p}^{\infty} \right) dx = \int_{\mathbb{R}^n} \int_{\sqrt[p]{s}>0: x \in \sqrt[p]{s}K} e^{-s} ds\ dx \\
&= \int_{\mathbb{R}^n} \int_{t>0: x \in tK} pt^{p-1} e^{-t^p} dt\ dx \\
\text{(Fubini)} \quad &= \int_0^{\infty} \int_{tK} pt^{p-1} e^{-t^p} dx\ dt \\
&= \int_0^{\infty} pt^{p-1} e^{-t^p} \mathrm{vol}(tK)\ dt = \mathrm{vol}(K) \int_0^{\infty} pt^{n+p-1} e^{-t^p} dt \\
&= \mathrm{vol}(K) \int_0^{\infty} s^{\frac{n}{p}} e^{-s} ds = \Gamma(\tfrac{n}{p}+1)\mathrm{vol}(K).
\end{aligned}$$

□

3.8 Theorem (Product formula).
Let $K \in \mathcal{K}_0^n, K' \in \mathcal{K}_0^m$ and $1 \leq p \leq \infty$. It holds

$$\mathrm{vol}(K \times_p K') = \frac{\Gamma(\frac{n}{p}+1)\mathrm{vol}(K)\Gamma(\frac{m}{p}+1)\mathrm{vol}(K')}{\Gamma(\frac{n+m}{p}+1)}.$$

Proof. For the case $p = \infty$, we use that $K \times_\infty K' = K \times K'$. So, we get

$$\mathrm{vol}(K \times_\infty K') = \mathrm{vol}(K \times K') = \mathrm{vol}(K)\mathrm{vol}(K').$$

Assume that $1 \leq p < \infty$. By combining Lemma 3.7 with the representation of the norm of $K \times_p K'$ in Proposition 3.4 iii) we obtain

$$\Gamma(\tfrac{n+m}{p}+1)\mathrm{vol}(K \times_p K') = \int_{\mathbb{R}^n \times \mathbb{R}^m} e^{-||(x,y)||_{K \times_p K'}^p} d(x,y)$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n \times \mathbb{R}^m} e^{-||x||_K^p - ||y||_{K'}^p} d(x,y) \\
&= \left(\int_{\mathbb{R}^n} e^{-||x||_K^p} dx\right)\left(\int_{\mathbb{R}^m} e^{-||y||_{K'}^p} dy\right) \\
&= \Gamma(\tfrac{n}{p}+1)\mathrm{vol}(K)\Gamma(\tfrac{m}{p}+1)\mathrm{vol}(K'),
\end{aligned}
$$

and the product formula follows. □

3.2 Kuperberg's arguments

We start by stating formulas for the Mahler volume of p-products. They are deduced with the help of Lemma 3.5, Theorem 3.8 and the associativity of the $\times_p$ relation.

3.9 Proposition. *Let $K_i \in \mathcal{K}_0^{n_i}$, $1 \le i \le s$, $n = n_1 + \ldots + n_s$ and furthermore $1 \le p, q \le \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, $K_1 \times_p K_2$ and $K_1 \times_q K_2$ have the same Mahler volume and it holds*

$$\mathbf{M}(K_1 \times_p K_2) = \frac{\Gamma(\frac{n_1}{p}+1)\Gamma(\frac{n_1}{q}+1)\Gamma(\frac{n_2}{p}+1)\Gamma(\frac{n_2}{q}+1)}{\Gamma(\frac{n_1+n_2}{p}+1)\Gamma(\frac{n_1+n_2}{q}+1)}\mathbf{M}(K_1)\mathbf{M}(K_2)$$

and

$$\mathbf{M}(K_1 \times_p \ldots \times_p K_s) = \frac{\prod_{i=1}^{s} \Gamma(\frac{n_i}{p}+1)\Gamma(\frac{n_i}{q}+1)\mathbf{M}(K_i)}{\Gamma(\frac{n}{p}+1)\Gamma(\frac{n}{q}+1)}.$$

Examples: Consider the interval $I = [-1, 1]$. We have $I^\star = I$ and therefore $\mathbf{M}(I) = 4$. For $1 \le p < \infty$ the p-norm is defined by $||x||_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$ and for $p = \infty$ by $||x||_\infty = \max_{1 \le i \le n} |x_i|$, $x \in \mathbb{R}^n$. We denote the unit ball of the p-norm with $B_p^n = \{x \in \mathbb{R}^n \mid ||x||_p \le 1\}$. In Proposition 3.4 iii) we have already seen that for $K \in \mathcal{K}_0^n$ and $K' \in \mathcal{K}_0^m$ the norm of $K \times_p K'$ is related to that of K and K' by

$$||(x,y)||_{K \times_p K'}^p = ||x||_K^p + ||y||_{K'}^p, \ (x,y) \in \mathbb{R}^n \times \mathbb{R}^m \text{ for } 1 \le p < \infty.$$

This can be extended to the p-product of more than two convex bodies. For $K_i \in \mathcal{K}_0^{n_i}$, $1 \le i \le s$, it holds

$$||(x^1, \ldots, x^s)||_{K_1 \times_p \ldots \times_p K_s}^p = \sum_{i=1}^{s} ||x^i||_{K_i}^p, \ (x^1, \ldots, x^s) \in \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_s}.$$

Now, we have $||x||_I = |x|, x \in \mathbb{R}$, which leads to

$$||(x_1, \dots, x_n)||^p_{I \times_p \dots \times_p I} = \sum_{i=1}^{n} |x_i|^p = ||x||^p_p, \; x \in \mathbb{R}^n.$$

This ends in a very useful description of the unit balls of the p-norm:

$$B^n_p = I \times_p \dots \times_p I, \text{ for } 1 \leq p < \infty.$$

Additionally, we also have $B^n_\infty = (B^n_1)^\star = (I \times_1 \dots \times_1 I)^\star = I \times_\infty \dots \times_\infty I$.

With help of Proposition 3.9 we now compute the Mahler volume of the Euclidean unit ball $B^n = B^n_2$, the cube $C_n = B^n_\infty$ and the crosspolytope $C^\star_n = B^n_1$. We have

$$\mathbf{M}(B^n) = \mathbf{M}(I \times_2 \dots \times_2 I) = \frac{\pi^n}{\Gamma(\frac{n}{2}+1)^2}$$

and

$$\mathbf{M}(C_n) = \mathbf{M}(C^\star_n) = \mathbf{M}(I \times_1 \dots \times_1 I) = \frac{4^n}{\Gamma(n+1)} = \frac{4^n}{n!}.$$

Note, that we used some properties of the Gamma function, i.e., $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(n+1) = n!$, for $n \in \mathbb{N}_0$, and $\Gamma(x+1) = x\Gamma(x)$, for $x > 0$.

3.10 Proposition. *For every centrally symmetric convex body $K \in \mathcal{K}^n_0$ we have*

i) $e(K)^{-n}\mathbf{M}(B^n) \leq \mathbf{M}(K) \leq e(K)^n\mathbf{M}(B^n)$ *and*

ii) $n^{-\frac{n}{2}}\mathbf{M}(B^n) \leq \mathbf{M}(K) \leq n^{\frac{n}{2}}\mathbf{M}(B^n)$.

Proof. i) The definitions of in- and circumradius of K yield

$$r(K)B^n \subset K \subset R(K)B^n$$

and by taking polars we get

$$\frac{1}{R(K)}B^n \subset K^\star \subset \frac{1}{r(K)}B^n.$$

This leads to

$$\mathbf{M}(K) = \text{vol}(K)\text{vol}(K^\star) \geq \text{vol}(r(K)B^n)\text{vol}(\frac{1}{R(K)}B^n) = e(K)^{-n}\mathbf{M}(B^n)$$

and analoguesly, we obtain $\mathbf{M}(K) \leq e(K)^n\mathbf{M}(B^n)$.
ii) By John's Theorem (see [6], Theorem 4.2.12) there is an invertible linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $B^n \subset TK \subset \sqrt{n}B^n$ and therefore, it holds $e(TK) \leq \sqrt{n}$. Due to the affine invariance of the Mahler volume this and i) yield the desired inequalities. □

3.11 Lemma. *For any body $K \in \mathcal{K}_0^n$ we have $\mathbf{M}(K) \geq 2^{-n}\mathbf{M}(K \cap_2 K^\star)$.*

Proof. The key for the proof will be to investigate the body $C := T(K \times_2 K^\star)$, where $T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the linear map defined by $T(x, y) := (x, x+y),\ x, y \in \mathbb{R}^n$, which has determinant 1. By using the product formula for the volume of p-products (see Theorem 3.8), we compute the volume of C as

$$\mathrm{vol}(C) = \mathrm{vol}(K \times_2 K^\star) = \frac{\Gamma(\frac{n}{2}+1)\mathrm{vol}(K)\Gamma(\frac{n}{2}+1)\mathrm{vol}(K^\star)}{\Gamma(n+1)} = \frac{\mathbf{M}(K)}{\binom{n}{\frac{n}{2}}}. \tag{3.1}$$

Note, that we adopt the notation of the binomial coefficients for arbitrary reals by writing $\binom{r}{s} = \frac{\Gamma(r+1)}{\Gamma(s+1)\Gamma(r-s+1)}$, for $0 \leq s \leq r$.

Next, we consider the projection map $\pi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,\ \pi(x, y) = y$.
Claim 1: i) $C \cap \pi^{-1}(0) = (K \cap_2 K^\star) \times \{0\}$ and ii) $\pi(C) = K +_2 K^\star$.

These two statements are consequences of Proposition 3.4. It holds

$$\begin{aligned} T(K \times_2 K^\star) \cap \pi^{-1}(0) &= \{(x, x+y) \mid ||x||_K^2 + ||y||_{K^\star}^2 \leq 1, x+y = 0\} \\ &= \{(x, 0) \mid ||x||_K^2 + ||x||_{K^\star}^2 \leq 1\} \\ &= (K \cap_2 K^\star) \times \{0\} \end{aligned}$$

and

$$\begin{aligned} \pi(T(K \times_2 K^\star)) &= \pi\left(\{(x, x+y) \mid ||x||_K^2 + ||y||_{K^\star}^2 \leq 1\}\right) \\ &= \{x + y \mid ||x||_K^2 + ||y||_{K^\star}^2 \leq 1\} \\ &= K +_2 K^\star. \end{aligned}$$

Now, we identify a dilated copy of $C \cap \pi^{-1}(0)$ in every affine section $C \cap \pi^{-1}(y), y \in \pi(C)$, of C.
Claim 2: For all $t \in [0, 1]$ and $y \in t\pi(C)$ there exists a $z \in \mathbb{R}^n \times \mathbb{R}^n$ such that $z + (1-t)(C \cap \pi^{-1}(0)) \subset C \cap \pi^{-1}(y)$.

For fixed $t \in [0, 1]$ and $(\bar{x}, \bar{y}) \in C$, let $y = t\pi(\bar{x}, \bar{y}) = t\bar{y}$. Now, for an arbitrary $(\tilde{x}, 0) \in C \cap \pi^{-1}(0)$ we have $t(\bar{x}, \bar{y}) + (1-t)(\tilde{x}, 0) \in C$, since C is convex. Moreover, $t(\bar{x}, \bar{y}) + (1-t)(\tilde{x}, 0) = (t\bar{x} + (1-t)\tilde{x}, y) \in \pi^{-1}(y)$, which verifies the claim with $z = t(\bar{x}, \bar{y})$.

Having these relations, we can estimate the volume of C in another way. The second equality comes from Cavalieri's principle.

$$\begin{aligned} \mathrm{vol}(C) &= \int_C d(x, y) = \int_{\pi(C)} \left(\int_{C \cap \pi^{-1}(y)} dx \right) dy \\ &= \int_0^1 \int_{\pi(C)} \int_{C \cap \pi^{-1}(y)} dx\ dy\ dt \end{aligned}$$

$$\begin{aligned}
&\geq \int_0^1 \int_{t\pi(C)} \int_{C\cap\pi^{-1}(y)} dx\ dy\ dt \\
\text{(Claim 2)}\quad &\geq \int_0^1 \int_{t\pi(C)} \int_{(1-t)(C\cap\pi^{-1}(0))} dx\ dy\ dt \\
&= \mathrm{vol}(\pi(C) \times_1 (C\cap\pi^{-1}(0))) \\
\text{(Claim 1)}\quad &= \mathrm{vol}((K +_2 K^\star) \times_1 (K \cap_2 K^\star \times \{0\})) \\
&= \mathrm{vol}((K +_2 K^\star) \times_1 (K \cap_2 K^\star)) \\
\text{(Theorem 3.8)}\quad &= \frac{\Gamma(n+1)^2}{\Gamma(2n+1)} \mathrm{vol}(K +_2 K^\star)\mathrm{vol}(K \cap_2 K^\star) \\
\text{(Lemma 3.5)}\quad &= \frac{\mathrm{vol}(K \cap_2 K^\star)\mathrm{vol}((K \cap_2 K^\star)^\star)}{\binom{2n}{n}} = \frac{\mathbf{M}(K \cap_2 K^\star)}{\binom{2n}{n}}.
\end{aligned}$$

When we look back to our first computation (3.1) of $\mathrm{vol}(C)$, the lemma will be shown as long as we have seen that

$$\binom{2n}{n} \leq 2^n \binom{n}{\frac{n}{2}}, \ n \in \mathbb{N}.$$

It is well known that $\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k}^2$, $\sum_{k=0}^n \binom{n}{k} = 2^n$ and $\binom{n}{\lfloor\frac{n}{2}\rfloor} \geq \binom{n}{k}$, for all $0 \leq k \leq n$. Thus, we estimate

$$\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k}^2 \leq \binom{n}{\lfloor\frac{n}{2}\rfloor} \sum_{k=0}^n \binom{n}{k} = 2^n \binom{n}{\lfloor\frac{n}{2}\rfloor}.$$

Note, that $\binom{n}{\lfloor\frac{n}{2}\rfloor} \leq \binom{n}{\frac{n}{2}}$ is equivalent to $\Gamma(\frac{n}{2}+1)^2 \leq \Gamma(\lfloor\frac{n}{2}\rfloor+1)\Gamma(n-\lfloor\frac{n}{2}\rfloor+1)$. The Gamma function is known to be log-convex, which means that for all $t \in [0,1]$ and $x, y > 0$ we have

$$\ln\Gamma((1-t)x+ty) \leq (1-t)\ln\Gamma(x) + t\ln\Gamma(y)$$

and so

$$\Gamma((1-t)x+ty) \leq \Gamma(x)^{1-t}\Gamma(y)^t.$$

If we pick $t = \frac{1}{2}$, $x = \lfloor\frac{n}{2}\rfloor+1$ and $y = n-\lfloor\frac{n}{2}\rfloor+1$, we have $(1-t)x+ty = \frac{n}{2}+1$ and therefore $\Gamma(\frac{n}{2}+1) \leq \Gamma(\lfloor\frac{n}{2}\rfloor+1)^{\frac{1}{2}}\Gamma(n-\lfloor\frac{n}{2}\rfloor+1)^{\frac{1}{2}}$. Eventually, we proved the desired statement, since

$$\mathbf{M}(K) \geq \binom{n}{\frac{n}{2}} \frac{\mathbf{M}(K \cap_2 K^\star)}{\binom{2n}{n}} \geq 2^{-n}\mathbf{M}(K \cap_2 K^\star).$$

□

We are now able to show the main theorem of this section.

3.12 Theorem. *Let $n \geq 4$ and $K \in \mathcal{K}_0^n$. Then*

$$\mathbf{M}(K) \geq (2\log_2 e(K))^{-n}\mathbf{M}(B^n).$$

In particular, from John's Theorem and the affine invariance of the Mahler volume, we have

$$\mathbf{M}(K) \geq (\log_2 n)^{-n}\mathbf{M}(B^n).$$

Proof. We will proceed by induction on $m \in \mathbb{N}$ such that $2^{2^{m-1}} < e(K) \leq 2^{2^m}$. At first, we may consider $K \in \mathcal{K}_0^n$ with $e(K) \leq 2$. In this case, the inequality in Proposition 3.10 i), i.e., $e(K)^{-n}\mathbf{M}(B^n) \leq \mathbf{M}(K)$, is sharper than the claim above, since $n \geq 4$ and therefore $e(K)^{-n} \geq 2^{-n} \geq (\log_2 n)^{-n}$.

Now, we assume that for all bodies $\bar{K} \in \mathcal{K}_0^n$ with $e(\bar{K}) \leq 2^{2^{m-1}}$ the theorem is proven and consider $K \in \mathcal{K}_0^n$ with $2^{2^{m-1}} < e(K) \leq 2^{2^m}$. By definition we have $r(K)B^n \subset K \subset R(K)B^n$, and so

$$\underbrace{\frac{r(K)}{r(K)^{\frac{1}{2}}R(K)^{\frac{1}{2}}}}_{=e(K)^{-\frac{1}{2}}} B^n \subset \frac{1}{r(K)^{\frac{1}{2}}R(K)^{\frac{1}{2}}} K \subset \underbrace{\frac{R(K)}{r(K)^{\frac{1}{2}}R(K)^{\frac{1}{2}}}}_{=e(K)^{\frac{1}{2}}} B^n$$

Since $e(tK) = e(K)$, for positive t, we can assume that $e(K)^{-n}B^n \subset K \subset e(K)^n B^n$. Taking polars, yields $e(K)^{-n}B^n \subset K^\star \subset e(K)^n B^n$. Define $\tilde{K} := K \cap_2 K^\star$. We see that

$$\tilde{K} \supset \bigcup_{t\in[0,1]} ((1-t)^{\frac{1}{2}} e(K)^{-\frac{1}{2}} B^n \cap t^{\frac{1}{2}} e(K)^{-\frac{1}{2}} B^n) \supset \frac{1}{\sqrt{2}} e(K)^{-\frac{1}{2}} B^n.$$

This means, that $r(\tilde{K}) \geq \frac{1}{\sqrt{2}} e(K)^{-\frac{1}{2}}$. For $x \in ((1-t)^{\frac{1}{2}} K \cap t^{\frac{1}{2}} K^\star)$ it holds

$$||x||^2 = x^\intercal x \leq (1-t)^{\frac{1}{2}} t^{\frac{1}{2}} \leq \frac{1}{2},$$

which leads to $R(\tilde{K}) \leq \frac{1}{\sqrt{2}}$. Combining these two inequalities yields

$$e(\tilde{K}) = \frac{R(\tilde{K})}{r(\tilde{K})} \leq e(K)^{\frac{1}{2}} \leq (2^{2^m})^{\frac{1}{2}} = 2^{2^{m-1}},$$

and by induction hypothesis we get

$$\mathbf{M}(\tilde{K}) \geq (2\log_2 e(\tilde{K}))^{-n}\mathbf{M}(B^n) \geq (\log_2 e(K))^{-n}\mathbf{M}(B^n).$$

From Lemma 3.11 we know $\mathbf{M}(K) \geq 2^{-n}\mathbf{M}(\tilde{K})$ and the claim follows. $\square$

4 Hanner polytopes and beyond

This chapter primary deals with *Hanner polytopes* which first occurred in a work by Olof Hanner [8]. These polytopes are related to the problem of Mahler, because they satisfy the conjectured lower bound in (2.2) with equality, which is shown below. We will also see, that the p-product construction of Kuperberg (see Definition 3.2) admits another description of a Hanner polytope. Basing ourselves on that, we investigate in the second section how the Mahler volume changes under taking p-products. The result is, that $K \times_p L$ satisfies the Mahler Conjecture for all choices of $p \in [1, \infty]$, if both K and L do so, in their respective dimensions.

4.1 Hanner polytopes

Considering a given convex body $K \in \mathcal{K}^n$, Hanner [8] was interested in the minimal number $I(K)$ of vectors $u_i \in \mathbb{R}^n$, such that the bodies $K+u_i$ pairwise meet, but the section of them all is empty. He proved that $I(K)$ is either $3, 4$ or nonexistent - we let $I(K)$ be infinity in this case. The polytopes of interest occurred in his work as examples of convex bodies with $I(K) > 3$. In 1981, Hansen and Lima [9] showed that Hanner polytopes are characterised by that property.

The *free sum* of two convex sets $K \in \mathcal{K}^n$ and $L \in \mathcal{K}^m$ is defined as the convex hull of $K \times \{0\}$ and $\{0\} \times L$.

4.1 Definition (Hanner polytope).
A Hanner polytope is obtained by successively applying Cartesian products and free sums to centered line segments in arbitrary order.

In the sequel, we will restrict to the interval $I = [-1, 1]$ as the involved line segment, since the Mahler volume is preserved under affine transformations. The p-product notation permits us to rewrite the Cartesian product of $K \in \mathcal{K}^n$ and $L \in \mathcal{K}^m$ as $K \times_\infty L = K \times L$ and the free sum as $K \times_1 L \cong (K \times \{0\}) +_1 (\{0\} \times L) = \operatorname{conv}\{(K \times \{0\}) \cup (\{0\} \times L)\}$. Therefore, every Hanner polytope in $\mathbb{R}^n$ is an affine image of $I \times_{p_1} \ldots \times_{p_{n-1}} I$, where p_i is either 1 or infinity, for $1 \leq i \leq n-1$.

The following proposition is a direct consequence of $I^\star = I$, the associativity of the p-product relation and Lemma 3.5.

4.2 Proposition. *Let $H = I \times_{p_1} \ldots \times_{p_{n-1}} I \in \mathcal{K}_0^n$ be a Hanner polytope and define $\bar{p}_i = 1$, if $p_i = \infty$, and $\bar{p}_i = \infty$, if $p_i = 1$, for $1 \le i \le n-1$. Then the polar body $H^\star$ is also a Hanner polytope and it holds $H^\star = I \times_{\bar{p}_1} \ldots \times_{\bar{p}_{n-1}} I$.*

Next, we show that Hanner polytopes attain the conjectured minimal Mahler volume.

4.3 Lemma. *Let $H \in \mathcal{K}_0^n$ be a Hanner polytope. Then, $\mathbf{M}(H) = \frac{4^n}{n!}$.*

Proof. Let $H = I \times_{p_1} \ldots \times_{p_{n-1}} I$ with $p_i \in \{1, \infty\}, 1 \le i \le n-1$. Proposition 3.9 states that $\mathbf{M}(K \times_1 L) = \mathbf{M}(K \times_\infty L)$ and together with the associativity of the p-product relation we obtain

$$\mathbf{M}(H) = \mathbf{M}(I \times_{p_1} (I \times_{p_2} \ldots \times_{p_{n-1}} I)) = \mathbf{M}(I \times_1 (I \times_{p_2} \ldots \times_{p_{n-1}} I)).$$

Inductively, this leads to $\mathbf{M}(H) = \mathbf{M}(I \times_1 \ldots \times_1 I) = \mathbf{M}(C_n^\star) = \frac{4^n}{n!}$. □

To conclude our consideration, we inspect Hanner polytopes in small dimensions up to affine transformations. In dimension one, there is only the line segment $I = [-1, 1]$.

For $n = 2$, we also have only one Hanner polytope, since the linear transformation given by the matrix $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ maps the diamond $C_2^\star = I \times_1 I$ to the square $C_2 = I \times_\infty I$.

This affine transformation of square and diamond implies, that in dimension $n = 3$ the polytope $I \times_\infty I \times_1 I$ is affinely isomorphic to the octahedron $C_3^\star$. Thus, there are two 3-dimensional Hanner polytopes, namely C_3 and $C_3^\star$.

Similarly, one gets four 4-dimensional Hanner polytopes - the cube C_4, the crosspolytope $C_4^\star$, $I \times_\infty I \times_\infty I \times_1 I$ and $I \times_1 I \times_1 I \times_\infty I$.

Observe, that when we add another $\times_1$ relation to $H = I \times_{p_1} \ldots \times_{p_{n-1}} I$, where $p_i \in \{1, \infty\}$, we end up with an $(n+1)$-dimensional Hanner polytope with two additional vertices. If we use another $\times_\infty$ relation, the number of vertices is doubled. For example, there is no 4-dimensional Hanner polytope with 14 vertices.

4.2 The transference property of $\times_p$

Suppose, we have two convex bodies $K \in \mathcal{K}_0^n$ and $L \in \mathcal{K}_0^m$, that fulfill the Mahler Conjecture. Is this property transferee to the p-product of K and L? A proof for the positive answer to this question is given in the sequel of this section.

In upcoming lemma, we write collections of elements, where the same element can occur various times as $[x_1, \dots, x_m]$. Moreover, we adopt the usual notation for the union of sets and write $[x_1, \dots, x_m] \cup [y_1, \dots, y_k] = [x_1, \dots, x_m, y_1, \dots, y_k]$.

4.4 Lemma. *Let $k, l \in \mathbb{N}$ and $M_k = \left[\frac{j}{k} \mid j = 1, \dots, k\right]$. There exists a bijection $f : M_k \cup M_l \to M_{k+l}$, such that $m \geq f(m)$, for all $m \in M_k \cup M_l$. Furthermore, there has to be an element $m' \in M_k \cup M_l$ with $m' > f(m')$.*

Proof. Let $M = M_k \cup M_l = [m_1, \dots, m_{k+l}]$ and $N = M_{k+l} = [n_1, \dots, n_{k+l}]$ be given in ascending order. Suppose $m_i \in M_k, i < k + l$, so there is an integer $z \in \{1, \dots, k\}$ with $m_i = \frac{z}{k}$. Recall the ceiling function $\lceil n \rceil$ of an integer n as the smallest integer greater than or equal to n. By definition we have $\lceil \frac{l}{k} z \rceil - 1 < \frac{l}{k} z \leq \lceil \frac{l}{k} z \rceil$ and therefore,

$$\frac{\lceil \frac{l}{k} z \rceil - 1}{l} < \frac{z}{k} \leq \frac{\lceil \frac{l}{k} z \rceil}{l}.$$

Apparently, there are $z - 1$ occurrences of elements from M_k in M which precede $\frac{z}{k}$. Thus, we identify the index i as the number $\lceil \frac{l}{k} z \rceil + z - 1$. It holds $\frac{l}{k} z \leq \lceil \frac{l}{k} z \rceil < \frac{l}{k} z + 1$, and equivalent transformations lead to

$$k \left\lceil \frac{l}{k} z \right\rceil + zk - k < zk + zl \leq k \left\lceil \frac{l}{k} z \right\rceil + zk.$$

This means, that

$$n_i = \frac{i}{k+l} = \frac{\lceil \frac{l}{k} z \rceil + z - 1}{k+l} < \frac{z}{k} = m_i \leq \frac{\lceil \frac{l}{k} z \rceil + z}{k+l} = \frac{i+1}{k+l} = n_{i+1}.$$

Similarly, we derive for $m_j \in M_l, j < k + l$, that $n_j < m_j \leq n_{j+1}$. Together this yields

$$n_1 < m_1 \leq n_2 < m_2 \leq \ldots \leq n_{k+l-2} < m_{k+l-2} \leq n_{k+l-1} < m_{k+l-1} \leq n_{k+l},$$

and furthermore we have $n_{k+l} = m_{k+l} = 1$. Thus, we can define a bijection $f : M \to N$ with the stipulated property, by mapping m_i to $f(m_i) = n_i$, for $1 \leq i \leq k+l$. The additional statement follows immediately by the definition of f, since $k + l \geq 2$. $\square$

This lemma enables us to extend an argument by Saint-Raymond [23], which he used to verify the Mahler Conjecture for the class of unit balls of p-norms. We get the following statement, that turns out to be the main ingredient for the proof of the transference property of $\times_p$.

4.5 Lemma. *Let $k, l \in \mathbb{N}$, and $1 \le p, q \le \infty$, such that $\frac{1}{p} + \frac{1}{q} = 1$. Then the following inequalities hold*

$$\frac{k! \cdot l!}{(k+l)!} \le \frac{\Gamma(\frac{k}{p}+1)\Gamma(\frac{l}{p}+1)\Gamma(\frac{k}{q}+1)\Gamma(\frac{l}{q}+1)}{\Gamma(\frac{k+l}{p}+1)\Gamma(\frac{k+l}{q}+1)} \le \frac{\Gamma(\frac{k}{2}+1)^2\Gamma(\frac{l}{2}+1)^2}{\Gamma(\frac{k+l}{2}+1)^2}.$$

Equality is attained on the left hand side, if and only if $p = 1$ or $q = 1$, and on the right hand side, if and only if $p = q = 2$.

Proof. Consider the well-known *digamma function* $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$ for $x > 0$. For the first derivative of ψ there is the following series representation (see [2])

$$\psi'(x) = \frac{d^2}{dx^2} \ln \Gamma(x) = \sum_{k=0}^{\infty} \frac{1}{(x+k)^2}.$$

For $x \ge 0$ define the function $g(x) := \frac{\Gamma(1+kx)\Gamma(1+lx)}{\Gamma(1+(k+l)x)}$. We obtain

$$\begin{aligned}
\frac{d^2}{dx^2} \ln g(x) &= k^2\psi'(1+kx) + l^2\psi'(1+lx) - (k+l)^2\psi'(1+(k+l)x) \\
&= \sum_{i=1}^{\infty} \frac{k^2}{(kx+i)^2} + \sum_{i=1}^{\infty} \frac{l^2}{(lx+i)^2} - \sum_{i=1}^{\infty} \frac{(k+l)^2}{((k+l)x+i)^2} \\
&= \sum_{i=1}^{\infty} \frac{1}{(x+\frac{i}{k})^2} + \sum_{i=1}^{\infty} \frac{1}{(x+\frac{i}{l})^2} - \sum_{i=1}^{\infty} \frac{1}{(x+\frac{i}{k+l})^2} \\
&= \sum_{i=0}^{\infty} \underbrace{\left(\sum_{j=1}^{k} \frac{1}{(x+i+\frac{j}{k})^2} + \sum_{j=1}^{l} \frac{1}{(x+i+\frac{j}{l})^2} - \sum_{j=1}^{k+l} \frac{1}{(x+i+\frac{j}{k+l})^2} \right)}_{=:s_{k,l}(i)}.
\end{aligned}$$

The sign of $s_{k,l}(i)$ is independent of x and i as long as $x+i \ge 0$ which is always fulfilled. By Lemma 4.4, we get $s_{k,l}(i) < 0$, for all $i \in \mathbb{N}_0$. Thus, $x \mapsto \ln g(x)$ is concave in $[0,1]$, and so is $\varphi(x) = \ln(g(x)g(1-x))$. Additionally, φ is symmetric around $x = \frac{1}{2}$, it attains its maximum at $x = \frac{1}{2}$ and its minima at $x = 0$ and $x = 1$, which gives us

$$\varphi(0) = \varphi(1) \le \varphi(x) \le \varphi(\frac{1}{2}).$$

That implies for $\frac{1}{p} + \frac{1}{q} = 1$,

$$\frac{k! \cdot l!}{(k+l)!} \le \frac{\Gamma(\frac{k}{p}+1)\Gamma(\frac{l}{p}+1)\Gamma(\frac{k}{q}+1)\Gamma(\frac{l}{q}+1)}{\Gamma(\frac{k+l}{p}+1)\Gamma(\frac{k+l}{q}+1)} \le \frac{\Gamma(\frac{k}{2}+1)^2\Gamma(\frac{l}{2}+1)^2}{\Gamma(\frac{k+l}{2}+1)^2},$$

where equality is attained if and only if $p = 1$ or $q = 1$ on the left hand side and if and only if $p = q = 2$ on the right hand side. □

4.6 Theorem. *Let* $K_i \in \mathcal{K}_0^{n_i}$, $1 \leq i \leq k$, *and* $n = n_1 + \ldots + n_k$. *If all* K_i *satisfy Mahler's Conjecture in their respective dimension, i.e.,* $\mathbf{M}(K_i) \geq \frac{4^{n_i}}{n_i!}$, *then for all* $p_1, \ldots, p_{k-1} \in [1, \infty]$, *we have*

$$\mathbf{M}(K_1 \times_{p_1} \ldots \times_{p_{k-1}} K_k) \quad \geq \quad \frac{4^n}{n!}. \tag{4.1}$$

Proof. We proceed by induction on the number k of involved bodies. There is nothing to show for $k = 1$, since we have $n = n_1$ and therefore, $\mathbf{M}(K_1) \geq \frac{4^n}{n!}$.

Let $k \geq 2$ and write $K = K_1 \times_{p_1} \ldots \times_{p_{k-1}} K_k$. Proposition 3.9 gives us

$$\mathbf{M}(K) = \frac{\Gamma(\frac{n_1}{p_1}+1)\Gamma(\frac{n-n_1}{p_1}+1)\Gamma(\frac{n_1}{q_1}+1)\Gamma(\frac{n-n_1}{q_1}+1)}{\Gamma(\frac{n}{p_1}+1)\Gamma(\frac{n}{q_1}+1)}\mathbf{M}(K_1)\mathbf{M}(K_2 \times_{p_2} \ldots \times_{p_{k-1}} K_k),$$

where q_1 satisfies $\frac{1}{p_1} + \frac{1}{q_1} = 1$. Lemma 4.5 and the induction hypothesis imply

$$\mathbf{M}(K) \geq \frac{n_1!(n-n_1)!}{n!} \cdot \frac{4^{n_1}}{n_1!} \cdot \frac{4^{n-n_1}}{(n-n_1)!} = \frac{4^n}{n!},$$

and we are done. □

4.7 Remark. *The equality characterisation in Lemma 4.5 yields the following addition to the above statement. Assume, that all the bodies fulfill* $\mathbf{M}(K_i) = \frac{4^{n_i}}{n_i!}$, *then equality holds in (4.1), if and only, if* p_j *is either* 1 *or infinity, for* $1 \leq j \leq k-1$.

Theorem 4.6 arises the question, whether we get new classes of centrally symmetric convex bodies, when applying p-products to bodies, that are known to satisfy the Mahler Conjecture. The following chapters deal with families of centrally symmetric convex bodies, for which the validity of Mahler's Conjecture is known. An interesting task for further work, would be to study the bodies that are given by p-products of members of such families.

5 Mahler's Conjecture and zonoids

In this chapter we investigate an approach on the Mahler Conjecture for the special class of zonoids. In [20] Reisner used probabilistic arguments involving random hyperplanes and a result on the number of vertices of certain polyhedra, to show that Mahler's Conjecture is valid for zonoids. Additionally, he characterised the equality case in [21]. In joint work with Gordon and Meyer [7] from 1987, he found a much more self-contained proof, that makes no use of probability and which is presented in the sequel.

The upcoming section will introduce the concept of zonoids. Several equivalent descriptions are given and a formula for the computation of the volume of such a convex body will be deduced. Subsequently, we will follow the details of the proof by Gordon, Meyer and Reisner.

5.1 Zonotopes and zonoids

5.1 Definition (Zonotope).
A zonotope $Z \in \mathcal{K}^n$ *is a finite Minkowski sum of line segments, i.e., there are* $a_i, b_i \in \mathbb{R}^n, 1 \leq i \leq m$, *such that* $Z = [a_1, b_1] + \ldots + [a_m, b_m]$.

By the definition of the support function we derive, that for $K, L \in \mathcal{K}^n$ and $t, s > 0$ it holds

$$h_{tK+sL}(x) \quad = \quad th_K(x) + sh_L(x), \quad \text{for all } x \in \mathbb{R}^n. \tag{5.1}$$

This property implies the following lemma.

5.2 Lemma. *Let* $Z \in \mathcal{K}_0^n$ *be a centrally symmetric zonotope. The support function of* Z *is given by*

$$h_Z(u) = \sum_{i=1}^{m} a_i |u^\intercal v_i|, \ \textit{for } u \in S^{n-1},$$

where $a_i > 0, v_i \in S^{n-1}, 1 \leq i \leq m$.

Proof. Pick $a_i > 0,\ v_i \in S^{n-1}$ and $w_i \in \mathbb{R}^n, 1 \leq i \leq m$, such that

$$Z = \sum_{i=1}^{m} (a_i[-v_i, v_i] + w_i).$$

The equation (5.1) yields

$$h_Z(u) = \sum_{i=1}^{m} (a_i h_{[-v_i,v_i]}(u) + h_{w_i}(u)) = \sum_{i=1}^{m} a_i |u^\intercal v_i| + \sum_{i=1}^{m} u^\intercal w_i.$$

Since, Z is centrally symmetric, we have $h_Z(u) = h_Z(-u)$, for all $u \in S^{n-1}$. This leads to $\sum_{i=1}^{m} u^\intercal w_i = 0$ and therefore to the desired form of h_Z. □

5.3 Lemma. *The support function gives an alternative way to compute the Hausdorff distance of $K_1, K_2 \in \mathcal{K}^n$. We have*

$$\delta(K_1, K_2) = \sup_{u \in S^{n-1}} |h_{K_1}(u) - h_{K_2}(u)| = ||h_{K_1} - h_{K_2}||_\infty.$$

Proof. Let $u \in S^{n-1}$. Then

$$\begin{aligned} h_{K_2 + ||h_{K_1} - h_{K_2}||_\infty B^n}(u) &= h_{K_2}(u) + ||h_{K_1} - h_{K_2}||_\infty h_{B^n}(u) \\ &= h_{K_2}(u) + ||h_{K_1} - h_{K_2}||_\infty \\ &\geq h_{K_2}(u) + |h_{K_1}(u) - h_{K_2}(u)| \ \geq \ h_{K_1}(u). \end{aligned}$$

This means $K_1 \subset K_2 + ||h_{K_1} - h_{K_2}||_\infty B^n$ and by changing the roles of K_1 and K_2, we also get $K_2 \subset K_1 + ||h_{K_1} - h_{K_2}||_\infty B^n$. In other words, it holds $\delta(K_1, K_2) \leq ||h_{K_1} - h_{K_2}||_\infty$.

To see the converse inequality, let $\varepsilon > 0$ such that $\delta(K_1, K_2) = \varepsilon$. By definition of the Hausdorff distance we have $K_1 \subset K_2 + \varepsilon B^n$ and $K_2 \subset K_1 + \varepsilon B^n$, or equivalently, $h_{K_1}(u) \leq h_{K_2}(u) + \varepsilon$ and $h_{K_2}(u) \leq h_{K_1}(u) + \varepsilon$, for all $u \in S^{n-1}$. Thus, $|h_{K_1}(u) - h_{K_2}(u)| \leq \varepsilon = \delta(K_1, K_2)$, for $u \in S^{n-1}$, and the lemma is proven. □

5.4 Definition (Zonoid).
A compact convex set $Z \subset \mathbb{R}^n$ is called a zonoid, *if it is the Hausdorff limit of a sequence of zonotopes, that is, there are zonotopes $Z_j \in \mathcal{K}^n, j \in \mathbb{N}$, such that $\lim_{j \to \infty} \delta(Z_j, Z) = 0$.*

5.5 Remark. *The class of zonotopes is invariant under Minkowski linear combinations and affine transformations. In particular, projections of zonotopes (zonoids) are again zonotopes (zonoids).*
Moreover, zonotopes are symmetric to a certain point in $\mathbb{R}^n$ and since this property is preserved under taking Hausdorff limits, zonoids are as well.

In the following, we state a characterisation of zonoids in terms of Borel measures in S^{n-1}.

5.6 Definition (Borel measure).
The smallest σ-algebra containing all open sets in $\mathbb{R}^n$ is said to be the family of Borel sets. *A* signed Borel measure *on $S \subset \mathbb{R}^n$ is a real-valued function μ defined on all Borel sets in S which is countably additive, i.e., for all pairwise disjoint Borel sets $A_j \subset S, j \in \mathbb{N}$, we have $\mu\left(\bigcup_{j\in\mathbb{N}} A_j\right) = \sum_{j\in\mathbb{N}} \mu(A_j)$.*
If the function μ is furthermore nonnegative, it is called a Borel measure *on S. A signed Borel measure μ on $S \subset \mathbb{R}^n$ is said to be* even [odd]*, if for all Borel sets A in S we have $\mu(-A) = \mu(A)$ $[= -\mu(A)]$.*

In the subsequent theorem we use a special case of Cauchy's projection formula (see [6], Appendix A.5), which states that for all $v \in S^{n-1}$ and $K \in \mathcal{K}_0^n$ it holds

$$h_{\Pi K}(v) = \mathrm{vol}_{n-1}(K|v^{\perp}) = \int_{S^{n-1}} |u^{\intercal}v| \; dS_{n-1}(K,u), \tag{5.2}$$

where $S_{n-1}(K,\cdot)$ is a finite Borel measure on the unit n-sphere S^{n-1}, the *surface area measure* of K. The surface area measure of $K = B^n$ coincides with the usual Lebesgue measure on S^{n-1}. Thus, we have

$$h_{\Pi B^n}(v) = \mathrm{vol}_{n-1}(B^n|v^{\perp}) = \int_{S^{n-1}} |u^{\intercal}v| \; du. \tag{5.3}$$

5.7 Theorem. *A compact $K \in \mathcal{K}^n$ is a centrally symmetric zonoid if and only if*

$$h_K(u) = \int_{S^{n-1}} |u^{\intercal}v| \; d\mu(v), \text{ for } u \in S^{n-1}, \text{ (and so for } u \in \mathbb{R}^n), \tag{5.4}$$

where μ is an even Borel measure on S^{n-1}.

Proof. As a start, let the support function of K be of the form (5.4). Then, K is centrally symmetric, since $h_K(u) = h_K(-u)$, for all $u \in S^{n-1}$. Fix $\varepsilon > 0$ and let $\delta = \frac{\varepsilon}{\mu(S^{n-1})} > 0$. Next, we pick a partition of the unit sphere, that is, a family $E_i \subset S^{n-1}, 1 \le i \le m$, such that $E_i \cap E_j = \emptyset$, whenever $i \neq j$, and $S^{n-1} = \bigcup_{i=1}^m E_i$. Furthermore, we assume that for $1 \le i \le m$ we have $\max_{x,y\in E_i} ||x-y|| < \delta$. Fix some $v_i \in E_i, 1 \le i \le m$. Then, for all $u \in S^{n-1}$,

$$\left| \int_{S^{n-1}} |u^{\intercal}v| \; d\mu(v) - \sum_{i=1}^m |u^{\intercal}v_i| \mu(E_i) \right| = \left| \sum_{i=1}^m \int_{E_i} (|u^{\intercal}v| - |u^{\intercal}v_i|) \; d\mu(v) \right|$$

$$\begin{aligned}
&\le \sum_{i=1}^m \int_{E_i} \Big| |u^\intercal v| - |u^\intercal v_i| \Big| \, d\mu(v) \;\le\; \sum_{i=1}^m \int_{E_i} |u^\intercal (v - v_i)| \, d\mu(v) \\
&\le \sum_{i=1}^m \int_{E_i} ||v - v_i|| \, d\mu(v) \;<\; \delta \sum_{i=1}^m \int_{E_i} d\mu(v) \\
&= \delta \sum_{i=1}^m \mu(E_i) \;=\; \delta \mu(S^{n-1}) \;=\; \varepsilon.
\end{aligned}$$

Note, that we used the famous Cauchy-Schwarz inequality here. This computation shows that the zonotope $Z = \sum_{i=1}^m \mu(E_i)[-v_i, v_i]$, with support function $h_Z(u) = \sum_{i=1}^m \mu(E_i)|u^\intercal v_i|$, and K have Hausdorff distance

$$\delta(K, Z) = ||h_K - h_Z||_\infty = \sup_{u \in S^{n-1}} |h_K(u) - h_Z(u)| < \varepsilon.$$

Since $\varepsilon > 0$ was chosen arbitrarily, K is the Hausdorff limit of zonotopes and therefore a zonoid.

In order to show, that the condition (5.4) on h_K is necessary for K to be a centrally symmetric zonoid, consider a sequence $(Z_j)_{j\in\mathbb{N}}$ of zonotopes in $\mathbb{R}^n$ converging to K. Without loss of generality we can assume that all Z_j are centrally symmetric, since K is. By Lemma 5.2 the support function of a $Z_j, j \in \mathbb{N}$, has the form

$$h_{Z_j}(u) = \sum_{i=1}^m a_i |u^\intercal v_i| = \int_{S^{n-1}} |u^\intercal v| \, d\mu(v), \;\; u \in S^{n-1},$$

where μ is the sum of point masses of weight $\frac{a_i}{2}$ at $\pm v_i, 1 \le i \le m$. Therefore, it suffices to show, that the set of compact $Z \in \mathcal{K}_0^n$ whose support functions have the desired form (5.4), is closed in $\mathcal{K}^n$.

To this end, let $K_j \in \mathcal{K}_0^n, j \in \mathbb{N}$, be compact with support function

$$h_{K_j}(u) = \int_{S^{n-1}} |u^\intercal v| \, d\mu_j(v), \forall u \in S^{n-1},$$

and μ_j even Borel measures on S^{n-1}. Suppose that $K_j \to K, \; j \to \infty$, in the Hausdorff metric. By the boundedness of K, there is an $r > 0$ and a $j_0 \in \mathbb{N}$ such that $K_j \subset rB^n$, for $j \ge j_0$. Consider the projection body ΠB^n of the Euclidean unit ball. Since, any projection of B^n on an $(n-1)$-dimensional subspace of $\mathbb{R}^n$ is an $(n-1)$-dimensional Euclidean unit ball, we have $h_{\Pi B^n}(u) = \mathrm{vol}_{n-1}(B^n | u^\perp) = \kappa_{n-1}$, for all $u \in S^{n-1}$. Combining equation (5.3) with Fubini's theorem leads to

$$2\kappa_{n-1}\mu_j(S^{n-1}) \;=\; 2\int_{S^{n-1}} h_{\Pi B^n}(v) \, d\mu_j(v) \;=\; \int_{S^{n-1}} \int_{S^{n-1}} |u^\intercal v| \, du \, d\mu_j(v)$$

$$\begin{aligned} &= \int_{S^{n-1}} \int_{S^{n-1}} |u^\intercal v| \; d\mu_j(v) \; du \;=\; \int_{S^{n-1}} h_{K_j}(u) \; du \\ &\leq \int_{S^{n-1}} h_{rB^n}(u) \; du \;=\; r \int_{S^{n-1}} du \;=\; rn\kappa_n, \text{ for } j \geq j_0. \end{aligned}$$

This means, that the sequence $(\mu_j(S^{n-1}))_{j\in\mathbb{N}}$ is bounded, which implies the existence of a subsequence $(\mu_{j_k})_{k\in\mathbb{N}} \subset (\mu_j)_{j\in\mathbb{N}}$ that converges weakly to an even Borel measure μ on S^{n-1} (see [25], the proof of Theorem 5.5.2). That is, for all bounded, continuous, real-valued functions on S^{n-1}

$$\int_{S^{n-1}} f(u) \; d\mu_{j_k}(u) \to \int_{S^{n-1}} f(u) \; d\mu(u), \; k \to \infty.$$

Since the scalar product is continuous, this leads to

$$h_{K_{j_k}}(u) = \int_{S^{n-1}} |u^\intercal v| \; d\mu_{j_k}(v) \to \int_{S^{n-1}} |u^\intercal v| \; d\mu(v), \; k \to \infty.$$

We also have $K_j \to K$, so by Lemma 5.3, the sequence $(h_{K_j})_{j\in\mathbb{N}}$ converges uniformly to h_K. Finally, by the uniqueness of the limit, we get $h_K(u) = \int_{S^{n-1}} |u^\intercal v| \; d\mu(v)$ for all $u \in S^{n-1}$. □

5.8 Remark. *The even Borel measure μ in the previous theorem is called the* generating measure *of the zonoid K. It is essentially unique, since from measure theory $\mu = 0$, if for all $u \in S^{n-1}$ it holds $\int_{S^{n-1}} |u^\intercal v| \; d\mu(v) = 0$. A proof can be found in [6], Appendix C.*

5.9 Proposition. *Let $K \in \mathcal{K}_0^n$ be a zonoid with generating measure μ. Then the volume of K is given by*

$$\operatorname{vol}(K) = \frac{1}{n} \int_{S^{n-1}} \operatorname{vol}_{n-1}(K|u^\perp) \; d\mu(u).$$

Proof. The surface area measure admits a formula (see [6], Theorem A.3.1) to compute the volume of K with the help of its support function h_K. It states that

$$\operatorname{vol}(K) = \frac{1}{n} \int_{S^{n-1}} h_K(u) \; dS_{n-1}(K, u).$$

With Theorem 5.7 and equation (5.2) we derive

$$\begin{aligned} \operatorname{vol}(K) &= \frac{1}{n} \int_{S^{n-1}} h_K(v) \; dS_{n-1}(K, v) \\ &= \frac{1}{n} \int_{S^{n-1}} \left(\frac{1}{2} \int_{S^{n-1}} |v^\intercal u| \; d\mu(u) \right) \; dS_{n-1}(K, v) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n}\int_{S^{n-1}} \left(\frac{1}{2}\int_{S^{n-1}} |v^\intercal u| \; dS_{n-1}(K,v)\right) \; d\mu(u) \\
&= \frac{1}{n}\int_{S^{n-1}} \operatorname{vol}_{n-1}(K|u^\perp) \; d\mu(u).
\end{aligned}
$$

□

As the following statement shows, projection bodies and zonoids describe exactly the same class of convex bodies, when we restrict to the centrally symmetric case.

5.10 Theorem. *Any projection body is a centrally symmetric zonoid. Conversely, every n-dimensional centrally symmetric zonoid in $\mathbb{R}^n$ is the projection body of a unique centrally symmetric convex body.*

Proof. Let $K \in \mathcal{K}_0^n$ and let A be a subset of S^{n-1}. The surface area measure fulfills $S_{n-1}(K,-A) = S_{n-1}(-K,A)$. Hence, from Cauchy's projection formula (5.2), we obtain

$$
\begin{aligned}
h_{\Pi K}(u) &= \frac{1}{2}\int_{S^{n-1}} |u^\intercal v| \; dS_{n-1}(K,v) \\
&= \frac{1}{4}\left(\int_{S^{n-1}} |u^\intercal v| \; dS_{n-1}(K,v) + \int_{S^{n-1}} |u^\intercal v| \; dS_{n-1}(-K,v)\right) \\
&= \int_{S^{n-1}} |u^\intercal v| \; d\mu(v),
\end{aligned}
$$

where $\mu = \frac{1}{4}(S_{n-1}(K,\cdot) + S_{n-1}(-K,\cdot))$ is a Borel measure on S^{n-1}. We are up to show, that μ is even. Again, by the above property of $S_{n-1}(K,\cdot)$, we get

$$
\begin{aligned}
\mu(-A) &= \frac{1}{4}(S_{n-1}(K,-A) + S_{n-1}(-K,-A)) \\
&= \frac{1}{4}(S_{n-1}(-K,A) + S_{n-1}(K,A)) \;=\; \mu(A).
\end{aligned}
$$

Thus, ΠK is a centrally symmetric zonoid by Theorem 5.7.

The remaining implication will not be shown here, since it uses some more concepts that are out of the focus of this work. The arguments can be found in Chapter 4 of the book by Gardner [6]. □

5.2 The Mahler volume of zonoids

In this section we illustrate that Mahler's Conjecture is true for the class of zonoids. As mentioned in the introduction to this chapter, we are following a work from 1988 by Gordon, Meyer and Reisner [7].

5.11 Lemma. *Let $K \in \mathcal{K}_0^n$ be a zonoid with generating measure μ. Then*

$$(n+1)\text{vol}(K) \int_{S^{n-1}} \int_{K^\star} |u^\intercal x| \; dx \; d\mu(u) = 2\text{vol}(K^\star) \int_{S^{n-1}} \text{vol}_{n-1}(K|u^\perp) \; d\mu(u).$$

Furthermore, for some $u_0 \in S^{n-1}$ we have

$$(n+1)\text{vol}(K) \int_{K^\star} |u_0^\intercal x| \; dx \geq 2\text{vol}(K^\star)\text{vol}_{n-1}(K|u_0^\perp).$$

Proof. Using Fubini's theorem and $h_K(\cdot) = ||\cdot||_{K^\star}$, we get

$$\begin{aligned}
\int_{S^{n-1}} \int_{K^\star} |u^\intercal x| \; dx \; d\mu(u) &= \int_{K^\star} \int_{S^{n-1}} |u^\intercal x| \; d\mu(u) \; dx \\
\text{(Theorem 5.7)} &= 2 \int_{K^\star} h_K(x) \; dx \; = \; 2 \int_{K^\star} ||x||_{K^\star} dx \\
&= 2 \int_{K^\star} \int_0^{||x||_{K^\star}} dt \; dx.
\end{aligned}$$

For the integration domain we have $\{(x,t) \in \mathbb{R}^{n+1} \mid x \in K^\star, 0 \leq t \leq ||x||_{K^\star}\} = \{(x,t) \in \mathbb{R}^{n+1} \mid 0 \leq t \leq ||x||_{K^\star} \leq 1\}$, and we continue the sequence of equalities by

$$\begin{aligned}
&= 2 \int_0^1 \text{vol}(\{x \in \mathbb{R}^n \mid t \leq ||x||_{K^\star} \leq 1\}) \; dt \\
&= 2 \int_0^1 (\text{vol}(K^\star) - \text{vol}(tK^\star)) \; dt \\
&= 2\text{vol}(K^\star) \int_0^1 (1 - t^n) \; dt \; = \; \frac{2n}{n+1}\text{vol}(K^\star) \\
\text{(Proposition 5.9)} &= \frac{2\text{vol}(K^\star)}{(n+1)\text{vol}(K)} \int_{S^{n-1}} \text{vol}_{n-1}(K|u^\perp) \; d\mu(u),
\end{aligned}$$

and the claim follows. The existence of a desired $u_0 \in S^{n-1}$, such that

$$(n+1)\text{vol}(K) \int_{K^\star} |u_0^\intercal x| \; dx \geq 2\text{vol}(K^\star)\text{vol}_{n-1}(K|u_0^\perp),$$

follows from the non-negativity of the generating measure μ of K. If there would be no such u_0, equality in the above formula could not hold. $\square$

The next step is to prove a technical lemma about estimating a certain integral.

5.12 Lemma. *Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be such that $0 < \int_0^\infty f(t)\, dt < \infty$ and $f(0) = 1$. Moreover, for some $p > 0$ let $f^{\frac{1}{p}}$ be concave. Then*

$$\int_0^\infty tf(t)\, dt \leq \frac{p+1}{p+2}\left(\int_0^\infty f(t)\, dt\right)^2$$

with equality if and only if there is an $a > 0$ such that for all real $t \geq 0$ we have $f(t) = (1-at)_+^p = \max\{(1-at)^p, 0\}$.

Proof. Consider $h : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ given by $h(a) = \frac{1}{a(p+1)}$. Clearly, h is surjective. By positivity of $\int_0^\infty f(t)\, dt$ there exists an $a > 0$ such that $\frac{1}{a(p+1)} = \int_0^\infty f(t)\, dt$. Hence

$$\begin{aligned}\int_0^\infty (1-at)_+^p\, dt &= \int_0^{\frac{1}{a}} (1-at)^p\, dt = -\frac{1}{a(p+1)}(1-at)^{p+1}\Big|_0^1 \\ &= \frac{1}{a(p+1)} = \int_0^\infty f(t)\, dt.\end{aligned}$$

Now, define $g(t) := f(t) - (1-at)_+^p$, where $t \geq 0$. Thus, $g(0) = 0$ and $\int_0^\infty g(t)\, dt = 0$. Let us show that there exists an $x_0 \geq 0$ such that $g(t) \geq 0$ for all $t \in [0, x_0]$ and $g(t) \leq 0$, for $t \geq x_0$.

In order to see this, we assume that there is an $\varepsilon > 0$ with $g(t) < 0$, whenever $t \in (0, \varepsilon]$. This implies for such $t \in (0, \varepsilon]$ that $f(t) < (1-at)_+^p$, or equivalently, $f(t)^{\frac{1}{p}} < (1-at)_+$. The concavity of $f^{\frac{1}{p}}$ and the fact, that $(1-at)_+$ is tangent on $f^{\frac{1}{p}}$ in $t = 0$, also lead to $f(t)^{\frac{1}{p}} \leq (1-at)_+$ for $t \geq \varepsilon$. We arrive at a contradiction, since, then $\int_0^\infty f(t)\, dt < \int_0^\infty (1-at)_+^p\, dt$, but the definition of g yields $\int_0^\infty g(t)\, dt = 0$. We come up with an $\bar{x} \geq 0$ fulfilling $g(t) \geq 0$, whenever $t \in [0, \bar{x}]$. The same argument shows, that $g(t) \geq 0$ cannot hold everywhere, thus, there has to be a desired $x_0 \geq 0$ with the stated properties.

That statement implies $\int_x^\infty g(t)\, dt \leq 0$, for all $x \geq 0$. Therefore,

$$\begin{aligned}\int_0^\infty tf(t)\, dt &= \int_0^\infty \int_0^t f(t)\, ds\, dt = \int_0^\infty \int_s^\infty f(t)\, dt\, ds \\ &\leq \int_0^\infty \int_s^\infty (1-at)_+^p\, dt\, ds = \int_0^{\frac{1}{a}} \int_s^{\frac{1}{a}} (1-at)^p\, dt\, ds \\ &= \int_0^{\frac{1}{a}} \frac{(1-as)^{p+1}}{a(p+1)}\, ds \\ &= \frac{1}{a^2(p+1)(p+2)} = \frac{p+1}{p+2}\left(\int_0^\infty f(t) dt\right)^2.\end{aligned}$$

Equality is attained if and only if for any nonnegative s it holds $\int_s^\infty f(t)\,dt = \int_s^\infty (1-at)_+^p\,dt$, that is, if and only if $f(t) = (1-at)_+^p$, for $t \geq 0$. □

5.13 Lemma. *Let $K \in \mathcal{K}_0^n$ and $u \in S^{n-1}$. Then*

$$\int_K |u^\intercal x|\,dx \;\leq\; \frac{n\text{vol}(K)^2}{2(n+1)\text{vol}_{n-1}(K\cap u^\perp)} \tag{5.5}$$

with equality if and only if $K = \text{conv}\{y, -y, K\cap u^\perp\}$, for some $y \in K$.

Proof. At first, we evaluate

$$\begin{aligned}\int_K |u^\intercal x|\,dx &= 2\int_0^\infty \int_{K\cap\{u^\intercal x = t\}} |u^\intercal x|\,dx\,dt \\ &= 2\int_0^\infty t\text{vol}_{n-1}(K\cap\{u^\intercal x = t\})\,dt \;=\; 2\int_0^\infty tg(t)\,dt,\end{aligned} \tag{5.6}$$

where $g(t) = \text{vol}_{n-1}(K\cap\{u^\intercal x = t\})$, for $t \in \mathbb{R}$. Since K is centrally symmetric, g is even and we have $g(t) = 0$, for $|t| > h_K(u) = ||u||_{K^\star}$. Moreover, it holds $g(0) = \text{vol}_{n-1}(K\cap u^\perp)$. The Brunn-Minkowski inequality (see [25], Theorem 6.1.1) yields

$$\begin{aligned}g(\lambda t + (1-\lambda)s)^{\frac{1}{n-1}} &= \text{vol}_{n-1}(K\cap\{u^\intercal x = \lambda t + (1-\lambda)s\})^{\frac{1}{n-1}} \\ &\geq \text{vol}_{n-1}(\lambda(K\cap\{u^\intercal x = t\}) + (1-\lambda)(K\cap\{u^\intercal x = s\}))^{\frac{1}{n-1}} \\ &\geq \lambda\text{vol}_{n-1}(K\cap\{u^\intercal x = t\})^{\frac{1}{n-1}} + (1-\lambda)\text{vol}_{n-1}(K\cap\{u^\intercal x = s\})^{\frac{1}{n-1}} \\ &= \lambda g(t)^{\frac{1}{n-1}} + (1-\lambda)g(s)^{\frac{1}{n-1}},\end{aligned}$$

for arbitrary $\lambda \in [0,1]$ and $s,t \in (-||u||_{K^\star}, ||u||_{K^\star})$. This means, that $g^{\frac{1}{n-1}}$ is concave on $(-||u||_{K^\star}, ||u||_{K^\star})$. Next, Fubini's theorem and the evenness of g give

$$\text{vol}(K) = \int_{-\infty}^\infty \text{vol}_{n-1}(K\cap\{u^\intercal x = t\})\,dt = 2\int_0^\infty g(t)\,dt. \tag{5.7}$$

Applying Lemma 5.12 to $f(t) = \frac{g(t)}{g(0)}, t \geq 0$, and $p = n-1$, we derive

$$\int_0^\infty t\frac{g(t)}{g(0)}\,dt \leq \frac{n}{n+1}\left(\int_0^\infty \frac{g(t)}{g(0)}\right)^2.$$

By definition of g this combines with (5.6) and (5.7) to

$$\int_K |u^\intercal x|\,dx \leq \frac{n\text{vol}(K)^2}{2(n+1)g(0)} = \frac{n\text{vol}(K)^2}{2(n+1)\text{vol}_{n-1}(K\cap u^\perp)}.$$

Let us now consider the case of equality. Lemma 5.12 states, that equality holds, if and only if, for all real $t \geq 0$, we have $\frac{g(t)}{g(0)} = \left(1 - \frac{t}{||u||_{K^\star}}\right)_+^{n-1}$. We can pick a $y \in K$, such that

$$||u||_{K^\star} = h_K(u) = \max_{x \in K} |u^\intercal x| = |u^\intercal y|.$$

Clearly, it holds, that $\mathrm{conv}\{y, -y, K \cap u^\perp\} \subset K$. Assume, that we have equality in (5.5). Then

$$\frac{1}{g(0)} \int_0^\infty g(t)\ dt = \int_0^{||u||_{K^\star}} \left(1 - \frac{t}{||u||_{K^\star}}\right)^{n-1} dt = \frac{||u||_{K^\star}}{n},$$

and thus

$$\begin{aligned} \mathrm{vol}(K) \ &\geq\ \mathrm{vol}(\mathrm{conv}\{y, -y, K \cap u^\perp\}) \ =\ \frac{2}{n}||u||_{K^\star}\mathrm{vol}_{n-1}(K \cap u^\perp) \\ &=\ 2\int_0^\infty g(t)\ dt \ =\ \mathrm{vol}(K). \end{aligned}$$

Therefore, equality must hold throughout and $K = \mathrm{conv}\{y, -y, K \cap u^\perp\}$, as desired.

Conversely, let $K = \mathrm{conv}\{y, -y, K \cap u^\perp\}$ and write $K_t = K \cap \{u^\intercal x = t\}$. For $t \in [0, 1]$, the bodies $K_0 = K \cap u^\perp$ and K_t are *homothetic*, i.e., there is a $v \in \mathbb{R}^n$ and a nonnegative constant s, such that $K_t = sK_0 + v$. Since, K is a double pyramid, the dilatation factor s is $\frac{|u^\intercal y| - t}{|u^\intercal y|} = (1 - \frac{t}{||u||_{K^\star}})$ and we obtain

$$\frac{g(t)}{g(0)} = \frac{\mathrm{vol}_{n-1}(K_t)}{\mathrm{vol}_{n-1}(K_0)} = \frac{\mathrm{vol}_{n-1}(sK_0 + v)}{\mathrm{vol}_{n-1}(K_0)} = s^{n-1} = \left(1 - \frac{t}{||u||_{K^\star}}\right)^{n-1}.$$

Because y is an apex of the double pyramid K, we have $g(t) = 0$, whenever $t > |u^\intercal y|$ and finally it holds $\frac{g(t)}{g(0)} = \left(1 - \frac{t}{||u||_{K^\star}}\right)_+^{n-1}$, for all t greater than or equal to zero, which yields equality in (5.5). □

We are now ready to prove the main theorem of this chapter.

5.14 Theorem. *Let $K \in \mathcal{K}_0^n$ be a zonoid. Then $\mathbf{M}(K) \geq \frac{4^n}{n!}$, with equality if and only if K is an n-cube.*

Proof. We proceed by induction on n. For $n = 1$ the result is clear, since every $K \in \mathcal{K}_0^1$ is just a line segment.

Assume that $n > 1$. From Lemma 5.11 and 5.13 we derive the existence of $u_0 \in S^{n-1}$ such that

$$2\mathrm{vol}(K^\star)\mathrm{vol}_{n-1}(K|u_0^\perp) \leq (n+1)\mathrm{vol}(K)\int_{K^\star} |u_0^\intercal x|\, dx \leq \frac{n\mathrm{vol}(K)\mathrm{vol}(K^\star)^2}{2\mathrm{vol}_{n-1}(K^\star \cap u_0^\perp)}.$$

For the polar body of the projection it holds $(K|u_0^\perp)^\star = K^\star \cap u_0^\perp$, which implies

$$\mathbf{M}(K) = \mathrm{vol}(K)\mathrm{vol}(K^\star) \geq \frac{4}{n}\mathrm{vol}(K|u_0^\perp)\mathrm{vol}((K|u_0^\perp)^\star) = \frac{4}{n}\mathbf{M}(K|u_0^\perp).$$

As mentioned in Remark 5.5, projections of zonoids are again zonoids. Therefore, by induction hypothesis we have

$$\mathbf{M}(K) \geq \frac{4}{n}\mathbf{M}(K|u_0^\perp) \geq \frac{4}{n}\frac{4^{n-1}}{(n-1)!} = \frac{4^n}{n!}.$$

In the case of equality, $\mathbf{M}(K) = \frac{4^n}{n!}$, it holds $\mathbf{M}(K|u_0^\perp) = \frac{4^{n-1}}{(n-1)!}$ and the induction hypothesis states that $K|u_0^\perp$ is an $(n-1)$-cube. The equality characterisation in Lemma 5.13 yields, that there is a $y_0 \in \mathbb{R}^n$ such that $K^\star = \mathrm{conv}\{y_0, -y_0, K^\star \cap u_0^\perp\}$, an n-dimensional crosspolytope. Thus, K is an n-cube. □

6 Polytopes with at most $2n + 2$ facets

In 1998, Lopez and Reisner [13] published a work on centrally symmetric polytopes in $\mathbb{R}^n$ with at most $2n + 2$ facets. They relied on a note by Ball [1], who stated that asking for the minimal Mahler volume among these polytopes is equivalent to find the minimal Mahler volume in all sections of the $(n+1)$-dimensional cube C_{n+1} by n-dimensional hyperplanes. By basing on that, Lopez and Reisner transformed the problem to a finite search in fixed dimension n. They used the computer to verify Mahler's Conjecture in dimensions $n \leq 8$ for these special polytopes.

Throughout this chapter we denote by P a centrally symmetric polytope in $\mathbb{R}^n$, having at most $2n + 2$ facets. Note, that $P^\star$ is a polytope with at most $2n + 2$ vertices.

The illustration of the proof of the following result is given in the two subsequent sections. We start by enlightening the mentioned equivalence for the polytopes of interest, and afterwards working through the arguments in [13] to transform the problem to a finite search.

6.1 Theorem. *Let $2 \leq n \leq 8$ and let $P \in \mathcal{K}_0^n$ be a polytope with at most $2n + 2$ facets. Then, we have $\mathbf{M}(P) \geq \frac{4^n}{n!}$.*

Note, that Lopez and Reisner also characterised the equality cases in the concerning dimensions. Again, the Hanner polytopes show up here. They found out that the minimum is attained if and only if up to affine transformations P is either the n-cube or $P = C_3^\star \times C_{n-3}$ for $3 \leq n \leq 8$. Since, $C_3^\star$ has 8 facets and C_{n-3} has $2(n-3) = 2n-6$, the latter body has $2n+2$ facets for all $n \geq 3$.

6.1 Equivalent description

We introduce the n-dimensional hyperplane $H_u = \{x \in \mathbb{R}^{n+1} \mid u^\intercal x = 0\}$ perpendicular to $u \in S^n$.

6.2 Lemma. *Let $P \in \mathcal{K}_0^n$ be a polytope with at most $2m$ facets. Then there exists a direction vector $u \in S^n$ and a linear map $A : \mathbb{R}^m \to \mathbb{R}^n$ such that $P = A(C_m \cap H_u)$.*

Proof. There are vectors $v_i \in \mathbb{R}^n \setminus \{0\}, 1 \leq i \leq m$, such that

$$P = \{x \in \mathbb{R}^n \mid -1 \leq v_i^\intercal x \leq 1, \text{ for } 1 \leq i \leq m\}.$$

Consider the linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ with $L(x) = \begin{pmatrix} v_1^\intercal \\ \vdots \\ v_m^\intercal \end{pmatrix} x$. We have

$$C_m \cap L(\mathbb{R}^n) = \left\{ \begin{pmatrix} v_1^\intercal \\ \vdots \\ v_m^\intercal \end{pmatrix} x \;\middle|\; x \in \mathbb{R}^n,\ |v_i^\intercal x| \leq 1,\ 1 \leq i \leq m \right\} = L(P).$$

Thus, the inverse map L^{-1} of $L : \mathbb{R}^n \to L(\mathbb{R}^n)$ enables us to write the polytope as $P = L^{-1}(C_m \cap L(\mathbb{R}^n))$ and since $L(\mathbb{R}^n)$ is an n-dimensional subspace of $\mathbb{R}^m$, there is a vector $u \in S^n$ with $L(\mathbb{R}^n) = H_u$. ☐

Putting $m = n+1$ in Lemma 6.2 and noticing, that $(C_{n+1} \cap H_u)^\star = C_{n+1}^\star | H_u$, implies

6.3 Proposition. *For the class of polytopes $P \in \mathcal{K}_0^n$ with at most $2n+2$ facets, Mahler's Conjecture is equivalent to*

$$\min_{u \in S^n} \operatorname{vol}(C_{n+1} \cap H_u) \cdot \operatorname{vol}(C_{n+1}^\star | H_u) \quad \geq \quad \frac{4^n}{n!}. \tag{6.1}$$

Note, that the minimum is independent on the length of the direction vector $u \in S^n$.

6.2 Transformation to a finite search

At first, we further investigate the minimisation problem (6.1). Consider the bodies $E = (\Pi C_{n+1}^\star)^\star$ and $F = IC_{n+1} \in \mathcal{K}_0^{n+1}$. By definition of projection and intersection body, we have for $v \in \mathbb{R}^{n+1} \setminus \{0\}$

$$\operatorname{vol}(C_{n+1}^\star | H_v) = h_{\Pi C_{n+1}^\star}\left(\frac{v}{||v||_2}\right) = \left|\left|\frac{v}{||v||_2}\right|\right|_{(\Pi C_{n+1}^\star)^\star} = \frac{||v||_E}{||v||_2},$$

and similarly

$$\operatorname{vol}(C_{n+1} \cap H_v) = \rho_{IC_{n+1}}\left(\frac{v}{||v||_2}\right) = \frac{||v||_2}{||v||_F}.$$

By utilising these two identities we rewrite (6.1) as

$$\frac{1}{\max\limits_{||v||_E=1} ||v||_F} = \frac{1}{\max\limits_{v \in \mathbb{R}^{n+1}\setminus\{0\}} \dfrac{||v||_F}{||v||_E}} = \min_{v \in \mathbb{R}^{n+1}\setminus\{0\}} \frac{||v||_E}{||v||_F} \quad \geq \quad \frac{4^n}{n!}.$$

Therefore, the minimum is attained at an extreme point of E, that is, the polar of the projection body of the $(n+1)$-dimensional crosspolytope.

6.4 Proposition. *Let $v \in \mathbb{R}^{n+1} \setminus \{0\}$. Then we have*

$$\mathrm{vol}(C_{n+1}^{\star}|H_v) = \frac{1}{2n!} \sum_{\varepsilon \in \{-1,1\}^{n+1}} |v^{\mathsf{T}}\varepsilon|.$$

Proof. The projection of the surface of $C_{n+1}^{\star}$ onto H_v covers almost all of $C_{n+1}^{\star}|H_v$ exactly twice. Furthermore, the boundary $\partial C_{n+1}^{\star}$ is composed of 2^{n+1} facets F_ε, indexed by their outer normal vectors $\varepsilon \in \{-1,1\}^{n+1}$. These facets are all regular simplices and hence of the same volume.

Denote with $\mathbb{1} = (1,\dots,1)$ and let $S = \mathrm{conv}\{0, e_1, \dots, e_{n+1}\}$ be the standard $(n+1)$-simplex. We compute

$$\frac{1}{(n+1)!} = \mathrm{vol}(S) = \frac{\mathrm{vol}(F_{\mathbb{1}})}{n+1}\frac{1}{\sqrt{n+1}}.$$

Thus, $\mathrm{vol}(F_\varepsilon) = \frac{\sqrt{n+1}}{n!}$, for all $\varepsilon \in \{-1,1\}^{n+1}$. Rescaling identifies $\frac{1}{\sqrt{n+1}}\varepsilon$ as the outer unit normal vector of the facet F_ε and hence, the volume of the projection of F_ε onto H_v is given by

$$\mathrm{vol}(F_\varepsilon|H_v) = \mathrm{vol}(F_\varepsilon)\frac{1}{\sqrt{n+1}}|v^{\mathsf{T}}\varepsilon| = \frac{1}{n!}|v^{\mathsf{T}}\varepsilon|.$$

In conclusion, this yields $\mathrm{vol}(C_{n+1}^{\star}|H_v) = \frac{1}{2n!} \sum_{\varepsilon \in \{-1,1\}^{n+1}} |v^{\mathsf{T}}\varepsilon|$, as desired. □

The next statement ensures, that the search for extreme points of E is indeed finite. Write $\mathcal{E} = \{\varepsilon \in \{-1,1\}^{n+1} \mid \varepsilon_1 = 1\}$ and consider $E' = n!E$ which is no restriction since we are only interested in direction vectors of extreme points of E.

6.5 Proposition. *A vector $v \in \mathbb{R}^{n+1}$ is an extreme point of E' if and only if $\sum_{\varepsilon\in\mathcal{E}} |v^{\mathsf{T}}\varepsilon| = 1$ and there are linearly independent vectors $\varepsilon_1, \dots, \varepsilon_n \in \mathcal{E}$, such that $v^{\mathsf{T}}\varepsilon_i = 0$, for $1 \le i \le n$.*

Proof. The definition of E' and Proposition 6.4 yield

$$||x||_{E'} = \sum_{\varepsilon\in\mathcal{E}} |x^{\mathsf{T}}\varepsilon|, \ x \in \mathbb{R}^{n+1}. \tag{6.2}$$

Next, consider the linear map $\iota : \mathbb{R}^{n+1} \to \mathbb{R}^{2^n}$ that takes a vector x to $\iota(x) = (x^{\mathsf{T}}\varepsilon)_{\varepsilon\in\mathcal{E}}$. Since, $\mathcal{E}$ surely contains n linear independent vectors, ι is injective, and we identify $\mathbb{R}^{n+1}$ with its image $M = \iota(\mathbb{R}^{n+1})$ under this embedding. From (6.2) we obtain $||x||_{E'} = ||\iota(x)||_{C_{2^n}^{\star}}$, and therefore v is an extreme point of E' if and only if $||v||_{E'} = \sum_{\varepsilon \in \mathcal{E}} |v^\intercal \varepsilon| = 1$ and v is an intersection point of the $(n+1)$-dimensional subspace M of $\mathbb{R}^{2^n}$ and a $(2^n - (n+1))$-dimensional face of $C^\star_{2^n}$.

An m-dimensional crosspolytope is given by $C^\star_m = \operatorname{conv}\{\pm e_1, \ldots, \pm e_m\}$, so we can identify each $(m-i)$-dimensional face of $C^\star_m$ as the set of all vectors $x \in \mathbb{R}^m$ with norm $||x||_{C^\star_m} = 1$ and equal sign pattern. Furthermore, exactly $i-1$ fixed coordinates of all such x always vanish. Therefore, any direction of an extreme point of E' is contained in the intersection of M with an $(2^n - n)$-dimensional subspace $N = \{x \in \mathbb{R}^{2^n} \mid x_{\varepsilon_i} = 0, 1 \le i \le n\}$ of $\mathbb{R}^{2^n}$, where $\varepsilon_1, \ldots, \varepsilon_n$ are distinct elements in $\mathcal{E}$. This property remains true, if we restrict to linearly independent subsets $\{\varepsilon_1, \ldots, \varepsilon_n\} \subset \mathcal{E}$. In this case, the intersection $M \cap N$ consists of all solutions $v \in \mathbb{R}^{n+1}$ of the system $v^\intercal \varepsilon_i = 0, 1 \le i \le n$, and hence is one-dimensional.

Summarising, we state that v is an extreme point of E' if and only if $\sum_{\varepsilon \in \mathcal{E}} |x^\intercal \varepsilon| = 1$ and v fulfills $v^\intercal \varepsilon_i = 0$, for some linearly independent $\{\varepsilon_1, \ldots, \varepsilon_n\}$, which we wished to show. □

Utilising this proposition we are now able to formulate an algorithm to find the minimum in (6.1). Note, that we neglect the norm condition, since we are only interested in the direction vectors.

(1) for all subsets $S \subset \mathcal{E}$ of size n do

  if $S = \{\varepsilon_1, \ldots, \varepsilon_n\}$ is linearly independent then collect S

(2) for all sets S collected in (1) do

  compute a nontrivial solution $v \in \mathbb{R}^{n+1}$ of the linear system

$$v^\intercal \varepsilon_i = 0, \varepsilon_i \in S$$

(3) for all v computed in (2) do

  calculate $\operatorname{vol}(C_{n+1} \cap H_v) \cdot \operatorname{vol}(C^\star_{n+1} | H_v)$

(4) return the minimum of the values calculated in (3)

Step (1) of this algorithm is very costly. Therefore, the search should be reduced by proving additional sufficient conditions for v to be a direction vector of an extreme point of E'.

At first, the symmetry of cube and crosspolytope to all coordinate hyperplanes implies that it is sufficient to consider only nonnegative vectors $v \in \mathbb{R}^{n+1}$, whose entries form a non-increasing sequence. A further reduction step assumes, that the inequality is valid for dimensions smaller than

n. Lopez and Reisner show, that a vector v which satisfies the conditions in Proposition 6.5 is parallel to an integer vector w, such that for every coordinate $1 \leq i \leq n+1$, it holds

$$1 \leq w_i \leq \frac{n^{\frac{n}{2}}}{2^{n-1}}. \tag{6.3}$$

Now, one has to find all non-increasing integer vectors w that satisfy (6.3), and check whether there are n linearly independent elements in $\mathcal{E}$, that are perpendicular to w. For the latter define a matrix U, whose rows are given by all vectors $\varepsilon \in \mathcal{E}$, such that $w^\intercal \varepsilon = 0$ and check if the rank of U is n. Note, that the rank cannot exceed n, since all elements in $\mathcal{E}$ have the same first coordinate.

The number of candidates still grows exponentially with the dimension, but up to $n = 8$ Lopez and Reisner could do the computations to obtain Theorem 6.1.

7 Mahler volume of 1-unconditional bodies

In this chapter, we prove Mahler's Conjecture for a third class of centrally symmetric convex bodies, namely those which admit a *1-unconditional basis.* This result was obtained by Saint-Raymond [23] in 1980.

7.1 Theorem (Saint-Raymond [23]).
Let $K \in \mathcal{K}_0^n$ be 1-unconditional, then $\mathbf{M}(K) \geq \frac{4^n}{n!}$ and equality holds if K is a Hanner polytope.

Reisner [22] showed in a work from 1987, that the equality case is indeed characterised by Hanner polytopes.

The upcoming section deals with the definition and some examples of *1-unconditional bodies.* Furthermore, we introduce a construction which the proof of Theorem 7.1 relies on and discuss its properties that will be used to derive the result in the second part.

7.1 Definition and preparation

7.2 Definition (1-unconditional body).
A centrally symmetric convex body $K \in \mathcal{K}_0^n$ is called 1-unconditional, *if there exists a basis $\{x_1, \dots, x_n\}$ of $\mathbb{R}^n$, such that for all scalars $a_i \in \mathbb{R}$ and signs $\varepsilon_i \in \{-1, 1\}, 1 \leq i \leq n$, we have* $\Big|\Big| \sum_{i=1}^n a_i x_i \Big|\Big|_K = \Big|\Big| \sum_{i=1}^n \varepsilon_i a_i x_i \Big|\Big|_K.$

Let $\{x_1, \dots, x_n\}$ be a basis of the 1-unconditional convex body $K \in \mathcal{K}_0^n$. Such a basis is called *1-unconditional basis* of K. Consider the unit vectors $e_i \in \mathbb{R}^n, 1 \leq i \leq n$, and the linear transformation A that maps x_i to e_i, for $1 \leq i \leq n$. Then, $\{e_1, \dots, e_n\}$ is a 1-unconditional basis of AK. To see this, we note that $||x||_{AK} = ||A^{-1}x||_K$, for $x \in \mathbb{R}^n$, and get

$$\begin{aligned} \Big|\Big| \sum_{i=1}^n a_i e_i \Big|\Big|_{AK} &= \Big|\Big| A^{-1} \left(\sum_{i=1}^n a_i e_i \right) \Big|\Big|_K = \Big|\Big| \sum_{i=1}^n a_i A^{-1} e_i \Big|\Big|_K \\ &= \Big|\Big| \sum_{i=1}^n a_i x_i \Big|\Big|_K = \Big|\Big| \sum_{i=1}^n \varepsilon_i a_i x_i \Big|\Big|_K = \Big|\Big| \sum_{i=1}^n \varepsilon_i a_i e_i \Big|\Big|_{AK}. \end{aligned}$$

Due to the affine invariance of the Mahler volume, we therefore can restrict to such 1-unconditional bodies with basis $\{e_1, \dots, e_n\}$. In this case the definition yields

$$||(y_1, \ldots, y_n)||_K = ||(|y_1|, \ldots, |y_n|)||_K, \; y \in \mathbb{R}^n,$$

which means that the body K is symmetric with respect to all coordinate hyperplanes.

Examples:

- By definition, the p-norm $||x||_p$ only depends on the absolute value of the coordinates of $x \in \mathbb{R}^n$. Thus, every p-ball B_p^n is a 1-unconditional body with basis $\{e_1, \ldots, e_n\}$.
- Let $K \in \mathcal{K}^n$ and $L \in \mathcal{K}^m$ be 1-unconditional bodies with basis $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$, respectively. The relation

  $$||(x, y)||^p_{K \times_p L} = ||x||^p_K + ||y||^p_L, \text{ for } (x, y) \in \mathbb{R}^n \times \mathbb{R}^m$$

  implies, that $K \times_p L$ is also 1-unconditional with basis

  $$\{(x_1, 0), \ldots, (x_n, 0), (0, y_1), \ldots, (0, y_m)\}.$$

  In particular, any Hanner polytope is a 1-unconditional body.

Next, we consider the mapping

$$T_n : \mathbb{R}^n_{>0} \to \mathbb{R}^n, \; T_n(t_1, \ldots, t_n) := (-\ln t_1, \ldots, -\ln t_n)$$

and for $K \in \mathcal{K}_0^n$ denote $T(K) := T_n(K \cap \mathbb{R}^n_{>0})$. In order to get an idea how the resulting set $T(K)$ looks like, we consider the unit ball B_p^n of the p-norm, for $1 \leq p < \infty$. We have

$$\begin{aligned} T(B_p^n) &= \{y \in \mathbb{R}^n \mid \exists (t_1, \ldots, t_n) \in B_p^n \text{ with } y = (-\ln t_1, \ldots, -\ln t_n)\} \\ &= \{y \in \mathbb{R}^n \mid (e^{-y_1}, \ldots, e^{-y_n}) \in B_p^n)\} \\ &= \left\{ y \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n e^{-p \cdot y_i} \leq 1 \right\} = \left\{ \frac{1}{p} y \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n e^{-y_i} \leq 1 \right\} \\ &= \frac{1}{p} T(C_n^\star). \end{aligned}$$

This means, that all the sets $T(B_p^n)$ are dilatations of $T(C_n^\star)$. Figure 6 illustrates the octahedron $C_3^\star$ and its image under T.

The following three statements, are essential properties of $T(K)$ and important ingredients for the proof of the main result.

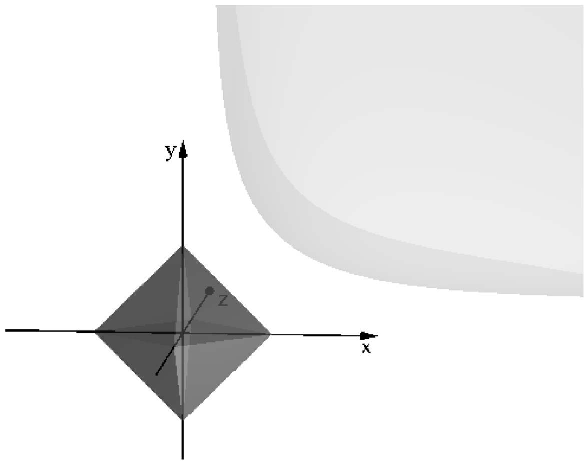

Figure 6: Octahedron $C_3^\star$ and $T(C_3^\star)$

7.3 Proposition. *Let $K \in \mathcal{K}_0^n$ be 1-unconditional with basis $\{e_1, \ldots, e_n\}$. Then $T(K)$ is convex.*

Proof. Pick some $s, t \in K \cap \mathbb{R}^n_{>0}$ with $s \le t$ and let $x = (-\ln s_1, \ldots, -\ln s_n)$ and $y = (-\ln t_1, \ldots, -\ln t_n) \in T(K)$. For all $\lambda \in [0,1]$ and $1 \le i \le n$ we have on the one hand

$$s_i \;\le\; s_i^\lambda\, t_i^{1-\lambda} \;\le\; t_i \tag{7.1}$$

and one the other hand

$$\lambda x_i + (1-\lambda) y_i \;=\; -\lambda \ln s_i - (1-\lambda)\ln t_i \;=\; -\ln(s_i^\lambda\, t_i^{1-\lambda}). \tag{7.2}$$

Since, K is symmetric with respect to all coordinate hyperplanes, the cube

$$Q_t = \operatorname{conv}\{(\varepsilon_1 t_1, \ldots, \varepsilon_n t_n) \mid \varepsilon \in \{-1,1\}^n\}$$

is contained in K. By (7.1) this yields

$$(s_1^\lambda\, t_1^{1-\lambda}, \ldots, s_n^\lambda\, t_n^{1-\lambda}) \in [s_1, t_1] \times \ldots \times [s_n, t_n] \subset Q_t \subset K.$$

Thus, with (7.2) we obtain $\lambda x + (1-\lambda) y \in T(K)$, which shows the convexity of $T(K)$. $\square$

7.4 Lemma. *For every $K \in \mathcal{K}_0^n$ it holds $T(K) + T(K^\star) = T(C_n^\star)$.*

Proof. First, observe that $T(C_n^\star) = \{u \in \mathbb{R}^n \mid \sum_{i=1}^n e^{-u_i} \le 1\}$ and $T(K) = \{u \in \mathbb{R}^n \mid e^{-u} = (e^{-u_1}, \ldots, e^{-u_n}) \in K\}$.

In order to see that $T(K) + T(K^\star) \subset T(C_n^\star)$, let $u \in T(K), v \in T(K^\star)$ and $w = u + v$. Thus, we have $e^{-u} \in K$ and $e^{-v} \in K^\star$. This gives us $\sum_{i=1}^n e^{-w_i} = \sum_{i=1}^n e^{-u_i} e^{-v_i} = (e^{-u})^\intercal (e^{-v}) \le 1$ and so $w \in T(C_n^\star)$.

For the converse, pick a $w \in T(C_n^\star)$ and write $\alpha_i = e^{-w_i}$, for $1 \le i \le n$. We clearly have $\sum_{i=1}^n \alpha_i \le 1$. Next, define a mapping $L : K \cap \mathbb{R}^n_{>0} \to \mathbb{R}$ by $L(x) = \sum_{i=1}^n \alpha_i \ln x_i$. For all $1 \le j \le n$ we have $\frac{\partial^2}{\partial x_i^2} L(x) = -\frac{\alpha_j}{x_j^2} < 0$, which means that L is strictly concave. Observe, that the unique supremum of L on $K \cap \mathbb{R}^n_{>0}$ is attained and does not lie "near" some of the coordinate hyperplanes, because $\ln x_i \to -\infty$, if $x_i \to 0$, and L is strictly increasing. Therefore, it exists an $\bar{x} \in K \cap \mathbb{R}^n_{>0}$ such that $\lambda := L(\bar{x}) = \sup_{x \in K \cap \mathbb{R}^n_{>0}} L(x)$. This implies, that we can find a $u \in T(K)$ with $\bar{x} = e^{-u}$.

Now, define an open set $U = \{x \in \mathbb{R}^n_{>0} \mid L(x) > \lambda\}$. It is convex due to the concavity of L. By the definition of U, it has no point in common with $K \cap \mathbb{R}^n_{>0}$, and the well-known separation theorem for distinct convex bodies (see [10], Chapter 3) yields a supporting hyperplane $H = \{x \in \mathbb{R}^n \mid a^\intercal x = 1\}$ at $\bar{x}$, such that $U \subset \{x \in \mathbb{R}^n \mid a^\intercal x > 1\}$ and $K \cap \mathbb{R}^n_{>0} \subset \{x \in \mathbb{R}^n \mid a^\intercal x \le 1\}$. Note, that a is forced to be strictly positive.

Write $f(x) = a^\intercal x$. Due to the maximality of $\bar{x}$ there exists a $\gamma > 0$ such that $\mathrm{grad} L(\bar{x}) = \gamma \mathrm{grad} f(\bar{x})$. This is equivalent to $(\frac{\alpha_1}{\bar{x}_1}, \dots, \frac{\alpha_n}{\bar{x}_n}) = \gamma(a_1, \dots, a_n)$, or $\alpha_i = \gamma a_i e^{-u_i}$, for $1 \le i \le n$. From here, we obtain

$$1 \ge \sum_{i=1}^n \alpha_i = \gamma \sum_{i=1}^n a_i e^{-u_i} = \gamma a^\intercal \bar{x} = \gamma$$

and, since $a > 0$, we have

$$||\gamma a||_{K^\star} = \gamma ||a||_{K^\star} = \gamma h_K(a) = \gamma \sup_{x \in K} a^\intercal x = \gamma \sup_{x \in K \cap \mathbb{R}^n_{>0}} a^\intercal x = \gamma \le 1.$$

In conclusion, we can find an element v of $T(K^\star)$ with $\gamma a_i = e^{-v_i}, 1 \le i \le n$, and we get $e^{-w_i} = \alpha_i = \gamma a_i e^{-u_i} = e^{-(u_i + v_i)}$. Thus, we end up with $w = u + v \in T(K) + T(K^\star)$ and the lemma is proven. □

7.5 Proposition. *For all $K \in \mathcal{K}_0^n$ there are $a, b \in \mathbb{R}^n$ such that*

$$a + \mathbb{R}^n_{\ge 0} \subset T(K) \subset b + \mathbb{R}^n_{\ge 0}.$$

Proof. Recall that $T(K) = \{u \in \mathbb{R}^n \mid e^{-u} \in K\} = \{u \in \mathbb{R}^n \mid ||e^{-u}||_K \le 1\}$. The equivalence of all norms on $\mathbb{R}^n$ gives us constants $\alpha, \beta \in \mathbb{R}$ such that

$$e^{-\beta} ||x||_\infty \le ||x||_K \le e^{-\alpha} ||x||_\infty, \ \forall x \in \mathbb{R}^n.$$

Putting $a = -(\alpha, \dots, \alpha)$ and $b = -(\beta, \dots, \beta) \in \mathbb{R}^n$, yields

$$\begin{aligned} a + \mathbb{R}^n_{\geq 0} &= a + T(C_n) = \{a + u \in \mathbb{R}^n \mid ||e^{-u}||_\infty \leq 1\} \\ &= \{u \in \mathbb{R}^n \mid ||e^{-(u-a)}||_\infty \leq 1\} = \{u \in \mathbb{R}^n \mid e^{-\alpha}||e^{-u}||_\infty \leq 1\} \\ &\subset \{u \in \mathbb{R}^n \mid ||e^{-u}||_K \leq 1\} = T(K), \end{aligned}$$

and similarly $T(K) \subset b + \mathbb{R}^n_{\geq 0}$. □

7.2 Details of the proof

Let $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable function. Then the *Laplace transform* $\mathcal{L}(f) : \mathbb{R}^n \to \mathbb{R}$ of f is defined by

$$\mathcal{L}(f)(t) = \int_{\mathbb{R}^n} e^{-u^\intercal t} f(u)\, du,\ t \in \mathbb{R}^n.$$

7.6 Proposition. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be integrable functions and $f * g$ be its* convolution *which is given by*

$$(f * g)(t) = \int_{\mathbb{R}^n} f(x) g(t - x)\, dx,\ t \in \mathbb{R}^n.$$

Then, we have $\mathcal{L}(f)\mathcal{L}(g) = \mathcal{L}(f * g)$.

Proof. Pick a $t \in \mathbb{R}^n$. Then

$$\begin{aligned} \mathcal{L}(f * g)(t) &= \int_{\mathbb{R}^n} e^{-u^\intercal t} (f * g)(u)\, du = \int_{\mathbb{R}^n} e^{-u^\intercal t} \int_{\mathbb{R}^n} f(x) g(u - x)\, dx\, du \\ \text{(Fubini)} &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-x^\intercal t} f(x) e^{-(u-x)^\intercal t} g(u - x)\, du\, dx \\ &= \int_{\mathbb{R}^n} e^{-x^\intercal t} f(x) \underbrace{\int_{\mathbb{R}^n} e^{-(u-x)^\intercal t} g(u - x)\, du}_{=\mathcal{L}(g)(t)}\, dx \\ &= \mathcal{L}(f)(t) \cdot \mathcal{L}(g)(t). \end{aligned}$$

□

The subsequent lemma provides a connection between the Mahler volume of K and the Laplace transforms of the characteristic functions of $T(K)$ and $T(K^\star)$.

7.7 Lemma. *Let $K \in \mathcal{K}^n_0$ be 1-unconditional with basis $\{e_1, \dots, e_n\}$. Then*

$$\mathbf{M}(K) = 4^n \mathcal{L}(\chi_{T(K)} * \chi_{T(K^\star)})(1, \dots, 1).$$

Proof. The symmetry of K with respect to all coordinate hyperplanes gives us

$$\mathrm{vol}(K) = 2^n \mathrm{vol}(K \cap \mathbb{R}^n_{>0}).$$

Let $k \in \mathbb{N}^n$. By substituting $x_i = e^{-u_i}, 1 \leq i \leq n$, we compute

$$\int_{K\cap\mathbb{R}^n_{>0}} x_1^{k_1-1} \cdot \ldots \cdot x_n^{k_n-1}\, dx = \int_{T_n(K\cap\mathbb{R}^n_{>0})} e^{-u_1 k_1} \cdot \ldots \cdot e^{-u_n k_n}\, du_1 \ldots du_n$$

$$= \int_{T(K)} e^{-u^\intercal k}\, du = \int_{\mathbb{R}^n} e^{-u^\intercal k} \chi_{T(K)}(u)\, du = \mathcal{L}(\chi_{T(K)})(k).$$

Choosing $k = (1, ..., 1)$ yields

$$\begin{aligned}
\mathbf{M}(K) &= 4^n \mathrm{vol}(K \cap \mathbb{R}^n_{>0}) \mathrm{vol}(K^\star \cap \mathbb{R}^n_{>0}) \\
&= 4^n \left(\int_{K\cap\mathbb{R}^n_{>0}} dx \right) \left(\int_{K^\star\cap\mathbb{R}^n_{>0}} dx \right) \\
&= 4^n \mathcal{L}(\chi_{T(K)})(1, \ldots, 1) \cdot \mathcal{L}(\chi_{T(K^\star)})(1, \ldots, 1) \\
\text{(Proposition 7.6)} \quad &= 4^n \mathcal{L}(\chi_{T(K)} * \chi_{T(K^\star)})(1, \ldots, 1),
\end{aligned}$$

as desired. □

The next step, is to relate the characteristic functions of two convex sets C_1 and C_2 to that of their Minkowski sum $C_1 + C_2$.

7.8 Lemma. *Let $C_1, C_2 \subset \mathbb{R}^n$ be two closed convex sets, such that there are $a, b \in \mathbb{R}^n$ fulfilling $a + \mathbb{R}^n_{\geq 0} \subset C_i \subset b + \mathbb{R}^n_{\geq 0}$ for $i = 1, 2$. Furthermore, we define the* cross-section *of $C_i, i = 1, 2$, at* altitude *$t \in \mathbb{R}$ as the set $C_i(t) = \{y \in \mathbb{R}^{n-1} \mid (y, t) \in C_i\}$. Then, for all $y \in \mathbb{R}^{n-1}$ and $r \in \mathbb{R}$ it holds*

$$\delta(y, r) := \int_{\mathbb{R}} \chi_{C_1(t)+C_2(r-t)}(y)\, dt \geq \int_{-\infty}^{r} \chi_{(C_1+C_2)(t)}(y)\, dt =: h(y, r).$$

Proof. First of all, we illustrate that we can restrict to C_1 and C_2 such that there is a $b \in \mathbb{R}^n$ with $\mathbb{R}^n_{\geq 0} \subset C_i \subset b + \mathbb{R}^n_{\geq 0}$. From the definition of the cross-section we get

$$(C_1 - a)(t) + (C_2 - a)(r - t) = C_1(t + a_n) + C_2(r + 2a_n - (t + a_n)) - 2a_{(n-1)}$$

and

$$(C_1 - a + C_2 - a)(t) = (C_1 + C_2)(t + 2a_n) - 2a_{(n-1)},$$

where $a_{(n-1)} = (a_1, \dots, a_{n-1}) \in \mathbb{R}^{n-1}$. This leads to

$$\begin{aligned}\int_{\mathbb{R}} \chi_{(C_1-a)(t)+(C_2-a)(r-t)}(y)\, dt &= \int_{\mathbb{R}} \chi_{C_1(t+a_n)+C_2(r+2a_n-(t+a_n))}(y+2a_{(n-1)})\, dt \\ &= \int_{\mathbb{R}} \chi_{C_1(t)+C_2(r+2a_n-t))}(y+2a_{(n-1)})\, dt \\ &= \delta(y+2a_{(n-1)}, r+2a_n)\end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^{r} \chi_{(C_1-a+C_2-a)(t)}(y)\, dt &= \int_{-\infty}^{r} \chi_{(C_1+C_2)(t+2a_n)}(y+2a_{(n-1)})\, dt \\ &= \int_{-\infty}^{r+2a_n} \chi_{(C_1+C_2)(t)}(y+2a_{(n-1)})\, dt \\ &= h(y+2a_{(n-1)}, r+2a_n),\end{aligned}$$

for all $y \in \mathbb{R}^{n-1}$ and $r \in \mathbb{R}$.

Next, we fix some $(y, r) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and abbreviate $\delta = \delta(y, r)$ and $h = h(y, r)$. Observe that, for $t \in \mathbb{R}$, we have $y \in C_1(t) + C_2(r-t)$ if and only if there exists a $y_1 \in \mathbb{R}^{n-1}$ such that $(y_1, t) \in C_1$ and $(y - y_1, r - t) \in C_2$. This is equivalent to $(C_1 \cap [(y, r) - C_2])(t) \neq \emptyset$ and therefore

$$\begin{aligned}\delta &= \operatorname{vol}(\{t \in \mathbb{R} \mid y \in C_1(t) + C_2(r-t)\}) \\ &= \operatorname{vol}(\{t \in \mathbb{R} \mid (C_1 \cap [(y,r) - C_2])(t) \neq \emptyset\}) \\ &= \operatorname{vol}(C_1 \cap [(y,r) - C_2] \mid H_n),\end{aligned}$$

where $H_n = \{x \in \mathbb{R}^n \mid x_1 = \ldots = x_{n-1} = 0\}$. So, δ is the length of an interval in $\mathbb{R}$, since C_i is convex for $i = 1, 2$.

In order to get $\delta \geq h$, it suffices to show $y \notin (C_1 + C_2)(t)$, whenever $t < r - \delta$, or equivalently $(y, t) \notin C_1 + C_2$. Suppose it holds the contrary $(y, t) \in C_1 + C_2$. Then there exists $(u, s) \in C_1$ such that $(y - u, t - s) = (y - u, r + t - s - r) \in C_2$. From the assumption $\mathbb{R}^n_{\geq 0} \subset C_i$, we get $C_i + \mathbb{R}^n_{\geq 0} \subset C_i, i = 1, 2$. And since $r - t > 0$, we obtain $(u, s + r - t) \in C_1$ and $(y - u, t - s + r - t) = (y - u, r - s) \in C_2$, which implies

$$(u, s) \in C_1 \cap [(y, r) - C_2] \text{ and } (u, s + r - t) \in C_1 \cap [(y, r) - C_2].$$

Thus, s and $s + r - t$ belong to the projection $(C_1 \cap [(y, r) - C_2]) \mid H_n$ and we have $s + r - t - s = r - t \leq \delta$, which contradicts our choice of $t < r - \delta$. □

7.9 Lemma. *Let $C_1, C_2 \subset \mathbb{R}^n$ be two closed convex sets, such that there are $a, b \in \mathbb{R}^n$ fulfilling $a + \mathbb{R}^n_{\geq 0} \subset C_i \subset b + \mathbb{R}^n_{\geq 0}$, for $i = 1, 2$. Then we have*

$$\chi_{C_1} * \chi_{C_2} \geq \chi_{C_1 + C_2} * \chi_{\mathbb{R}^n_{\geq 0}}.$$

Proof. We proceed by induction on the dimension n. In the case $n = 1$, there are $\alpha_i \in \mathbb{R}$ such that $C_i = [\alpha_i, \infty)$, $i = 1, 2$. This yields $C_1 + C_2 = [\alpha_1 + \alpha_2, \infty)$ and for all $x \in \mathbb{R}$

$$\begin{aligned}
\chi_{C_1} * \chi_{C_2}(x) &= \int_{\mathbb{R}} \chi_{C_1}(t)\chi_{C_2}(x-t)\ dt \ = \ \int_{\mathbb{R}} \chi_{C_1}(t)\chi_{x-C_2}(t)\ dt \\
&= \int_{\mathbb{R}} \chi_{C_1 \cap (x-C_2)}(t)\ dt \ = \ \int_{\mathbb{R}} \chi_{[x-\alpha_2, \alpha_1]}(t)\ dt \\
&= \max\{\alpha_1 + \alpha_2 - x, 0\}.
\end{aligned}$$

Similarly, we get

$$\chi_{C_1+C_2} * \chi_{\mathbb{R}_{\geq 0}}(x) = \int_{\mathbb{R}} \chi_{[x, \alpha_1+\alpha_2]}(t)\ dt = \max\{\alpha_1 + \alpha_2 - x, 0\}.$$

Thus, in dimension one it even holds equality.

Next, we assume $n > 1$. Recall from Lemma 7.8, that for $t \in \mathbb{R}$ and $i = 1, 2$ we write $C_i(t) = \{y \in \mathbb{R}^{n-1} \mid (y, t) \in C_i\}$. Pick $(x, r) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and denote $y_{(n-1)} = (y_1, \ldots, y_{n-1})$. It holds

$$\chi_{C_1} * \chi_{C_2}(x, r) \ = \ \int_{\mathbb{R}^n} \chi_{C_1}(y)\chi_{C_2}((x, r) - y)\ dy$$

$$\begin{aligned}
&= \int_{\mathbb{R}^n} \chi_{C_1}(y_{(n-1)}, y_n)\chi_{C_2}(x - y_{(n-1)}, r - y_n)\ dy \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \chi_{C_1(y_n)}(y_{(n-1)})\chi_{C_2(r-y_n)}(x - y_{(n-1)})\ dy_{(n-1)}\ dy \\
&= \int_{\mathbb{R}} \chi_{C_1(y_n)} * \chi_{C_2(r-y_n)}(x)\ dy_n \\
&\geq \int_{\mathbb{R}} \chi_{C_1(y_n)+C_2(r-y_n)} * \chi_{\mathbb{R}^{n-1}_{\geq 0}}(x)\ dy_n \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \chi_{C_1(y_n)+C_2(r-y_n)}(y_{(n-1)})\chi_{\mathbb{R}^{n-1}_{\geq 0}}(x - y_{(n-1)})\ dy_{(n-1)}\ dy_n \\
&= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \chi_{C_1(y_n)+C_2(r-y_n)}(y_{(n-1)})\ dy_n \right) \chi_{\mathbb{R}^{n-1}_{\geq 0}}(x - y_{(n-1)})\ dy_{(n-1)} \\
&\geq \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{r} \chi_{(C_1+C_2)(y_n)}(y_{(n-1)})\ dy_n \right) \chi_{\mathbb{R}^{n-1}_{\geq 0}}(x - y_{(n-1)})\ dy_{(n-1)} \\
&= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \chi_{(C_1+C_2)(y_n)}(y_{(n-1)})\chi_{\mathbb{R}^n_{\geq 0}(r-y_n)}(x - y_{(n-1)})\ dy_n\ dy_{(n-1)} \\
&= \int_{\mathbb{R}} \chi_{(C_1+C_2)(y_n)} * \chi_{\mathbb{R}^n_{\geq 0}(r-y_n)}(x)\ dy_n
\end{aligned}$$

$$= \int_{\mathbb{R}^n} \chi_{C_1+C_2}(y)\chi_{\mathbb{R}^n_{\geq 0}}((x,r)-y)\,dy$$
$$= \chi_{C_1+C_2} * \chi_{\mathbb{R}^n_{\geq 0}}(x,r).$$

The first inequality above comes from the induction hypothesis and the second one from Lemma 7.8. □

Eventually, we are well prepared to show the main theorem.

Proof of Theorem 7.1. Since, $M(C_n^\star) = \frac{4^n}{n!}$ and because of Lemma 7.7 it suffices to show that

$$\mathcal{L}(\chi_{T(K)} * \chi_{T(K^\star)})(1,\ldots,1) \geq \mathcal{L}(\chi_{T(C_n^\star)} * \chi_{T(C_n)})(1,\ldots,1). \tag{7.3}$$

We have

$$\begin{aligned} T(C_n) &= \{y \in \mathbb{R}^n \mid \exists (t_1,\ldots,t_n) \in C_n \text{ with } y = (-\ln t_1,\ldots,-\ln t_n)\} \\ &= \{y \in \mathbb{R}^n \mid (e^{-y_1},\ldots,e^{-y_n}) \in C_n)\} \\ &= \{y \in \mathbb{R}^n \mid e^{-y_i} \leq 1,\ 1 \leq i \leq n\} \ = \ \mathbb{R}^n_{\geq 0}, \end{aligned}$$

and by Lemma 7.4, it holds $T(K) + T(K^\star) = T(C_n^\star)$. In Proposition 7.3 we have seen that the set $T(K)$ is always convex for 1-unconditional K. By definition it is also closed and with Lemma 7.9 we obtain

$$\chi_{T(K)} * \chi_{T(K^\star)} \geq \chi_{T(K)+T(K^\star)} * \chi_{\mathbb{R}^n_{\geq 0}} = \chi_{T(C_n^\star)} * \chi_{T(C_n)}.$$

This verifies (7.3) and therefore, we have indeed $\mathbf{M}(K) \geq \frac{4^n}{n!}$. □

Index

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