# Ext on affine toric varieties

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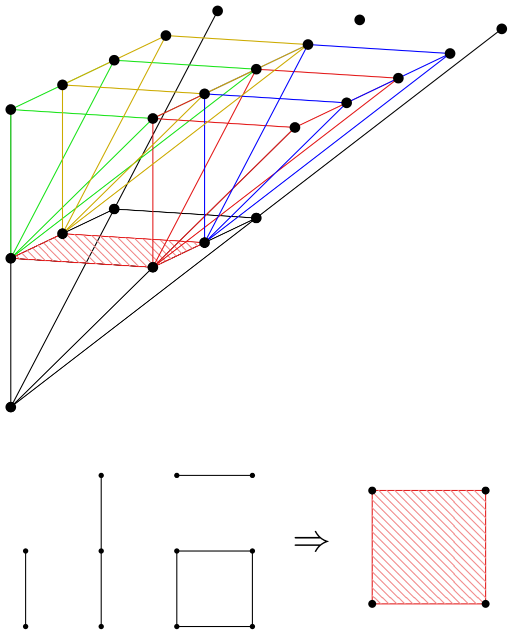
Dissertation



eingereicht am

Fachbereich Mathematik und Informatik
der Freien Universität Berlin

3. Juni 2015

Die vorliegende Dissertation wurde von Prof. Dr. Klaus Altmann betreut.

## Eidesstattliche Erklärung

Ich versichere, diese Dissertation selbständig verfaßt, alle verwendeten Hilfsmittel sowie Hilfen angegeben und die Arbeit nicht in einem früheren Promotionsverfahren eingereicht zu haben.

Lars Kastner, Berlin, 3. Juni 2015.

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## Abstract

We want to compute $\mathrm{Ext}^i$ of two $T$-invariant Weil divisors $D$ and $D'$ on an normal affine toric variety. Our goal is a combinatorial criterion for maximal Cohen-Macaulayness of $D$. The main tool is a generalization of the Taylor resolution of monomial ideals in polynomial rings to monomial ideals in toric rings. Then the question translates to computation of $\mathrm{Ext}^i$ of two divisorial ideals. We arrive at a spectral sequence giving both a sufficient criterion for vanishing of higher $\mathrm{Ext}^i$ and a superset of the support of $\mathrm{Ext}^1$ in combinatorial terms.

After the general construction we restrict to the case of cyclic quotient singularities. Here we can give an explicit combinatorial description of $\mathrm{Ext}^1(D, D')$. Furthermore we show a relationship between the continued fraction giving the cyclic quotient singularity and the dimensions of the $\mathrm{Ext}^1(D, D')$. We conclude by giving a homogeneous basis of the algebra $\mathrm{Ext}(D)$, giving rise to a combinatorial description of the multiplication.

# Contents

# Introduction

On smooth varieties, all Weil divisors are Cartier. On singular varieties this is no longer true and translates to the fact that the sheaf $\mathcal{O}(D)$ need not be locally free anymore, for $D$ a Weil divisor.

Our goal is to investigate those sheaves for a normal affine toric variety $X$. In particular, we are interested in finding a combinatorial answer for the question, whether the divisorial ideal corresponding to a $T$-invariant Weil divisor is maximal Cohen-Macaulay (MCM), or even special MCM (sMCM) for surfaces. Both properties can be expressed via the vanishing of certain Ext-modules: MCM is equivalent to the vanishing of $\operatorname{Ext}^i(D, K_X)$ for $i > 0$, and by a result of Wemyss and Iyama ([IW10]) sMCM means being MCM and an additional vanishing of $\operatorname{Ext}^1(D, R)$. Our Ansatz to compute $\operatorname{Ext}^i(D, D')$ is to resolve $D$ projectively. Hence, the main task is to resolve certain monomial ideals in semigroup rings.

As a prototype we use the Taylor resolution ([Tay66]) of monomials ideals. In the smooth case, the Taylor resolution gives a finite free resolution of any monomial ideal, due to existence of a least common multiple in a polynomial ring.

In the singular case, the result is a complex with at least one free module. All other appearing modules are direct sums of divisorial ideals. Applying the Taylor resolution to the single summands yields an algorithmic way to obtain a resolution of $D$, which is free up to any desired length. Now we take a projective Cartan-Eilenberg resolution of the Taylor resolution of $D$. Applying $\operatorname{Hom}(\bullet, D')$ yields a spectral sequence, which then allows us to study the support of $\operatorname{Ext}^1(D, D')$. Furthermore, we can derive a sufficient criterion for the vanishing of $\operatorname{Ext}^i(D, D')$ for $i > 0$.

**Main Result 1** (Proposition 3.8)**.** *Let $X$ be an affine toric variety and $D$, $D'$ two Weil divisors. Then there is a subset $P(D)$ of the $T$-invariant Weil divisors such that*

$$\operatorname{Ext}^1_X(G, D') = 0 \ \forall G \in P(D) \ \Rightarrow \ \operatorname{Ext}^i_X(D, D') = 0 \text{ for } i > 0.$$

*In the case of $X$ coming from a simplicial cone, the set $P(D)$ modulo linear equivalence is finite.*

The last part is easy to deduce, since for simplicial $X$ the class group is finite. An example for $X$ and $D$ such that $P(D)$ modulo linear equivalence is infinite is unknown to us.

The first interesting class of normal affine toric varieties are cyclic quotient singularities (CQS). On the algebro-geometric side, these are just quotients of $\mathbb{C}^2$ by finite groups. On the side of toric geometry, one starts with a two dimensional cone. Here, our generalization of the Taylor resolution yields a set of short exact sequences and a description of $\mathrm{Ext}^1(D, D')$ as a certain set of lattice points. Furthermore we obtain a recursive representation of $\mathrm{Ext}^n$ as a direct sum of $\mathrm{Ext}^1$'s and a duality with the Tor-modules.

**Main Result 2** (Proposition 6.11, Theorem 7.5)**.** *For $D$ and $D'$ being two $T$-invariant Weil divisors on a CQS $X$ we have the following two equations:*

$$\mathrm{Ext}^i_X(D, K_X - D') \;=\; \mathrm{Ext}^i_X(D', K_X - D)$$

*and*

$$(\mathrm{Tor}_i^X(D, D'))^* = \mathrm{Ext}_X^{i+2}(D, K_X - D'),$$

*where $(\bullet)^* = \mathrm{Hom}(\bullet, \mathbb{C})$ and $i > 0$.*

An interesting feature of these varieties is their close relationship with continued fractions. For example Stevens ([Ste91]) and Christophersen ([Chr91]) used these to analyze the deformation theory of CQS, by building certain continued fractions representing zero. These represent the components of the versal deformation. Furthermore, in [Ste91] Stevens describes a correspondence between the P-resolutions, that Kollár and Shepherd-Barron introduced in [KS88] for the purpose of studying the deformation theory of CQS as well. Thus it comes to no surprise that continued fractions yield an easy way for computing $\dim_{\mathbb{C}} \mathrm{Ext}^1$ in the case of CQS.

**Main Result 3** (Theorem 6.26, Algorithm 6.30)**.** *Starting with a CQS $X$ coming from a continued fraction $\underline{a}$, denote by*

$$\mathcal{E}_1(\underline{a}) \;:=\; (\dim_{\mathbb{C}} \mathrm{Ext}^1_X(D, D'))_{D,D' \in \mathrm{Cl}(X)}.$$

*Then there are two other continued fractions $\underline{\tilde{a}}$ and $\underline{a}'$ such that we can compute $\mathcal{E}_1(\underline{a})$ from $\mathcal{E}_1(\underline{\tilde{a}})$ and $\mathcal{E}_1(\underline{a}')$. In particular, $\underline{\tilde{a}}$ and $\underline{a}'$ are "smaller" than $\underline{a}$, yielding a recursive algorithm to compute $\mathcal{E}_1(\underline{a})$.*

The combinatorial description of Ext for CQS culminates in a combinatorial description of the algebra $\mathrm{Ext}(D) = \oplus_{i \geq 0} \mathrm{Ext}^i(D, D)$. We provide an explicit basis of $\mathrm{Ext}(D)$ as a $\mathbb{C}$-vector space, consisting of homogeneous elements. Then we use Yoneda's description of Ext to formulate the multiplication in $\mathrm{Ext}(D)$ in discrete mathematics.

**Main Result 4** (Theorem 8.11, Corollary 8.22)**.** *The homogeneous basis of $\mathrm{Ext}(D)$ mentioned above allows a fast combinatorial algorithm for computing the product of two elements of $\mathrm{Ext}(D)$.*

Let us give an overview of the structure of this thesis:

The preliminaries contain a brief overview on the toric objects and methods we use. We show how to obtain generators for divisorial ideals. Furthermore we describe the link between Ext-modules and MCM'ness in more detail.

Before restricting to CQS, we introduce a generalization of the taylor resolution for toric rings in Chapter 3. This generalization gives rise to a spectral sequence containing descriptions of all $\mathrm{Ext}^i(D, D')$ in terms of direct sums of $\mathrm{Ext}^j(G, D')$, with $j < i$ and $G$ some other Weil divisor. In particular we can now state the vanishing condition of Main Result 1. Furthermore we provide a basic description of the support of $\mathrm{Ext}^1(D, D')$.

The remaining chapters deal with the case of CQS exclusively. First we describe the toric construction of CQS and $T$-invariant divisors on CQS in Chapter 4. Furthermore we introduce the notion of continued fraction expansions and explain its relation to CQS. In Chapter 5 we come back to the generalized Taylor resolution and apply it to the case of torus invariant divisors on CQS. We arrive at a quiver, which completely encodes the information to freely resolve any divisorial ideal.

In Chapter 6 we describe the finite-dimensional multigraded vector spaces $\mathrm{Ext}^1(D, D')$ combinatorially. Since there are only finitely many Weil divisors on a CQS up to linear equivalence, we can write down the respective dimensions $\dim_{\mathbb{C}} \mathrm{Ext}^1(D, D')$ in a matrix. We then proceed by constructing this matrix from the sole knowledge of the continued fraction representation of $n/q$.

We go on in Chapter 7 by explaining how to compute the dimensions of higher $\mathrm{Ext}^i(D, D')$ modules using the combinatorial description of Chapter 6 and the quiver of Section 5.2. Since the $\mathrm{Tor}_i(D, D')$ modules exhibit a very similar behaviour as the $\mathrm{Ext}^i(D, D')$ modules it is naturally to compare these in greater detail. Finally we arrive at the second formula of Main Result 2.

In Chapter 8 we investigate the structure of the algebras $\mathrm{Ext}(D)$ closer. These have a $\mathbb{Z} \times M$ grading. We describe a homogeneous basis of $\mathrm{Ext}(D)$ as a $\mathbb{C}$-vector space, as mentioned above. We finish the chapter with the combinatorial description of the multiplication.

Many experiments and examples were computed using the software framework `polymake` ([GJ00]) and the computer algebra system **Singular** ([Dec+15]). In Appendix A we give sample code for some of the experiments and provide code to test possible counter-examples to conjectures within the thesis. Furthermore we give an overview of the functionality of the code now contained in `polymake` that has been written in context of this thesis.

# Preliminaries

In this chapter we will give the necessary definitions from both algebraic and toric geometry, and commutative algebra to develop the methods of the latter chapters. Since our main focus is on translating Ext from commutative algebra into combinatorics in the toric setting, the first part is about toric geometry. We then provide some context on how the Ext-functor may be used to find maximal Cohen-Macaulay modules.

Throughout this thesis we will work over the complex numbers $\mathbb{C}$.

## 2.1. Toric geometry

In this section we will define the rings and modules that we will work with throughout the remaining chapters.

### Affine toric varieties

First we fix a lattice $N$ and the corresponding $\mathbb{Q}$-vector space $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$. The dual lattice of $N$ is given via $M := \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. We denote the pairing by

$$\langle \bullet, \bullet \rangle : \begin{array}{ccc} M \times N & \to & \mathbb{Z} \\ (u, v) & \mapsto & \langle u, v \rangle := u(v) \end{array} .$$

The pairing extends linearly to $N_{\mathbb{Q}} \times M_{\mathbb{Q}} \to \mathbb{Q}$. Whenever we use coordinates we will denote the elements of $N$ in parentheses $(\bullet)$ and the elements of $M$ in square brackets $[\bullet]$.

For a given cone $\sigma \subseteq N_{\mathbb{Q}}$, we can build its dual cone $\sigma^{\vee}$ via

$$\sigma^{\vee} := \{u \in M_{\mathbb{Q}} \mid \langle u, x \rangle \geq 0 \ \forall x \in \sigma\}.$$

This enables to give the basic definition of a normal affine toric variety:

**Definition 2.1** (Normal affine toric variety)**.** The *toric variety* associated to a cone $\sigma$ is given as

$$\operatorname{TV}(\sigma) := \operatorname{Spec}(\mathbb{C}[\sigma^{\vee} \cap M]).$$

This definition turns TV into a covariant functor from cones to varieties which is described in more detail in for example [Ful93] or [CLS11].

The interesting part for us is the ring $R := \mathbb{C}[\sigma^{\vee} \cap M]$. It is an $M$-graded $\mathbb{C}$-algebra. Let us assume that $\sigma$ is pointed and full-dimensional. Then the dual cone $\sigma^{\vee}$ will have these properties as well. Thus the semigroup $\sigma^{\vee} \cap M$ has a unique minimal finite system of generators $H \subseteq \sigma^{\vee} \cap M$ which translate into a minimal system of homogeneous generators of $R$ as a $\mathbb{C}$-algebra.

**Definition 2.2** (Hilbert basis)**.** The minimal generating set $H$ of the semigroup $\sigma^\vee \cap M$ is called the *Hilbert basis* of $\sigma^\vee$.

Hence, for $H = \{h_1, \dots, h_n\}$ we have a surjection

$$\begin{array}{rccc} \varphi : & \mathbb{C}[x_1, \dots, x_n] & \twoheadrightarrow & R \\ & x_i & \mapsto & x^{h_i} \end{array}.$$

**Definition 2.3** (Toric ideal)**.** The *toric ideal* of $H$ is defined to be the kernel of $\varphi$.

**Example 2.4.** We identify $N$ and $latv$ with $\mathbb{Z}^2$ via the usual scalar product in order for them to have the same picture. Subsequently we have $N_\mathbb{Q} = \mathbb{Q}^2 = M_\mathbb{Q}$. Take the cone

$$\sigma := \operatorname{cone}\{(1,0), (-3,7)\} \subseteq \mathbb{Q}^2.$$

The dual cone is then spanned by the vectors $[0,1]$ and $[7,3]$, i.e. the rays orthogonal to one one of the rays of $\sigma$ and evaluating positively on the other.

The green cone denotes $\sigma$ and the red cone denotes $\sigma^\vee$. The Hilbert basis of $\sigma^\vee$ consists of the points

$$\{[0,1],\ [1,1],\ [2,1],\ [7,3]\},$$

as depicted in Example 4.1, and for the toric ideal we get

$$I = (y^4 - xz, x^7 - w^4 z, xy^3 - wz, x^3y^2 - w^2z, x^5y - w^3z, -x^2 + wy) \subseteq \mathbb{C}[w,x,y,z].$$

## Torus invariant divisors

The modules we want to examine come from torus invariant Weil divisors. Such a divisor $D$ allows us to define a sheaf $\mathcal{O}(D)$. Taking global sections we end up with an $R$-module. We follow the notation of [CLS11] closely. Let us recall the basic construction and facts of [CLS11] and [Ful93].

Assume that $\sigma$ is given via primitive ray generators

$$\sigma = \operatorname{cone}\{v^1, \dots, v^n\}, \text{ with } v_i \in N,$$

and take $R = \mathbb{C}[\sigma^\vee \cap M]$ and $X := \operatorname{TV}(\sigma) = \operatorname{Spec} R$. Furthermore denote by $\rho^i$ the ray spanned by $v^i$. The rays of $\sigma$ yield exactly the codimension one torus orbits of $X$ and, taking the closure thereof, we obtain the torus invariant prime divisors $D_{\rho^i}$. Now any $T$-invariant Weil divisor $D$ on $X$ may be written as

$$D \;=\; \sum_{\rho \in \sigma[1]} a_\rho D_\rho \;=\; \sum_{i=1}^{n} a_i D_{\rho^i}$$

with $a_i \in \mathbb{Z}$. Taking the polyhedron

$$P_D \;:=\; \{u \in M_{\mathbb{Q}} \mid \langle u, v^i \rangle \geq -a_i\}$$

we obtain an $M$-graded $R$-module via taking the global sections of the sheaf $\mathcal{O}(D)$ according to [CLS11, Prop. 4.3.3]:

$$H^0(X, D) \;=\; \Gamma(X, \mathcal{O}(D)) \;=\; \mathbb{C} \cdot \{x^u \mid u \in P_D \cap M\} \subseteq \mathbb{C}[M],$$

where the multiplication is induced via the inclusion $R \subseteq \mathbb{C}[M]$. This module is a so-called fractional ideal. Furthermore, $H^0(X, D)$ is a divisorial ideal, as described in [BG09; BH98], i.e. it is reflexive as an $R$-module, or, equivalently, it comes from a Weil divisor.

Thus there are three ways to describe a $T$-invariant Weil divisor $D$:

1. As an element $D \in \mathbb{Z}^{\#\sigma[1]}$,
2. as the divisorial ideal $H^0(X, D) \subseteq \mathbb{C}[M]$, and
3. as the polyhedron $P_D$ described above.

We assume $\sigma$ to be full-dimensional and pointed, thus, the dual cone $\sigma^\vee$ has these properties as well. This implies that $P_D$ has $\sigma^\vee$ as its tailcone and hence $P_D \neq \emptyset$. Thus, we get a one-to-one correspondence between all three representations. We will denote all of these by the same letter $D$. Furthermore for $P_D \subseteq \sigma^\vee$ the module $H^0(X, D)$ even becomes an honest ideal of $R$. Additionally we can always move $P_D$ inside $\sigma^\vee$ adding some lattice vector. Hence every $H^0(X, D)$ is isomorphic to an ideal of $\sigma^\vee$ via an $M$-graded isomorphism.

Please be aware that the addition of divisors does not become the multiplication of divisorial ideals in $\mathbb{C}[M]$. In general it is true that

$$H^0(X, D + D') \supseteq H^0(X, D) \cdot H^0(X, D').$$

Equality does not hold in general, take for example the divisors of Example 4.7.

*Remark* 2.5. The sheaf $\mathcal{O}(D)$ is coherent, because we work on an affine variety, thus we may use [Har83, III, Exercise 6.7] to deduce

$$\operatorname{Ext}^i_R(D, D') := \operatorname{Ext}^i_X(\mathcal{O}(D), \mathcal{O}(D')) = \operatorname{Ext}^i_R(H^0(X, D), H^0(X, D')).$$

However, keep in mind that the sheaves $\mathcal{O}(D)$ are not necessarily locally free.

In order to compute Ext it is necessary to resolve the modules $H^0(X, D)$ injectively or projectively. In our setting we go for the free and hence projective resolution. Thus we need a way to compute generators of $H^0(X, D)$. We are especially interested in homogeneous generators, since we want to translate everything to discrete mathematics.

In our affine setting we can obtain a homogeneous generating set for $H^0(X, D)$ in the following way:

**Proposition 2.6.** Let $C_D$ be the cone

$$C_D := \operatorname{cone}\left\{\{0\} \times \sigma^\vee, \{1\} \times P_D\right\} \;\subseteq \mathbb{Q} \times M_{\mathbb{Q}}.$$

Then we obtain a minimal homogeneous generating set of $H^0(X, D)$ by taking $\{x^u\}$ where $(1, u)$ is an element of the Hilbert basis of $C_D$.

*Proof.* We obtain homogenicity by constrution and minimality follows from the Hilbert basis property. Since $D$ is canonically embedded in $C_D$ at height one and since there are no elements with negative height, a lattice point $w \in D \cap M$ can be written as sum

$$(1, w) = (1, u) + \sum_{i=0}^{r} a_i \cdot (0, v_i), \ u, v_i \in M, \ a_i \in \mathbb{Z}_{\geq 0},$$

and all the right hand side elements are in the Hilbert basis of $C_D$. The $v_i$ correspond to monomials in $R$, thus $x^w$ is just an $R$-multiple of $x^u$ and we are done. □

**Definition 2.7** (Minimal set of generators)**.** We will denote the *minimal set of generators* computed by Proposition 2.6 as $G(D)$. We define its set of exponents to be

$$\mathrm{Supp}(G(D)) := \{u \in M \mid x^u \in G(D)\}.$$

For the two-dimensional case, $\mathrm{Supp}(G(D))$ consists of the vertices of $D \cap M$. This is not true in higher dimensions, though. Take for example the divisorial ideal generated by the points on the long edge of the rectangle in Example 3.3. Here one needs the middle point on the edge as well, despite it not being a vertex of $D \cap M$.

By the previous convention we obtain the equality

$$D \cap M = \mathrm{Supp}(G(D)) + \sigma^\vee \cap M = \{u \in M \mid x^u \in D\},$$

where of course the left hand side $D$ refers to the polyhedron $D$, while the left hand side $D$ denotes the submodule of $\mathbb{C}[M]$ and the middle one is the divisor $D$. This notation extends the definition of the support of a graded module given later in Definition 2.11 to $G(D)$ in a natural way. We will often omit the Supp in front of $G(D)$ and instead write $G(D)$ for the lattice points and for the generators of the divisorial ideal, synonymously.

**Example 2.8.** Let us continue the example from above. There the cone $\sigma$ was spanned by two primitive vectors

$$v^1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } v^2 := \begin{pmatrix} -3 \\ 7 \end{pmatrix}.$$

We pick the following two divisors:

$$E^1 = -1 \cdot [\langle v^1 \rangle_{\mathbb{Q}_{\geq 0}}] \text{ and } E^3 := -3 \cdot [\langle v^1 \rangle_{\mathbb{Q}_{\geq 0}}].$$

Then the polyhedra of global sections are just

$$P_{E^1} = \left[1, \frac{3}{7}\right] + \sigma^\vee \text{ and } P_{E^1} = \left[3, \frac{9}{7}\right] + \sigma^\vee.$$

For a picture we refer to Example 4.7.

### Graded commutative algebra

In the setting with $R := \mathbb{C}[\sigma^\vee \cap M] = \bigoplus_{u \in \sigma^\vee \cap M} \mathbb{C} \cdot x^u$ being a $M$-graded algebra, the divisorial ideals $H^0(X, D)$ for $T$-invariant $D$ on $X := \operatorname{Spec} R$ become $M$-graded $R$-modules. Thus let us fix some notation regarding $M$-graded modules.

We start with defining the shift of an $M$-graded module. This definition differs from the usual definition by sign, as for example in [BG09], but it allows us not to worry about the sign of the shifting parameter anymore.

**Definition 2.9** ($M$-graded module, shift)**.** An $M$-graded $R$-module is an $R$-module $W$ equipped with a grading $W = \oplus_{u \in M} W_u$ such that $R_v \cdot W_u \in W_{v+u}$ for all $u, v \in M$. For an element $u \in M$ we define the shifted module $W[u]$ piecewise by $W[u]_v := W_{v-u}$.

**Example 2.10.** Of course $R$ itself is an $M$-graded $R$-module. Since $\sigma$ is full-dimensional, $\sigma^\vee$ becomes pointed and thus $R_0 = k$. Shifting by $u$ here corresponds to adding $u$ to $\sigma^\vee$.
As claimed before the $H^0(X, D)$ are $M$-graded modules as well.

We can view $D$ as a polyhedron and hence we can add a vector to it using Minkowski-addition, for example a lattice vector $u \in M$. We denote this by $D + u$. Instead we could also shift $D$ by $u$, because it is $M$-graded and obtain $D[u]$. Luckily the shift is designed in such a way that both procedures yield the same, i.e. shifting the module by $u$ and then considering the associated polyhedron is the same as adding $u$ to the polyhedron $D$. We write

$$x^u \cdot D = D[u] = D + u.$$

To switch from $M$-graded modules to polyhedra one uses the support:

**Definition 2.11** (support)**.** The *support* of an $M$-graded module $W$ is the set

$$\operatorname{Supp} W := \{u \in M \mid W_u \neq 0\}.$$

This is not to be confused with the support in the sense of commutative algebra, i.e. all those prime ideals $p \in \operatorname{Spec} R$ such that $W_p \neq 0$. We will only use the latter in Proposition 2.28.

*Remark* 2.12. For the ring $R$ itself we have $\operatorname{Supp} R = \sigma^\vee \cap M$. For the divisors $D$ we obtain

$$\operatorname{Supp}(H^0(X, D)) = P_D \cap M.$$

There is a one-to-one correspondence between $M$-graded fractional ideals $I$ and their supports $\operatorname{Supp}(I)$. If there exists a polyhedron $P \subseteq M_{\mathbb{Q}}$ with the same facet vectors as $\sigma^\vee$ such that $\operatorname{Supp}(I) = P \cap M$, then $I$ is divisorial and hence, reflexive. Otherwise finding the inclusion-wise smallest such $P$ that contains $\operatorname{Supp}(I)$ corresponds to finding the reflexive hull of $I$ on the algebraic side.

Shifts provide a great tool to describe how standard commutative algebra operations preserve the $M$-grading on modules. But first let us explain how Hom obtains a grading.

**Definition 2.13** (homogeneous morphism)**.** Let $f : W \to V$ be an $R$-module homomorphism of $M$-graded $R$-modules. Then $f$ is *homogeneous of degree* $u \in M$ if $f(W_{v-u}) \subseteq V_v$ for all $v \in M$.

In general $\mathrm{Hom}_R(W,V)$ will not be graded, i.e. not every morphism decomposes as a finite sum of homogeneous morphsims, but for $W$ finitely generated this is true, see for example [BG09, 6.B]. Since we will deal with finitely generated modules exclusively, we may assume $\mathrm{Hom}_R(W,V)$ to be canonically $M$-graded.

*Remark* 2.14. Let $V$ and $W$ be a finitely generated $M$-graded $R$-modules. Then we have the isomorphism

$$\mathrm{Hom}_R(W[u],V[v]) = \mathrm{Hom}_R(W,V)[v-u] \text{ for all } u,v \in M$$

of $M$-graded $R$-modules.

□

A finitely generated free $M$-graded $R$-module is an $R$-module of the form $W = \oplus_{i=0}^r R[w^i]$, with $w^i \in M$. Denote by $\{e_W^i\}$ the standard generators of $W$. Then the degree of $e_W^i$ is exactly $\deg(e_W^i) = w^i \in M$. A homogeneous element $w \in W$ of degree $\deg w = u \in M$ can be written in the following way:

$$w = \sum_{i=0}^r a_i \cdot x^{u-\deg e_W^i} \cdot e_W^i = \sum_{i=0}^r a_i \cdot x^{u-w^i} \cdot e_W^i, \; a_i \in k.$$

Hence $w$ is completely determined by its degree $u$ and its coefficient vector $\underline{a} = (a_0,\ldots,a_r) \in k^{r+1}$. For a given homogeneous morphism of degree 0 of finitely generated free $M$-graded $R$-modules

$$f: \; W := \oplus_{i=0}^r R[w^i] \to \oplus_{j=0}^s R[v^j] =: V,$$

the image of a homogeneous element is completely determined by the coefficient vector of the image, i.e. for the standard generators $\{e_W^i\}$ of $W$ and $\{e_V^j\}$ of $V$ we have

$$f(e_W^i) = \sum_{j=0}^s a_{ji} \cdot x^{w^i - v^j} \cdot e_V^j.$$

Thus, it is enough to write down the matrix $A := (a_{ji})$ with entries in $\mathbb{C}$, and $a_{ji} = 0$ if $w^i - v^j \notin \sigma^\vee$, i.e. if a suitable monomial does not exist in $R$.

**Example 2.15.** In Example 5.9 consider the matrix $d_1$. With

$$R = \mathbb{C}[y, xy, x^7y^3] = \mathbb{C}[x^{[0,1]}, x^{[1,1]}, x^{[7,3]}]$$

it describes a map

$$d_1 : R[[4,3]] \oplus R[[5,3]] \oplus R[[10,5]] \oplus R[[7,4]] \oplus R[[8,4]] \oplus R[[11,5]] \to R[[3,2]] \oplus R[[4,2]] \oplus R[[7,3]]$$

given by the following matrix in the 'usual' notation:

$$\begin{pmatrix} xy & xy^2 & x^7y^3 & 0 & 0 & 0 \\ -y & -xy & -x^6y^3 & x^3y^2 & x^4y^2 & x^7y^3 \\ 0 & 0 & 0 & -y & -xy & -x^4y^2 \end{pmatrix},$$

i.e. the entries of this matrix are $a_{ji}\, x^{w^i - v^j}$. Setting $x = y = 1$ yields the easier format, as depicted in Example 5.9. The loss of information in the resulting matrix is compensated by the $M$-grading of the modules and by knowing that $d_1$ is a morphism of degree 0.

We now know that $\mathrm{Hom}_R(W,V)$ is canonically graded. It is easy to see that quotients of graded modules are again graded. Hence, $\mathrm{Ext}_R^i(W,V)$ inherits a canonical grading as well.

## 2.2. Commutative algebra

First we recall the notion of (maximal) Cohen-Macaulayness. In the second part we discuss an Ext based criterion for Cohen-Macaulayness of torus invariant divisors.

### CM and MCM

Our notation follows the book [BH98]. Throughout this section we denote by $R = (R, \mathfrak{m})$ a Noetherian local ring. We will summarize some basic properties of and results on Cohen-Macaulay modules along the lines of [BH98] and [Yos90].

First let us deal with the notions of Cohen-Macaulay (CM) and maximal Cohen-Macaulay (MCM). These involve the depth and the dimension of $R$-modules. The depth of a module $M$ is defined as the maximum length of a $M$-regular sequence in $R$ and the dimension of $M$ is the Krull dimension of the quotient ring $R/\operatorname{Ann}(M)$. We obtain the following chain of inequalities ([BH98, Prop 1.2.12])

$$\operatorname{depth} M \leq \dim M \leq \dim R,$$

giving rise to the following definition:

**Definition 2.16** (CM,MCM [BH98])**.** An $R$-module $M$ is called *Cohen-Macaulay (CM)* if and only if

$$\operatorname{depth} M = \dim M.$$

If $M$ is CM and furthermore $\dim M = \dim R$, then $M$ is called *maximal Cohen-Macaulay* (MCM).

The ring $R$ is called CM (MCM) if and only if it is CM (MCM) as an module over itself.

Let $R$ be a CM ring. For such an $R$ we have the notion of a canonical module $\omega_R$:

**Definition 2.17.** [Yos90, Def. 1.10] The canonical module $\omega_R$ is a CM module over $R$ such that

$$\operatorname{Ext}^i_R(k, \omega_R) = \begin{cases} 0 & i \neq d \\ k & i = d \end{cases},$$

with $k = R/\mathfrak{m}$ and $d = \dim R$.

Sometimes this is denoted as $R = (R, \mathfrak{m}, \omega_R)$. If $R$ has a canonical module, then $\omega_R$ is unique up to isomorphism ([BH98, Thm 3.3.4]), which is why we define 'the' canonical module, instead of 'a' canonical module.

Provided $R$ admits a canonical module, we have the following proposition linking CM modules with Ext:

**Proposition 2.18.** [Yos90, Cor. 1.13] Let $M$ be a CM module over the CM ring $R = (R, \mathfrak{m})$ and assume $R$ has the canonical module $\omega_R$. Then $\operatorname{Hom}_R(M, \omega_R)$ is a CM module as well and there is an isomorphism $M \cong \operatorname{Hom}_R(\operatorname{Hom}_R(M, \omega_R), \omega_R)$. Moreover $\operatorname{Ext}^i_R(M, \omega_R) = 0$ for all $i > 0$.

In particular, $M$ being MCM implies vanishing of $\operatorname{Ext}^i_R(M, \omega_R)$ for all $i > 0$. If we assume $M$ to be finitely generated, the converse becomes true as well, i.e. $M$ is MCM if $\operatorname{Ext}^i_R(M, \omega_R) = 0$ for all $i > 0$. This is a consequence of the following proposition:

**Proposition 2.19.** [BH98, Cor. 3.5.11] Let $(R, \mathfrak{m}, k = R/\mathfrak{m})$ be a CM local ring of dimension $n$ with canonical module $\omega_R$, and $M$ a finitely generated $R$-module of depth $t$ and dimension $d$. Then

(a) $\operatorname{Ext}^i_R(M, \omega_R) = 0$ for $i < n - d$ and $i > n - t$,

(b) $\operatorname{Ext}^i_R(M, \omega_R) \neq 0$ for $i = n - d$ and $i = n - t$,

(c) $\dim \operatorname{Ext}^i_R(M, \omega_R) \leq n - i$ for all $n - d \leq i \leq n - t$.

To prove the last part of Proposition 2.18, insert $n = d = t$ in Proposition 2.19.

Conversely if $\operatorname{Ext}^i_R(M, \omega_R) = 0$ for all $i > 0$, we use Proposition 2.19 as well. Inserting that $i = 0$ is the only possibility for $\operatorname{Ext}^i_R(M, \omega_R) \neq 0$ we obtain

$$i = 0 = \dim M - \operatorname{depth} M = \dim R - \operatorname{depth} M.$$

Proposition 2.19 itself is a consequence Grothendiecks local duality theorem. In the setting of this theorem the canonical module exists, since $R$ is additionally assumed to be complete ([BH98, Cor. 3.3.8]). Please note that in the following theorem $\mathbb{C}$ denotes the residue field $R/\mathfrak{m}$.

**Theorem 2.20.** *[BH98, Thm 3.5.8] Let $(R, \mathfrak{m}, k)$ be a local complete CM-ring of dimension $d$. Then for all finite $R$-modules $M$ and all integers $i$ there exist natural isomorphisms*

$$\begin{aligned} H^i_{\mathfrak{m}}(M) &\cong \operatorname{Hom}_R(\operatorname{Ext}^{d-i}_R(M, \omega_R), E(k)) \\ \operatorname{Ext}^i_R(M, \omega_R) &\cong \operatorname{Hom}_R(H^{d-i}_{\mathfrak{m}}(M), E(k)). \end{aligned}$$

Here $E(k)$ denotes the injective hull of $\mathbb{C}$ and $H^{d-i}_{\mathfrak{m}}$ is the local cohomology with respect to $\mathfrak{m}$. For $(R, \mathfrak{m})$ with $\dim R = 0$ we have $E(k) \cong \omega_R$ ([Eis95, Prop. 21.1]). Since $\omega_R$ and $\operatorname{Ext}^i$ commute with completion in our setting one can easily deduce Proposition 2.19. Furthermore we see how Ext and local cohomology are related.

*Remark* 2.21. Additionally we can deduce the following alternative formula for computing the depth of a module over $R = (R, \mathfrak{m}, k)$ from Proposition 2.19:

$$\operatorname{depth} M = \dim R - \sup\{i \geq 0 \mid \operatorname{Ext}^i_R(M, \omega_R) \neq 0\},$$

which is used for example in [IW10].

### (s)MCM for torus invariant divisors

As before, let $\sigma \subseteq N_{\mathbb{Q}}$ be a full-dimensional, pointed cone and let $R = \mathbb{C}[\sigma^\vee \cap M]$ be the associated semigroup algebra. Let $D$ be a torus invariant Weil divisor on $\operatorname{Spec} R$, to be precise, denote by $D$ the corresponding fractional ideal $D \subseteq \mathbb{C}[M]$.

Since the annihilator $\operatorname{Ann}(D)$ of a torus invariant divisor $D$ is zero, we immediately have the equation

$$\dim(D) = \dim(R/\operatorname{Ann}(D)) = \dim(R)$$

and hence for $D$ being Cohen-Macaulay and maximal Cohen-Macaulay is the same.

A special class of divisors is MCM by default:

**Definition 2.22** ($\mathbb{Q}$-Cartier)**.** We call the divisor $D$ $\mathbb{Q}$-Cartier if and only if $P_D = v + \sigma^\vee$ for some $v \in M_\mathbb{Q}$.

This property implies being 'conic' in the sense of [BG09], though conic is weaker, as we are allowed to choose $v \in M_\mathbb{R}$ in the above definition. The classes of the $\mathbb{Q}$-Cartier divisors are exactly the torsion elements in $\operatorname{Cl}\operatorname{Spec}(R)$.

*Remark* 2.23. According to [BG09, Cor. 6.68] conic fractional ideals are automatically Cohen-Macaulay, i.e. a conic divisor $D$ is automatically MCM. The idea is to use Hochster's theorem (Theorem 2.25) and then display $D$ as a direct summand of a finitely generated CM $R$-module.

However the converse is not true. Thus, for $\sigma$ being simplicial all divisors will be conic. Hence, we cannot expect any two-dimensional examples of non-conic MCM divisors. Still [BG09, Remark after Thm 6.69, Fig. 6.5] gives an example of MCM non-conic divisors in higher dimension.

*Remark* 2.24. Another interesting non-trivial fact is [BG09, Cor 6.72], stating that the number of CM divisor classes is finite, and hence, the number of MCM divisor classes is finite as well. The proof relies on [BG09, Thm 6.71], stating that for given $m \in \mathbb{Z}_{\geq 0}$ there are only finitely many divisor classes with at most $m$ generators, and Serre's numerical Cohen-Macaulay criterion.

We want to demonstrate how the toric setting fits into the setting of Section 2.2. The first important thing here is Hochster's theorem:

**Theorem 2.25.** *[BG09, Thm 6.10] Let $S$ be a normal affine monoid. Then $\mathbb{C}[S]$ is Cohen-Macaulay for every field $\mathbb{C}$.*

In particular the $M$-graded $\mathbb{C}$-algebra $R = \mathbb{C}[\sigma^\vee \cap M]$ is Cohen-Macaulay. Since we assume $\sigma^\vee$ to be pointed and full-dimensional, $R$ has a unique homogeneous maximal ideal

$$\mathfrak{m} := \ (x^u \mid u \in \sigma^\vee \cap M \backslash \{0\}).$$

Following the thoughts of [BH98], the ring $R$ is closely related to a local ring, since it has a unique homogeneous maximal ideal $\mathfrak{m}$. Thus one can introduce the homogeneous equivalent of a local ring, namely $R$ is *local with *maximal ideal $\mathfrak{m}$. Furthermore the ideal generated by the interior $\operatorname{int}(\sigma^\vee) \cap M$ gives us the *canonical module $\omega_R$.

Localization at $\mathfrak{m}$ yields the ring $(R_\mathfrak{m}, \mathfrak{m})$ which is a local Cohen-Macaulay $\mathbb{C}$-algebra with canonical module $\omega_{R_\mathfrak{m}}$, and thus it fits into the setting of Proposition 2.19, as well as the localized toric divisors $D_\mathfrak{m}$.

*Remark* 2.26. Let $D$ be a toric divisor on $\operatorname{Spec} R$. Since $R$ is not local we say that $D$ is CM/MCM, if $D_\mathfrak{m}$ is CM/MCM. Usually one would have to check all localizations, but since $D$ is a $M$-graded module it is enough only to check $\mathfrak{m}$ ([BG09, Prop. 6.7]).

Of course, vanishing of $\operatorname{Ext}^i_R(D, D')$ implies vanishing of $\operatorname{Ext}^i_{R_\mathfrak{m}}(D_\mathfrak{m}, D'_\mathfrak{m})$, since $D$ and $D'$ are finitely generated and localization is a flat functor (see [Eis95, Prop 2.10]) and hence

$$(\operatorname{Ext}^i_R(D, D'))_\mathfrak{m} = \operatorname{Ext}^i_{R_\mathfrak{m}}(D_\mathfrak{m}, D'_\mathfrak{m}).$$

Conversely, at first glance, it is not necessarily true that $\operatorname{Ext}^i_R(D, D')$ vanishes if $\operatorname{Ext}^i_{R_\mathfrak{m}}(D_\mathfrak{m}, D'_\mathfrak{m}) = 0$.

However, since $D$ and $D'$ are $T$-invariant, $\operatorname{Ext}^i_R(D,D')$ is $M$-graded. Thus, the annihilator $\operatorname{Ann}(\operatorname{Ext}^i_R(D,D'))$ is a monomial ideal. In particular it will always be contained in $\mathfrak{m}$. Thus $\mathfrak{m}$ will always be in the support of $\operatorname{Ext}^i_R(D,D')$, unless it is zero. Therefore, $\operatorname{Ext}^i_R(D,D')=0$ is equivalent to $\operatorname{Ext}^i_{R_\mathfrak{m}}(D_\mathfrak{m},D'_\mathfrak{m})=0$.

Before we give our final proposition on Ext for an isolated singularity, let us make a few remarks on the completion $\hat{R}$ of $R$ with respect to $\mathfrak{m}$. Since $R$ is CM $\hat{R}$ is a complete local Cohen-Macaulay ring with maximal ideal $\hat{\mathfrak{m}}=\mathfrak{m}\otimes_R\hat{R}$ (see [BH98, Cor. 2.1.8, Thm. 2.1.9]). Since $\omega_R$ and $D$ are finitely generated we obtain their completed counterparts by tensoring with $\hat{R}$ over $R$ (see [AM69, Prop. 10.13] and [BH98, Thm. 3.3.5]). In particular $\omega_R\otimes_R\hat{R}$ is the canonical module of $\hat{R}$. Furthermore $\hat{R}$ is a flat $R$-module and this allows us to state the following lemma:

**Lemma 2.27.**
$$\operatorname{Ext}^i_{\hat{R}}(\hat{D},\hat{D}')=\operatorname{Ext}^i_R(D,D')\otimes_R\hat{R}.$$

*Proof.* Let
$$F_\bullet:\ \ldots\to R^{n_2}\to R^{n_1}\to R^{n_0}\twoheadrightarrow D\to 0$$
be a free resolution of $D$. As mentioned above, $\hat{R}$ is a flat $R$-module and thus
$$F_\bullet\otimes_R\hat{R}:\ \ldots\to\hat{R}^{n_2}\to\hat{R}^{n_1}\to\hat{R}^{n_0}\twoheadrightarrow\hat{D}\to 0$$
is a free resolution of $\hat{D}=D\otimes_R\hat{R}$ over $\hat{R}$. Now we apply $\operatorname{Hom}_{\hat{R}}(\bullet,\hat{D}')$ to this complex and take cohomology. The result is the left hand side of the equation. To link this with the right hand side we use [Eis95, Prop. 2.10]: For a ring $R$, $S$ a flat $R$-algebra and $M$, $N$ finitely generated $R$-modules we have
$$\operatorname{Hom}_R(M,N)\otimes_R S\ \cong\ \operatorname{Hom}_S(M\otimes_R S,N\otimes_R S).$$
Insert $S:=\hat{R}$ to obtain
$$\operatorname{Hom}_R(F_\bullet,D')\otimes_R\hat{R}\ \cong\ \operatorname{Hom}_{\hat{R}}(F_\bullet\otimes_R\hat{R},\hat{D}').$$
as complexes of $\hat{R}$-modules. Taking cohomology concludes the proof. $\square$

Let $\operatorname{Spec}R$ be an isolated singularity, i.e. assume that all proper faces of $\sigma$ are smooth. This allows us to identify Ext with its completed counterpart as $R$-modules:

**Proposition 2.28.** Let $\sigma$ be the cone of an isolated toric singularity. Let $R=\mathbb{C}[\sigma^\vee\cap M]$ and $\mathfrak{m}$ the unique homogeneous maximal ideal. Let $D$ and $D'$ be two torus invariant Weil divisors. Then, for $i>0$,

1. $\dim_k\operatorname{Ext}^i_R(D,D')<\infty$; and
2. $\operatorname{Ext}^i_R(D,D')=\operatorname{Ext}^i_{\hat{R}}(\hat{D},\hat{D}')$ as $R$-modules.

*Proof.* Localization in a maximal ideal $\mathfrak{m}'\in\operatorname{Spec}R$ is an exact functor and thus
$$(\operatorname{Ext}^i_R(D,D'))_{\mathfrak{m}'}=\operatorname{Ext}^i_{R_{\mathfrak{m}'}}(D_{\mathfrak{m}'},D'_{\mathfrak{m}'}).$$
Since $\sigma$ is an isolated singularity, we know that $R_{\mathfrak{m}'}$ is regular for all maximal ideals $\mathfrak{m}'\neq\mathfrak{m}$. Hence $D_{\mathfrak{m}'}\cong R_{\mathfrak{m}'}$ and
$$(\operatorname{Ext}^i_R(D,D'))_{\mathfrak{m}'}\cong\operatorname{Ext}^i_{R_{\mathfrak{m}'}}(R_{\mathfrak{m}'},R'_{\mathfrak{m}'})=0\quad\forall\mathfrak{m}'\neq\mathfrak{m}.$$

Thus $\operatorname{Ext}^i_R(D, D')$ is only supported in $\mathfrak{m}$ and the formula ([AM69, p46 19.])

$$\{\mathfrak{m}\} = \operatorname{Supp} \operatorname{Ext}^i_R(D, D') = V(\operatorname{Ann} \operatorname{Ext}^i_R(D, D'))$$

yields $\sqrt{\operatorname{Ann} \operatorname{Ext}^i_R(D, D')} = \mathfrak{m}$. Consequently there must be an integer $n > 0$ such that

$$\mathfrak{m}^n \subseteq \operatorname{Ann} \operatorname{Ext}^i_R(D, D'), \text{i.e.} \quad \mathfrak{m}^n \cdot \operatorname{Ext}^i_R(D, D') = 0.$$

This yields the first part of the proposition.

Denote by $A := \operatorname{Ext}^i_R(D, D')$ and use the first part to pick $n > 0$ such that $\mathfrak{m}^n \cdot \operatorname{Ext}^i_R(D, D') = 0$. Following the construction of the completion via projective limits as in [AM69] we build the chain

$$A \subseteq \mathfrak{m} \cdot A \subseteq \mathfrak{m}^2 \cdot A \subseteq \ldots \subseteq \mathfrak{m}^n \cdot A = 0 = \mathfrak{m}^{n+1} A = \ldots$$

and thus taking quotients yields

$$0 \leftarrow A/\mathfrak{m} \cdot A \leftarrow A/\mathfrak{m}^2 \cdot A \leftarrow \ldots \leftarrow A = A/\mathfrak{m}^n \cdot A = A = \ldots$$

Hence, the projective limit $\varprojlim_k A/\mathfrak{m}^k \cdot A$ must be $A$. $\square$

# Generalizing the Taylor resolution

The aim of this chapter is to generalize the construction of [Tay66] for monomial ideals in toric rings. We will develop the notation along the lines of [Eis95, p443, exercise 17.11.].

## 3.1. The generalized Taylor complex

Let $\sigma \subseteq N_{\mathbb{Q}}$ be a pointed full-dimensional cone. Let $R := \mathbb{C}[\sigma^\vee \cap M]$ be the associated semigroup ring. Let $D$ be a Weil divisor on $X = \mathrm{TV}(\sigma) = \operatorname{Spec} R$ and assume that $D = H^0(X, D)$ is an ideal of $R$. Then $D$ is generated by monomials, thus take

$$D = (x^{u^0}, x^{u^1}, \dots, x^{u^r}) \subseteq R$$

with the $u^i$ obtained by the method described in Proposition 2.6.

If we assume $\sigma^\vee \not\cong (\mathbb{Z}_{\geq 0})^d$, i.e. $\sigma$ is not smooth, then by the graded version of the Auslander-Buchsbaum-Serre theorem ([Eis95, Thm. 19.12]) there are ideals with infinite free resolutions. In fact most divisorial ideals $H^0(X, D)$ will have an infinite free resolution. The goal of this chapter is to give finite polyhedral data encoding these infinite free resolutions. In the case of cyclic quotient singularities this data will become a finite quiver, as we will discuss in Section 5.2. The vertices of this quiver correspond to the elements of the class group of the cyclic quotient singularity. For higher dimensions a quiver does not encode enough information and since there might be infinitely many divisor classes, we also lose finiteness again. Still we conjecture that for completely encoding a graded free resolution of a divisorial ideal one only needs finitely many divisor classes, see Conjecture 3.10.

Now remember that $R$ is $M$-graded, and that an element $u \in M$ allows to shift $R$ by $u$. We denote the divisorial ideal $R[u] \subseteq \mathbb{C}[M]$ as $(x^u)$. For $u \in \sigma^\vee \cap M$, then $(x^u) \subseteq R$ is even an honest ideal of $R$.

The first part of a free resolution of $D$ typically looks like

$$F_0(D) := \bigoplus_{i=0}^{r} R[u^i] \twoheadrightarrow D \to 0.$$

Now take $I \subseteq \{0, \dots, r\}$ and define

$$P_D^I := \bigcap_{i \in I} (u^i + \sigma^\vee) \text{ and } m_I := \bigcap_{i \in I} (x^{u^i}) = (x^u \mid u \in P_D^I \cap M).$$

Furthermore we set $P_D^{\emptyset} := P_D$. In the setting of $\sigma$ being smooth, $R$ is isomorphic to the polynomial ring in $d$ variables. Then $m_I$ becomes a principal ideal generated by

$\operatorname{lcm}\{x^{u^i} \mid i \in I\}$. For singular $\sigma$, $m_I$ does not have to be a principal ideal. One can think of $m_I$ as generalizing the concept of the least common multiple.

Next for each $k = 1, \ldots, r$ we define

$$S_k := \bigoplus_{I \subseteq \{0,\ldots,r\},\ \#I = k+1} m_I.$$

Every $S_k$ is a direct sum of homogeneous ideals. In particular, $S_k$ is an $M$-graded $R$-module corresponding to the set of polyhedra $\{P_D^I \mid \#I = k+1\}$. Denote by $x_I^u$, $\#I = k+1$ the element of degree $u \in M$ in the $I$-component of $S_k$, i.e. $x_I^u$ is $x^u$ in $m_I$. Assume $I = \{i_0, \ldots, i_k\}$ with ascending entries $i_0 < \ldots < i_k$. For $J \subseteq \{0, \ldots, r\}$, $\#J = k$, define

$$c_{I,J} := \begin{cases} 0 & J \not\subset I \\ (-1)^m & I = J \cup \{i_m\} \text{ for some } m. \end{cases}$$

Finally we define $M$-graded morphisms

$$d_k : S_k \to S_{k-1},\ x_I^u \mapsto \sum_J c_{I,J}\ x_J^u.$$

**Definition 3.1** (generalized Taylor complex)**.** The *generalized Taylor complex* of a Weil divisor $D$ is the sequence

$$S(D):\ 0 \to S_r \to S_{r-1} \to \ldots \to S_2 \to S_1 \twoheadrightarrow D \to 0.$$

Sometimes we will write $S_i(D)$ for the module $S_i$, if several $D$'s are being resolved simultaneously.

*Remark* 3.2. Of course the generalized Taylor complex exists for arbitrary monomial ideals. All ideals $m_I$ in the resolution come from Weil divisors. To see this directly take [BH98, p 315]. The toric way is to consider the support polyhedra: We get $\operatorname{Supp}(m_I)$ by intersecting the polyhedra $u^i + \sigma^\vee$ for $i \in I$. This intersection has the same facet vectors as $\sigma^\vee$ and hence it corresponds to a $T$-invariant Weil divisor.

This even holds if we resolve an arbitrary monomial ideal instead of a divisorial one: Only $S_0 = m_\emptyset$ will then not be divisorial and for computation of derived functors we forget exactly this part.

Some easy observations are:

1. The sequence $S(D)$ is exact. This can be proven along the lines of the proof of the usual Taylor resolution, by induction over the number of generators. Additionally to replacing the lcm, one needs to replace the expression $m_i/\gcd(m, m_i)$ by the quotient of ideals $(m_i : m)$, for $m_i$ and $m$ being monomial ideals. Furthermore, all monomials appearing should be considered as divisorial ideals. One then recognizes the Taylor complex as the mapping cylinder of a map of certain Taylor complexes with one generator less.

2. If $\sigma = \mathbb{Q}^n_{\geq 0}$ we obtain the traditional Taylor resolution.

3. The module $S_1$ is always a free $R$-module and we have $F_0(D) \cong S_1$.

4. The modules $S_k$ do not have to be free for $k > 1$.

Observation 4 demonstrates the main flaw of this construction. Of course we would like all $S_k$ to be free, but this already fails in the case of cyclic quotient singularities, as we will see later (Section 5.2): There the minimal free resolutions will be infinite, while $S(D)$ is finite.

Since $S_k$ is a direct sum of ideals $m_I$ which again come from Weil divisors $D_I$ one approach is to replace $m_I$ with $F_0(D_I)$, and subsequently to replace $S_k$ by $\oplus F_0(D_I)$. Then the problem becomes the map $d_{k+1} : S_{k+1} \to S_k$, which does not necessarily lift to a map $S_{k+1} \to \oplus F_0(D_I)$. i.e. $S_{k+1}$ might not be projective. There is one exception: We can always replace $S_r$, the last non-zero module, with its generalized Taylor resolution, since $S_{r+1} = 0$.

**Example 3.3.** Take $\sigma^\vee$ to be generated by a $2 \times 1$ rectangle at height 1, i.e.

$$R = \mathbb{C}[x^{[0,0,1]}, x^{[1,0,1]}, x^{[0,1,1]}, x^{[1,1,1]}, x^{[0,2,1]}, x^{[1,2,1]}].$$

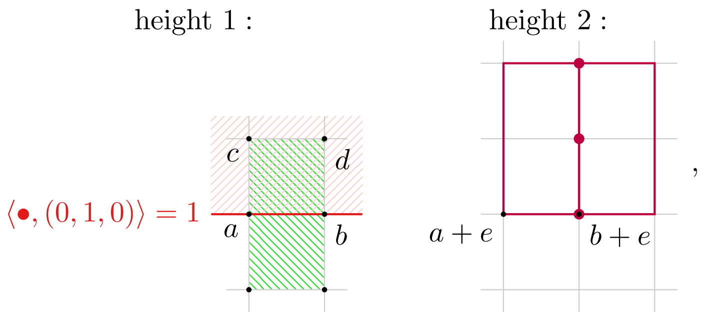
The generators are the lattice points inside the green rectangle in the height one picture below.

Take $D$ the Weil divisor such that $D = (x^a := x^{[0,1,1]}, x^b := x^{[1,1,1]}, x^c := x^{[0,2,1]}, x^d := x^{[1,2,1]})$. To illustrate this, we compute $\sigma$:

$$\sigma = \text{cone}\{(1,0,0), (0,1,0), (-1,0,1), (0,-1,2)\}.$$

Then $D$ corresponds to moving the hyperplane given by $(0,1,0)$ inside $\sigma$ by one, i.e. in the language of [CLS11] $D = -D_{(0,1,0)}$. This hyperplane is indicated by as a red line in the height one picture below, the halfspace $\langle \bullet, (0,1,0) \rangle > 1$ is the shaded area above this hyperplane.

The rectangle is normal, i.e. lattice points in multiples of the rectangle are sums of lattice points of the rectangle. Thus we can compute the intersection ideals by attaching the rectangle to each point and calculating the lattice points in the intersection of these rectangles, while carefully accounting for the height. Let us demonstrate this for $a$ and $b$, i.e. the first two generators of $D$:




where we denote by $e = [0,0,1]$. The points of the intersection yield that $m_{\{1,2\}} = (x^a) \cap (x^b) = (x^{[1,1,2]}, x^{[1,2,2]}, x^{[1,3,2]})$. Thus we can construct the map

$$\begin{array}{rcl} m_{\{1,2\}} & \to & (x^a_{\{1\}}) \oplus (x^b_{\{2\}}) \oplus (x^c_{\{3\}}) \oplus (x^d_{\{4\}}) = S_0(D) \\ x^u_{\{1,2\}} & \mapsto & x^u_{\{1\}} - x^u_{\{2\}}. \end{array}$$

Obviously the composition of this map with the surjection $(x^a_{\{1\}}) \oplus (x^b_{\{2\}}) \oplus (x^c_{\{3\}}) \oplus (x^d_{\{4\}}) \twoheadrightarrow D$ is zero.

In the same manner we can construct the other direct summands of $S_1(D)$ and build the map $d_1 : S_1(D) \to S_0(D)$ from its restriction to the direct summands of $S_1(D)$. Taking higher intersections gives the other parts of the generalized Taylor resolution $S(D)$.

One can imagine several approaches to utilize the above setting for algorithmically resolving $D$ freely up to any given (finite) point. Let us briefly discuss one possibility:

Assume $S_n$ to be a free module. We want to build a complex $S'(D)$ which is quasiisomorphic to $S(D)$, has $D$ at position $-1$ and a free module at position $n-1$. Unfortunately the new complex might not have a free module at position $n$ anymore. Let $S_{n-1} = \oplus_I D_I$. We take

$$S'(D)_i = \begin{cases} S(D)_i & i = 0, \dots, n-2 \\ \oplus_I S_0(D_I) & i = n-1 \\ S(D)_i \oplus_I S_{i-n+1}(D_I) & \text{else} \end{cases}.$$

As differentials $d'_i$ in $S'(D)$ we take the direct sums of the differentials in the respective complexes. Since $S_n$ is assumed to be free, we may lift $d_n : S_n \to S_{n-1}$ to a morphism $S_n \to \oplus_I S_0(D_I)$. For $d'_{n-1}$ we take the composition $\oplus_I S_0(D_I) \twoheadrightarrow S_{n-1} \to S_{n-2}$.

Starting with turning $S_r$ into a free module and then applying this principle, one can turn $S_1$ into a free module. Since the complex stays bounded at all times, we may now again start at its end and turn $S_2$ into a free module and so on. Repeatedly applying this principle, we can construct a free resolution up to any given (finite) length, with the drawback of this becoming huge and far from minimal.

*Remark* 3.4. One can get a little algorithmic speedup if one is just interested in the free resolution up to a certain position, lets say $n$. In that case, one cuts the resolution behind $n$, i.e. while applying the principle above, one would add

$$S''(D)_i = \begin{cases} S'(D)_i & i \leq n \\ 0 & \text{else} \end{cases}.$$

A maybe more promising approach is given in Section 3.2: Since the sequence $S(D)$ is not a short exact sequence, we cannot build the long exact sequence of cohomology to compute Ext. Instead we turn it into a spectral sequence, taking its Cartan-Eilenberg resolution. This allows some interesting statements about the support of $\mathrm{Ext}^1(D, D')$.

### Reduction and the Scarf complex

Combinatorially the generalized Taylor complex corresponds to mapping the vertices of the $r-1$-simplex onto the $r$ generators of $D$ and taking the lcm $m_I$ for faces $\mathrm{conv}(I) \subseteq \Delta_{r-1}$.

This approach is described in more detail in [BPS98] for the case of $R$ being the polynomial ring: For every simplicial complex with $r$ vertices one can build a sequence as in Definition 3.1. However, this sequence needs not be exact as for the full simplex. In [BPS98, Lemma 2.1] the authors state a criterion for a simplicial complex yielding an exact sequence for a given monomial ideal in the polynomial ring. This criterion can be simplified for the case of a simplicial tree as described in [Far14]. These criteria may be generalized to our setting of toric rings if one replaces the term "$m_I$ divides $m$" by $m_I \supseteq m$ for our generalized lcm. The subsequent paper [BS98] generalizes this approach for so-called monomial modules and for lattice ideals.

Thus, the topic of reducing the size of this complex is well-covered and we will only describe how to reduce the size of the complex $S(D)$ for a very specific case: Pick three generators $x^{u^1}$, $x^{u^2}$ and $x^{u^3}$ of $D$. Furthermore assume

$$m_{\{1,3\}} \subseteq m_{\{1,2\}} \cap m_{\{2,3\}} = m_{\{1,2,3\}}.$$

Then all relations of $x^{u^1}$ and $x^{u^3}$ decompose into relations of $x^{u^1}$ with $x^{u^2}$ and $x^{u^2}$ with $x^{u^3}$. Hence, one does not need $m_{\{1,3\}}$ as a direct summand of $S_2(D)$. This corresponds to removing the edge $\operatorname{conv}\{1,3\}$ from the $r$-simplex and consequently also dropping all faces of $\Delta_{r-1}$ that contained $\operatorname{conv}\{1,3\}$. We will omit the proof at this point since this case will be treated in greater detail in Theorem 5.8 for the two-dimensional case. Assuming the above equation everything can easily be generalized for higher dimensions.

*Remark* 3.5. Note that we only need one edge-path in the two-dimensional case, i.e. all $> 2$-dimensional faces disappear via Lemma 5.2. Thus $S_3 = 0$ and we obtain the exact sequence

$$0 \to S_2 \to F_0(D) \to D \to 0,$$

which is exactly the sequence described in Theorem 5.8.

We can apply the same principle to the so-called Scarf complex (see [MS05]). One obtains this complex by taking the subsets $I$ of $\{1, \dots, n\}$ such that the least common multiple is unique, i.e. if we have $m_I = m_J$, then $I = J$. By [MS05, Proposition 6.12] every free resolution contains the Scarf complex as a subcomplex, in the case of resolving a monomial ideal in a polynomial ring. This corresponds to the case of a smooth toric variety, where we replace $D$ by an arbitrary monomial ideal. Taking the generalized Taylor resolution, it will always contain the generalized version of the Scarf complex. In the two-dimensional case the Scarf complex is exactly the short exact sequence of the above remark.

## 3.2. Building a spectral sequence from $S(D)$

We follow the notation of spectral sequences found in [Wei94]. In the case of $S(D)$ being a short exact sequence, the following construction of a spectral sequence will result in a long exact sequence of cohomology, yielding a statement like Theorem 5.16 for this specific divisor $D$.

Let

$$0 \to S_n \to S_{n-1} \to \ldots \to S_1 \to S_0 \twoheadrightarrow D \to 0$$

be the generalized Taylor resolution of $D$. Take the projective Cartan Eilenberg resolution

$P_{\bullet,\bullet}$ of $S(D)$:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \cdots & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_{n,3} \arrow[r] \arrow[d] & P_{n-1,3} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_{3,3} \arrow[r] \arrow[d] & P_{2,3} \arrow[r] \arrow[d] & P_{1,3} \arrow[r] \arrow[d] & P_{0,3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_{n,2} \arrow[r] \arrow[d] & P_{n-1,2} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_{3,2} \arrow[r] \arrow[d] & P_{2,2} \arrow[r] \arrow[d] & P_{1,2} \arrow[r] \arrow[d] & P_{0,2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_{n,1} \arrow[r] \arrow[d] & P_{n-1,1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_{3,1} \arrow[r] \arrow[d] & P_{2,1} \arrow[r] \arrow[d] & P_{1,1} \arrow[r] \arrow[d] & P_{0,1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_{n,0} \arrow[r] \arrow[d] & P_{n-1,0} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_{3,0} \arrow[r] \arrow[d] & P_{2,0} \arrow[r] \arrow[d] & P_{1,0} \arrow[r] \arrow[d] & P_{0,0} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_n \arrow[r, hook] & S_{n-1} \arrow[r] & \cdots \arrow[r] & S_2 \arrow[r] & S_1 \arrow[r] & S_0 \arrow[r, two heads] & D \arrow[r] & 0
\end{tikzcd},$$

where we denote as $d^v$ the vertical differentials and $d^h$ the horizontal ones. In particular, the complex $P_{i,\bullet}$ is a projective resolution of $S_i$. Now we apply the functor $\mathrm{Hom}(\bullet, D')$ to $P_{\bullet,\bullet}$ with the vertical differentials and obtain a spectral cohomology sequence with

$$E_0^{p,q} := \mathrm{Hom}(P_{p,q}, D'),\ d := \mathrm{Hom}(d^v, D').$$

Taking cohomology we find the next layer modules to be

$$E_1^{p,q} = \mathrm{Ext}^i(P_{p,q}, D'),$$

giving rise to the following diagram:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \iddots \\
0 \arrow[r] & \mathrm{Ext}^3(D, D') \arrow[r] & 0 \arrow[r] & \mathrm{Ext}^3(S_1, D') \arrow[r] & \mathrm{Ext}^3(S_2, D') \arrow[r] & \cdots \\
0 \arrow[r] & \mathrm{Ext}^2(D, D') \arrow[r] & 0 \arrow[r] & \mathrm{Ext}^2(S_1, D') \arrow[r] & \mathrm{Ext}^2(S_2, D') \arrow[r] & \cdots \\
0 \arrow[r] & \mathrm{Ext}^1(D, D') \arrow[r] & 0 \arrow[r] & \mathrm{Ext}^1(S_1, D') \arrow[r] & \mathrm{Ext}^1(S_2, D') \arrow[r] & \cdots \\
0 \arrow[r] & \mathrm{Hom}(D, D') \arrow[r] & \mathrm{Hom}(S_0, D') \arrow[r] & \mathrm{Hom}(S_1, D') \arrow[r] & \mathrm{Hom}(S_2, D') \arrow[r] & \cdots
\end{tikzcd},$$

where we already included the information that $S_0$ is free. This spectral sequence converges to the hypercohomology of the complex $S(D)$. But $S(D)$ was exact and bounded which means that the hypercohomology is zero. With this information we go on to analyze the next layer:

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
0 \arrow[drr] & 0 \arrow[drr] & \operatorname{Ext}^3(D,D') \arrow[drr] & 0 \arrow[drr] & \ker d^1_{2,3} \arrow[drr] & E_2^{3,3} & \cdots \\
0 \arrow[drr] & 0 \arrow[drr] & \operatorname{Ext}^2(D,D') \arrow[drr] & 0 \arrow[drr] & \ker d^1_{2,2} \arrow[drr] & E_2^{3,2} & \cdots \\
0 \arrow[drr] & 0 \arrow[drr] & \operatorname{Ext}^1(D,D') \arrow[drr] & 0 \arrow[drr] & \ker d^1_{2,1} \arrow[drr] & E_2^{3,1} & \cdots \\
0 \arrow[drr] & 0 \arrow[drr] & E_2^{0,0} \arrow[drr] & E_2^{1,0} \arrow[drr] & E_2^{2,0} \arrow[drr] & E_2^{3,0} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{tikzcd}
$$

By vanishing of the hypercohomology we immediately obtain $E_2^{0,0} = 0$ and $E_2^{1,0} = 0$. The basis for computing $\operatorname{Ext}^1(D, D')$ though, is the following proposition:

**Proposition 3.6.** In the spectral sequence developed above we have

$$\operatorname{Ext}^1(D, D') \cong E_2^{2,0}.$$

*Proof.* Take the map

$$d_2^{0,1} : \operatorname{Ext}^1(D, D') \to E_2^{2,0}.$$

The next layer yields the equations

$$E_3^{0,1} = \ker d_2^{0,1} \text{ and } E_3^{2,0} = \operatorname{coker} d_2^{0,1}.$$

Since the spectral sequence lives in the first quadrant only, we obtain

$$E_3^{0,1} = E_4^{0,1} = \ldots = E_\infty^{0,1}$$

and the same holds for $E_3^{2,0}$. Thus vanishing of the hypercohomology yields $E_3^{0,1} = E_3^{2,0} = 0$. Hence $d_2^{0,1}$ must be an isomorphism. $\square$

*Remark* 3.7. In the case that $n = 2$ we get exactly the recursion described in Theorem 5.16.

We want to have a criterion for $\operatorname{Ext}^i(D, D')$ to vanish for $i > 0$. This can be derived using the spectral sequence of page 30. Fix a divisor $D$, then we have the generalized Taylor complex $S(D)$ of Remark 3.2. Now we collect all involved divisors in a set $R(D)$:

$$R(D) := \{H \mid \exists i : \ H \text{ is a direct summand of } S_i(D)\}.$$

We continue this recursively:

$$P(D) := R(D) \cup \bigcup_{H \in R(D)} P(H).$$

Having this we can state the following proposition:

**Proposition 3.8.** Assume $\mathrm{Ext}^1(H, D') = 0$ for all $H \in P(D)$. Then $\mathrm{Ext}^i(H, D') = 0$ for all $i > 0$ and all $H \in P(D)$.

*Proof.* We will prove this by induction on $i$. For $i = 1$, we have $\mathrm{Ext}^1(H, D') = 0$ for all $H \in P(D)$ by assumption.

Now assume that $\mathrm{Ext}^j(H, D') = 0$ for all $H \in P(D)$ and all $j \leq i - 1$. We can build the spectral sequence of 30 for each $H \in P(D)$. In the second layer we insert zero for all rows below $\mathbb{C}$. Considering the convergence of the spectral sequence this automatically yields $\mathrm{Ext}^i(H, D') = 0$ for all $H \in P(D)$ and we are done. $\square$

Of course we can restate the proposition for just one divisor.

**Corollary 3.9.** Assume $\mathrm{Ext}^1(H, D') = 0$ for all $H \in P(D)$. Then $\mathrm{Ext}^i(D, D') = 0$ for all $i > 0$.

In general $P(D)$ will be infinite, which can already be seen in the two-dimensional case. This reflects the fact that free resolutions become infinite. However $P(D)$ considered modulo linear equivalence may be finite:

*Conjecture* 3.10. The set

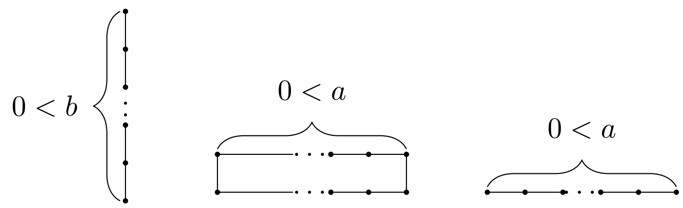
$$\bar{P}(D) = P(D)/\sim,$$

where $\sim$ denotes linear equivalence, is finite.

In Section A.2 we have some code snippets for testing whether for a given $D$ the set $\bar{P}(D)$ is finite.

The finiteness of $\bar{P}(D)$ in the two-dimensional case is reflected in the quiver of Definition 5.10. For $\sigma$ being simplicial the class group is finite and hence $\bar{P}(D)$. So far we do not know an example where $\bar{P}(D)$ has been proven infinite.

**Example 3.11.** Going back to Example 3.3, we notice that given any $D \not\sim 0$ the convex hull of the generators $\mathrm{conv}(G(D)) \subseteq M_{\mathbb{Q}}$ always assumes one of the following three shapes:




Considering the intersections for building the generalized Taylor resolution of Definition 3.1, one gets a restriction on the size (i.e. length and width) of the involved divisors. Thus we can deduce that in this case $\bar{P}(D)$ is finite for any $D$. This is noteworthy, since the cone over the rectangle yields a toric variety with infinite class group. Please also see subsection A.2.1 and the picture on the titlepage for the divisor generated by a square at height one.

### The support of $\mathrm{Ext}^1$

We can now use Proposition 3.6 in order to compute the support of $\mathrm{Ext}^1(D, D')$. Let $D$ be generated by

$$D = (x^{u^1}, \dots, x^{u^n}) \subseteq \mathbb{C}[M]$$

and denote by $(\bullet)^* := \mathrm{Hom}(\bullet, D')$. By Remark 3.2 we have the exact sequence

$$\bigoplus_{1=i<j<k}^{n} D_{ijk} \to \bigoplus_{1=i<j}^{n} D_{ij} \to \bigoplus_{i=1}^{n} D_i,$$

where we set

$$D_i := (x^{u^i}) \subseteq \mathbb{C}[M] \text{ and } D_I := \bigcap_{i \in I} D_i.$$

Applying $(\bullet)^*$ to this sequence we get

$$\bigoplus_{1=i<j<k}^{n} D^*_{ijk} \leftarrow \bigoplus_{1=i<j}^{n} D^*_{ij} \leftarrow \bigoplus_{i=1}^{n} D^*_i \tag{3.1}$$

and by Proposition 3.6 the cohomology in the middle is exactly $\mathrm{Ext}^1(D, D')$. Now as polytopes we have

$$D^*_I = \{x \in M_{\mathbb{Q}} \mid x + D_I \in D'\}.$$

We will use this description together with the sequence above to derive some facts about the support of $\mathrm{Ext}^1(D, D')$. Let us assume that $n \geq 2$, since for $n = 1$ we have $D \cong R$ as $R$-module and thus $\mathrm{Ext}^1(D, D') = 0$.

For $n = 2$ we only have to consider the three ideals $D_1$, $D_2$ and $D_{12}$. Writing down the sequence of Equation 3.1 we obtain for the support

$$\mathrm{Supp}\,\mathrm{Ext}^1(D, D') = \mathrm{Supp}\, D^*_{12} \setminus (\mathrm{Supp}\, D^*_1 \cup \mathrm{Supp}\, D^*_2).$$

We now proceed by constructing the sequence of Equation 3.1 inductively, i.e. we assume we have the sequence for $n-1$ generators and show what happens when adding an $n$-th generator. Define the following matrices

$$A_n := \left(\begin{array}{c|c} A_{n-1} & 0 \\ \hline & -1 \\ Id & \vdots \\ & -1 \end{array}\right) \text{ and } B_n := \left(\begin{array}{c|c} B_{n-1} & 0 \\ \hline Id & -A_{n-1} \end{array}\right),$$

where we set

$$A_2 := (1 \ -1) \text{ and } B_3 = (1 \ -1 \ 1).$$

Then the sequence of Equation 3.1 can be viewed as

$$\begin{pmatrix} \bigoplus_{1=i<j<k}^{n-1} D^*_{ijk} \\ \oplus \\ \bigoplus_{1=i<j}^{n-1} D^*_{ijn} \end{pmatrix} \xleftarrow{B_n} \begin{pmatrix} \bigoplus_{1=i<j}^{n-1} D^*_{ij} \\ \oplus \\ \bigoplus_{i=1}^{n-1} D^*_{in} \end{pmatrix} \xleftarrow{A_n} \begin{pmatrix} \bigoplus_{i=1}^{n-1} D^*_i \\ \oplus \\ D^*_n \end{pmatrix}$$

**Definition 3.12.** Let
$$I \subseteq \{(i,j) \mid 1 \leq i < j \leq n\}$$
be a subset of the edges of the simplex $\Delta_{n-1}$. Then we define $B_n^I$ to be the matrix with columns
$$\mathrm{col}_{(ij)}(B_n^I) = \begin{cases} \mathrm{col}_{(ij)}(B) & (ij) \in I \\ 0 & \text{else} \end{cases}.$$

**Lemma 3.13.** Assume that we can decompose $I \subseteq \Delta_{n-1}[1]$ into $I = J \cup J'$ such that
$$\mathrm{conv}(J) \cap \mathrm{conv}(J') = \emptyset.$$
Then we have
$$\mathrm{rk}(B_n^I) = \mathrm{rk}(B_n^J) + \mathrm{rk}(B_n^{J'}).$$

*Proof.* The condition on $J$ and $J'$ yields that there is no 2-face of $\Delta_{n-1}$ containing an edge of both $J$ and $J'$: Since a 2-face is a triangle it has to contain an edge from the $\mathrm{conv}(J)$ to $\mathrm{conv}(J')$ and vice versa. This shows that two of the three edges cannot be contained in either of the two polytopes. This results in the non-zero rows of $B_n^J$ being disjoint to the non-zero rows of $B_n^{J'}$. The non-zero columns were already disjoint by default, since $J \cap J' = \emptyset$. This results in a decomposition of $B_n^I$ into two blocks of the respective non-zero parts, and hence we are done. □

**Lemma 3.14.** Let $I \subseteq \Delta_{n-1}[2]$ such that $\dim \mathrm{conv}(I) < n$, i.e. $\mathrm{conv}(I)$ is strictly contained in a face of $\Delta_{n-1}$. Then $B_n^I$ has rank $\#I$.

*Proof.* Assume that $\mathrm{conv}(I)$ is contained in the facet of $\Delta_{n-1}$ opposite of the vertex $n$. Then the only non-zero columns of $B_n^I$ come from the first $\binom{n}{2}$ columns of $B_n$. This is exactly the part of $B_n$ with $Id$ at the bottom. This yields the desired result for $\mathrm{conv}(I)$ being contained in the facet opposite of the vertex $n$. Considering permutations of the vertices results in the lemma. □

Now we can derive the final theorem on the support of $\mathrm{Ext}^1_R(D, D')$:

**Theorem 3.15.** *Take $u \in M$ and let*
$$I := \{(i,j) \in \Delta_{n-1}[2] \mid u \in \mathrm{Supp}(D_{ij}^*)\}.$$
*Then either of the three conditions implies that $u \notin \mathrm{Supp}(\mathrm{Ext}^1_R(D, D'))$:*

1. $\mathrm{conv}\, I \neq \Delta_{n-1}$.
2. $I = J \cup J'$ *such that* $\mathrm{conv}(J) \cap \mathrm{conv}(J') = \emptyset$ *with* $J, J' \neq \emptyset$.
3. $u \in \mathrm{Supp}(D_i^*)$ *for all* $i = 1, \ldots, n$.

*Proof.* The assumption of 1 yields that $\mathrm{conv}\, I$ is contained in a proper face of $\Delta_{n-1}$. Thus we use Lemma 3.14 and obtain that the rank of $B_n^I$ is exactly $\#I$. But this is also the number of direct summands that have $u$ in their support. Hence the number of independent equations and the number of variables is the same.

The second claim can be derived from the first, considering $J$ and $J'$ separately, by using Lemma 3.13. The assumption of 2 now says that both $\operatorname{conv}(J)$ and $\operatorname{conv}(J')$ are contained in proper faces of $\Delta_{n-1}$. Thus we are in the setting of 1.

For an element $u \in M$ as in 3 we consider the matrices $A_n$ and $B_n$ as maps between $\mathbb{C}$-vector spaces. Since taking the dual of a sequence of finite dimensional vector spaces is exact, we know that the degree $u$ cohomology vanishes. □

We can derive a corollary and a remark from this theorem. Take again $u \in M$ and let $I$ be as in the theorem. Then $I$ is also a subset of the edges of the complete graph on $n$ points.

**Corollary 3.16.** Assume that $u \in \operatorname{Supp}(\operatorname{Ext}^1_R(D, D'))$. Then the set of edges $I$ is connected and contains a spanning tree.

*Proof.* This is an easy consequence of 1 and 2 of the theorem. □

The remark verifies that $\operatorname{Ext}^1_R(D, D')$ is concentrated in the general orbit:

*Remark* 3.17. Let $s \in \operatorname{Ext}^1_R(D, D')$ be a homogeneous element of degree $u \in M$. Then there exists $v \in \operatorname{int}(\sigma^\vee) \cap M$ such that $x^v \cdot s = 0$.

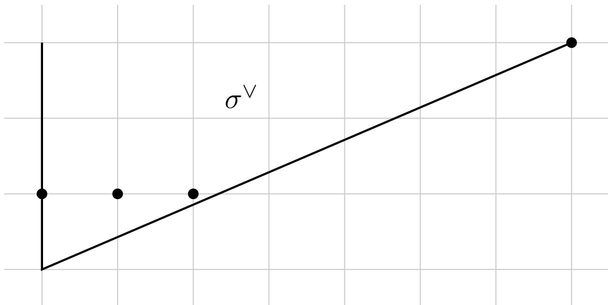
One just needs to choose $v$ such that $u + v \in \operatorname{Supp}(D_i^*)$ for all $i = 1, \ldots, n$. This is possible since $\sigma^\vee$ is full-dimensional and all $\operatorname{Supp}(D_i^*)$ have $\sigma^\vee$ as their tailcone. Then use 3 of the theorem.

# Cyclic quotient singularities

## 4.1. Toric construction

Fix two integers $n$ and $q$, such that $\gcd(n, q) = 1$ and $0 < q < n$. Let $\sigma = \langle \rho^0, \rho^1 \rangle$ be the cone belonging to $X = Y_{n,q}$, where $\rho^0 = (1, 0)$ and $\rho^1 = (-q, n)$. Take the coordinate ring $R = R(n, q) = \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2]$ of $X$. Then $R$ is generated as an $\mathbb{C}$-algebra by the elements $\{x^h \mid h \in H\}$, where $H$ denotes the Hilbert basis of $\sigma^\vee$.

**Example 4.1.** Let $n = 7$ and $q = 3$. Then the Hilbert basis of the cone $\sigma^\vee$ has four elements, indicated by the dots in the picture.




Thus $R(7, 3) = \mathbb{C}[x^{[0,1]}, x^{[1,1]}, x^{[2,1]}, x^{[7,3]}]$ or, if we label the axes $x$ and $y$, $R(7, 3) = \mathbb{C}[y, xy, x^2y, x^7y^3]$.

### Continued fractions

In the two-dimensional case a Hilbert basis of $\sigma^\vee$ can easily be obtained, using continued fractions:

**Definition 4.2** (Continued fraction expansion)**.** The *continued fraction expansion* of a rational number $\frac{p}{q} > 0$, $p, q \in \mathbb{Z}_{>0}$, is a sequence $\underline{a} = [a_1, \ldots, a_s]$ of integers $a_i \geq 2$ such that

$$\frac{p}{q} = a_1 - \frac{1}{[a_2, \ldots, a_s]}, \text{ where } [a_1] := a_1.$$

Let $\underline{a} := [a_1, \ldots, a_s]$ be the continued fraction expansion of $\frac{n}{n-q}$, and let $H = \{b^0 = [0, 1], b^1, \ldots, b^{s+1} = [n, q]\}$ be the Hilbert basis of $\sigma^\vee$ ordered by ascending first coordinate. Then we have the equations

$$b^{i-1} + b^{i+1} = a_i \cdot b^i \quad \forall i = 1, \ldots, s.$$

Equivalently the continued fraction expansion of $\frac{n}{q}$ corresponds to the Hilbert basis of $\sigma$.

**Example 4.3.** In the example we have

$$\frac{7}{7-3} = [2,4] \text{ and } \frac{7}{3} = [3,2,2].$$

For $\sigma^\vee$ we obtain the following equations:

$$[0,1] + [2,1] = 2 \cdot [1,1] \text{ and } [1,1] + [7,3] = 4 \cdot [2,1].$$

*Remark* 4.4. The relationship of continued fractions and cyclic quotient singularities has already been thoroughly studied: Stevens ([Ste91]) and Christophersen ([Chr91]) discovered their relationship with the versal deformation of a cyclic quotient singularity. In [Alt98] Altmann relates so-called P-resolutions to cyclic quotient singularities and [Ilt08] uses these observations to calculate Milnor numbers. Thus for proofs of the above claims we refer to these discussions. Of course the strong relationship suggests that continued fractions play an important role for computing Ext as well.

### The class group of a CQS

Having fixed the rays $\rho^0 = (1,0)$ and $\rho^1 = (-q,n)$ of $\sigma$, we can now explicitly write down the class group sequence [CLS11, Thm. 4.1.3]:

$$0 \longrightarrow M \xrightarrow{A := \begin{pmatrix} 1 & 0 \\ -q & n \end{pmatrix}} \mathbb{Z}^2 \longrightarrow \operatorname{Cl} X \longrightarrow 0 \tag{4.1}$$

Hence we obtain that $\operatorname{Cl}(X) \cong \mathbb{Z}/n\mathbb{Z}$. Additionally we see that the images of $(-j,0)$, $j = 0,\dots,n-1$, form a system of representatives of the class group $\operatorname{Cl}(X)$. We will denote the corresponding divisors by $E^j = (-j,0)$. The advantage of this set of generators is that $E^j \subseteq \sigma^\vee$, in particular $H^0(X, E^j)$ is an ideal of $R$. Using Proposition 2.6 we can compute the generators $G(E^j)$.

*Remark* 4.5. Just a reminder to be careful in this case: Since $D$ can be expressed as an element of $\mathbb{Z}^2$ it is tempting just to add an element $u \in M = \mathbb{Z}^2$ to it coordinate wise. This is wrong. The correct way is to take for the divisor $D - Au$, for the module $D[u]$ and for the polyhedron $D + u$. Most of the time we will use the last two notations, so keep in mind, never to add $D$ and $u$ as vectors, but instead add the polyhedron $D$ and the point $u \in \mathbb{Z}^2$ by Minkowski summation. To stress this we will denote by $[a,b]$ elements of $M = \mathbb{Z}^2$ and divisors $D$ as $(a,b)$.

The two-dimensional case yields another advantage:

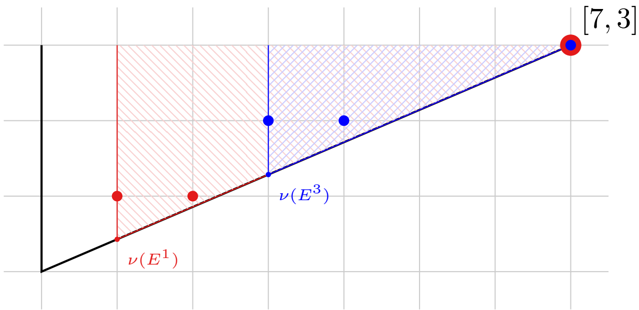
**Definition 4.6** (vertex of $D$)**.** Let $D$ be a $T$-invariant Weil divisor, then $D$ is $\mathbb{Q}$-Cartier which means that we can write the polyhedron $D$ as $\nu(D) + \sigma^\vee$ with $\nu(D) \in \mathbb{Q}^2$. The vector $\nu(D)$ is called the *vertex of* $D$.

Remember that $\mathbb{Q}$-Cartier implies conic in the sense of [BG09], and essentially yields that such a divisor is MCM. It also simplifies Minkowski summation as in $D + u$:

$$D + u = \nu(D) + \sigma^\vee + u = (\nu(D) + u) + \sigma^\vee.$$

This observation will be used implicitly throughout Section 5.1 and following chapters, since it gives rise to an easy method of determining which $E^i$ is linearly equivalent to a given $D$.

**Example 4.7.** We will consider the divisors $E^1$ and $E^3$. First we draw the corresponding polyhedra of global sections.




Then we compute the generators of $H^0(X, E^1)$ and $H^0(X, E^3)$. They are denoted as dots of the respective colors. Both modules are generated by 3 elements. To compute the different Ext-modules $\mathrm{Ext}^i_R(E^3, E^1)$ we will either have to find an injective resolution of $H^0(X, E^1)$ or an projective resolution $H^0(X, E^3)$. We choose the second possibility, since there is a straight forward way to build a free resolution of such a module that also provides several other insights.

### Generators and Inequalities

Using Proposition 2.6 we can calculate a generating system $G$ for $D$ as an $R$-module for $D = a \cdot [\rho^0] + b \cdot [\rho^1] = (a, b)$ an arbitrary Weil divisor. Since $\sigma$ is pointed $G(D) = \{x^{u^0}, \ldots, x^{u^r}\}$ is unique. Furthermore we assume that $G(D)$ is ordered with respect to the first coordinate of the $u^i$, i.e. from the left to the right. Since $[0, 1] \in \sigma^\vee$ it is safe to say that all $u^i$ have different first coordinate, justifying this assumption.

Thus we obtain:

**Proposition 4.8.** Let $G(D)$ be the generating set of $D$ ordered as above, i.e. we have $u^i = [a_i, b_i]$ and

$$a_0 < a_1 < \ldots < a_r.$$

Then we obtain

$$\langle \rho^0, u^0 \rangle < \langle \rho^0, u^1 \rangle < \ldots < \langle \rho^0, u^r \rangle$$

and

$$\langle \rho^1, u^r \rangle < \langle \rho^1, u^{r-1} \rangle < \ldots < \langle \rho^1, u^0 \rangle,$$

where $\rho^0 = (1, 0)$ and $\rho^1 = (-q, n)$ are the primitive ray generators of $\sigma$.

*Proof.* By Proposition 2.6 we know $u^i \notin u^j + \sigma^\vee$ for all $i \neq j$. Since we sorted the $u^i$ by ascending first coordinate the first chain of inequalities is trivial.

For the second chain assume that $\langle \rho^1, u^0 \rangle \leq \langle \rho^1, u^1 \rangle$. This immediately implies $u^1 \in u^0 + \sigma^\vee$, a contradiction. The other inequalities can be obtained in the same manner. $\square$

### The canonical divisor

Let $D = a[\rho^0] + b[\rho^1]$, $a, b \in \mathbb{Z}$ be a torus invariant Weil divisor. Proposition 4.8 yields a handy way for to find $i$ such that $D$ and $E^i$ are linearly equivalent. First we observe that it is enough to know the first and the last generator $u^0$ and $u^r$ to uniquely determine the polyhedron of global sections. Second observation is that there is only one $E^i$ with same distance between first and last generator. Hence we obtain

$$D = E^{n-a_r-a_0} + ([n, q] - u^r).$$

In particular we can explicitly compute the vertex of the canonical divisor $K_X := -[\rho^0] - [\rho^1]$.

*Remark* 4.9. By the inequalities of Proposition 4.8 we immediately see that $b^1 = [1, 1]$ and $b^s := [a, b]$ are the first and last generator of the ideal corresponding to $K_X$. Thus we have

$$\nu(K_X) = [1, \frac{q+1}{n}].$$

One way to see this is to use the hyperplanes of $\sigma^\vee$:

$$\langle \nu(K_X), \rho^0 \rangle = 1 = \langle b^1, \rho^0 \rangle.$$

By construction we have the equation $\langle [a, b], (-q, n) \rangle = -qa + bn = 1$ and we get for $\rho^1$:

$$\langle \nu(K_X), \rho^1 \rangle = -q + n \cdot \frac{q+1}{n} = 1 = \langle b^s, \rho^1 \rangle.$$

## 4.2. Invariant construction

Another viewpoint on the cyclic quotient singularities $Y_{n,q}$ is as quotient of $\mathbb{C}^2$ by the group

$$G := \langle \begin{pmatrix} \xi_n & 0 \\ 0 & \xi_n^q \end{pmatrix} \rangle,$$

with $\xi_n$ being an $n$-th root of unity. It is well-known ([Ful93]) that taking invariants we get

$$\mathbb{C}[x, y]^G = \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2].$$

*Remark* 4.10. The articles [Nak13] and [Wun87] work with the completed version. Since $G$ is a finite reductive subgroup of $\mathrm{GL}(2, \mathbb{C})$, taking $G$-invariants becomes an exact functor and we get

$$\mathbb{C}[[x, y]]^G = (\mathbb{C}[x, y]^G)^\wedge = \mathbb{C}[[\sigma^\vee \cap \mathbb{Z}^2]]$$

if we take the completion in the *maximal ideal $\mathfrak{m}$.

The discussion of Section 2.2 now shows that computing Ext on the completed rings is the same as working with the toric rings.

# Resolving torus invariant divisors on CQS

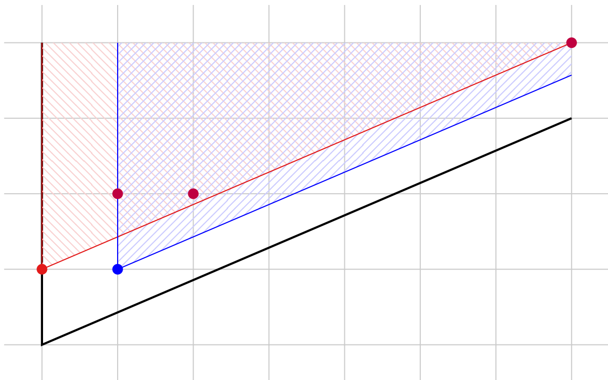
We stay in the two-dimensional setting: Let $0 < q < n$ be two coprime integers and let $\sigma \subseteq \mathbb{Q}^2$ be the cone generated by the rays through $(1,0)$ and $(-q,n)$. Then we denote by $R$ the ring $R := \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2]$ and by $X$ the associated toric variety $X = \operatorname{Spec} R = \operatorname{TV}(\sigma)$.

## 5.1. A short exact sequence

Let $D = a \cdot [\rho^0] + b \cdot [\rho^1]$, $a, b \in \mathbb{Z}$ be an arbitrary Weil divisor on $X$. In this section we will apply the construction of the generalized Taylor resolution developed in Section 3.1. This will result in a resolution of $D$, which is *minimal, i.e. for the differentials $d_i : F_i \to F_{i-1}$ we have $d_i(F_i) \subseteq \mathfrak{m}F_{i-1}$. If we localize at $\mathfrak{m}$, this resolution becomes minimal in the usual sense.

Let us start with an example demonstrating the main point of the generalized Taylor resolution, namely the generalized lcm:

**Example 5.1.** We stay in the setting of $n = 7$ and $q = 3$. We want to consider the elements $y$ and $xy$ in $\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2]$. The intersection of the two principal ideals generated by these elements is generated by the monomials corresponding to the purple dots in the picture.



But considering the lcm as the generators of the intersection ideal also solves our problem as we will see in a moment. The syzygies of homogeneous elements can be decomposed into homogeneous summands. Hence, it is enough to give a generating set for the homogeneous relations.

Another key observation is that it is enough to consider relations for neighbouring elements of the generating set $G$.

**Lemma 5.2.** Let $G = \{x^{u^i} \mid i = 0, \dots, r\}$ generate the module $D$. Let $\underline{a} \in R^{r+1}$ be a syzygy, i.e.

$$\sum_{i=0}^{r} a_i x^{u^i} = 0.$$

We call the vector $(a_0, a_1, \dots, a_r) \in \mathbb{C}^{r+1}$ the *coefficient vector* of $\underline{a}$. Then $\underline{a}$ can be decomposed into a sum of homogeneous elements of $R^{r+1}$ whose coefficient vectors are exactly $(e^i - e^{i+1})$, $i = 0, \dots, r-1$, i.e. the coefficient vector is $(0, \dots, 0, 1, -1, 0, \dots, 0)$.

*Proof.* Recall that $R^{r+1}$ is $M$-graded via $\deg e^i = u^i$. Since the $x^{u^i}$ are homogeneous of $\deg x^{u^i} = u^i$ $\underline{a}$ can be decomposed into its homogeneous parts, thus without loss of generality we assume that $\underline{a}$ is homogeneous. Hence, let $\deg \underline{a} = u \in M$. We will explicitly decompose a homogeneous relation of three elements, relations of more elements can then be decomposed inductively with the same principle. Thus, let

$$\underline{a} = a \cdot e^0 + b \cdot e^1 + c \cdot e^2, \text{ with } a, b, c \in R \backslash \{0\}, \text{ i.e. } a \cdot x^{u^0} + b \cdot x^{u^1} + c \cdot x^{u^2} = 0.$$

Each summand is homogeneous of degree $u$, thus we can determine the degrees of $a$, $b$ and $c$ and since there is exactly one monomial of each degree in $R$. For example we have $\deg a x^{u^0} = u$ and thus $\deg a = u - u^0$. Therefore $a$ must be a multiple of $x^{u-u^0}$.

We can now write

$$\underline{a} = a' \cdot x^{u-u^0} \cdot e^0 + b' \cdot x^{u-u^1} \cdot e^1 + c' \cdot x^{u-u^2} \cdot e^2, \text{ with } a', b', c' \in k \backslash \{0\}.$$

Now we want to cut off $a' \cdot x^{u-u^0} \cdot e^0$, thus we just subtract the relation $a' \cdot x^{u-u^0} \cdot e^0 - a' \cdot x^{u-u^1} \cdot e^1$. Using this principle we can decompose nearly all relations into the conjectured elements, except relations with zero entries between non-zero entries. In the above case this corresponds to $b' = 0$ and the problem becomes to determine whether $u - u^1$ still corresponds to a monomial of $R$.

Thus, assume

$$\underline{a} = a' \cdot x^{u-u^0} \cdot e^0 + c' \cdot x^{u-u^2} \cdot e^2, \text{ with } a', c' \in k \backslash \{0\}.$$

First we show that $u \in u^1 + \sigma^\vee$. We already know that $u \in (u^0 + \sigma^\vee) \cap (u^2 + \sigma^\vee)$. By Proposition 4.8 we obtain the following chains of inequalities

$$\langle \rho^0, u^0 \rangle < \langle \rho^0, u^1 \rangle < \langle \rho^0, u^2 \rangle < \langle \rho^0, u \rangle$$

since $u \in u^2 + \sigma^\vee$ and

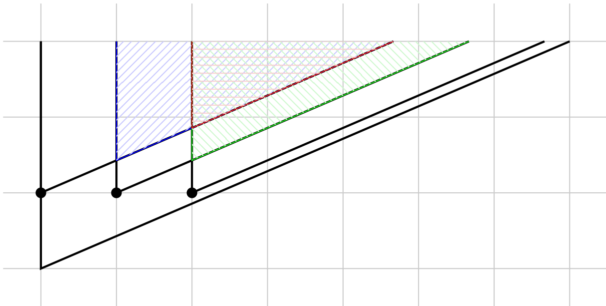
$$\langle \rho^1, u^2 \rangle < \langle \rho^1, u^1 \rangle < \langle \rho^1, u^0 \rangle < \langle \rho^1, u \rangle$$

since $u \in u^0 + \sigma^\vee$. This immediately yields $u \in u^1 + \sigma^\vee$ and thus the monomial $x^{u-u^1}$ exists in $R$. Therefore we may subtract the relation $a' \cdot x^{u-u^0} \cdot e^0 - a' \cdot x^{u-u^1} \cdot e^1$ from $\underline{a}$, thus closing the gap.

Using the above principles one can decompose any relation. □

*Remark* 5.3. This lemma relies heavily on the observation of Proposition 4.8, i.e. that we can sort the generators of our ideal/module. Thus being in dimension 2 is crucial.

**Example 5.4.** We illustrate the above proof with a picture. A relation (or rather its degree) of the three elements $y$, $xy$ and $x^2y$ has to lie in the red cone and thus also lies in the green cone and the blue cone. Hence, it can be decomposed into a sum of two relations of the desired form.



This picture fails in higher dimensions, take for example the divisor of Example 3.3.

Finally we state how to explicitly compute the relations for two elements.

Let $x^u$ and $x^v$ with $u, v \in \sigma^\vee \cap M$. Then their homogeneous relations correspond exactly to the lattice points of the polyhedron $Q := (u + \sigma^\vee) \cap (v + \sigma^\vee)$. Using the observations of subsubsection 4.1 we want to recognize $Q$ as some shifted $E^j$. Of course, this is only a technical detail, but for later algorithms it will obliterate several computations of intersections of polyhedra and lead to a significant speed-up. Remember that

$$E^j = \sigma^\vee \cap ([j, 0] + \sigma^\vee) = [j, \frac{jq}{n}] + \sigma^\vee, \text{ i.e. } \nu(E^j) = [j, \frac{jq}{n}].$$

Hence, to find $j$ for $Q$, only the distance of the $x$-coordinates of $u$ and $v$ matters:

**Proposition 5.5.** Let $x^u, x^v \in R$, with $u = [u_0, u_1]$ and $v = [v_0, v_1]$ such that $0 < v_0 - u_0 < n$ and $v \notin u + \sigma^\vee$. In the setting of the above lemma we obtain

$$Q = u + E^{v_0 - u_0} = E^{v_0 - u_0}[u].$$

Thus the syzygy module of $x^u$ and $x^v$ is equal to $H^0(X, E^{v_0 - u_0}[u])$ as an $M$-graded module.

*Proof.* We can describe $Q$ as

$$Q = \{w \in M \mid \langle x, w\rangle \geq \langle x, u\rangle \text{ and } \langle x, w\rangle \geq \langle x, v\rangle \ \forall x \in \sigma\}.$$

Using the assumption on $u$ and $v$ we can reduce this to

$$\begin{aligned} Q &= \{w \in M \mid \langle \rho^0, w\rangle \geq \langle \rho^0, v\rangle \text{ and } \langle \rho^1, w\rangle \geq \langle \rho^1, u\rangle\} \\ &= \{[w_0, w_1] \in M \mid w_0 \geq v_0 \text{ and } -qw_0 + nw_1 \geq -qu_0 + nu_1\}. \end{aligned}$$

On the other hand we have

$$E^{v_0 - u_0} = \{[w_0, w_1] \in M \mid w_0 \geq v_0 - u_0 \text{ and } -qw_0 + nw_1 \geq 0\}.$$

Adding $[u_0, u_1]$ to all points of this polyhedron obviously yields the desired equation. □

*Remark* 5.6. Of course one can generalize the above proposition to situations where $v_0 - u_0 > n$ by computing modulo $n$ at the appropriate places. However the sorted generators of $D$ satisfy this condition.

Now we join all the above preliminaries into one theorem on how to build a free resolution of $D$.

Let $G := \{x^{u^0}, \ldots, x^{u^r}\} \subseteq D$ be the minimal sorted homogeneous generating set of $D$ as an $R$-module. Let $u^i = [a_i, b_i]$. Then we have $0 < a_j - a_i < n$ for all $i < j$, since a further distance of two neighbouring generators would violate the convexity of $D$.

Hence we can describe the syzygies of neighbouring generators in the following way:

**Definition 5.7.** Let $G := G(D)$ denote the generators of $D$. Define by

$$F_0(D) := R^G := \bigoplus_{i=0}^{r} R[u^i]$$

the first module in the graded free resolution of $D$. Furthermore for $i = 1, \ldots, r$ define a map

$$E^{a_i - a_{i-1}}[u^{i-1}] \to F_0(D),\ x^u \mapsto x^{u-u^{i-1}} e^{i-1} - x^{u-u^i} e^i.$$

By construction this morphism is homogeneous of degree $0 \in M$.

The next natural step is now to take the direct sum of all these maps into $F_0(D)$. This yields a short exact sequence giving us complete understanding and control over the free resolutions of any $D$.

**Theorem 5.8.** *For $D$ as above we have the short exact sequence*

$$S(D):\ 0 \to \bigoplus_{i=1}^{r} E^{a_i - a_{i-1}}[u^{i-1}] \to F_0(D) \to D \to 0.$$

*Proof.* The proof serves as a nice summary of the above results: First we obtain by Lemma 5.2 that it is enough to consider two-term relations of neighbouring elements $x^{u^i}$ and $x^{u^{i-1}}$ for determining the syzygies of $D$. Proposition 5.5 describes the syzygies of the elements $x^{u^i}$ and $x^{u^{i-1}}$ as

$$\operatorname{lcm}(x^{u^i}, x^{u^{i-1}}) = E^{a_i - a_{i-1}}[u^{i-1}].$$

Thus we have proven that the first map is surjective onto the kernel of the second map.

Injectivity is a consequence of Lemma 5.2, since an element $x^u$ of $\operatorname{lcm}(x^{u^i}, x^{u^{i-1}})$ maps to an element of the same degree in $F^0(D)$ with coefficient vector $e^{i-1} - e^i$ and thus the different coefficient vectors for varying $i$ are linearly independent, turning the whole map injective. $\square$

We will derive several consequences of this theorem: In Section 5.2 we describe its use in computing free resolutions and in Section 6.1 we describe its application in computing $\operatorname{Ext}^1$ of two Weil divisors.

## 5.2. The resolution quiver

For a Weil divisor $D$ the short exact sequence of Theorem 5.8 also encodes its free resolution completely. For sake of simplicity let us denote the sequence as

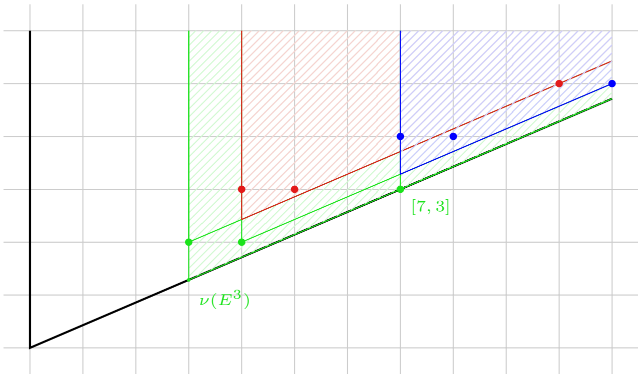
$$S(D):\ 0 \to \bigoplus_{i=1}^{r} D^i \to F_0(D) \to D \to 0.$$

Then we can replace the $D^i$ by their free resolutions and set

$$F_k(D) := \bigoplus_{i=1}^{r} F_{k-1}(D^i),\ k \geq 2,\ \text{with } F_0(D^i) = \bigoplus_{u \in G(D^i)} R[u]$$

as initial data, where we again assume $G(D^i)$ to be sorted as in Proposition 4.8.

**Example 5.9.** In our example $D = E^3$ is an ideal of $R$, generated by $x^3y^2$, $x^4y^2$ and $x^7y^3$. We obtain the following sequence




$$F_1(E^3) = F_0(E^1[3,2]) \oplus F_0(E^3[4,2]) \to F_0(E^3) \twoheadrightarrow H^0(X, E^3) \to 0$$

with

$$d_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix},\ d_0 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

The upper matrix already appeared in Example 2.15 in the usual notation. One also recognizes the intersections of the cones at neighbouring generators as the shifted $E^i$ we described above.

By computing the short exact sequence of Theorem 5.8 for every Weil divisor $D$ and inserting arrows from $D^i \to D$ we obtain a quiver. If we only consider divisor classes, identifying multiple arrows, this quiver becomes finite. This finiteness comes with a price: We lose the information about the grading.

We fix this problem in the following way: Apply Theorem 5.8 to all our $E^i$, $i = 1, \ldots, n$ and obtain

$$S(E^i):\ 0 \to \oplus_{j=1}^{r_i} E^{k_j}[u^j] \to F_0(E^i) \twoheadrightarrow E^i \to 0.$$

Now build a quiver $\mathcal{R}$ with vertices $E^i$ and arrows $E^{k_j} \to E^i$ and label these arrows with the corresponding shifting degree $u^j$. This procedure partly reverses the identification of multiple arrows, as there can be several arrows between two vertices with different labels.

**Definition 5.10.** We call $\mathcal{R}$ the *resolution quiver* of $Y_{n,q}$.

As we will see later, the quiver $\mathcal{R}$ possibly contains loops and multiple arrows. If there are multiple arrows, they cannot carry the same label by construction.

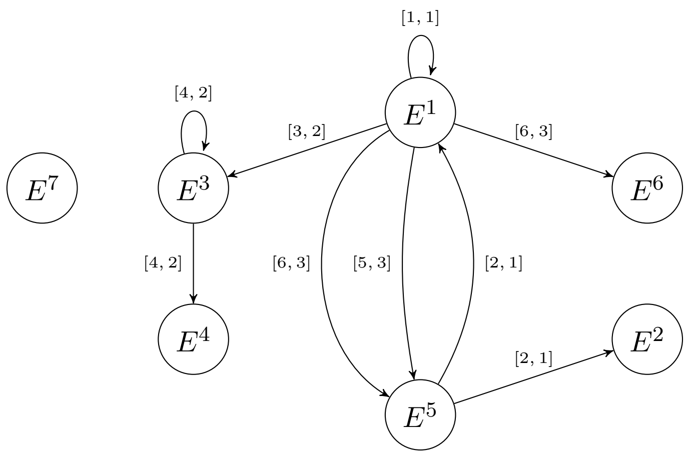
*Remark* 5.11. Every other Weil divisor $D$ is linearly equivalent to some $E^i$ and can be written as $D = E^i + u$ for some $u \in \mathbb{Z}^2$. Thus one can just resolve $E^i$ and shift the resolution by $u$ afterwards. Applying the recursive definition from above we see that $\mathcal{R}$ completely controls the minimal multigraded resolution of any Weil divisor. In a compact formula this means:

$$F_{n+1}(E^i) = \bigoplus_{j=1}^{r_i} F_n(E^{k_j})[u^j],$$

in the setting of the above exact sequence.

One can even resolve arbitrary monomial ideals $(x^{v^0}, \dots, x^{v^r})$, $v^i \in \sigma^\vee \cap M$ of $R$: First assume the set of generators ordered and minimal. Now each intersection $(v^{i-1} + \sigma^\vee) \cap (v^i + \sigma^\vee)$ corresponds to the section polyhedron of some shifted $E^i$.

**Example 5.12.** In the running example with $n = 7$ and $q = 3$ the quiver $\mathcal{R}$ looks as follows:




One immediately recognizes the decomposition of $F_1(E^3)$ given in Example 5.9. Using this quiver we can for example provide a formula for $F_2(E^3)$:

$$\begin{aligned} F_2(E^3) &= F_1(E^1)[3,2] \oplus F_1(E^3)[4,2] \\ &= \big(F_0(E^1)[1,1] \oplus F_0(E^5)[2,1]\big)\,[3,2] \oplus \big(F_0(E^1)[3,2] \oplus F_0(E^3)[4,2]\big)\,[4,2] \end{aligned}$$

As one can see, $\mathcal{R}$ is never connected, since $E^0 \sim E^n$ corresponds to $R$ which can be resolved trivially. Also $\mathcal{R}$ can have multiple edges and loops. To simplify notation in the future, we have the following definition:

**Definition 5.13.** For a Weil divisor $D$ we denote by $\mathrm{in}_{\mathcal{R}}(D)$ the sources of arrows of $\mathcal{R}$ that end in the vertex of $\mathcal{R}$ which is linearly equivalent to $D$. I.e. if $D = E^i[u]$ and there are arrows $E^{k_j} \to E^i$ labelled by $u_j \in M$ for $j = 1, \dots, r$, then

$$\mathrm{in}_{\mathcal{R}}(D) \;=\; \{E^{k_j}[u + u_j] \mid j = 1, \dots, r\}.$$

*Remark* 5.14. By construction we have the following relation between the number of generators $r+1$ of $D$ and the number of elements of $\mathrm{in}_{\mathcal{R}}(D)$:

$$\#\,\mathrm{in}_{\mathcal{R}}(D) \;=\; r \;= G(D)-1.$$

**Example 5.15.** Have a look at the quiver in Example 5.12. We can now easily write down $\mathrm{in}_{\mathcal{R}}(D)$ for e.g. $E^1$ and $E^3$:

$$\mathrm{in}_{\mathcal{R}}(E^1) = \{E^1[1,1],\ E^5[2,1]\},\ \mathrm{in}_{\mathcal{R}}(E^3) = \{E^1[3,2],\ E^3[4,2]\}.$$

**Theorem 5.16.** *For $E^i$ denote by $E^{k_1},\dots,E^{k_r}$ the sources of the incoming arrows. Then for any Weil divisor $D$ on $X$*

$$\mathrm{Ext}_R^{m+1}(E^i,D) \;=\; \bigoplus_{j=1}^{r}\mathrm{Ext}_R^m(E^{k_j},D)[-u^{k_j}],\ m\geq 1.$$

*The formula remains valid if replacing $D$ by an arbitrary $R$-module.*

*Proof.* Collecting the sources $E^{k_1},\dots,E^{k_r}$ of the incoming arrows together with their shifts $u^{k_j}$ yields the short exact sequence of Theorem 5.8 for $E^i$. Now one applies the functor $\mathrm{Hom}(\bullet,D)$ and takes the long exact sequence of cohomology. Since the middle module $F_0(E^i)$ is free we have $\mathrm{Ext}^n(F_0(E^i),D)=0$ for $n\geq 1$. This yields isomorphisms

$$\mathrm{Ext}_R^{n+1}(E^i,D)\cong \mathrm{Ext}_R^n\left(\bigoplus_{j=1}^{r_i}E^{k_j}[u^j],D\right)\ \text{ for } n\geq 1.$$

Since Ext commutes with direct sums and since we know how to compute Hom of graded modules, we can transform the right hand side into the desired form. □

*Remark* 5.17. Using Definition 5.13, Theorem 5.16 becomes the following for general $D$ and $D'$:

$$\mathrm{Ext}_R^{n+1}(D,D') \;=\; \bigoplus_{G\in\mathrm{in}_{\mathcal{R}}(D)}\mathrm{Ext}_R^n(G,D'),\ n\geq 1.$$

From an algorithmic perspective it is more efficient to have the shifts outside of $\mathrm{Ext}^n(\bullet,\bullet)$. One needs just compute $\mathrm{Ext}^1(E^i,E^j)$ for all $i,j=1,\dots,n$. Then we use shifts and the recursion from above to compute arbitrary $\mathrm{Ext}^n(D,D')$.

We will examine two special cases here, namely the case of $q=n-1$ and the case of $q=1$.

**Example 5.19.** Let $0<q=1<n\in\mathbb{Z}$, i.e. $Y_{n,q}$ is the cone over a rational normal curve. Then it is easy to see that the generators of the ideal $E^i$ are exactly

$$[i,1],[i+1,1],\dots,[n,1].$$

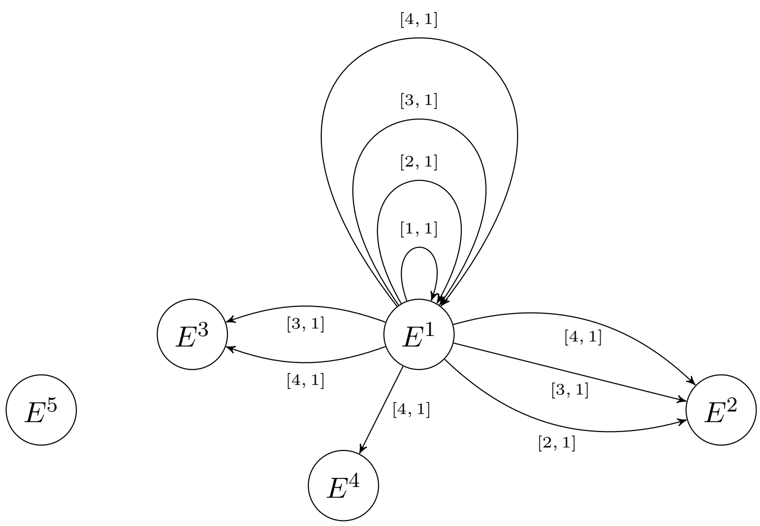
Thus neighbouring generators have difference 1 on the $x$-axis and by Proposition 5.5 we deduce that the syzygies of these generators are isomorphic to $E^1$. Hence we have for a fixed $E^i$ exactly $n-i$ arrows from $E^1$ to $E^1$ which are labeled by

$$[i,1],[i+1,1],\dots,[n-1,1].$$

**Algorithm 5.18:** Computing the resolution quiver $\mathcal{R}$

**Input**: $n, q \in \mathbb{Z}$ with $0 < q < n$ and $\gcd(n, q) = 1$
**Output**: The quiver $\mathcal{R}$ for $Y_{n,q}$ as described above
Initialize $\mathcal{R}$ as a new quiver with vertices $\{E^0, \dots, E^{n-1}\}$ and no edges;
**for** $i = 1, \dots, n-1$ **do**
  Let $G = \{u^0, \dots, u^r\}$ be the generators of $E^i$;
  **for** $j = 1, \dots, r$ **do**
    $source := u_0^j - u_0^{j-1}$;
    $shift := u^{j-1}$;
    Add the edge $E^{source} \to E^i$ with label $shift$ to $\mathcal{R}$;
**return** $\mathcal{R}$

To give a sample picture we pick $n = 5$:



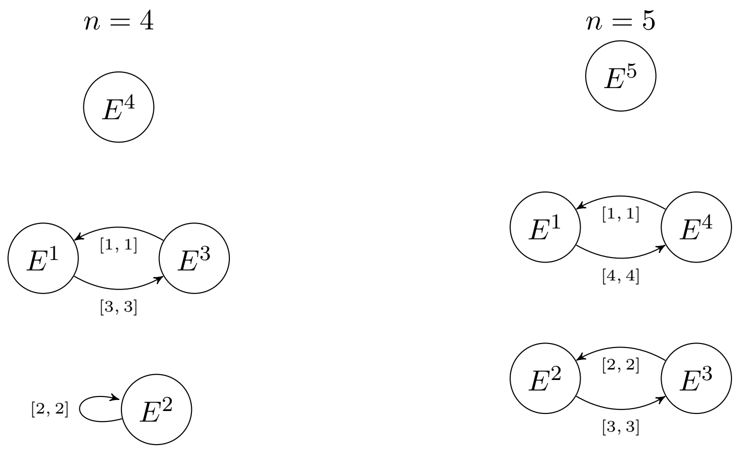

**Example 5.20.** Let $0 < q = n - 1 < n \in \mathbb{Z}$, then $Y_{n,q}$ is a $A_{n-1}$-singularity. In particular, $Y_{n,q}$ is a hypersurface, which is reflected in the fact that the Hilbert basis of $\sigma^\vee$ consists of three elements.

We proceed as in the previous case. The divisor $E^i$ is generated by the two elements

$$[i, i] \text{ and } [n, n-1]$$

for $i \neq n$. Thus there is an arrow from $E^{n-i}$ to $E^i$ labeled with $[i, i]$. Applying the principle to $E^{n-i}$ we also get an arrow from $E^i$ to $E^{n-i}$ labeled with $[n-i, n-i]$. If $n$ is even we have a loop at $E^{n/2}$, otherwise the quiver $\mathcal{R}$ consists of $\frac{n-1}{2}$ disjoint 2-cycles.

Consider the following two examples:




## 5.3. Comparing $\mathcal{R}$, the AR-quiver, and the McKay-quiver

Starting with the associated local ring $(R_\mathfrak{m}, \mathfrak{m})$ to the cyclic quotient singularity $Y_{n,q} = \operatorname{Spec} R$ one can define Auslander-Reiten sequences (AR-sequences), see for example [Yos90, Chapter 2]. These are short exact sequences of the following shape

$$0 \to N \to E \to M \to 0,$$

with both $N$ and $M$ indecomposable. Since the sequences of Section 5.1 have $N$ being a direct sum with more than one summand in most cases, it is already clear at this point that they will mostly not be AR-sequences.

This becomes even more imminent considering the Auslander-Reiten-quiver (AR-quiver): The AR-quiver has isomorphism classes of indecomposable modules for its vertices and the number of edges between two vertices $[N]$ and $[M]$, with $N$ and $M$ as above may be read of from $E$ ([Yos90, Lemma 5.5]). For an isolated singularity this yields a finite quiver. In the case of cyclic quotient singularities this yields at most one edge from $[N]$ to $[M]$ (see [Nak13]).

On the other hand we have the description of cyclic quotient singularities via a finite group acting on $V = \mathbb{C}^2$, given in Section 4.2. This gives rise to the McKay quiver. Let us briefly recall the construction given in [Yos90]. The vertices are the classes of non-isomorphic irreducible representations of $G$. Taking two such representations $V_i$ and $V_j$ we have exactly

$$\dim_\mathbb{C} \operatorname{Hom}_{\mathbb{C}G}(V_i, V \otimes_\mathbb{C} V_j)$$

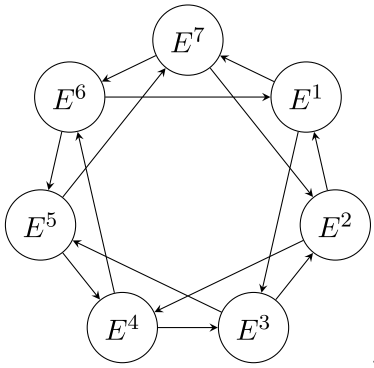
arrows from $V_i$ to $V_j$.

In the case of CQS, the McKay quiver and the AR-quiver become the same. This is due to a result by Auslander ([Aus86, Section 2], which we state in a slightly reformulated way:

**Theorem 5.21.** *[Yos90, Thm. 10.14] If $G \subset \mathrm{GL}(2, \mathbb{C})$ has no pseudo-reflection but the identity, then the invariant subring $R = \mathbb{C}\{x, y\}^G$ is always of finite representation type, and the AR-quiver of $R$ coincides with the McKay quiver of $(V, G)$.*

Let us have a look at the example:

**Example 5.22.** In the case $n = 7$ and $q = 3$ the AR-quiver looks like




.

Comparing this to $\mathcal{R}$ in Example 5.12 we already see the differences: The AR-quiver is connected, while $\mathcal{R}$ has an isolated vertex. Also $\mathcal{R}$ lacks the high symmetry of the AR-quiver.

Thus the AR-quiver is fundamentally different from the quiver $\mathcal{R}$. In fact the AR-quiver of a cyclic quotient singularity will always consist of two directed cycles involving all MCM-modules. This is due to [Wun87] and explained in our notation in [Nak13]. On the other side, $\mathcal{R}$ will always have the isolated point $E^n$ and in most cases also has sinks. In the case $q = 1$, $\mathcal{R}$ looks like a star with $E^1$ at the center, see Example 5.19. In the Gorenstein case, $\mathcal{R}$ is disconnected, for $q > 2$, see Example 5.20.

This difference is also caused by the way in which $\mathcal{R}$ and the AR-quiver arise. While we can build the AR-quiver for higher dimensional toric singularities as well, the short exact sequences of Section 5.1 tend not to be short for higher dimensions, see Chapter 3. This indicates that a quiver is not the right object for $\mathcal{R}$, only in the very specific case of two-dimensional varieties. Additionally the class group becomes infinite in the non-simplicial case. Thus $\mathcal{R}$ has infinitely many vertices. On the other side the AR-quiver stays finite, since there are only finitely many MCM-classes.

There are different possible categories for $\mathcal{R}$ that all have quivers as a subcategory. But before making a choice one needs to solve the problem of getting the minimal free resolution from the generalized Taylor complex of Chapter 3, i.e. one has to reduce the exact sequence $S(D)$. This step is crucial, because only then we get short exact sequences in the two-dimensional case. Having $\mathcal{R}$ a quiver will then be a property of the underlying toric variety.

# $\mathrm{Ext}^1$

## 6.1. A combinatorial formula for $\mathrm{Ext}^1$

Computing $\mathrm{Ext}^1$ is now very easy using Theorem 5.8.

For a Weil divisor $D$ we obtain the short exact sequence

$$S(D):\ 0 \to \bigoplus_{G \in \mathrm{in}_{\mathcal{R}}(D)} G \to F_0(D) \to D \to 0$$

and we apply $\mathrm{Hom}(\bullet, D')$ to it, with $D'$ a second Weil divisor. We obtain a long exact sequence of cohomology of which we will only consider the first part at this point. Since $F_0(D)$ is a free module, $\mathrm{Ext}^1(F_0(D), D') = 0$ and we have

$$0 \to \mathrm{Hom}(D, D') \to \mathrm{Hom}(F_0(D), D') \to \mathrm{Hom}(\bigoplus_{G \in \mathrm{in}_{\mathcal{R}}(D)} G, D') \to \mathrm{Ext}^1(D, D') \to 0.$$

For the second to last summand we have

$$\mathrm{Hom}(\bigoplus_{G \in \mathrm{in}_{\mathcal{R}}(D)} G, D') = \bigoplus_{G \in \mathrm{in}_{\mathcal{R}}(D)} \mathrm{Hom}(G, D').$$

Thus we have the following formula

$$\mathrm{Ext}^1(D, D') = {\bigoplus_{G \in \mathrm{in}_{\mathcal{R}}(D)} \mathrm{Hom}(G[u^{i-1}], D')} \Big/ {\mathrm{Hom}(F_0(D), D')},$$

where the denominator refers to the image of the third map. We can use this description to obtain a combinatorial formula for $\mathrm{Ext}^1$:

**Theorem 6.1.** *To two Weil divisors $D$ and $D'$ on $Y_{n,q}$ define the following set*

$$\mathrm{ext}(D, D') := \left( \bigcup_{G \in \mathrm{in}_{\mathcal{R}}(D)} (-\nu(G) + D') \right) \setminus \big( (-u^0 + D') \cup (-u^r + D') \big),$$

*where $u^0, u^1, \ldots, u^r \in M$ denote the sorted generators $G(D)$ of $D$. Then we have the following isomorphism*

$$\mathrm{Ext}^1_R(D, D') = \bigoplus_{u \in \mathrm{ext}(D,D') \cap M} \mathbb{C} \cdot \bar{x}^u$$

*of $M$-graded $\mathbb{C}$-vector spaces, where the multiplication is given as*

$$x^v \cdot \bar{x}^u = \begin{cases} \bar{x}^{v+u} & v+u \in \operatorname{ext}(D,D') \\ 0 & \textit{else}, \end{cases}$$

*with $x^v \in R$.*

*Proof.* We start by examining the module of homomorphisms between two divisorial ideals $D$ and $D'$. We claim that it is completely determined by its support $\hom(D,D')$, i.e.

$$\operatorname{Hom}(D,D')) \;=\; \bigoplus_{\hom(D,D')\cap M} \mathbb{C}\cdot x^u, \text{ where } \hom(D,D') = -\nu(D)+\nu(D')+\sigma^\vee.$$

Since both $D$ and $D'$ are finitely generated $M$-graded modules, their module of homomorphisms is canonically $M$-graded as well. Both are divisorial ideals contained in $\mathbb{C}[M]$. Assume we are given a non-trivial homogeneous morphism $f : D \to D'$, sending an element $x^u \in D$ to $f(x^u) = a \cdot x^w \in D'$, with $a \in \mathbb{C}^*$. In particular, we have $\deg f = w - u$. Taking any other element $x^v \in D$, we may write down the following equation:

$$f(x^v) = f(x^u \cdot x^{v-u}) = x^{v-u} \cdot f(x^u) = a \cdot x^{v-u+w}.$$

This does not follow by $R$-linearity of $f$ alone, rather one has to take the relations of the elements of both $D$ and $D'$into account. In particular this implies that $v-u+w \in D'$, otherwise we can construct torsion in the image of $D$ under $f$, but $D'$ has no torsion. Thus we deduce for the polyhedra that

$$(D \cap M) + w - u \subseteq (D' + M).$$

Take $u^0$ and $u^r$ to be the first and last generators of $D$, and name those of $D'$ $v^0$ and $v^s$. Then this implies

$$\langle v^0, \rho^0\rangle \le \langle u^0, \rho^0\rangle \text{ and } \langle v^s, \rho^s\rangle \le \langle u^r, \rho^1\rangle.$$

Since the hyperplanes through these points are the bounding hyperplanes of $D$, $D'$ respectively, the containment remains valid when omitting the intersection with $M$. Hence it is enough to find all lattice points $u \in M$ such that $D + u \subseteq D'$. But these are exactly the lattice points of $\hom(D,D')$. In particular, $\operatorname{Hom}(D,D')$ is a divisorial ideal itself, corresponding to the polyhedron $-\nu(D) + D'$. Hence we can speak of $x^u \in \operatorname{Hom}(D,D')$.

Using this observation, we construct

$$\operatorname{Supp}\left(\bigoplus_{G\in \operatorname{in}_{\mathcal{R}} D} \operatorname{Hom}(G,D')\right) = \bigcup_{G\in \operatorname{in}_{\mathcal{R}} D} \hom(G,D')\cap M = \left(\bigcup_{G\in \operatorname{in}_{\mathcal{R}} D} -\nu(G)+\nu(D')+\sigma^\vee\right)\cap M.$$

This motivates the first term of $\operatorname{ext}(D,D')$.

Next we compute the denominator in the representation of $\operatorname{Ext}^1(D,D')$ as a quotient:

$$\operatorname{Hom}(F^0(D),D') = \operatorname{Hom}(\bigoplus_{u\in G(D)} R[u], D') = \bigoplus_{u\in G(D)} \operatorname{Hom}(R[u],D') = \bigoplus_{u\in G(D)} D'[-u].$$

Take $G(D)$ to be $u^0, u^1, \ldots, u^r \in M$. The numerator in the quotient representation has exactly $r$ terms, these are the $\mathrm{Hom}(G, D')$ for $G \in \mathrm{in}_{\mathcal{R}}(D)$. Our claim is that the $D'[-u^i]$ glue two of the $\mathrm{Hom}(G, D')$ together for $i = 1, \ldots, r-1$, while the boundary $D'[-u^0]$ and $D'[-u^r]$ are torsion. This explains the formula for $\mathrm{ext}(D, D')$ completely.

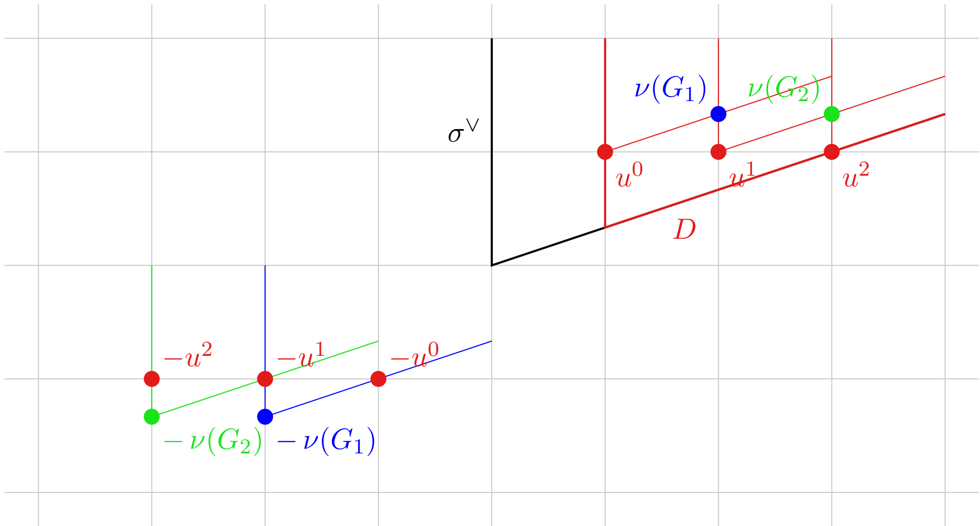
For the sake of simplicity, we will prove this in the case $r = 2$. Here, we have

$$F_0(D) = R[u^0] \oplus R[u^1] \oplus R[u^2], \text{ and } \mathrm{in}_{\mathcal{R}}(D) = \{G_1, G_2\}.$$

As described in Section 5.1 we obtain $G_i$ by intersecting

$$G_i \ = \ (u^{i-1} + \sigma^\vee) \cap (u^i + \sigma^\vee) \ = \ \nu(G_i) + \sigma^\vee.$$

Let us illustrate the situation using a picture. The red shifted cone depicts our divisor $D$, the red dots depict the generators $u^i$.




The picture already illustrates the main point of the first part, namely

$$(-\nu(G_1) + \nu(D') + \sigma^\vee) \cap (-\nu(G_2) + \nu(D') + \sigma^\vee) = -u^1 + \nu(D') + \sigma^\vee.$$

All one needs is to multiply the equation

$$u^1 = \nu(G_1) - a \cdot [0, 1] = \nu(G_2) - b \cdot [n, q], \text{ with } a, b > 0$$

by $(-1)$ to immediately obtain that $u^1$ lies on the extremal rays of both polyhedra $-\nu(G_i) + \sigma^\vee$.

Next we compute the image of the direct summand $\mathrm{Hom}(R[u^1], D')$ in $\mathrm{Hom}(G_1, D') \oplus \mathrm{Hom}(G_2, D')$. As discussed above, a homogeneous morphism $R[u^1] \to D'$ is determined by an element $u \in D'[-u^1] \cap M$, i.e. $1 \mapsto x^{u+u^1} \in D'$. Now we take the composition of morphisms

$$G_1 \oplus G_2 \to R[u^0] \oplus R[u^1] \oplus R[u^2] \to D',$$

i.e. we embed $f$ in $\mathrm{Hom}(F_0(D), D')$ and then compose it with the differential. Concretely this yields for the homogeneous elements

$$(x^v, x^w) \mapsto (x^{v-u^0}, x^{w-u^1} - x^{v-u^1}, -x^{w-u^2}) \mapsto x^{u+w} - x^{u+v}.$$

If we restrict this composed map to either $G_1$ or $G_2$, it yields a morphism $G_i \to D'$ of degree $u$ in both cases. Thus for the points we get

$$\begin{array}{ccc} D'[-u^1] & \to & \mathrm{Hom}(G_1, D') \oplus \mathrm{Hom}(G_2, D') \\ x^u & \mapsto & (x^u, -x^u) \end{array}.$$

Hence, in $\mathrm{Ext}^1(D, D')$ $x^u \in \mathrm{Hom}(G_1, D')$ is the same as $x^u \in \mathrm{Hom}(G_2, D')$, since $(x^u, -x^u)$ is zero. This explains the glueing part.

We will finish this proof by discussing the image of $D'[-u^0] = \mathrm{Hom}(R[u^0], D')$ in $\mathrm{Hom}(G_1, D') \oplus \mathrm{Hom}(G_2, D')$. Proceeding as in the previous part we obtain

$$\begin{array}{ccc} D'[-u^0] & \to & \mathrm{Hom}(G_1, D') \oplus \mathrm{Hom}(G_2, D') \\ x^u & \mapsto & (x^u, 0) \end{array}.$$

This completely cuts off $D'[-u^0]$ from $\mathrm{Hom}(G_1, D')$.

We may proceed analogously for $D'[-u^r]$.

For $r > 2$ we obtain several gluings. Then every $D'[-u^k]$ for $0 < k < r$ glues with its respective neighbours as $D'[-u^1]$. Again, $D'[-u^0]$ and $D'[-u^r]$ are cut off. $\square$

*Remark* 6.2. For sake of completeness one can recognize the degrees of $\mathrm{Hom}(D, D')$ as the lattice points of $D'[-u^0] \cap D'[-u^r]$. This explains the first term in the long exact cohomology sequence.

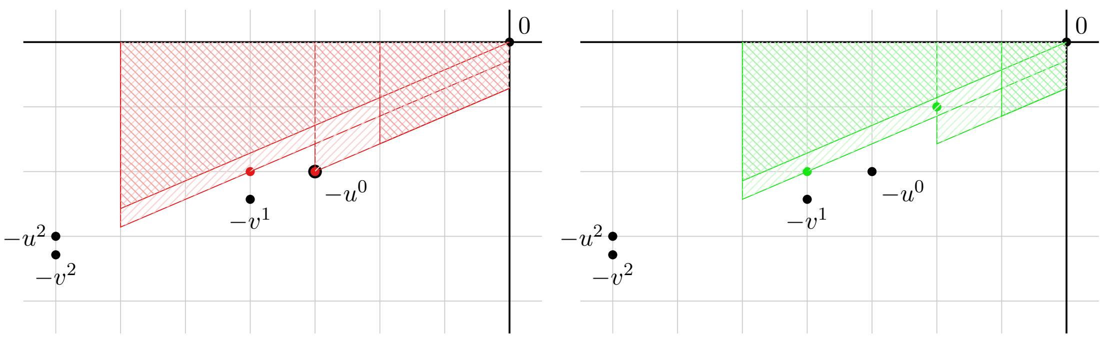
**Example 6.3.** We continue our example with $q = 3$ and $n = 7$ from above and compute $\mathrm{Ext}^1_R(E^3, E^1)$ and $\mathrm{Ext}^1_R(E^3, E^2)$. Let us first identify all variables needed for the theorem:

$$u^0 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \; u^1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \; u^2 = \begin{bmatrix} 7 \\ 3 \end{bmatrix}, \; v^1 = \begin{bmatrix} 4 \\ \frac{17}{7} \end{bmatrix}, \; v^2 = \begin{bmatrix} 7 \\ \frac{23}{7} \end{bmatrix},$$

and the polyhedra of global sections:

$$E^1 = \begin{bmatrix} 1 \\ \frac{3}{7} \end{bmatrix} + \sigma^\vee, \; E^2 = \begin{bmatrix} 2 \\ \frac{6}{7} \end{bmatrix} + \sigma^\vee.$$

Now we draw the corresponding point sets $\mathrm{ext}(D, D')$:




Thus we obtain

$$\mathrm{Ext}^1_R(E^3, E^1) = \mathbb{C} \cdot \bar{x}^{[-4,-2]} \oplus \mathbb{C} \cdot \bar{x}^{[-3,-2]} \text{ and } \mathrm{Ext}^1_R(E^3, E^2) = \mathbb{C} \cdot \bar{x}^{[-4,-2]} \oplus \mathbb{C} \cdot \bar{x}^{[-2,-1]}.$$

As an immediate observation we may complement the first part of Proposition 2.28 with a combinatorial explanation in our case of cyclic quotient singularities in the following two remarks:

*Remark* 6.4. Since a cyclic quotient singularity is isolated, Proposition 2.28 implies that for two Weil divisors $D$, $D'$ on a cyclic quotient singularity the $\mathbb{C}$-dimension of $\mathrm{Ext}^1_R(D, D')$ as a $\mathbb{C}$-vector space is finite.

We verify this by showing that $\mathrm{ext}(D, D')$ in Theorem 6.1 is bounded. The ordering on $\{u^0, \ldots, u^r\}$ implies the same ordering on $\{v^1, \ldots, v^r\}$. Multiplying with $-1$ just reverses the ordering. Thus we obtain

$$\langle \rho^0, u^r \rangle = \langle \rho^0, v^r \rangle \text{ and } \langle \rho^1, u^0 \rangle = \langle \rho^1, v^1 \rangle,$$

which results in cutting off everything above the hyperplanes at these values given by $\rho^0$ and $\rho^1$. Since $\mathrm{ext}(D, D')$ was bounded from below by default, we obtain the desired result. For an illustration please see Example 6.3.

This explains $\mathrm{Ext}^1$ and via Theorem 5.16 we extend Remark 6.4 to general $n$:

*Remark* 6.5. In full generality, the first part of Theorem 6.1 states that the $R$-module $\mathrm{Ext}^n_R(D, D')$ is a finite dimensional $\mathbb{C}$-vector space for any $n \geq 1$.

One can now use induction: Theorem 5.16 yields the step, since every higher $\mathrm{Ext}^i$ is just a direct sum of $\mathrm{Ext}^1$'s, and Remark 6.4 is the anchor.

## 6.2. First consequences of the $\mathrm{Ext}^1$ formula

A helpful observation is that we can relate the set $\mathrm{ext}(D, D')$ to a combinatorial invariant of $D$. This invariant is the following:

**Definition 6.6.** For a divisor $D$ with generators $\{u^0, \ldots, u^r\} = G(D)$ define the following subset

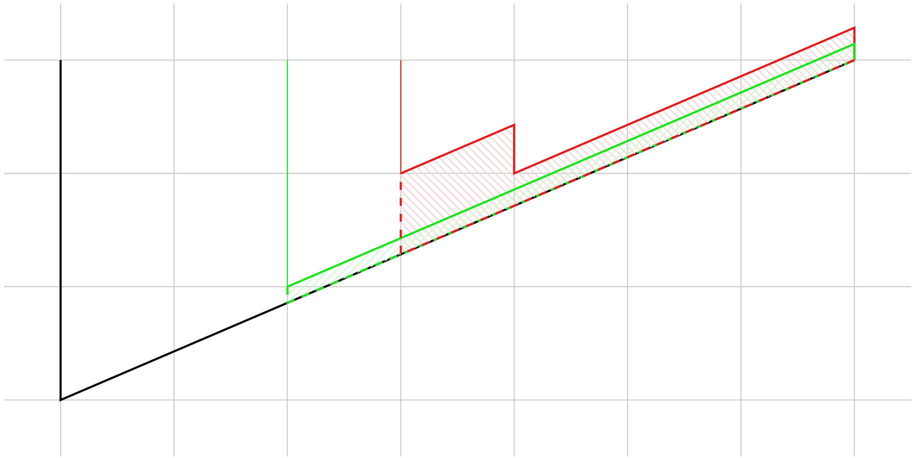
$$\mathrm{below}(D) := \mathrm{int}(D) \setminus \bigcup_{i=0}^{r} \mathrm{int}(u^i + \sigma^\vee),$$

where $\mathrm{int}(\bullet)$ denotes the relative interior.

One may think of $\mathrm{below}(D)$ measuring how far $D$ is from being lattice-equivalent to $\sigma^\vee$. If $\mathrm{below}(D)$ is empty, then $D \sim E^n$. Let us illustrate this using a picture:

**Example 6.7.** We draw the sets $\mathrm{below}(E^i)$ for $E^3$ and $E^2$ in the running example with

$n = 7$ and $q = 3$:



Please remember that the leftmost and the bottom boundary do not belong to the below sets, this is indicated by the dashed lines. Since $E^2$ only has two generators, its below set looks like a parallelepiped. On the other hand, $E^3$ is generated by three elements and hence, its below set has one 'dent'.

Now let us link the sets below$(D)$ and ext$(D, D')$:

**Proposition 6.8.** We have the following two equations:

$$\operatorname{ext}(D, D') \;=\; \operatorname{ext}(D, 0) + \nu(D'), \text{ and } -\operatorname{ext}(D, 0) \;=\; \operatorname{below}(D).$$

*Proof.* The first equation follows immediately from the definition of ext$(D, D')$ in Theorem 6.1. Please also look at the diagram in the proof of Theorem 6.1 for an illustration of the situation.

For the second equation we start on the left hand side:

$$-\operatorname{ext}(D, 0) \;=\; \left( \bigcup_{G \in \mathrm{in}_{\mathcal{R}}(D)} (\nu(G) - \sigma^\vee) \right) \setminus \big( (u^0 - \sigma^\vee) \cup (u^r - \sigma^\vee) \big)$$

This means that all the rays of the $\nu(G) - \sigma^\vee$ point "downwards", in particular they pass through the generators $u^i$. For $u^0$ and $u^r$ being the first and last generator of $D$ is equivalent to

$$(u^0 - \sigma^\vee) \cap (u^r - \sigma^\vee) \;=\; \nu(D) - \sigma^\vee.$$

When we cut off $(u^0 - \sigma^\vee) \cup (u^r - \sigma^\vee)$ from a $\nu(G) - \sigma^\vee$, the remaining part will live in the parallelepiped spanned by $\nu(G)$, $u^0$, $u^r$ and $u^r + u^0 - \nu(G)$. Thus, the set resulting after the cutoff will not contain any boundary of $D$. Hence, cutting off $(u^0 - \sigma^\vee) \cup (u^r - \sigma^\vee)$ becomes the same as intersecting with int$(D)$ for $\nu(G) - \sigma^\vee$.

Next consider the ordered elements $\{G_1, \ldots, G_r\} = \mathrm{in}_{\mathcal{R}}(D)$. Analogously to the previous observation we have

$$(\nu(G_i) - \sigma^\vee) \cap (\nu(G_{i+1}) - \sigma^\vee) \;=\; u^i - \sigma^\vee.$$

Using this intersection point, we conclude that

$$-\operatorname{ext}(D,0) \;\subseteq\; \operatorname{int}(D)\backslash(u^i+\operatorname{int}(\sigma^\vee)).$$

Taking the intersection on the right hand side for all i yields $-\operatorname{ext}(D,0)\subseteq \operatorname{below}(D)$. By construction we already know

$$(u^i+\sigma^\vee)\cap(u^{i+1}+\sigma^\vee) \;=\; \nu(G_{i+1})+\sigma^\vee \;=\; G_{i+1}.$$

This yields that $\operatorname{below}(D)$ is covered by the $\nu(G_{i+1})-\sigma^\vee$ and thus, we get equality. $\square$

Additionally we can give a generator free description of $\operatorname{below}(D)$:

$$\operatorname{below}(D)=\operatorname{int}(D)\backslash\bigcup_{u\in\operatorname{int}(D)\cap\mathbb{Z}^2}\operatorname{int}(u+\sigma^\vee).$$

To determine the dimension of $\mathrm{Ext}^1(D,D')$ we count the lattice points of

$$\operatorname{ext}(D,D')=\nu(D')+\operatorname{ext}(D,0)=-(\operatorname{below}(D)-\nu(D')).$$

Since such a $\nu(D')$ will always have an integral first coordinate we just need to analyze the behaviour of $\operatorname{below}(D)$ when moving it up and down in steps of $\frac{q}{n}$ or $\frac{1}{n}$, since $\gcd(n,q)=1$. We will proceed in this direction in Section 6.3.

**Proposition 6.9.** By construction, $\operatorname{below}(D)$ does not contain any lattice points in its interior and we obtain

$$\dim\mathrm{Ext}^1(D,0)=\#(\operatorname{below}(D)\cap\mathbb{Z}^2)=r-2$$

since all generators except $u^0$ and $u^r$ are contained in the boundary of $\operatorname{below}(D)$.

$\square$

The following proposition is already known ([BG09]) since our divisors or ideals are $\mathbb{Q}$-Cartier (or 'conic' in [BG09]), i.e. they come from rational translates of the original cone $\sigma^\vee$. Nevertheless the proof of the proposition serves as a check for correctness of our construction.

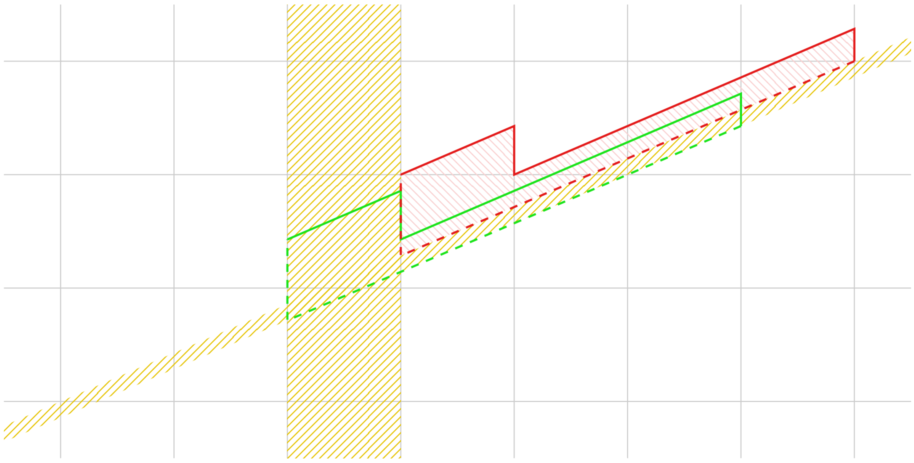
**Proposition 6.10.** For any Weil divisor $D$ on $Y_{n,q}$ we have

$$\mathrm{Ext}^1(D,K)=0.$$

*Proof.* From Remark 4.9 we already know that the vertex of $K$ is $[1,\frac{q+1}{n}]$. Thus all we have to show is that

$$(\operatorname{below}(D)-\nu(K))\cap M \;=\; \emptyset.$$

Let us start by illustrating the situation:



The red area depicts below$(D)$ and the green area is below$(D) - \nu(K)$. We will proceed in two steps: First we argue that below$(D) - \nu(K)$ cannot contain any lattice points other than the generators of $D$ using Proposition 6.9. Second we show explicitly that below$(D) - \nu(K)$ does not contain any generator of $D$.

For the first step we need the golden area. This is exactly

$$\begin{aligned} G \ &:= \{u \in M_{\mathbb{Q}} \mid \langle \nu(D), \rho^0 \rangle - 1 < \langle u, \rho^0 \rangle < \langle \nu(D), \rho^0 \rangle\} \\ &\cup \ \{u \in M_{\mathbb{Q}} \mid \langle \nu(D), \rho^1 \rangle - 1 < \langle u, \rho^1 \rangle < \langle \nu(D), \rho^1 \rangle\}. \end{aligned}$$

Remember that $\rho^0 = (1, 0)$, $\rho^1 = (-q, n)$ are the primitive ray generators of $\sigma$. Thus $G \cap M$ is empty. Furthermore, we have the containment equation

$$(\mathrm{below}(D) - \nu(K)) \backslash \overline{\mathrm{below}(D)} \subset G,$$

where the line denotes the closure, i.e. including the leftmost and the bottom boundary. Thus any lattice points of below$(D) - \nu(K)$ must be contained in $\overline{\mathrm{below}(D)}$. But the lattice points of below$(D)$ must be generators of $D$ by Proposition 6.9 and thus, taking the closure, we get

$$\overline{\mathrm{below}(D)} \ = \ \{u^0, \dots, u^r\} \ = G(D).$$

If there was another lattice point on the leftmost or bottom boundary of $\overline{\mathrm{below}(D)}$ this would contradict the statement that $\{u^0, \dots, u^r\}$ generate $D$. Hence we get

$$(\mathrm{below}(D) - \nu(K)) \cap M \ = \ G(D) \cap (\mathrm{below}(D) - \nu(K)).$$

All that is left is to observe closely what happens with the generators when moving below$(D)$. We start by moving the below$(D)$ to the left parallel to its top and bottom edge directions, i.e. consider

$$\mathrm{below}(D) - [1, \frac{q}{n}].$$

This set does not contain $u^r$. It does contain $u^0, \dots, u^{r-1}$, but since we moved below$(D)$ to the left, parallel to the bottom and top edge directions, these points are now on the top boundary. In particular, these points do not form the inverse apexes of below$(D)$

anymore. Thus, by moving the new set down by $[0, -\frac{1}{n}]$ we loose all those points, and since

$$\mathrm{below}(D) - [1, \frac{q}{n}] - [0, \frac{1}{n}] \;=\; \mathrm{below}(D) - \nu(K)$$

we obtain the desired result. □

The previous proposition yields an interesting symmetry or duality of certain $\mathrm{Ext}^1$. From a combinatorial point of view this is remarkable, because the sets $\mathrm{below}(E^i)$ and $\mathrm{below}(E^j)$ can vary widely in size and still contain the exact same lattice points under certain shifts.

**Proposition 6.11.** For given $D$ and $D'$ we have

$$\mathrm{Ext}^1(D, K - D') \;=\; \mathrm{Ext}^1(D', K - D).$$

*Proof.* Since it simplifies notation, we will show

$$\mathrm{Ext}^1(E^i, K - E^j) \;=\; \mathrm{Ext}^1(E^j, K - E^i),$$

and the claim follows, because the $E^i$ form a system of representatives for the class group. We do not have to worry about shifts, because of the minus sign in the second argument.

We can express both sides in terms of the below sets introduced above. Ultimately we have to show that

$$(\mathrm{below}(E^i) + \nu(E^j) - \nu(K)) \cap M = (\mathrm{below}(E^j) + \nu(E^i) - \nu(K)) \cap M.$$

For simplicity denote by $C_{ij} := \mathrm{below}(E^i) + \nu(E^j)$. We want to show that lattice points of the above sets can only appear in the intersection, i.e.

$$(C_{ij} - \nu(K)) \cap M \;=\; ((C_{ij} - \nu(K)) \cap (C_{ji} - \nu(K))) \cap M,$$

where we may interchange $i$ and $j$.

Thus let us take everything but the intersection and cover it in a nice way:

$$(C_{ij} \cup C_{ji}) \backslash (C_{ij} \cap C_{ji}) \subseteq \left( \bigcup_{u \in G(E^i)} (u + \mathrm{below}(E^j)) \right) \cap \left( \bigcup_{u \in G(E^j)} (u + \mathrm{below}(E^i)) \right).$$

Now the right hand side consists entirely of lattice shifts of $\mathrm{below}(E^i)$ and $\mathrm{below}(E^j)$ and thus shifting those using $\nu(K)$ will turn them empty when intersecting with $\mathbb{Z}^2$ by Proposition 6.10. Hence lattice points of $C_{ij} + \nu(K)$ live in the intersection $C_{ij} \cap C_{ji}$. □

**Definition 6.12** (Ext matrix)**.** Let $X := Y_{n,q}$ be given by the continued fraction $\underline{a} = [a_0, \ldots, a_r]$. Then we define a matrix

$$\mathcal{E}_k(\underline{a}) := \Big( m_{ij} := \dim \mathrm{Ext}^k_R(E^j, K_X - E^i) \Big)$$

for $i, j = 1, \ldots, n$ and $k > 0$.

*Remark* 6.13. Utilizing Proposition 6.11 and Proposition 6.10 we can already examine part of the structure of $\mathcal{E}_1(\underline{a})$: Proposition 6.11 tells us that the matrix is symmetric and thus Proposition 6.10 yields that the last row and column must be zero. The reason why we let $\mathcal{E}_1$ depend on the continued fraction rather than on $n$ and $q$ lies in the next chapter where we give an explicit algorithm for computing $\mathcal{E}_1$.

**Example 6.14.** We finish this chapter with an example for the previous remark. Namely if we know the $\mathcal{E}_1$ for $\underline{a}'$ and $\underline{a}''$, where we obtain $\underline{a}'$ and $\underline{a}''$ by omitting or decreasing the last entry of $\underline{a}$, respectively, we can reconstruct $\mathcal{E}_1(\underline{a})$.

In our ongoing example for $n = 7$ and $q = 3$ we have $\underline{a} = [2, 4]$ and for $\mathcal{E}_1([2, 4])$ we get

$$\begin{pmatrix} 2&1&2&1&2&1&0\\ 1&1&1&1&1&0&0\\ 2&1&3&2&2&1&0\\ 1&1&2&1&1&0&0\\ 2&1&2&1&2&1&0\\ 1&0&1&0&1&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} = \begin{pmatrix} 1&1&1&1&1&1&0\\ 1&1&1&1&1&0&0\\ 1&1&1&1&0&0&0\\ 1&1&1&0&0&0&0\\ 1&1&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} + \begin{pmatrix} 1&0&1&0&1&0&0\\ 0&0&0&0&0&0&0\\ 1&0&2&1&2&1&0\\ 0&0&1&1&1&0&0\\ 1&0&2&1&2&1&0\\ 0&0&1&0&1&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

where we recognize $\mathcal{E}_1([2, 3])$ and $\mathcal{E}_1([2])$ on the right hand side.

Large scale tests reveal a recursive way to compute $\mathcal{E}_1(\underline{a})$ which will be proven and discussed in Section 6.3. In fact this recursion does not only hold on the level of computing dimensions, but even of the level of computing the lattice points of the shifted below-sets, or the degrees of $\operatorname{Ext}^1$.

## 6.3. A recursive formula for $\dim \operatorname{Ext}^1$

We already observed that in this setting all appearing $\operatorname{Ext}^i$ were finite dimensional $\mathbb{C}$-vector spaces. If just the dimension matters, the formula given in Theorem 6.1 can be simplified and yields a very fast algorithm to determine dimensions of arbitrary $\operatorname{Ext}^i$.

In particular we will give a recursive algorithm for computation of the $\mathcal{E}_1$ depending only on the continued fraction $\underline{a} = [a_1, \ldots, a_s]$ associated to the given cyclic quotient singularity. One has to treat two different cases, depending on the last entry of the continued fraction, namely the case $a_s > 2$ and $a_s = 2$.

The strategy in both cases is the same: First we show that a subsquare of the matrix $\mathcal{E}_1(\underline{a})$ already determines the matrix completely, these are the statements of Lemma 6.19 and Lemma 6.23. Then we show that this subsquare is exactly the $\mathcal{E}_1(\underline{\tilde{a}})$ of another continued fraction $\underline{\tilde{a}}$, these are Theorem 6.22 and Theorem 6.25.

For the notation: There will be two rings, $\Lambda$ and $\Gamma$. What we want to compute is always $\operatorname{Ext}_\Lambda$ while we assume that $\operatorname{Ext}_\Gamma$ is known. Everything with an $\tilde{}$ on top lives on the $\Gamma$ side and everything without lives on the side of $\Lambda$. Since $\Lambda$ and $\Gamma$ appear as subscripts we decided against using $\tilde{\Lambda}$ for the sake of readability.

Before we start, we simplify the notation a little:

**Definition 6.15.** Let $R := R(n, q)$, then we define

$$E_R(i, j) := \operatorname{below}_R(E^i) - \nu(K_R) + \nu(E^j).$$

We can even make this more concrete by inserting the corresponding vertices:

$$E_R(i,j) = \mathrm{below}_R(E^i) - [1, \frac{q+1}{n}] + [j, \frac{jq}{n}].$$

To relate this with the formulas for Ext discussed in Section 6.1, especially in Theorem 6.1, observe that

$$-E_R(i,j) = -\,\mathrm{below}_R(E^i) + \nu(K_R) - \nu(E^j) = \mathrm{ext}(E^i, K_R - E^j)$$

and hence we have

$$\mathrm{Ext}^1_R(E^i, K_R - E^j) = \bigoplus_{u \in E_R(i,j) \cap \mathbb{Z}^2} \mathbb{C} \otimes \bar{x}^{-u}.$$

Of course we can also relate these sets to our matrix $\mathcal{E}_1$ via

$$m_{ij} = \#(E_R(i,j) \cap \mathbb{Z}^2).$$

$a_s > 2$

The goal of this part is to reduce the last entry of our continued fraction by 1. Let us first give an overview on how we will proceed: First we state all the details of the setup which we want to work within. Second we show that it is enough to consider only a part of the matrix $\mathcal{E}_1$, which we call the lower right square. And third we prove a theorem relating this quadrant of the $\mathcal{E}_1$ with the $\mathcal{E}_1$ associated to the continued fraction $\underline{\tilde{a}}$ which only differs from $\underline{a}$ in the last entry.

### The setup

Computing $\mathcal{E}_1$ from $\underline{a}$ recursively corresponds to changing $\underline{a}$. On the algebraic side this corresponds to changing the ring. Let us make clear in what way our ring changes:

The continued fraction $\underline{a}$ corresponds to the Hilbert basis

$$H(n,q) = \{b^0 = [0,1], b^1, \ldots, b^s, b^{s+1} = [n,q]\}$$

of $\sigma^\vee$ via the equations

$$b^{i-1} + b^{i+1} = a_i \cdot b^i \text{ for } i = 1, \ldots, r.$$

Since $n > q$ we immediately obtain $b^1 = [1,1]$. Now take $\underline{\tilde{a}} = [a_1, \ldots, a_{s-1}, a_s - 1]$. This defines a new cyclic quotient singularity which in turn can be given by two numbers $\tilde{n}$ and $\tilde{q}$. Of course we will again want $\tilde{n} > \tilde{q} > 0$ with $\gcd(\tilde{n}, \tilde{q}) = 1$. Thus we have a cone

$$\tilde{\sigma}^\vee = \mathrm{cone}([0,1], [\tilde{n}, \tilde{q}])$$

and we already know most of its Hilbert basis

$$H(\tilde{n}, \tilde{q}) = \{\tilde{b}^0 = b^0, b^1, \ldots, \tilde{b}^r = b^s, \tilde{b}^{s+1} = [\tilde{n}, \tilde{q}]\},$$

except for the last vector $\tilde{b}^{s+1}$. Taking the equation

$$\tilde{b}^{s-1} + \tilde{b}^{s+1} = (a_s - 1) \cdot \tilde{b}^r$$

and inserting $\tilde{b}^r = b^s$, $\tilde{b}^{s-1} = b^{s-1}$ we obtain

$$\tilde{b}^{s+1} = a_s \cdot b^s - b^s - b^{s-1}.$$

But we already know

$$b^{s+1} + b^{s-1} = a_s \cdot b^s$$

and thus obtain

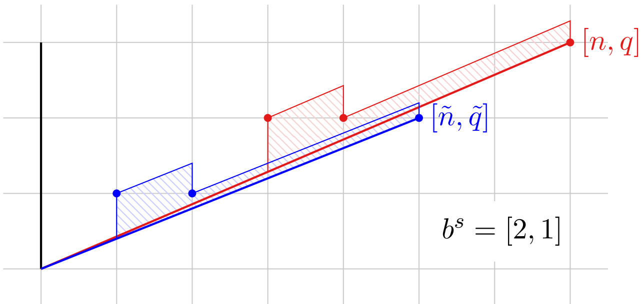
$$[\tilde{n}, \tilde{q}] = [n, q] - b^s.$$

Since $\{[n, q], b^s\}$ was a lattice basis, $\{[\tilde{n}, \tilde{q}], b^s\}$ will be a lattice basis as well, which proves that $[\tilde{n}, \tilde{q}]$ is primitive. In particular we obtain that the pairwise determinants of the vectors $b^s$, $b^{s+1}$ and $\tilde{b}^{s+1}$ are $\pm 1$ which yields that the corresponding parallelepipeds do not contain any lattice points in the interior.

We obtain an inclusion $\sigma^\vee \hookrightarrow \tilde{\sigma}^\vee$ and this yields an injection of rings

$$\Lambda := R(n, q) \hookrightarrow R(\tilde{n}, \tilde{q}) =: \Gamma.$$

From now on we will have to deal with $E^i$'s living in $\Lambda$ or in $\Gamma$ and thus they receive an additional label, i.e. $E^i_\Lambda$ and $E^i_\Gamma$. A consequence of the observation that the parallelepiped of $b^{s+1}$ and $\tilde{b}^{s+1}$ does not have any lattice points in the interior is illustrated below:




i.e. for $n - \tilde{n} < i \leq n$ we have

$$G(E^i_\Lambda) = b^s + G(E^{i-n+\tilde{n}}_\Gamma).$$

which really means that the ideals corresponding to $E^i_\Lambda$ and $E^{i-n+\tilde{n}}_\Gamma + b^s$ in the respective rings are generated by the same monomials of $\mathbb{C}[\mathbb{Z}^2]$. As an initial observation we immediately obtain

$$\dim \operatorname{Ext}^1_\Lambda(E^i_\Lambda, E^0_\Lambda) = \dim \operatorname{Ext}^1_\Gamma(E^{i-n+\tilde{n}}_\Gamma, E^0_\Gamma)$$

by the observations at the beginning of Section 6.2.

**Example 6.16.** In our running example with $n = 7$ and $q = 3$ we know $b^s = [2, 1]$ and thus obtain $\tilde{n} = 5$ and $\tilde{q} = 2$. The following picture illustrates the situation by depicting the weight cones of $\Lambda$ and $\Gamma$:

We have $\Lambda = R(7, 3)$ and $\Gamma = R(5, 2)$. The associated continued fractions are

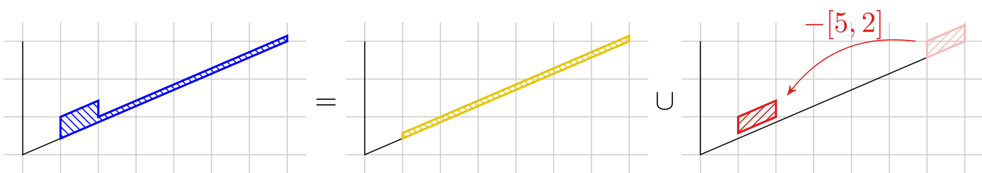
$$\frac{7}{7-3} = [2, 4] =: \underline{a} \text{ and } \frac{5}{5-2} = [2, 3] =: \underline{\tilde{a}}.$$

### Reduction to the lower right square

The ring $\Gamma$ will not play a role during the reduction part, so it is ok to ignore the subscript at the $E^i$'s.

As one might already have guessed, the lower right square of $\mathcal{E}_1(\underline{a})$ is the square corresponding to $n - \tilde{n} < i, j \leq n$. Thus assume that we are given $1 \leq i \leq n - \tilde{n}$ and $1 \leq j \leq n$. Naturally the set $G(E^i_\Lambda)$ will contain both $b^s$ and $b^{s+1}$.

**Example 6.17.** Let us analyze the set $\text{below}(E^i_\Lambda)$ for our running example:




The red sets indicates below $E^6_\Lambda$ and the translate below $E^6_\Lambda - [5, 2]$. The yellow subset $S$ of $\text{below}(E^i_\Lambda)$ is exactly the difference of below $E^1_\Lambda$ and below $E^6_\Lambda - [5, 2]$. Indeed we obtain the following equation:

$$\text{below } E^1_\Lambda = S \cup (\text{below } E^6_\Lambda - [5, 2]).$$

Requiring this equation to hold also defines exactly which boundaries should be contained in $S$ and which boundaries should be open. Thus the union on the right hand side is even a disjoint one. Hence we obtain

$$\dim \operatorname{Ext}^1(E^1_\Lambda, E^j_\Lambda) = \dim \operatorname{Ext}^1(E^6_\Lambda, E^j_\Lambda) + \#[(S - \nu(E^j_\Lambda)) \cap \mathbb{Z}^2].$$

We denote this $S$ as $S(1)$. Similarly one can obtain the formula

$$\dim \operatorname{Ext}^1(E^2_\Lambda, E^j_\Lambda) = \dim \operatorname{Ext}^1(E^7_\Lambda, E^j_\Lambda) + \#[(S(2) - \nu(E^j_\Lambda)) \cap \mathbb{Z}^2].$$

Of course this formula can be generalized, but first let us have a closer look at the set $S$ or $S(i)$. By construction or namely the equation we impose on $S(i)$, the upper and the right boundary belong to $S(i)$, the left and the lower boundary do not. But what we want to look at now is the height of $S(i)$, i.e. the distance between two boundary points with the same $x$-coordinate. We already know that the upper boundary contains $b^s = [n-\tilde{n}, q-\tilde{q}]$. The point below it on the lower boundary is $[n-\tilde{n}, (n-\tilde{n})\cdot\frac{q}{n}]$. Thus for the height we have

$$q - \tilde{q} - (n-\tilde{n})\cdot\frac{q}{n} = \frac{qn - \tilde{q}n - nq + \tilde{n}q}{n} = \frac{\langle[-\tilde{n}, -\tilde{q}], (-q, n)\rangle}{n} = \frac{1}{n}.$$

This means that $S(i)$ is the thinnest slice we can build using the hyperplane $(-q, n)$ and since we also know its wideness, we can easily control its behaviour under translations by $\nu(E^j_\Lambda)$):

**Proposition 6.18** (The behaviour of the thin slice $S$)**.** Let $0 < i, j \leq n$. Now take

$$S(i) := \{u \in \mathbb{Q}^2 \mid i < \langle u, \rho^0\rangle \leq n \text{ and } 0 < \langle u, \rho^1\rangle \leq 1\}.$$

the half open parralelepiped with vertices $[i, \frac{iq}{n}], [n, q], [n, q+\frac{1}{n}], [i, \frac{iq+1}{n}]$. Then we have the following formula

$$\Big[S(i) - \nu(K_\Lambda) + \nu(E^j_\Lambda)\Big] \cap \mathbb{Z}^2 = \begin{cases} [n, q] & j \leq n-i \\ \emptyset & \text{else.} \end{cases}$$

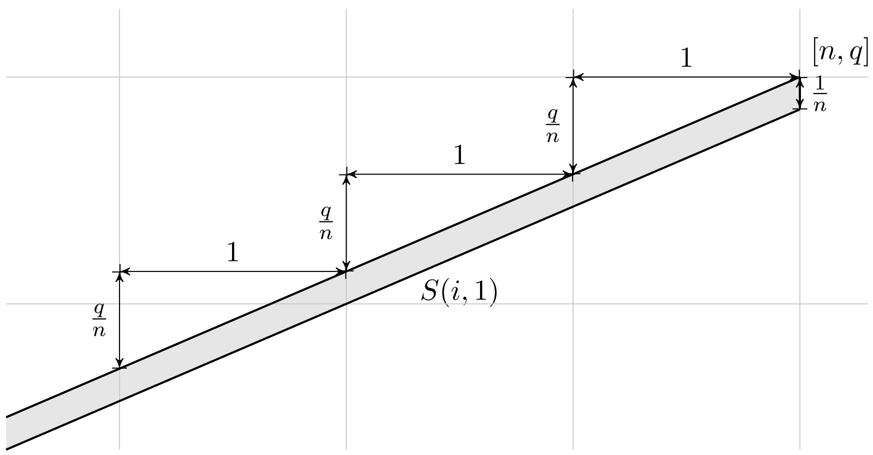
*Proof.* Denote by

$$S(i,j) := S(i) - \nu(K_\Lambda) + \nu(E^j_\Lambda) = S(i) - [1, \frac{q+1}{n}] + [j, \frac{jq}{n}].$$

For $j = 1$ the upper right vertex becomes

$$[n, q + \frac{1}{n}] - [1, \frac{q+1}{n}] + [1, \frac{q}{n}] = [n, q].$$

Now for increasing $j$ the top edge of $S(i,j)$ will slide along this lattice point in the direction $[1, q/n]$, i.e. the point on the top edge of $S(i, 1)$ which is one step to the left of $[n, q]$ has exactly $[n-1, q-\frac{q}{n}]$ as coordinates and adding $[1, q/n]$ to $S(i, 1)$ turns it into a lattice point:



Thus two questions remain: How often can we increase $j$ and keep $[n, q]$ inside $S(i, j)$? And: Why can there not be any other lattice point in $S(i, j)$?

The first question can be answered, knowing that the length of $S(i, j)$ in terms of the $x$-coordinate is $n - i$. This implies that there are $n - i + 1$ possible positions for $[n, q]$ on the top edge of $S(i, 1)$. But since the right boundary does not belong to $S(i, 1)$, we have to exclude one point and thus obtain $j \leq n - i$.

For the second question we first observe that any lattice point in $S(i, j)$ must of course evaluate to an integer with both $\rho^0$ and $\rho^1$. Thus candidates for such points must be on the boundary. Since the lower and the right boundary are excluded this only leaves the top edge. On the top edge the $x$-distance between two lattice points is always $n$ and thus there cannot be another lattice point, since $n - i \leq n - 1$. □

Coming from this we obtain the following lemma for our below-sets:

**Lemma 6.19.** Let $1 \leq i \leq n - \tilde{n}$ and $1 \leq j \leq n$. Then the following formula holds

$$E_\Lambda(i, j) \cap \mathbb{Z}^2 = \begin{cases} (E_\Lambda(i + \tilde{n}, j) - [\tilde{n}, \tilde{q}]) \cap \mathbb{Z}^2 \cup \{[n, q]\} & j \leq n - i \\ (E_\Lambda(i + \tilde{n}, j) - [\tilde{n}, \tilde{q}]) \cap \mathbb{Z}^2 & \text{else} \end{cases}$$

*Proof.* We have

$$\operatorname{below}(E^i) = S(i) \sqcup (\operatorname{below}(E^{i+\tilde{n}} - [\tilde{n}, \tilde{q}]).$$

Applying Proposition 6.18 yields the desired formula. □

*Remark* 6.20. As a small remark we have the inequality

$$n - (n - \tilde{n}) > n - \tilde{n}$$

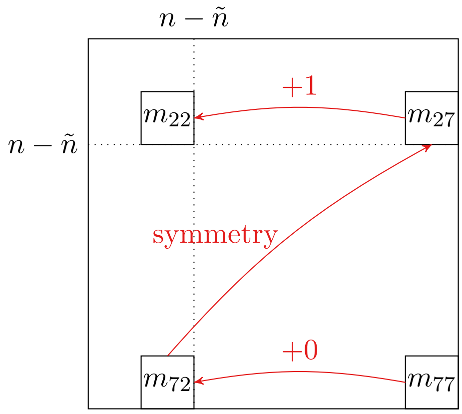
by the equations induced from the continued fractions. Essentially this means that the square of $\mathcal{E}_1$ corresponding to $n - \tilde{n} < i, j \leq n$ is bigger than the square $1 \leq i, j \leq n - \tilde{n}$.

A consequence of this is that we can compute the whole matrix $\mathcal{E}_1$ by just knowing the lower right square. We can express the columns $1 \leq i \leq n - \tilde{n}$ by the columns $\tilde{n} \leq i + \tilde{n} \leq n$. By the symmetry of Proposition 6.11 we can do the same thing for the columns. In particular, if we have $1 \leq i, j \leq n - \tilde{n}$, i.e. an entry of the upper left square, we obtain

$$\#(E_\Lambda(i, j) \cap \mathbb{Z}^2) = \begin{cases} \#(E_\Lambda(i + \tilde{n}, j + \tilde{n}) \cap \mathbb{Z}^2) + 2 & i + j \leq n - \tilde{n} \\ \#(E_\Lambda(i + \tilde{n}, j + \tilde{n}) \cap \mathbb{Z}^2) + 1 & n - \tilde{n} < i + j \leq n \\ \#(E_\Lambda(i + \tilde{n}, j + \tilde{n}) \cap \mathbb{Z}^2) & \text{else} \end{cases}$$

**Example 6.21.** We illustrate this principle in our running example:

$\mathcal{E}_1 =$




### The lower right square theorem

**Theorem 6.22** (Lower right square for $a_s > 2$)**.** *Let $n - \tilde{n} < i, j \leq n$, then the following formula holds*

$$E_\Lambda(i,j) \cap \mathbb{Z}^2 = \begin{cases} (E_\Gamma(i-n+\tilde{n}, j-n+\tilde{n}) + 2b^s) \cap \mathbb{Z}^2 \cup \{[n,q]\} & i+j \leq n \\ (E_\Gamma(i-n+\tilde{n}, j-n+\tilde{n}) + 2b^s) \cap \mathbb{Z}^2 & \textit{else.} \end{cases}$$

*Proof.* The strategy of the proof is the following: First we want to choose $j$ such that $\text{below}_\Gamma(E_\Gamma^{i-n+\tilde{n}}) + b^s \subseteq \text{below}_\Lambda(E_\Lambda^i)$, i.e. all other terms should cancel out. Then we observe what happens if we move our below-sets from that position by in- and decreasing $j$ such that we cover the whole spectrum needed.

Thus let us determine $j$ such that $-\nu(K_\Lambda) + \nu(E_\Lambda^j) \in \mathbb{Z}^2$. Since we now know that $b^s = [n-\tilde{n}, q-\tilde{q}]$ we can use the observations of Remark 4.9 to find that

$$K_\Lambda = E_\Lambda^{n-(n-\tilde{n})+1} - [n,q] + [n-\tilde{n}, q-\tilde{q}] = E_\Lambda^{\tilde{n}+1} - [\tilde{n}, \tilde{q}].$$

Hence, our $j$ should be $\tilde{n}+1$. For further computation we also need $K_\Gamma$ which can be determined in the same manner:

$$K_\Gamma = E_\Gamma^{\tilde{n}-(n-\tilde{n})+1} - [\tilde{n}, \tilde{q}] + [n-\tilde{n}, q-\tilde{q}] = E_\Gamma^{2\tilde{n}-n+1} - [2\tilde{n}-n, 2\tilde{q}-q].$$

Now we utilize the equation $[n,q] = [\tilde{n}, \tilde{q}] + b^s$ to determine where the points $[n,q]$ on the $\Lambda$-side and the point $[\tilde{n}, \tilde{q}]$ on the $\Gamma$-side are mapped to, for $j = \tilde{n}+1$:

$\Lambda$:

$$[n,q] - [1, \frac{q+1}{n}] + [\tilde{n}+1, \frac{(\tilde{n}+1)q}{n}] = [n+\tilde{n}, \frac{qn - 1 + \tilde{n}q - \tilde{q}n + \tilde{q}n}{n}] = [n+\tilde{n}, q+\tilde{q}].$$

$\Gamma$:

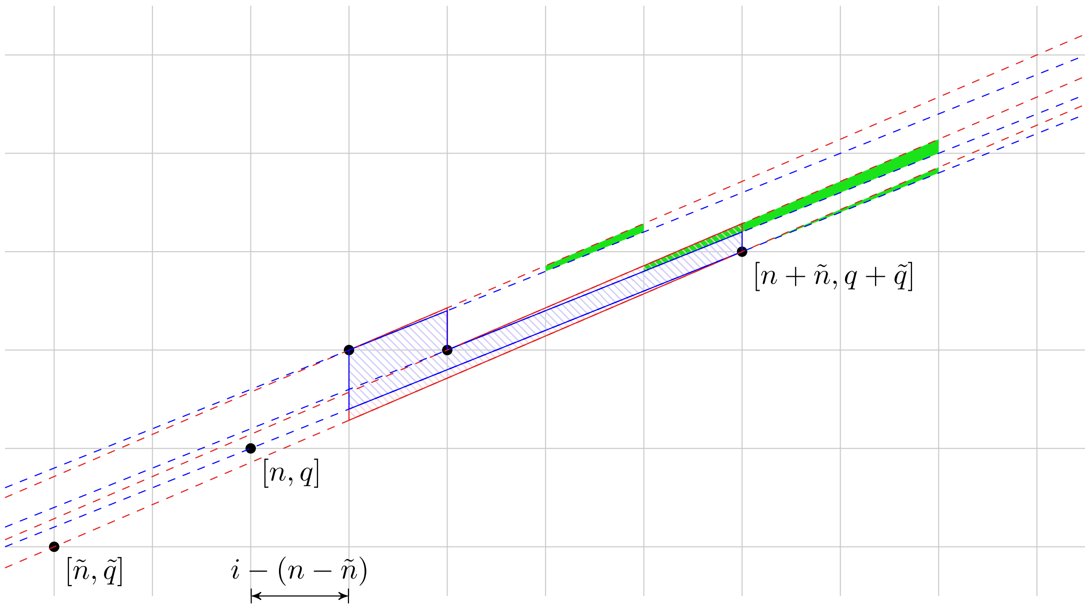
$$\begin{aligned}[\tilde{n},\tilde{q}] + 2b^s - [1,\frac{\tilde{q}+1}{\tilde{n}}] + [\tilde{n}+1-n+\tilde{n},\frac{(\tilde{n}+1-n+\tilde{n})\tilde{q}}{\tilde{n}}] \\ = [n+\tilde{n},\frac{2q\tilde{n}+\tilde{q}\tilde{n}-\tilde{q}n-1}{\tilde{n}}] \\ = [n+\tilde{n},\frac{2q\tilde{n}+\tilde{q}\tilde{n}-\tilde{q}n+\tilde{q}n-q\tilde{n}}{\tilde{n}}] \\ = [n+\tilde{n},q+\tilde{q}]\end{aligned}$$

Thus we see that for $j=\tilde{n}+1$ we have

$$E_\Lambda(i,\tilde{n}+1)\cap\mathbb{Z}^2 = [E_\Gamma(i-n+\tilde{n},2\tilde{n}-n+1)+2b^s]\cap\mathbb{Z}^2$$

In particular the left hand side is contained in the right hand side before intersecting with $\mathbb{Z}^2$. This is our starting point, we will now in- and decrease $j$ and observe the difference of the below-sets.

Let us visualize the situation:




First we calculate how far we have to move in each direction. The situation is illustrated for $j=\tilde{n}+1$ and we want $j$ to run from $n-\tilde{n}$ up to $n$. This means we have to decrease $j$ $2\tilde{n}-n$ times and we have to increase it $n-\tilde{n}-1$ times.

Decreasing $j$ in the picture means that the blue set corresponding to the right hand side of the equation will move to the left along the blue dashed lines and the red set will move along the red lines. If we increase $j$ the sets will move to the right along the dashed lines.

The important thing is that we can measure the difference of the translated sets in terms of the parallelepiped

$$P := \operatorname{conv}\{[0,0],[n,q],[\tilde{n},\tilde{q}],[n+\tilde{n},q+\tilde{q}]\}.$$

shifted by lattice points of $\mathbb{Z}^2$. By difference we mean the complement of the intersection of the sets in their union. As an example we have illustrated this difference in the picture for $j = \tilde{n} + 1 + 2$. One immediately observes that every segment of this difference is contained in a translation of $P$.

Let us do an example calculation: Starting from the lattice point $[n + \tilde{n}, q + \tilde{q}]$ the next (to the right) lattice point on the blue line is $[n + 2\tilde{n}, q + 2\tilde{q}]$ and on the red line we have $[2n + \tilde{n}, 2q + \tilde{q}]$. Since we know that $2\tilde{n} > n$ we have

$$n + \tilde{n} + (n - \tilde{n}) = 2n < n + 2\tilde{n},$$

which means that by moving $n - \tilde{n} - 1$ to the right we will never reach any of those lattice points. It also means that the difference below our red and blue set will always be contained in $P + [n + \tilde{n}, q + \tilde{q}]$ for increasing $j$ up to $n$ and since $P$ does not contain any lattice points besides its vertices we will not get any new lattice points in the blue set that lie below the red set.

One can apply similar arguments to the differences occurring above our sets for both increasing and decreasing $j$.

The only pitfall happens at the point $[n, q]$: If we decrease $j$ enough it will be on the boundary of the blue set which means that it is not contained. Since the blue sets boundary moves below the blue dashed line, it will contain $[n, q]$. After decreasing $j$ by $i - (n - \tilde{n})$ the point is on the boundary of both sets, decreasing $j$ further will move $[n, q]$ inside the red set and these $j$ can be described by

$$j \leq \tilde{n} + 1 - (i - (n - \tilde{n})) - 1 = n - i \quad \Longleftrightarrow \quad i + j \leq n.$$

Further points cannot occur for decreasing $j$ by arguing with translates of $P$ again. □

$a_s = 2$

### The setup

In this case we again have a continued fraction $\underline{a} = [a_1, \ldots, a_s = 2]$ and this time we want to forget about the last entry, i.e. $\underline{\tilde{a}} = [a_1, \ldots, a_{s-1}]$. Thus it is immediately obvious that $[\tilde{n}, \tilde{q}] = b^s$. But the inclusion of rings now goes in the other direction:

$$\Gamma := R(\tilde{n}, \tilde{q}) \hookrightarrow R(n, q) =: \Lambda$$

as one can easily observe on the level of cones. For the Hilbert bases this of course implies

$$H(\tilde{n}, \tilde{q}) = H(n, q) \backslash \{b^{s+1}\}.$$

An immediate observation is that for $i \leq \tilde{n}$ we have

$$G(E^i_\Lambda) = G(E^i_\Gamma) \cup \{b^{s+1}\}.$$

The strategy to follow is closely related to the $a_s > 2$ methods. First we can reduce to the upper left square of the matrix $\mathcal{E}_1$. Second we can translate the entries of the upper left square from the ring $\Lambda$ to the ring $\Gamma$.

### Reduction to the upper left square

Interestingly we can use Proposition 6.18 in this case as well. But this time we use it in the other direction, i.e. we want to reduce to the upper left square of the matrix $\mathcal{E}_1$. We obtain the following lemma:

**Lemma 6.23.** Let $\tilde{n} < i \leq n$ and $1 \leq j \leq n$. Then the following formula holds

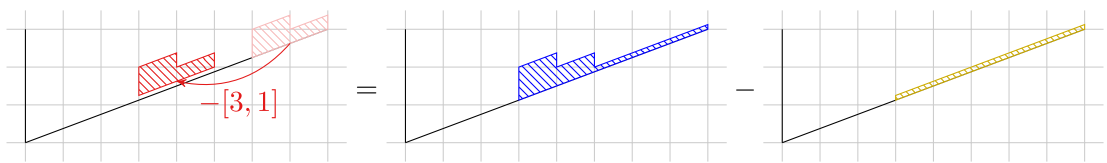
$$(E_\Lambda(i,j) - [n-\tilde{n}, q-\tilde{q}]) \cap \mathbb{Z}^2 = \begin{cases} (E_\Lambda(i-(n-\tilde{n}), j)) \cap \mathbb{Z}^2 \backslash \{[n,q]\} & j \leq n-i \\ (E_\Lambda(i-(n-\tilde{n}), j)) \cap \mathbb{Z}^2 & \text{else.} \end{cases}$$

*Proof.* We have

$$\text{below}_\Lambda(E^i_\Lambda) - [n-\tilde{n}, q-\tilde{q}] = (\text{below}_\Lambda(E^{i-(n-\tilde{n})}_\Lambda) \backslash S(i-(n-\tilde{n})).$$

Applying Proposition 6.18 yields the desired formula. □

**Example 6.24.** We illustrate this lemma with a new example which we will use throughout the remaining section. Of course this example has $a_s = 2$: As continued fraction we chose $\frac{n}{n-q} = \underline{a} = [2,3,2]$ and obtain $n = 8$ and $q = 3$. The Hilbert basis of the cone $\sigma^\vee$ has the elements $[0,1]$, $[1,1]$, $[2,1]$, $[5,2]$ and $[8,3]$. To illustrate the situation consider $E^6$ and $E^6 - [3,1]$:




### The upper left square theorem

**Theorem 6.25** (upper left square for $a_s = 2$)**.** *Let $1 \leq i, j \leq \tilde{n}$, then we have the formula*

$$E_\Lambda(i,j) \cap \mathbb{Z}^2 = \begin{cases} E_\Gamma(i,j) \cap \mathbb{Z}^2 \cup \{b^{s+1} = [n,q]\} & i+j \leq n \\ E_\Gamma(i,j) \cap \mathbb{Z}^2 & \textit{else.} \end{cases}$$

*Proof.* First we again want to move both sides into a position such that containment becomes obvious. Thus we choose $j$ such that $\nu(K_\Lambda) = \nu(E^j_\Lambda)$. The generators of $K_\Lambda$ are $[1,1]$ and $b^s = [\tilde{n}, \tilde{q}]$. Hence we get $j = n - \tilde{n} + 1$. Keeping the equation

$$\langle [\tilde{n}, \tilde{q}, (-q, n)] \rangle = -q\tilde{n} + \tilde{q}n = 1$$

in mind, let us compute where the generator $b^s$, which must be contained in both below sets, is mapped:

$\Lambda$:

$$\begin{aligned}[\tilde{n}, \tilde{q}] - [1, \frac{q+1}{n}] + [n-\tilde{n}+1, \frac{(n-\tilde{n}+1)q}{n}] &= [n, \frac{\tilde{q}n + q - 1 + nq - \tilde{n}q + q}{n}] \\ &= [n, \frac{\tilde{q}n + q\tilde{n} - \tilde{q}n + nq - \tilde{n}q}{n}] = [n,q]\end{aligned}$$

$\Gamma$:

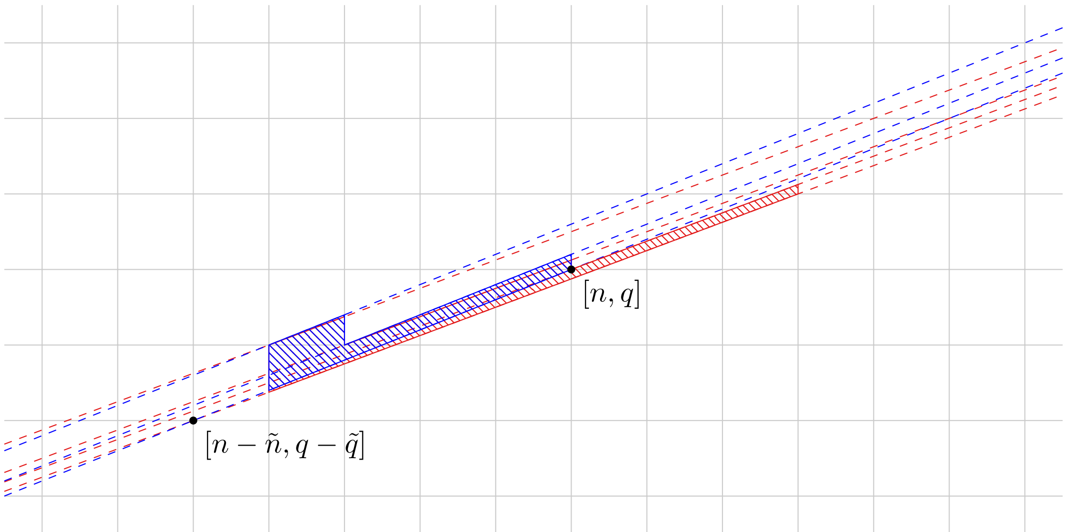
$$[\tilde{n}, \tilde{q}] - [1, \frac{\tilde{q}+1}{\tilde{n}}] + [n - \tilde{n} + 1, \frac{(n-\tilde{n}+1)\tilde{q}}{\tilde{n}}] = [n, \frac{\tilde{q}\tilde{n} - \tilde{q} - 1 + n\tilde{q} - \tilde{n}\tilde{q} + \tilde{q}}{\tilde{n}}]$$
$$= [n, \frac{q\tilde{n} - \tilde{q}n + n\tilde{q}}{\tilde{n}}] = [n, q]$$

In this situation the below-sets on both sides are simply shifted by a lattice vector and we are in the situation of Section 6.2. We only have to look at the generators and we observe that on the $\Lambda$-side there is one more generator and hence $b^s$ is contained in the right hand side and not in the left hand side.

The rest of the proof is analogous to the proof of Theorem 6.22: One has the parallelepiped

$$P := \operatorname{conv}\{[0,0], [\tilde{n}, \tilde{q}], [n, q], [\tilde{n}+n, \tilde{q}+q]\}$$

which does not contain any lattice points in the interior. The difference when shifting to one or the other side can be observed in terms of $P$ shifted by lattice vectors. The only lattice point which is contained in the $\Lambda$ side sometimes, but never in the $\Gamma$ side, is $b^s$. Let us illustrate this with a picture in the example for $i = 1$:




Again the dashed red lines mark the rails on which the red set moves when changing $j$ and the same for the blue lines. Once again one can compute how much we need to in- and decrease $j$ and then deduce that the appearing differences can be reformulated as subsets of $P$ shifted by some $u \in \mathbb{Z}^2$ and thus the only difference will always be the point $[n, q]$. Then analogously to the proof of Proposition 6.18 one controls the behaviour of this point to be the one stated in the theorem. □

### Summary and the algorithm

First we observe that Lemma 6.19 and Lemma 6.23 allow us to cover the matrix $\mathcal{E}_1(\underline{a})$ completely in both cases, namely $a_s > 2$ and $a_s = 2$. Second and surprisingly we see that in order to apply Theorem 6.25 we never needed the $a_s = 2$ condition. This is only necessary if we want the theorem to cover all of $\mathcal{E}_1(\underline{a})$, but if $a_s > 2$ we can still use

Theorem 6.25 to determine what the upper left square looks like. Of course we run into a problem if $s = 1$, i.e. if there is only one entry in the continued fraction.

Equivalently we are allowed to apply Theorem 6.22 to the $a_s = 2$ case to determine the lower right square. However, taking $a_s - 1$ in the new continued fraction turns it invalid, since it does not correspond to the Hilbert basis of the cone spanned by $[0,1]$ and $b^{s+1} - b^s$ anymore. But we can read off the desired continued fraction from the actual Hilbert basis of this cone. One could of course also just 'collapse' the 1 at the end of the continued fraction onto the previous entry:

$$[a_1, \dots, a_{s-1}, 1] = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \frac{1}{a_{s-1} - \frac{1}{1}}}} = [a_1, \dots, a_{s-1} - 1].$$

On the cone level this corresponds to having the equation

$$b^{n-1} + b^{n+1} = b^n \rightsquigarrow b^{n+1} = b^n - b^{n-1}$$

and thus $b^n$ is not needed for the Hilbert basis anymore. One can check that the theorem stays true. Still we run into a problem if our continued fraction consists only of 2's.

To solve both problems we define $\mathcal{E}_1([]) = \mathcal{E}_1([1]) = (0)$ to be the $1 \times 1$ zero matrix. The first case corresponds to our Hilbert basis having only two generators, namely $n = 1 > q = 0$ and thus the cone is smooth and the only divisor to consider is the trivial one. Applying the previous considerations, $[1]$ corresponds to taking $[2]$ and then modifying the Hilbert basis such that the last element becomes $[2,1] - [1,1] = [1,0]$. Again $[1,1]$ becomes superfluous and we are in the smooth case.

Thus we obtain the following theorem:

**Theorem 6.26.** *Let $\underline{a} = [a_1, \dots, a_s] = \frac{n}{q-n}$ be the continued fraction expansion of $\frac{n}{q-n}$. Then the matrix $\mathcal{E}_1(\underline{a})$ can be determined as*

$$\mathcal{E}_1(\underline{a}) = \begin{pmatrix} \mathcal{E}_1([a_1, \dots, a_{s-1}]) & A \\ A^T & \mathcal{E}_1([a_1, \dots, a_s - 1]) \end{pmatrix} + \begin{pmatrix} 1 & \cdots & 1 & 0 \\ \vdots & \iddots & \iddots & \vdots \\ 1 & \iddots & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix},$$

*where the matrix $A$ is completely determined by the larger quadratic matrix, see Theorem 6.28.*

*Proof.* The upper left matrix is a consequence of Theorem 6.25 and the lower right is a consequence of Theorem 6.22. □

**Definition 6.27** ($\mathcal{D}(n)$)**.** To have a short notation we define the second summand of this theorem to be $\mathcal{D}(n)$, where $n$ denotes the size of this quadratic matrix.

**Theorem 6.28.** *In the setting of Theorem 6.26, denote by*

$$\frac{n'}{n' - q'} = [a_1, \dots, a_{s-1}] \text{ and } \frac{n''}{n'' - q''} = [a_1, \dots, a_s - 1].$$

*Assume that* $a_s = 2$, *then*

$$A \;=\; (\mathcal{E}_1([a_1,\dots,a_{s-1}])_{ij}),\ \textit{where } i = 1,\dots,n',\ j = n'-n'',\dots,n'.$$

*Otherwise if,* $a_s > 2$

$$A \;=\; (\mathcal{E}_1([a_1,\dots,a_s-1])_{ij}),\ \textit{where } i = n''-n',\dots,n'',\ j = 1,\dots,n''.$$

*Proof.* This is a direct consequence of Theorem 6.25 and Theorem 6.22 as well. Please note that both $\mathcal{E}_1([a_1,\dots,a_{s-1}])$ and $\mathcal{E}_1([a_1,\dots,a_s-1])$ will always give a part of $A$. The bigger one is then able to give all entries of $A$. □

**Example 6.29.** The simplest example is $n = 2$ and $q = 1$. We obtain

$$\mathcal{E}_1([2]) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{E}_1([]) & A \\ A^T & \mathcal{E}_1([1]) \end{pmatrix}.$$

Both quadratic matrices on the diagonal have the same size, since we are in both boundary cases (Section 7.3) $q = 1$ and $q = n-1$ at the same time. Luckily they agree on that $A$ should be zero and we get

$$\mathcal{E}_1([2]) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Having this we can extend this example in two ways: By adding a 2 as last entry or by increasing the last entry by 1.

$$\mathcal{E}_1([2,2]) = \begin{pmatrix} \mathcal{E}_1([2]) & A \\ A^T & \mathcal{E}_1([2,1]) \end{pmatrix} + \mathcal{D}(3).$$

We notice that for the lower right corner we have to collapse the 1 onto the previous entry.

$$\mathcal{E}_1([2,2]) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

For the other case we obtain

$$\mathcal{E}_1([3]) = \begin{pmatrix} \mathcal{E}_1([]) & A \\ A^T & \mathcal{E}_1([2]) \end{pmatrix} + \mathcal{D}(3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We summarize this theorem in an algorithm. For the notation: If $A$ is a matrix, then $A[\bullet,\bullet]$ denotes the submatrix of $A$ given by lists of indices for the rows and columns.

Furthermore numRows$(A)$ denotes the number of rows of $A$.

**Algorithm 6.30:** Computing the matrix $\mathcal{E}_1(\underline{a})$

```
Input: a = [a_1, ..., a_s] the continued fraction representing n/(n-q)
Output: E_1(a)
begin
    if a = [1] or a = [] then
        // Recursion anchor
        return [[0]];

    if a_s = 1 then
        // The given continued fraction has a trailing 1 which we
           collapse onto the previous entry
        return E_1([a_1, ..., a_{s-1} - 1]);

    upperLeft := E_1([a_1, ..., a_{s-1}]);
    lowerRight := E_1([a_1, ..., a_s - 1]);
    if numRows(upperLeft) > numRows(lowerRight) then
        // The upper left corner matrix is bigger
        start := numRows(upperLeft) - numRows(lowerRight);
        end := numRows(upperLeft) - 1;
        A := upperLeft[All, start..end];
    else
        start := numRows(lowerRight) - numRows(upperLeft);
        end := numRows(lowerRight) - 1;
        A := lowerRight[start..end, All];

    size := numRows(upperLeft) + numRows(lowerRight);
    result := ( upperLeft   A          )
              ( A^T         lowerRight ) + D(size);
    return result
```

*Remark* 6.31. Theorem 6.25 and Theorem 6.22 can also be formulated via the continued fraction expansion of $\frac{n}{q} = \underline{a}$ which essentially just means switching the conditions. If $\underline{a} := \frac{n}{n-q}$ as usual, then removing the last entry of $\underline{a}$ corresponds to decreasing the last entry of $\underline{a}$ by one and vice versa. Hence also the algorithm may be formulated in terms of $\underline{a}$ instead of $\underline{a}$.

# First applications of the combinatorial method

## 7.1. The dimension $\dim \operatorname{Ext}^i$ for higher $i$

Combining the algorithm of Section 6.3 and the insights of Section 5.2 we can now compute the dimensions of higher Ext. Essentially it comes down to multiplying the incidence matrix of the quiver $\mathcal{R}$ with the matrix $\mathcal{E}_1$ containing the $\operatorname{Ext}^1$-dimensions. Since one has to be careful to keep the labeling consistent, let us be a bit more precise. From the quiver $\mathcal{R}$ we obtain the following formula:

$$\dim \operatorname{Ext}^2(E^i, D) = \sum_{j=1}^{n} \#\{\text{Arrows } E^j \to E^i\} \cdot \dim \operatorname{Ext}^1(E^j, D).$$

Thus let $\mathcal{I}$ be the matrix

$$\mathcal{I} = \left(a_{ij} := \#\{\text{Arrows } E^j \to E^i\}\right)_{i,j=1,\dots,n}.$$

**Definition 7.1.** We call $\mathcal{I}$ the *incidence matrix* of the quiver $\mathcal{R}$.

Now we obtain

$$\mathcal{I} \cdot \mathcal{E}_1 = \mathcal{E}_2.$$

Even better, replacing 2 by $k+1$ and 1 by $\mathbb{C}$ in the above formula, with $k > 0$, we get in general

$$\mathcal{I}^k \cdot \mathcal{E}_1 = \mathcal{E}_{k+1}.$$

Thus we can compute the dimension of arbitrary $\operatorname{Ext}^k$ by just multiplying matrices.

**Example 7.2.** Recalling the quiver displayed in Example 5.12 we obtain in the running example

$$\mathcal{I} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and we can now compute

$$\mathcal{E}_2 = \mathcal{I} \cdot \mathcal{E}_1 = \begin{pmatrix} 4 & 2 & 4 & 2 & 4 & 2 & 0 \\ 2 & 1 & 2 & 1 & 2 & 1 & 0 \\ 4 & 2 & 5 & 3 & 4 & 2 & 0 \\ 2 & 1 & 3 & 2 & 2 & 1 & 0 \\ 4 & 2 & 4 & 2 & 4 & 2 & 0 \\ 2 & 1 & 2 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

## 7.2. Ext and Tor

**Definition 7.3** (Tor matrix)**.** Let us define by

$$\mathcal{T}_k := \big(t_{ij} := \dim \operatorname{Tor}_k(E^i, E^j)\big)_{i,j=1,\dots,n},$$

the matrix containing the dimensions of $\operatorname{Tor}_1$.

In particular, $\mathcal{T}_k$ is symmetric for all $k > 0$, and higher $\operatorname{Tor}_k$ can be computed as $\mathcal{T}_{k+1} = \mathcal{I}^k \cdot \mathcal{T}_1$ as well.

Another observation is the following corollary:

**Corollary 7.4.** In the special CQS setting one has

$$\mathcal{I}^2 \cdot \mathcal{E}_1 = \mathcal{T}_1,$$

which is an immediate consequence of the following theorem:

**Theorem 7.5.** *For two $T$-invariant Weil divisors $D$ and $D'$ on $Y_{n,q}$ we have*

$$\operatorname{Ext}^3(D, K - D') = \operatorname{Tor}_1(D, D')^*,$$

*where $(\bullet)^* := \operatorname{Hom}_{\mathbb{C}}(\bullet, \mathbb{C})$ denotes the dual as graded $\mathbb{C}$-vector spaces.*

The strategy of the proof will be as follows: First we resolve $D$ freely, second we determine the kernel and image at index 1 in this resolution. Then we will tensor with $D'$ and compute the homology. This will culminate in a lemma combinatorially describing $\operatorname{Tor}_1(D, D')$.

Take the free resolution of Section 5.2 and resolve $D$:

$$F_2(D) \xrightarrow{d_2} F_1(D) \xrightarrow{d_1} F_0(D) \twoheadrightarrow D \to 0.$$

We want to tensor this complex with $D'$ and then take cohomology. This is easy, since the $F_i$ are free modules. Let us have a closer look at the kernels and images of the maps involved. Take the exact sequence of Theorem 5.8 to be

$$0 \to \bigoplus_{j=1}^{s} D^j \hookrightarrow F_0(D) \twoheadrightarrow D \to 0$$

then we can rewrite the above resolution as

$$\bigoplus_{D^j \in \mathrm{in}_{\mathcal{R}} D} F_1(D^j) \xrightarrow{d_2} \bigoplus_{D^j \in \mathrm{in}_{\mathcal{R}} D} F_0(D^j) \xrightarrow{d_1} F_0(D) \twoheadrightarrow D \to 0.$$

Now we determine the kernel of the map $d_1$ by applying the sequence of Theorem 5.8 to each $D^j$ separately:

$$0 \hookrightarrow \bigoplus_{D^j \in \mathrm{in}_{\mathcal{R}} D} \left( \bigoplus_{D^{jk} \in \mathrm{in}_{\mathcal{R}} D^j} D^{jk} \right) \to \bigoplus_{D^j \in \mathrm{in}_{\mathcal{R}} D} F_0(D^j) \xrightarrow{d_1} F_0(D) \twoheadrightarrow D \to 0.$$

Thus the kernel of $d_1$ is isomorphic to the direct sum of the $D^{jk}$.

Now fix one $D^{jk}$ and assume it arises as the kernel of a map

$$R[u] \oplus R[v] \to R, \ e^1 \mapsto x^u, \ e^2 \mapsto x^v,$$

which is a component of $d_1$. Denote by $u^0, \ldots, u^s$ the generators of $D^{jk}$. We can continue this complex

$$\bigoplus_{i=0}^{s} R[u^i] \to R[u] \oplus R[v] \to R.$$

Next we tensor with $D'$, which just means replacing $R$ by $D'$, since all modules involved are free. In particular, the kernel of the restricted $d_1 \otimes D'$ is just $\nu\, D' + D^{jk}$, and the image of $d_2$ consists of the $D'[u^i] = \nu\, D' + R[u^i]$, $i = 0, \ldots, s$. In our usual notation we would just add the vertex $\nu(D')$ to each $R$. Especially the kernel now becomes $\nu(D') + D^{jk}$ and we can compute the homology as

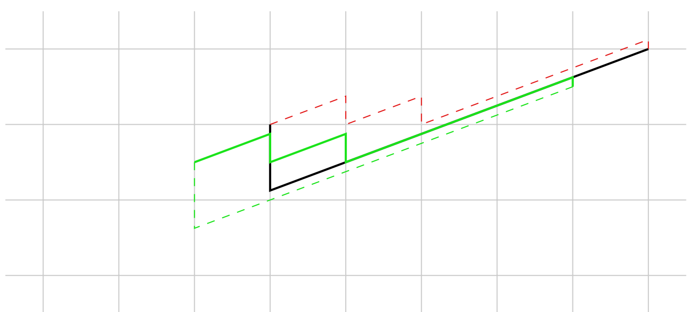
$$\left[ (\nu(D') + D^{jk}) \backslash \left( \bigoplus_{i=0}^{s} (\nu(D') + u^i + \sigma^\vee) \right) \right] \cap \mathbb{Z}^2.$$

Factoring out $\nu(D')$ gives us the following combinatorial invariant of a Weil divisor $Q$:

**Definition 7.6** (abelow-set)**.**

$$\mathrm{abelow}(Q) := Q \backslash \left( \bigoplus_{u \in Q \cap \mathbb{Z}^2} u + \sigma^\vee \right).$$

The abelow-set is the opposite of the below-set in the following sense: It contains the upper ragged boundary, but not the lower and left boundary. In the following picture we draw both the abelow and the below-set for the same divisor:



Note that in the picture we have moved the sets below and abelow in such a position that lattice points can only appear in their intersection. This will be explained in greater detail later and is exactly the key to the theorem.

The homology can now be rewritten as $(\nu(D') + \mathrm{abelow}(D^{jk})) \cap \mathbb{Z}^2$. Finally we are ready to compute $\mathrm{Tor}_1$.

**Lemma 7.7.** Let

$$T(Q, D') := \big(\nu(D') + \mathrm{abelow}(Q)\big) \cap \mathbb{Z}^2,$$

then

$$\mathrm{Tor}_1(D, D') = \bigoplus_{D^j \in \mathrm{in}_{\mathcal{R}} D} \left[ \bigoplus_{D^{jk} \in \mathrm{in}_{\mathcal{R}} D^j} \left( \bigoplus_{u \in T(D^{jk}, D')} \mathbb{C}\bar{x}^u \right) \right].$$

*Proof.* Consider the construction of abelow given above. To obtain $\mathrm{Tor}_1$ we need to take the direct sum over all $D^{jk}$ and these are obtained recursively from the quiver $\mathcal{R}$ exactly as in the Ext-case. □

*Proof of Theorem 7.5.* Remember that for the Ext computation we could proceed as follows:

$$\begin{aligned} \mathrm{Ext}^3(D, K - D') &= \bigoplus_{D^j \in \mathrm{in}_{\mathcal{R}} D} \mathrm{Ext}^2(D^j, K - D') \\ &= \bigoplus_{D^j \in \mathrm{in}_{\mathcal{R}} D} \left( \bigoplus_{D^{jk} \in \mathrm{in}_{\mathcal{R}} D^j} \mathrm{Ext}^1(D^{jk}, K - D') \right). \end{aligned}$$

Each of these modules can be expressed via the $\mathrm{below}(D^{jk})$-sets. If we again define

$$\mathrm{ext}(Q, K - D') := \big(\nu(K) - \nu(D') - \mathrm{below}(Q)\big) \cap \mathbb{Z}^2$$

then $\mathrm{Ext}^3$ is given as

$$\mathrm{Ext}^3(D, K - D') = \bigoplus_{D^j \in \mathrm{in}_{\mathcal{R}} D} \left[ \bigoplus_{D^{jk} \in \mathrm{in}_{\mathcal{R}} D^j} \left( \bigoplus_{u \in S(D^{jk}, D')} \mathbb{C}\bar{x}^u \right) \right].$$

Now we compare summandwise and observe that

$$\begin{aligned} -\,\mathrm{ext}(Q, K - D') &= [-\,\nu(K) + \nu(D') + \mathrm{below}(Q)] \cap \mathbb{Z}^2 \\ &= [\nu(D') + \mathrm{abelow}(Q)] \cap \mathbb{Z}^2 = T(Q, D'). \end{aligned}$$

This is due to the borders of the sets, the set $\mathrm{abelow}(Q)$ does contain its lower and left border but not the upper one, while the set $\mathrm{below}(Q)$ contains the upper and right border and not the lower one, since abelow was constructed from $\sigma^\vee$ and below was constructed from $\mathrm{int}(\sigma^\vee)$. Actually that is the only difference in the construction of these sets. What happens next is that these sets get shifted in such a way that lattice points can only live in their intersection.

Looking at the picture after the definition of abelow, we see: Shifting both these sets by $\nu(D')$ comes down to moving up or down in steps of $\frac{1}{n}$, since the first coordinate will always be an integer. The dashed green boundary does not belong to the sets, neither does the dashed red boundary. Hence we can obtain the green set by subtracting $[1, \frac{1}{n}]$ from the red set and then changing the containment of the boundaries. Thus one deduces that any lattice point contained in either of these sets must be contained in their intersection. □

The recursion of Theorem 5.16, that also holds in the case of Ext, allows us to generalize this result for higher $i$.

**Corollary 7.8.** In the situation of Theorem 7.5 we have

$$\mathrm{Ext}^{i+2}(D, K - D') = \mathrm{Tor}_i(D, D')^*,$$

for $i > 0$.

□

Thus our algorithm for computing $\mathcal{E}_1$, together with the algorithm for the quiver $\mathcal{R}$ also yields an efficient way to compute $\mathcal{T}_1$. Another interesting consequence of the theorem is that $\mathcal{E}_k$ must be symmetric for all $k > 0$.

*Remark* 7.9. In Section 8.5 we will describe the multiplication of the algebra $\mathrm{Ext}(D, D)$. Of course one could ask whether this is related to the algebra structure of Tor as described in [Eis95]. There are several reasons why this does not work: First of all we would need $D \otimes D \cong D$ to stay inside the algebra. Additionally if we consider the map

$$\mathrm{Tor}_1(D, D) \times \mathrm{Tor}_1(D, D) \to \mathrm{Tor}_2(D, D)$$

and then insert Ext via the isomorphism of Theorem 7.5 we see that this map becomes

$$\mathrm{Ext}^3(D, K - D) \times \mathrm{Ext}^3(D, K - D) \to \mathrm{Ext}^4(D, K - D).$$

The main problem now is that the first two terms are not compatible for the Ext-multiplication and on the right hand side we need $\mathrm{Ext}^6$ instead of $\mathrm{Ext}^4$, since the Ext-multiplication respects the grading.

## 7.3. The boundary cases $q = 1$ and $q = n - 1$

$q = 1$

Let $n, q \in \mathbb{Z}_{\geq 0}$ such that $n > q = 1$. The cone $\sigma^\vee$ is generated by the rays $[0, 1]$ and $[n, 1]$ and has the Hilbert basis

$$H(n, 1) = \{[0, 1], [1, 1], [2, 1], \ldots, [n - 1, 1], [n, 1]\}.$$

The Hilbert basis has exactly $n + 1$ elements and we can deduce that the continued fraction expansion is

$$\frac{n}{n-1} = [2, 2, \ldots, 2] =: [a_1, a_2, \ldots, a_{n-1}] = \underline{a}.$$

Using Theorem 6.26 we compute $\mathcal{E}_1(\underline{a})$:

$$\mathcal{E}_1(\underline{a}) = \begin{pmatrix} 1 & \cdots & 1 & 0 \\ \vdots & \iddots & \iddots & \vdots \\ 1 & \iddots & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} + \left(\begin{array}{ccc|c} & & & 0 \\ & \mathcal{E}_1([2,\dots,2=a_{n-1}]) & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right),$$

continuing recursively we obtain

$$\mathcal{E}_1(\underline{a}) = \begin{pmatrix} n-1 & n-2 & \cdots & 1 & 0 \\ n-2 & \iddots & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & & \vdots \\ 1 & \iddots & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

We can deduce the entries of the matrix $\mathcal{I}$ from the observations of Example 5.19:

$$\mathcal{I} = \begin{pmatrix} n-1 & 0 & \dots & 0 \\ n-2 & 0 & \dots & 0 \\ \vdots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

$q = n-1$

Let $n, q \in \mathbb{Z}_{\geq 0}$ such that $n > q = n-1$. The cone $\sigma^\vee$ is generated by the rays $[0,1]$ and $[n, n-1]$ and has the Hilbert basis

$$H(n,1) = \{[0,1],[1,1],[n,n-1]\}.$$

The Hilbert basis has exactly 3 elements and we can deduce that the continued fraction expansion is

$$\frac{n}{1} = [n] = [a_1] = \underline{a}$$

Again we compute $\mathcal{E}_1(\underline{a})$ via Theorem 6.26:

$$\mathcal{E}_1([n]) = \begin{pmatrix} 1 & \cdots & 1 & 0 \\ \vdots & \iddots & \iddots & \vdots \\ 1 & \iddots & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} + \left(\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \mathcal{E}_1([n-1]) & \\ 0 & & & \end{array}\right),$$

continuing recursively we obtain

$$\mathcal{E}_1([n]) = \begin{pmatrix} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 \\ \vdots & 2 & \cdots & \cdots & \cdots & 2 & \vdots & \vdots \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & & \cdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ \vdots & 2 & \cdots & \cdots & \cdots & 2 & \vdots & \vdots \\ 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

The graph $\mathcal{R}$ for this case has already been discussed as Example 5.20. Hence we obtain for $\mathcal{I}$:

$$\mathcal{I} = \begin{pmatrix} 0 & \cdots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \cdots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Next we identify the divisor $K_X$. Using Remark 4.9 we compute its vertex to be

$$\nu(K_X) = [1, \frac{q+1}{n}] = [1, \frac{n-1+1}{n}] = [1, 1]$$

and obtain $K_X \sim E^n \sim E^0$. Conveniently this yields

$$\mathrm{Ext}^i(D, R) = \mathrm{Ext}^i(D, K_X - E^0) = \mathrm{Ext}^i(E^0, K_X - D) = 0$$

and thus every divisor is special MCM for $q = n - 1$.

## 7.4. Classification of special MCM divisors

On cyclic quotient singularities there is a subclass of MCM modules that are interesting, the so-called special MCM modules.

**Definition 7.10** ([Nak13; IW10]: sMCM)**.** We call an MCM $R$-module *special* if and only if $(M \otimes_R \omega_R)/torsion$ is MCM.

Again we can establish a link with the Ext functor:

**Theorem 7.11** ([IW10])**.** *For a CM R-module M the following are equivalent:*

1. *M is sMCM*
2. $\mathrm{Ext}^1_R(M, R) = 0$.

If we assume $D$ to be a torus invariant divisor on a cyclic quotient singularity, we have the formula

$$\dim_k \operatorname{Ext}^1_R(D,R) = G(D) - 2.$$

Thus being sMCM for a torus invariant divisor $D$ means having at most two generators. If we assume $D$ to be non-trivial, i.e. $D \ncong R$, then $D$ has exactly two generators. This also agrees with [Wun87, Lemma 5].

Keeping in mind the Gorenstein case $q = n - 1$ we obtain the following classification of special MCM divisors on cyclic quotient singularities:

**Theorem 7.12.** *The following statements are equivalent:*

1. *$Y_{n,q}$ is Gorenstein.*
2. *All Weil divisors are special MCM.*

*Proof.* The direction $1 \Rightarrow 2$ has been shown in the boundary case examples.

Now assume that $Y_{n,q}$ is not Gorenstein. Then, by the continued fraction expansion of subsubsection 4.1, we obtain that the Hilbert basis of $\sigma^\vee$ has more than three elements. Considering the construction of Proposition 2.6 with $D = E^1$, we see that $E^1$ has at least three generators, exactly the points of the Hilbert basis of $\sigma^\vee$ without $[0,1]$. This is due to $[0,0,1]$ not being in the Hilbert basis of $C_D$. $\square$

# The Ext-algebra

In this chapter we want to describe the multiplication in the algebra $\operatorname{Ext}(D)$. We start by repeating Yoneda's description of $\operatorname{Ext}^i(D, D')$ as exact sequences. Subsequently we are able to describe the elements of $\operatorname{Ext}^i(D, D')$ as a path in $\mathcal{R}$ and a degree $u \in M$.

This reminds one of the construction of path algebras, though the final result Corollary 8.22 shows that the condition of path algebras for paths to concatenate well becomes a larger containment condition. Additionally the degree has to be considered as well. Thus $\operatorname{Ext}(D)$ is not a path algebra. Since not every quiver is the resolution quiver of a CQS, it is also false that the $\operatorname{Ext}(D)$ generalize path algebras. A concept containing both path algebras and the algebras $\operatorname{Ext}(D)$ needs to generalize toric rings as well, since $\mathbb{C}[\sigma^\vee \cap M] = \operatorname{Ext}(R)$. Furthermore it needs to take the thoughts of Section 5.3 into account.

Finally we will have a look at how one can find generators of $\operatorname{Ext}(D)$ and finish our example.

## 8.1. Yoneda's interpretation of Ext

Let us first review the construction in general following the procedures of [Eis95]. Given two $R$-modules $M$ and $N$, and an element $e \in \operatorname{Ext}^k_R(M, N)$ we obtain an exact sequence in the following manner: Take a free resolution

$$F_\bullet : \; \ldots \to F_{k+1} \to F_k \to F_{k-1} \to \ldots \to F_1 \to F_0 \twoheadrightarrow M \to 0$$

of $M$, then applying $\operatorname{Hom}_R(\bullet, N)$ and taking the $\mathbb{C}$-th cohomology yields $\operatorname{Ext}^k_R(M, N)$. Let us name the differentials $d_k : F_k \to F_{k-1}$. Then $e$ is an element of

$$\ker(d^*_{k+1} : \operatorname{Hom}(F_k, N) \to \operatorname{Hom}(F_{k+1}, N)) \big/ \operatorname{im}(d^*_k : \operatorname{Hom}(F_{k-1}, N) \to \operatorname{Hom}(F_k, N)) .$$

Hence we pick a representative for $e$ in $\operatorname{Hom}(F_k, N)$ which we will denote by $e$ as well. Now we obtain the following diagram:

$$\begin{tikzcd}
F_{k+1} \arrow[r, "d_{k+1}"] & F_k \arrow[r, "d_k"] \arrow[d, "e"] & F_{k-1} \\
& N &
\end{tikzcd}$$

The morphism $e$ being in the kernel of $d^*_{k+1}$ means $e \circ d_{k+1} = 0$. Thus $e$ is defined on classes modulo the image of $d_{k+1}$ and we obtain by exactness of the free resolution:

$$F_k \big/ \operatorname{im}(d_{k+1}) = F_k \big/ \ker(d_k) = \operatorname{im}(d_k) = \ker(d_{k-1}).$$

Therefore $e$ is a map $e : \ker(d_{k-1}) \to N$. The exact sequence associated to $e$ is the second row in the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & \ker(d_{k-1}) \arrow[r, "i"] \arrow[d, "e"] & F_{k-1} \arrow[r, "d_{k-1}"] & F_{k-2} \arrow[r, "d_{k-2}"] & \cdots \arrow[r, "d_1"] & F_0 \arrow[r, "d_0"] & M \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[r] & \operatorname{coker}(-e, i) \arrow[r] & F_{k-2} \arrow[r, "d_{k-2}"] & \cdots \arrow[r, "d_1"] & F_0 \arrow[r, "d_0"] & M \arrow[r] & 0
\end{tikzcd}.$$

Of course we chose a representative for $e$ and other choices will yield different exact sequences. Identifying these sequences means factoring out by Yoneda equivalence:

**Definition 8.1** (Yoneda equivalence)**.** Two exact sequences of $R$-modules are said to be Yoneda equivalent if they fit into the below commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, equal] & A_k \arrow[r] \arrow[d] & \cdots \arrow[r] & A_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & B_k \arrow[r] & \cdots \arrow[r] & B_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}.$$

*Remark* 8.2. As already stated in [Eis95] Yoneda equivalence does not pose an equivalence relation in that it is not symmetric. Instead we will consider the equivalence relation generated by Yoneda equivalence.

One can then define an $R$-module structure on the space of sequences of constant length $\mathbb{C}$ modulo Yoneda equivalence, yielding an isomorphism with $\mathrm{Ext}^k_R(M, N)$. Since this has already been described thoroughly in [Eis95] we will just remark that we already have an $R$-module structure on $\mathrm{Hom}(F_k, N)$ which translates to the space of exact sequences of length $\mathbb{C}$ modulo Yoneda equivalence seamlessly via the above construction.

Next we obtain a multiplication map

$$\mu : \ \mathrm{Ext}^n_R(B, C) \times \mathrm{Ext}^m_R(A, B) \to \mathrm{Ext}^{n+m}_R(A, C)$$

defined by

$$([0 \to C \to \underline{X} \xrightarrow{p} B \to 0], [0 \to B \xrightarrow{i} \underline{Y} \to A \to 0]) \mapsto [0 \to C \to \underline{X} \xrightarrow{i \circ p} \underline{Y} \to A \to 0].$$

In particular this construction equips the direct sum

$$\mathrm{Ext}(D, D) := \bigoplus_{i \geq 0} \mathrm{Ext}^i(D, D)$$

with the structure of a $\mathbb{C}$-algebra.

## 8.2. The homogeneous elements of $\mathrm{Ext}^k(D, D')$

In Section 6.1 we described $\mathrm{Ext}^1(D, D')$ as a $\mathbb{Z}^2$-graded vector space. We want to combine this description with the recursion developed in Section 5.2. The goal is to obtain a description of the monomials of the ring $\mathrm{Ext}(D)$.

Let $S \subseteq \mathbb{Z}^2$ such that

$$\operatorname{Ext}^1(D, D') = \bigoplus_{u \in S} \mathbb{C} \cdot \bar{x}^u.$$

Then, for each homogeneous generator $\bar{x}^u$, we obtain an exact sequence

$$\bar{x}^u : \; 0 \to D' \to X(\bar{x}^u) \to D \to 0,$$

which is unique, up to Yoneda equivalence, i.e. we pick a representative and $X(\bar{x}^u)$ denotes a suitable module.

Knowledge of $S$ immediately yields $\operatorname{Ext}^1(D, D')$, but the exact sequences contain more information: They tell us which $\operatorname{Ext}^1$ they belong to. Thus we add this information by considering $(u, D, D')$ instead of $u$. This approach immediately suggests how to represent the generators of higher $\operatorname{Ext}^k(D, D')$.

Take for example $k = 2$: There is a short exact sequence

$$0 \to \bigoplus_{i=1}^{s} D^i \to R^s \twoheadrightarrow D \to 0$$

and hence

$$\operatorname{Ext}^2(D, D') = \bigoplus_{i=1}^{s} \operatorname{Ext}^1(D^i, D').$$

We already know how to describe the generators on the right hand side individually, but they lack the information which $D$ they correspond to, i.e. the information that they are considered as elements of an $\operatorname{Ext}^2$ and not $\operatorname{Ext}^1$. Now each element $(u, D^i, D') \in \operatorname{Ext}^2(D, D')$ arises via a path $D^i \to D$ in the graph $\mathcal{R}$. Thus we add this path as information instead of $D^i$ and obtain $(u, D^i \to D, D') \in \operatorname{Ext}^2(D, D')$.

Of course $D$ does not have to be a vertex of $\mathcal{R}$. But there is an $E^j$ that is linearly equivalent to $D$ and thus we write $D = E^j + u$. Now this $E^j$ has incoming arrows from several $E^i$ that are labeled with $u^i \in \mathbb{Z}^2$, respectively. These are the candidates for the $D^i$ mentioned, but instead of storing the shift with the arrow we move it back into $E^i$. Thus the arrow $E^i \to E^j$ labeled with $u^i$ translates to the arrow $D^i := (E^i + u^i + u) \to (E^j + u) = D$.

We can generalize this approach to higher $\operatorname{Ext}^k$ by taking longer paths. The description for $\operatorname{Ext}^1$ fits well into this strategy: If the path has length 0 we get an element of $\operatorname{Ext}^1$ representing a class of short exact sequences. It all comes down to the following definition:

**Definition 8.3** (Path of $\mathcal{R}$)**.** A path of $D^k \to \ldots \to D^1 \to D$ is a sequence of $T$-invariant Weil divisors such that there is the labeled path in $\mathcal{R}$

$$E^{i_k} \xrightarrow{u^{i_k}} E^{i_{k-1}} \xrightarrow{u^{i_{k-1}}} \cdots \xrightarrow{u^{i_3}} E^{i_2} \xrightarrow{u^{i_2}} E^{i_1} \xrightarrow{u^{i_1}} E^{i_0} \quad ,$$

where we have

$$D = E^{i_0} + u \text{ for some } u := u^{i_0} \in \mathbb{Z}^2$$

and

$$D^j = E^{i_j} + \sum_{i=0}^{j} u^{i_j}.$$

Simply put, this means that $F_0(D^j)$ is a direct summand of $F_1(D^{j-1})$. Recursively we then obtain that $F_0(D^j)$ is a direct summand of $F_j(D)$.

**Definition 8.4** (Monomial of $\mathrm{Ext}^{k+1}(D, D')$)**.** We call the elements

$$(u, D^k \to \ldots \to D^1 \to D, D') \in \mathrm{Ext}^{k+1}(D, D')$$

*monomials* of degree $k + 1$.

This resembles the fact that in a polynomial ring the monomials are the homogeneous generators of the ring as a $\mathbb{C}$-vector space. Since $\mathrm{Ext}^0(D, D) = R$ we now have two different kinds of monomials in $\mathrm{Ext}(D)$, namely those of $R$, that now live in degree 0, and those described above. Furthermore, there is a second finer grading on $\mathrm{Ext}(D)$ than the $\mathbb{Z}$-grading: Since all modules $\mathrm{Ext}^i(D, D)$ are $M$-graded, we get a grading of $\mathrm{Ext}(D, D)$ by $\mathbb{Z} \times M$. It is clear that the multiplication respects the $\mathbb{Z}$-grading.

Thus, the question is, whether the monomials really behave like monomials in a polynomial ring. This question splits into the following parts:

1. Is the product of two monomials again $\mathbb{Z} \times M$-homogeneous?
2. And, if the first answer is positive, is the product of two monomials again a monomial?
3. Is the multiplication commutative?

*Remark* 8.5. Alternatively to the approach via Yoneda equivalence classes of exact sequences one can characterize Ext in terms of derived categories, see for example [Wei94]. There elements of Ext correspond to morphisms of complexes. For an element to be homogeneous means for the morphisms to be homogeneous. Hence, with this approach it immediately becomes clear that the product of homogeneous elements stays homogeneous. However, in order to find an answer to the second question one encounters exactly the same calculations as for the Yoneda approach.

## 8.3. $\mathrm{Ext}^1$ as short exact sequences

In order to answer these questions we will first have to find exact sequences that serve as representatives of the equivalence classes defined by monomials. We will start with degree 1 monomials.

Please note that Lemma 8.6 re-explains the gluing of Theorem 6.1 in terms of short exact sequences and Yoneda equivalence. Furthermore, Example 8.8 additionally explains the vanishing for the boundary generators $u^0$ and $u^s$ in $G(D)$. Everything but the construction in this section serves the purpose to demonstrate that we get the right thing.

Assume we are given a monomial $(u, D, D')$. This corresponds to the element $\bar{x}^u \in \mathrm{Ext}^1(D, D')$. In order to construct a short exact sequence from this element we need to find out which map $\ker(d_0) \to D'$ is associated to $\bar{x}^u$. Thus we revisit the construction of $\mathrm{Ext}^1$ in 6.1

First we assume $D$ to have the sorted generators $\{u^0, \dots, u^s\}$ and receive a short exact sequence

$$0 \to \bigoplus_{i=1}^{s} D^i \to F_0(D) := \bigoplus_{i=0}^{s} R[u^i] \twoheadrightarrow D \to 0$$

which encodes the first step of the free resolution of $D$. Applying $\mathrm{Hom}(\bullet, D')$ yields the sequence

$$0 \to \mathrm{Hom}(D, D') \to \mathrm{Hom}(F_0(D), D') \to \mathrm{Hom}(\bigoplus_{i=1}^{s} D^i, D') \to \mathrm{Ext}^1(D, D') \to 0$$

and thus the formula

$$\mathrm{Ext}^1(D, D') = \left. \mathrm{Hom}(\bigoplus_{i=1}^{s} D^i, D') \middle/ \mathrm{Hom}(F_0(D), D') \right. .$$

The degree $u$ yields $\bar{x}^u$ which should give us a map $\varphi^u : \bigoplus_{i=1}^{s} D^i \to D'$. We already observed that

$$\mathrm{Hom}(\bigoplus_{i=1}^{s} D^i, D') = \bigoplus_{i=1}^{s} \mathrm{Hom}(D^i, D') = \bigoplus_{i=1}^{s} (-\nu(D^i) + D').$$

Let us assume for the moment that $u \in (-\nu(D^1)+D')$ but not in any other $(-\nu(D^i)+D')$, $i \geq 2$. The map defined by $\varphi^u|_{D^1} : D^1 \to D'$ is given by multiplication with the monomial $\bar{x}^u \in \mathbb{C}[\mathbb{Z}^2]$, in other words, by a degree shift of $u$. The other maps $\varphi^u|_{D^i} : D^i \to D'$ must be zero and this completes the construction. In particular if $u$ is only contained in one of the $(-\nu(D^i) + D')$ we can apply the same principle.

Let us modify this representation a little by transferring the degree from the map to $D'$ and considering $D'[-u]$ instead of $D'$. For such $u$, $\varphi^u|_{D^1} : D^1 \hookrightarrow D'[-u]$ becomes the canonical embedding and remains zero on the other $D^i$.

Having found a description of $\varphi^u$ as a map

$$\varphi^u : \bigoplus_{i=1}^{s} D^i = \ker(d_0) \to D'[-u]$$

we go on defining the associated short exact sequence $\alpha(\varphi^u)$

$$\begin{tikzcd}
 & 0 \arrow[r] & \ker(d_0) = \bigoplus_{i=1}^{s} D^i \arrow[r, "i"] \arrow[d, "\varphi^u"] & F_0(D) \arrow[r, "d_0"] \arrow[d] & D \arrow[r] & 0 \\
\alpha(\varphi^u) : & 0 \arrow[r] & D'[-u] \arrow[r] & \mathrm{coker}(-\varphi^u, i) \arrow[r, "\bar{d}_0"] & D \arrow[r] & 0
\end{tikzcd}$$

The middle module in the sequence $\alpha(\varphi^u)$ can be computed as

$$\begin{aligned}
\mathrm{coker}(-\varphi^u, i) &= \left. D'[-u] \oplus F_0(D) \middle/ \mathrm{im}(-\varphi^u, i) \right. \\
&= \left. D'[-u] \oplus F_0(D) \middle/ \left\{ \begin{array}{c|c} (-x^u, x^{u-u^{i-1}} e^{i-1} - x^{u-u^i} e^i) & x^u \in D^i,\ \varphi^u|_{D^i} : D^i \hookrightarrow D'[-u] \\ (0, x^{u-u^{i-1}} e^{i-1} - x^{u-u^i} e^i) & x^u \in D^i,\ \text{all other } D^i \end{array} \right\} \right.
\end{aligned}$$

We will use the following shorthand notation for future reference:

$$\operatorname{coker}(-\varphi^u, i) = D'[-u] \oplus F_0(D) \Big/ \{(-D^i, D^i)\}.$$

Keep in mind, that there is more in the denominator, namely $(0, D^j)$ for all $j \neq i$.

What if $u$ is contained in several of the $(-\nu(D^i) + D')$? Simply choose one of these $i$ and set $\varphi^u|_{D^i} : D^i \hookrightarrow D'[-u]$. Then set $\varphi^u|_{D^j} = 0$ for all other $j \neq i$. The following lemma explains why this is independent of the choice of $i$:

**Lemma 8.6.** Assume $u \in (-\nu(D^1) + D') \cap (-\nu(D^2) + D')$. We obtain two maps

$$\begin{array}{cccc} \varphi_1^u : & D^1 \oplus D^2 \oplus \left(\bigoplus_{i=3}^s D^i\right) & \to & D'[-u] \\ & (a, b, \underline{c}) & \mapsto & a \end{array}$$

and

$$\begin{array}{cccc} \varphi_2^u : & D^1 \oplus D^2 \oplus \left(\bigoplus_{i=3}^s D^i\right) & \to & D'[-u] \\ & (a, b, \underline{c}) & \mapsto & b \end{array}.$$

Then the sequences $\alpha(\varphi_1^u)$ and $\alpha(\varphi_2^u)$ are Yoneda equivalent.

*Proof.* It is enough to prove this statement for $D$ having three generators, just as in the proof of Theorem 6.1. Thus we have an exact sequence

$$0 \to D^1 \oplus D^2 \to R[u^0] \oplus R[u^1] \oplus R[u^2] \twoheadrightarrow D \to 0.$$

Having the two maps $\varphi_1^u$ and $\varphi_2^u$ yields on the polyhedral side $D^1 \subseteq D'[-u]$ and $D^2 \subseteq D'[-u]$. Since $D'[-u] = \nu(D') - u + \sigma^\vee$ this implies $R[u^1] \subseteq D'[-u]$, because $R[u^1]$ is the minimal translate of $\sigma^\vee$ containing both $D^1$ and $D^2$. Hence we obtain a map

$$f : R[u^0] \oplus R[u^1] \oplus R[u^2] \to D'[-u],\ (a, b, c) \mapsto b,$$

where this map is meant homogeneous of degree 0, i.e. $(0, 1, 0)$ gets mapped to $x^{u^1}$. Finally we can construct a map $F : \operatorname{coker}(-\varphi_1^u, i) \to \operatorname{coker}(-\varphi_2^u, i)$:

$$\begin{array}{cccc} F : & D'[-u] \oplus F_0(D) \Big/ \{(-D^1, D^1)\} & \longrightarrow & D'[-u] \oplus F_0(D) \Big/ \{(-D^2, D^2)\} \\ & (g, (a, b, c)) & \mapsto & (g - f(b), (a, b, c)) \end{array}.$$

What remains to check is the well-definedness. Fix a homogeneous element of the denominator $(-x^v, (x^{v-u^0}, -x^{v-u^1}, 0))$.

$$\begin{aligned} F(-x^v, (x^{v-u^0}, -x^{v-u^1}, 0)) &= (-x^v - f(-x^{v-u^1}), (x^{v-u^0}, -x^{v-u^1}, 0)) \\ &= (-x^v + x^v, (x^{v-u^0}, -x^{v-u^1}, 0)) \\ &= (0, (x^{v-u^0}, -x^{v-u^1}, 0)). \end{aligned}$$

Thus homogeneous elements of $(-D^1, D^1)$ get mapped to $(0, D^1)$ which is zero. The other type of homogeneous elements are those of $(0, D^2)$.

$$\begin{aligned} F(0, (0, x^{v-u^1}, -x^{v-u^2})) &= (-f(x^{v-u^1}), (0, x^{v-u^1}, -x^{v-u^2})) \\ &= (-x^v, (0, x^{v-u^1}, -x^{v-u^2})). \end{aligned}$$

Therefore elements of type $(0, D^2)$ get mapped to elements of type $(-D^2, D^2)$ which are zero on the right hand side. Hence we have proven that $F$ is well-defined. It is obvious that $F$ commutes with the identities of the Yoneda equivalence diagram and we are done. □

Of course it was already clear on an abstract level that the sequences associated to $\varphi_1^u$ and $\varphi_2^u$ had to be Yoneda equivalent by construction of $\mathrm{Ext}^1$. To be precise, considering the difference map $\varphi_1^u - \varphi_2^u$, this map is exactly $f$ concatenated with the embedding $D^1 \oplus D^2 \hookrightarrow R[u^0] \oplus R[u^1] \oplus R[u^2]$. Hence $\varphi_1^u - \varphi_2^u$ is an element of the image $\mathrm{Hom}(F_0(D), D')$ and thus becomes zero in $\mathrm{Ext}^1$.

We illustrate some details of the this proof in Example 8.8.

*Remark* 8.7. The lemma does not depend on the particular indices $D^1$ and $D^2$. We can apply it for different $D^i$ and $D^{i+1}$ as well as for $u$ being contained more than two $D^i$.

The lemma demonstrates that the map that associates to a degree $u \in \mathrm{Ext}^1(D, D')$ an equivalence class of short exact sequences $\alpha(u) = [\alpha(\varphi^u)]$ is well-defined, which we already knew beforehand, by Yoneda's description of Ext.

The situation described in the lemma can indeed occur. Let us have a look at an example and illustrate the Yoneda equivalence of the sequences we obtain if we find a degree $u$ satisfying the precondition of the lemma.

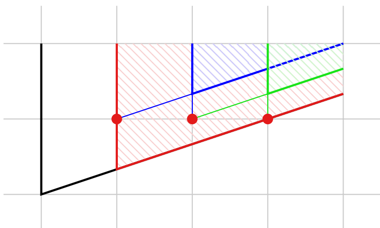
**Example 8.8.** Take $n = 3$, $q = 1$. Furthermore fix $D = E^1$ and $D' = E^0$. Then $D$ is generated by three monomials

$$x^{u^0}, x^{u^1} \text{ and } x^{u^2}, \text{ with } u^0 = [1, 1],\ u^1 = [2, 1],\ u^2 = [3, 1].$$

Set $D^1 := E^1[1, 1]$ and $D^2 := E^1[2, 1]$, then we obtain an exact sequence

$$0 \to D^1 \oplus D^2 \to R[1, 1] \oplus R[2, 1] \oplus R[3, 1] \twoheadrightarrow D \to 0.$$

The situation is illustrated in the following picture.



With the method of 6.1 we obtain

$$\mathrm{Ext}^1(E^1, E^0) = \mathbb{C} \cdot \bar{x}^{[-2,-1]}$$

and hence we pick $u = [-2, -1]$. Now we calculate the vertices of all divisors involved.

$$\begin{aligned}
\nu(E^1) &= [1, \tfrac{1}{3}] \\
\nu(D^1) &= \nu(E^1) + [1, 1] = [2, \tfrac{4}{3}] \\
\nu(D^2) &= \nu(E^1) + [2, 1] = [3, \tfrac{4}{3}].
\end{aligned}$$

Adding the degree $u$ to the last two vertices clearly demonstrates that both $D^1$ and $D^2$ are contained in $E^0[-u]$. We are in the situation which allows us two choose between two different maps for building the exact sequence $\alpha(u)$. Thus we demonstrate that both choices yield Yoneda equivalent short exact sequences. First define the two maps:

$$\begin{array}{rccc} \varphi_1^u: & D^1 \oplus D^2 & \to & D'[-u] \\ & (a,b) & \mapsto & a \end{array} \quad \text{and} \quad \begin{array}{rccc} \varphi_2^u: & D^1 \oplus D^2 & \to & D'[-u] \\ & (a,b) & \mapsto & b \end{array} .$$

One immediately notices that the difference of these two maps lifts to a map $F_0(D) \to E^0[2,1]$ and thus we write down the following diagram to demonstrate the Yoneda-equivalence of the two resulting exact sequences:

$$\begin{tikzcd}
\alpha(\varphi_1^u): & 0 \arrow[r] & E^0[2,1] \arrow[r, hook] \arrow[d, "Id"] & E^0[2,1] \oplus F_0(D) \big/ \{(-D^1, D^1)\} \arrow[r, two heads] \arrow[d, "{\substack{(a,\, b_0e^0+b_1e^1+b_2e^2) \\ \downarrow \\ (a - b_1x^{u^1},\, b_0e^0+b_1e^1+b_2e^2)}}"] & D \arrow[r] \arrow[d, "Id"] & 0 \\
\alpha(\varphi_2^u): & 0 \arrow[r] & E^0[2,1] \arrow[r, hook] & E^0[2,1] \oplus F_0(D) \big/ \{(-D^2, D^2)\} \arrow[r, two heads] & D \arrow[r] & 0
\end{tikzcd},$$

where we denote by $b_0e^0 + b_1e^1 + b_2e^2$ an element of $F_0(D)$ with $\deg(e^i) = u^i$. The only thing left to check is the well-definedness of the map in the center. For example we have $(x^u, -x^{u^0-u}e^0 + x^{u^1-u}e^1) = 0$ for $u \in D^1$ in the first row. The image under the central map is

$$\begin{aligned} (x^u, -x^{u^0-u}e^0 + x^{u^1-u}e^1) &\mapsto (x^u - x^{u^1-u}x^u, -x^{u^0-u}e^0 + x^{u^1-u}e^1) \\ &= (0, -x^{u^0-u}e^0 + x^{u^1-u}e^1), \end{aligned}$$

which is zero in the second row.

Another thing we want to demonstrate is that $\alpha(u)$ splits for choices outside of $\operatorname{Ext}^1(D, D')$. If $u \notin (-\nu(D^1) + D') \cup (-\nu(D^2) + D')$ this is easy, since then $\varphi^u = 0$. Hence we pick $u = [-1,-1]$. This choice yields the inclusion $D^1 \hookrightarrow E^0[1,1]$. Looking at the picture from the beginning of this example, we realize that even $R[-u^0] \subseteq E^0[1,1]$. Hence we can define the map

$$\begin{array}{ccc} E^0[1,1] \oplus \operatorname{coker} d_1^D & \longrightarrow & E^0[2,1] \oplus F_0(D) \big/ \{(-D^1, D^1)\} \\ \\ (a, b_0e^0 + b_1e^1 + b_2e^2) & \longmapsto & (a - b_0x^{u^0}, b_0e^0 + b_1e^1 + b_2e^2) \end{array} .$$

Again we have to check well-definedness. As an example take

$$(0, -x^{u-u^0}e^0 + x^{u-u^1}e^1) \;\mapsto\; (x^u, -x^{u-u^0}e^0 + x^{u-u^1}e^1).$$

## 8.4. Higher $\mathrm{Ext}^n$ as exact sequences

Given a monomial

$$(u, D^n \to \ldots \to D^1 \to D, D') \in \mathrm{Ext}^{n+1}(D, D')$$

we want to find a representative of the associated Yoneda equivalence class of exact sequences. We want to proceed inductively along the path $D^n \to \ldots \to D^1$: In Section 8.3 we constructed a representative for the monomial $(u, D^n, D') \in \mathrm{Ext}^1(D^n, D')$ which is our induction hypothesis. The construction involved a map $\varphi^u : \ker d_0^{D^n} \to D'[-u]$. Having a path $D^n \to D^{n-1}$ tells us that $F_0(D^n)$ is a direct summand of $F^1(D^{n-1})$ and we canonically get a projection $F^1(D^{n-1}) \twoheadrightarrow F_0(D^n)$. Applying this principle repeatedly produces a diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker d_n^D \arrow[r, hook] \arrow[d, "\pi^{D^1}"'] \arrow[ddddd, bend left=40, "\varphi_n^u"] & F_n(D) \arrow[r] & F_{n-1}(D) \arrow[r] & F_{n-2}(D) \arrow[r] & \cdots \arrow[r] & F_1(D) \arrow[r] & F_0(D) \arrow[r, two heads] & D \arrow[r] & 0 \\
0 \arrow[r] & \ker d_{n-1}^{D^1} \arrow[r, hook] \arrow[d, "\pi^{D^2}"'] \arrow[dddd, bend left=40, "\varphi_{n-1}^u"] & F_{n-1}(D^1) \arrow[r] & F_{n-2}(D^1) \arrow[r] & F_{n-3}(D^1) \arrow[r] & \cdots \arrow[r] & F_0(D^1) \arrow[r, two heads] & D^1 \arrow[r] & 0 \\
\vdots & \vdots \arrow[d, "\pi^{D^{n-1}}"'] \arrow[ddd, bend left=40, "\varphi_2^u"] & \vdots & & \vdots & \vdots & \iddots \\
0 \arrow[r] & \ker d_1^{D^{n-1}} \arrow[r, hook] \arrow[d, "\pi^{D^n}"'] \arrow[dd, bend left=40, "\varphi_1^u"] & F_1(D^{n-1}) \arrow[r] & F_0(D^{n-1}) \arrow[r, two heads] & D^{n-1} \arrow[r] & 0 \\
0 \arrow[r] & \ker d_0^{D^n} \arrow[r, hook] \arrow[d, "\varphi^u"'] & F_0(D^n) \arrow[r, two heads] & D^n \arrow[r] & 0 \\
0 \arrow[r] & D'[-u] \arrow[r, hook] \arrow[d, equal] & \mathrm{coker}(-\varphi^u, i^{D^n}) \arrow[r, two heads] & D^n \arrow[r] & 0 \\
0 \arrow[r] & D'[-u] \arrow[r, hook] & \mathrm{coker}(-\varphi_n^u, i^{D^n}) \arrow[r] & F_{n-1}(D) \arrow[r] & F_{n-2}(D) \arrow[r] & \cdots \arrow[r] & F_1(D) \arrow[r] & F_0(D) \arrow[r, two heads] & D \arrow[r] & 0
\end{tikzcd}$$

This means we construct $\varphi_n^u$ recursively:

$$\varphi_0^u := \varphi^u, \text{ and } \varphi_{i+1}^u := \varphi_i^u \circ \pi^{D^{n-i}}.$$

Hence we have found a representative for the monomial $(u, D^n \to \ldots \to D^1 \to D, D')$. But with this description we can derive even more.

**Lemma 8.9.** The monomials

$$(u, D^n \to \ldots \to D^1 \to D, D') \text{ and } (v, H^m \to \ldots \to H^1 \to D, D')$$

are Yoneda equivalent if and only if the following conditions hold

1. $m = n$,

2. $H^i = D^i$ for all $i = 1, \dots, n$ and

3. $(u, D^n, D')$ and $(v, H^n, D')$ are Yoneda equivalent.

*Proof.* The proof already follows by construction of the exact sequences. Alternatively one has the observation of Section 8.2 that the monomials form a $\mathbb{C}$-basis of the Ext-modules. □

*Remark* 8.10. Of course the monomials

$$(u, D^n, D') \text{ and } (v, H^n, D')$$

are Yoneda equivalent if and only if $u = v$ and $D^n = H^n$. The third condition just means for both monomials to be contained in same $\mathrm{Ext}^1(D^n, D')$.

## 8.5. The multiplication

Throughout this chapter we will work with maps representing elements of certain $\mathrm{Ext}^1$'s. One should keep in mind that Yoneda equivalence will always flatten out any choices we make.

$\mathrm{Ext}^1 \times \mathrm{Ext}^1$

**Theorem 8.11.** *Let $(w, B, C)$ and $(u, A, B)$ be two monomials of degree 1. Assume there exists $E \in \mathrm{in}_{\mathcal{R}}(A)$ such that*

$$u + w \in \mathrm{Ext}^1(E, C) \text{ and } E \subseteq B[u].$$

*Then*

$$(w, B, C) \cdot (u, A, B) = (w + u, E \to A, C).$$

*Otherwise the product is zero.*

Assume we are given two elements $\bar{x}^u \in \mathrm{Ext}^1(A, B)$ and $\bar{x}^w \in \mathrm{Ext}^1(B, C)$. The corresponding exact sequences arise via maps $\varphi^u : \ker d_0^A \to B[-u]$ and $\varphi^w : \ker d_0^B \to C[-w]$ as described in Section 8.3. Hence we obtain sequences

$$\alpha(\varphi^u) : \; 0 \longrightarrow B[-u] \hookrightarrow \mathrm{coker}(\varphi^u, i) \twoheadrightarrow A \longrightarrow 0$$

and

$$\alpha(\varphi^w) : \; 0 \longrightarrow C[-w] \hookrightarrow \mathrm{coker}(\varphi^w, i) \twoheadrightarrow B \longrightarrow 0 \quad .$$

The goal is now to study the map

$$\begin{array}{rccc} \mu : & \mathrm{Ext}^1(B, C) \times \mathrm{Ext}^1(A, B) & \to & \mathrm{Ext}^2(A, C) \\ & ([\alpha(\varphi^w)],\ [\alpha(\varphi^u)]) & \mapsto & [\alpha(\varphi^w)] \cdot [\alpha(\varphi^u)] \end{array}$$

Figure 8.12.: General construction.

$$
\begin{tikzcd}
0 \arrow[r] & \ker d_1^A \arrow[r, hook, "i_1^A"] \arrow[ddd, bend right, "h"'] \arrow[dd, bend left, "h'"] & F_1(A) \arrow[rr, "d_1^A"] \arrow[dr, two heads, "\pi"] \arrow[dd, bend right, "\Phi"'] \arrow[ddr, bend left, "\varphi^u \circ \pi"] & & F_0(A) \arrow[r, two heads, "d_0^A"] & A \arrow[r] & 0 \\
 & & & \ker d_0^A \arrow[ur, hook, "i_0^A"'] \arrow[d, "\varphi^u"] & & & \\
0 \arrow[r] & \ker d_0^{B[-u]} \arrow[r, hook, "i^{B[-u]}"] \arrow[d, "\varphi^w"] \arrow[d, bend right] & F_0(B[-u]) \arrow[r, two heads, "d_0^{B[-u]}"] & B[-u] \arrow[r] & 0 & & \\
0 \arrow[r] & C[-u-w] \arrow[r, hook] \arrow[d, equal] & \operatorname{coker}(-h, i_1^A) \arrow[rr, "\tau"] \arrow[d, "a"] & & F_0(A) \arrow[r, two heads] \arrow[d, "b"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C[-u-w] \arrow[r, hook] & \operatorname{coker}(-\varphi^w[-u], i^{B[-u]}) \arrow[rr, "\sigma"] & & \operatorname{coker}(-\varphi^u, i_0^A) \arrow[r, two heads] & A \arrow[r] & 0
\end{tikzcd}
$$

in combinatorial terms. In particular the product should be homogeneous of degree $u+w$. Since the degree-$(u+w)$-homogeneous elements of $\mathrm{Ext}^2(A, C)$ arise from maps $h : \ker d_1^A \to C[-u-w]$ we want to construct such a map from $\varphi^u$ and $\varphi^w$. Hence we get a diagram as depicted in Figure 8.12.

The golden maps are maps we need to construct in order to get to the desired $h : \ker d_1^A \to C[-u-w]$. At this point only commutative algebra is needed. Let us walk through them in the order they arise.

$\varphi^u \circ \pi$: The free module $F_1(A)$ surjects onto the kernel of $d_0^A$ via $\pi$, and composition with the given map $\varphi^u : \ker d_0^A \to B[-u]$ yields this map.

$\Phi$: Since $F_1(A)$ is a free module, and thereby projective, any map $f : M \to B[-u]$ gives rise to a map $\Phi : F^1(A) \to M$ such that $f \circ \Phi = \varphi^u \circ \pi$. In this case we pick as $f$ the map $d_0^{B[-u]} : F_0(B[-u]) \twoheadrightarrow B[-u]$ and obtain a $\Phi : F_1(A) \to F_0(B[-u])$ making the diagram commute. The map $\Phi$ does not have to be unique.

$h'$: Let $k \in \ker d_1^A$, then $\pi \circ i_1^A(k) = 0$. This implies by commutativity of the diagram

$$\varphi^u \circ \pi \circ i_1^A(k) = 0 = d_0^{B[-u]} \circ \Phi \circ i_1^A(k).$$

Exactness of the middle row yields a unique preimage of $\Phi \circ i_1^A(k)$ under $i^{B[-u]}$ and we set $h'(k)$ to be this preimage, i.e.

$$h' \;:=\; (i^{B[-u]})^{-1} \circ \Phi \circ i_1^A.$$

$h$: At this point the hardest part is already done and we simply set $h = \varphi^w \circ h'$.

Next we construct the purple maps $a$ and $b$ in order to verify that the sequence originating from $h$ is indeed Yoneda equivalent to the product sequence. The product sequence is the last row in the diagram, the row above it is the sequence defined by $h$. Let us write down a detailed version of the box in the center of the last two rows:

$$\begin{tikzcd} C[-u-w]\oplus F_1(A)\big/\mathrm{im}(-h,i^A) \arrow[r, "{(0,d_1^A)}", "\tau"'] \arrow[d, "a", "{\begin{pmatrix} id_{C[-u-w]} & 0 \\ 0 & \Phi\end{pmatrix}}"'] & F_0(A) \arrow[d, "b"', "{\begin{pmatrix} 0 \\ id_{F_0(A)}\end{pmatrix}}"] \\ C[-u-w]\oplus F_0(B[-u])\big/\mathrm{im}(-\varphi^w[-u], i^{B[-u]}) \arrow[r, bend right, "\sigma", "{\begin{pmatrix} 0 & d_0^{B[-u]} \\ 0 & 0\end{pmatrix}}"'] & B[-u]\oplus F_0(A)\big/\mathrm{im}(-\varphi^u, i_0^A) \end{tikzcd}$$

Firstly we prove the well-definedness of $a$. Take an element $k \in \ker d_1^A$, map it to $(-h(k), i_1^A(k))$ in the upper left corner and apply $a$:

$$a(-h(k), i_1^A(k))) = (-h(k), \Phi \circ i_1^A(k)) = (-\varphi^w \circ h'(k), i^{B[-u]} \circ h'(k))$$

which is zero in the lower left corner.

For commutativity of the diagram take an element $(d, f)$ in the upper left corner and follow its path:

$$\begin{aligned}\sigma \circ a(d, f) &= (d_0^{B[-u]} \circ \Phi(f), 0) = (\varphi^u \circ \pi(f), 0)\\ &= (0, i_0^A \circ \pi(f)) = (0, d_1^A(f))\\ &= b \circ \tau(d, f).\end{aligned}$$

This procedure yields no information on whether the product $[\alpha(\varphi^w)] \cdot [\alpha(\varphi^u)]$ is a monomial or not, and, if yes, which one. We need to supplement the single steps with the combinatorial details of $\varphi^u$ and $\varphi^w$.

At first we write down the exact sequence of Theorem 5.8 for $A$:

$$0 \longrightarrow \ker d_0^A = \bigoplus_{i=1}^s D^i \hookrightarrow F_0(A) \twoheadrightarrow A \longrightarrow 0.$$

The initial considerations of Section 8.3 imply that we can pick $\varphi^u : \ker d_0^A \to B[-u]$ in such a way that it is zero on all the summands but one, on which it is just the canonical injection. Let this summand be $E$. As described above we construct several maps, all of which fit into the diagram Figure 8.13.

The golden maps come from above and can be obtained as composition of the other maps in the diagram. Since the diagram commutes, any choice will yield the same map.

What remains is the construction of the two green maps. Since $F_0(E)$ is free and the third row is exact, we can deduce existence as above. Moreover $h_0$ is completely determined by $\Phi_0$. Of course $\Phi_0$ will not be unique. Still we require $\Phi_0$ to be homogeneous of degree 0. In particular, we want $e^i$ to be mapped to an element of the form $x^u \cdot e^j$ with $\deg e^i = u + \deg e^j$. Since we have $E \subseteq B[-u]$ these requirements can certainly be satisfied, but they still leave us with a choice.

Assume we are given two divisors $D$ and $E$ with sorted generators

$$\operatorname{Supp}(G(D)) = \{u^0, \ldots, u^{s(D)}\} \text{ and } \operatorname{Supp}(G(E)) = \{v^0, \ldots, v^{s(E)}\}$$

Furthermore, assume $E \subseteq D$. Then naturally each $v^l$ is contained in some $R[u^i]$. Now we shrink $D$ by moving its bounding hyperplanes. Thus we may assume $v^0 \in R[u^0]$ and $v^{s(E)} \in R[u^{s(D)}]$. This setting allows discussing the distribution of the $v^l$ in the $R[u^i]$.

**Lemma 8.14.** Let $E \subseteq D$ and $v^0 \in R[u^0]$, $v^{s(E)} \in R[u^{s(D)}]$. Then for every $i = 0, \ldots, s(D)$ there exists $l \in 0, \ldots, s(E)$ such that $v^l \in R[u^i]$.

*Proof.* By assumption it is clear that the statement is true for $i = 0$ and $i = s(D)$. Thus assume $0 < i < s(D)$. We want to verify the statement by contradiction, hence assume that no $v^l \in R[u^i]$ for the chosen $i$. Now choose $l$ such that

$$\langle v^l, \rho^0 \rangle < \langle u^i, \rho^0 \rangle \leq \langle v^{l+1}, \rho^0 \rangle.$$

Since $v^{l+1} \notin R[u^i]$ and since $v^l \in R[u^k]$ for some $k < i$ we also get

$$\langle v^{l+1}, \rho^1 \rangle < \langle u^i, \rho^1 \rangle \leq \langle v^l, \rho^1 \rangle.$$

This immediately implies $u^i \in \operatorname{below} D'$ for $D'$ the divisor generated by $v^l$ and $v^{l+1}$, which contradicts Proposition 6.9. $\square$

Figure 8.13.: Construction for CQS.

$$
\begin{tikzcd}
0 \arrow[r] & \ker d_1^{A} \arrow[r, hook, "i^{A}"] \arrow[dd, equal] \arrow[ddddddd, bend right, "h"'] & F_1(A) \arrow[rr, "d_1^{A}"] \arrow[dr, "\pi"] \arrow[dd, equal] \arrow[dddd, bend left, "\Phi"] & & F_0(A) \arrow[r, two heads, "d_0^{A}"] & A \arrow[r] & 0 \\
& & & \ker d_0^{A} \arrow[ur, hook] \arrow[d, equal] \arrow[ddd, bend left, "\varphi^u"] & & & \\
0 \arrow[r] & \bigoplus_{i=1}^{s(A)} \ker d_0^{A^i} \arrow[r, hook] \arrow[d, two heads, "p_E"] & \bigoplus_{i=1}^{s(A)} F_0(A^i) \arrow[r, two heads] \arrow[d, two heads, "p_E"] & \bigoplus_{i=1}^{s(A)} A^i \arrow[r] \arrow[d, two heads, "p_E"] & 0 & & \\
0 \arrow[r] & \bigoplus_{i=1}^{s(E)} E^i \arrow[r, hook] \arrow[d, "h_0"] & F_0(E) \arrow[r, two heads] \arrow[d, "\Phi_0"] & E \arrow[r] \arrow[d, hook] & 0 & & \\
0 \arrow[r] & \ker d_0^{B[-u]} \arrow[r, hook, "i^{B[-u]}"] \arrow[d, equal] \arrow[ddd, bend left, "\varphi^w"] & F_0(B[-u]) \arrow[r, two heads, "d_0^{B[-u]}"] & B[-u] \arrow[r] & 0 & & \\
& \bigoplus_{i=1}^{s(B)} B^i \arrow[d, two heads, "p_F"] & & & & & \\
& F \arrow[d, hook] & & & & & \\
0 \arrow[r] & C[-u-w] \arrow[r, hook] & \operatorname{coker}(-h, i^{A}) \arrow[rr] & & F_0(A) \arrow[r, two heads] & A \arrow[r] & 0
\end{tikzcd}
$$

*Remark* 8.15. One should note that Lemma 8.14 does not imply $s(E) = s(D)$, since a $v^l$ may be contained in several of the $R[u^i]$.

Now we are finally in the position to discuss the lemmata leading to the construction of $\Phi_0$. Let us fix generators for $B[-u]$ and $E$ such that

$$F_0(B[-u]) = \bigoplus_{k=0}^{s(B[-u])} R[u^k] \text{ and } F_0(E) = \bigoplus_{l=0}^{s(E)} R[v^l],$$

with the ordering of Proposition 4.8.

**Lemma 8.16.** Let $F$ be a direct summand of $\ker d_0^{B[-u]}$ such that $F \subseteq C[-u-w]$. Furthermore, let $u + w \in \operatorname{Ext}^1(E, C)$. Then $\ker d_0^E$ has a direct summand $E^l$ that is contained in $F$.

*Proof.* The assumption $u + w \in \operatorname{Ext}^1(E, C)$ and $w \in \operatorname{Ext}^1(B[-u], C[-u])$ implies that the elements $v^0, v^{s(E)}, u^0, u^{s(B[-u])}$ must not be contained in $C[-u-w]$ by Theorem 6.1. Otherwise we could map for example $R[v^0]$ into $C[-u-w]$ and the sequence of degree $u+w$ would become trivial just as in Example 8.8. By Yoneda equivalence this would eliminate the degree $u + w$ from the support of $\operatorname{Ext}^1(E, C)$ which contradicts the assumption.

This yields the inequalities

$$\langle u^0, \rho^0\rangle < \langle C[-u-w], \rho^0\rangle < \langle u^{s(B[-u])}, \rho^0\rangle \text{ and } \langle u^{s(B[-u])}, \rho^1\rangle < \langle C[-u-w], \rho^1\rangle < \langle u^0, \rho^1\rangle$$

and the same set of inequalities for $v^0$ and $v^{s(E)}$.

Next assume that $F$ is the intersection

$$F = (u^{k-1} + \sigma^\vee) \cap (u^k + \sigma^\vee) \text{ for some } k \in \{1, \dots, s(B[-u])\}.$$

The containment $F \subseteq C[-u-w]$ yields the inequalities

$$\langle C[-u-w], \rho^0\rangle \leq \langle u^k, \rho^0\rangle \text{ and } \langle C[-u-w], \rho^1\rangle \leq \langle u^{k-1}, \rho^1\rangle.$$

Using Lemma 8.14 we choose $l-1$ such that $v^{l-1} \in R[u^{k-1}]$. Furthermore we may require that $v^l \notin R[u^{k-1}]$ since we know

$$\langle v^{s(E)}, \rho^1\rangle \leq \langle C[-u-w], \rho^1\rangle \leq \langle u^{k-1}, \rho^1\rangle.$$

This immediately implies $\langle v^l, \rho^0\rangle \geq \langle u^k, \rho^0\rangle$, since $v^l$ has to be contained in some $R[u^{k'}]$ for $k' > k-1$. Thus pick

$$E^l = (v^{l-1} + \sigma^\vee) \cap (v^l + \sigma^\vee)$$

to obtain

$$\langle E^l, \rho^0\rangle = \langle v^l, \rho^0\rangle \geq \langle F, \rho^0\rangle = \langle u^k, \rho^0\rangle \text{ and}$$
$$\langle E^l, \rho^1\rangle = \langle v^{l-1}, \rho^1\rangle \geq \langle u^{k-1}, \rho^1\rangle = \langle F, \rho^1\rangle.$$

∎

**Lemma 8.17.** In the setting of Lemma 8.16 we may pick $E^l$ such that

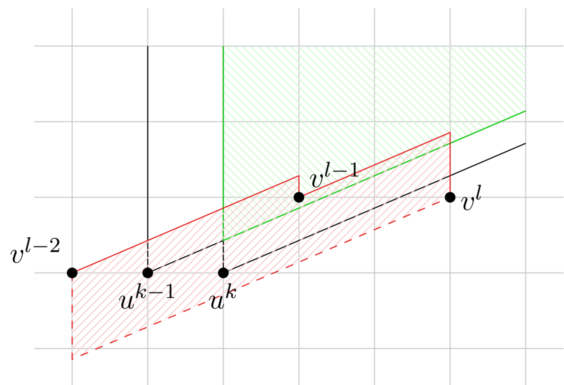
$$E^l = (v^{l-1} + \sigma^\vee) \cap (v^l + \sigma^\vee), \text{ and}$$
$$F = (u^{k-1} + \sigma^\vee) \cap (u^k + \sigma^\vee),$$

with $v^{l-1} \in R[u^{k-1}]$ and $v^l \in R[u^k]$.

*Proof.* By Lemma 8.14 there must be an $v^{l-1} \in R[u^{k-1}]$. If it is not unique, pick the right-most one in terms of the first coordinate, i.e. such that $v^l \notin R[u^{k-1}]$. Furthermore, it is safe to assume that $v^l \notin R[u^k]$, because $w \in \mathrm{Ext}^1(B[-u], C[-u])$. In the case that $k = s(B[-u])$ this immediately yields a contradiction and we are done. Otherwise we know there must be some $v^i \in R[u^k]$ and we obtain $v^{l-1} \in R[u^k]$. In the same way we assume $v^{l-2} \notin R[u^{k-1}]$ in order to get a contradiction. Now take $D'$ to be the divisor generated by $D' = (v^{l-2}, v^{l-1}, v^l)$. Then

$$u^{k-1}, u^k, v^{l-1} \in \mathrm{below}(D')$$

which is a contradiction to Proposition 6.9. The situation is illustrated in the following picture:




The red area denotes the set below$(D')$. As one can see it contains at least the three lattice points $u^{k-1}$, $u^k$ and $v^{l-1}$, but according to Proposition 6.9 it should only contain $v^{l-1}$. ☐

**Lemma 8.18.** Assume $E^l$ as in Lemma 8.16 is unique. Then

$$E^l = (v^{l-1} + \sigma^\vee) \cap (v^l + \sigma^\vee), \text{ and}$$
$$F = (u^{k-1} + \sigma^\vee) \cap (u^k + \sigma^\vee),$$

with $v^{l-1} \in R[u^{k-1}] \backslash R[u^k]$ and $v^l \in R[u^k] \backslash R[u^{k-1}]$.

*Proof.* The trick is that if a generator $v^l$ is contained in $F$, then both neighbouring intersections are contained in $F$ and thus $E^l$ was not unique. ☐

Finally we are in the situation that we can construct the map $\Phi_0$, and thus, we prove the main theorem:

*Proof of Theorem 8.11.* Let $E^l$, $u^{k-1}, u^k, v^{l-1}$ and $v^l$ as in Lemma 8.17. By assumption there is a map

$$\varphi^{u+w} : \bigoplus_{i=1}^{s(E)} E^i \to C[-u-w],$$

which is the inclusion on $E^l$ and zero on the other summands. This map, concatenated with $p_E$, yields the monomial on the right hand side of the equation.

We want to construct

$$\Phi_0 : \; F_0(E) = \bigoplus_{l=0}^{s(E)} R[v^l] \;\to\; F_0(B[-u]) = \bigoplus_{k=0}^{s(B[-u])} R[u^k],$$

such that

$$\varphi^{u+w} \circ p_E = \varphi^w \circ h_0 \circ p_E = h.$$

The main requirement is that $\Phi_0$ is homogeneous of degree 0. Thus we pick

$$\Phi_0(x^{v^{l-1}}) := x^{v^{l-1}-u^{k-1}} \cdot x^{u^{k-1}} \text{ and } \Phi_0(x^{v^l}) := x^{v^l-u^k} \cdot x^{u^k}.$$

Furthermore we send $x^{v^i}$ for $i < l-1$ into $R[u^j]$, where $j \leq k-1$. For $i > l$ we send $x^{v^i}$ into $R[u^j]$, where $j \geq k$.

Thus $h_0$ sends $E^l$ into $F$. By construction no other intersection will be send to any element of $F$. Hence the composition $\varphi^w \circ h_0$ is the canonical injection on $E^l$ and zero on the other direct summands. □

**Theorem 8.19.** *Assume $u + w \notin \mathrm{Ext}^1(E, C)$. Then*

$$(w, B, C) \cdot (u, A, B) = 0.$$

*Proof.* We still assume $E \subseteq C[-u]$. Hence we can still construct all maps appearing in the diagram Figure 8.13. The easiest case is when we are able to choose $\Phi^0$ such that we completely miss one of the direct summands $R[u^{k-1}]$ and $R[u^k]$. Then the image of $\ker d_0^E$ under $h_0$ completely avoids $F$ and the map $h$ becomes zero.

For the other cases we may construct $\Phi^0$ exactly as in the proof of Theorem 8.11. Again we have the formula

$$\varphi^{u+w} \circ p_E = \varphi^w \circ h_0 \circ p_E = h.$$

Hence as a result of the multiplication we get an honest homogeneous element of $\mathrm{Ext}^1(E, C)$. Even better: It is constructed exactly as Section 8.4 suggests. Thus, since we already know that there is no degree $u + w$ in $\mathrm{Ext}^1(E, C)$, this element must be Yoneda equivalent to zero. □

*Remark* 8.20. Remember that one might have a choice if several of the summands $A^i$ are contained in $B[-u]$. Thus assume we are given $A^i$ and $A^{i+1}$, such that both are contained in $B[-u]$. According to Theorem 8.11 the product $(w, B, C) \cdot (u, A, B)$ now has the choice to either be

$$(w + u, A^i \to A, C) \text{ or } (w + u, A^{i+1} \to A, C).$$

But those live in different direct summands of

$$\mathrm{Ext}^2(A, C) = \bigoplus_{i=1}^{s(A)} \mathrm{Ext}^1(A^i, C)$$

which is a contradiction. Hence the degree $w + u$ ceases to exist in both summands.

To provide a more thorough explanation for this phenomenon fix $A^i$ and assume $i : A^i \hookrightarrow B[-u]$ to be the injection. Since also $A^{i+1}$ is contained, we know that the direct summand $R[u^{i+1}]$ of $F_0(A)$ is contained in $B[-u]$. This yields the following chain inclusions of polyhedra:

$$A^i \subseteq R[u^{i+1}] \subseteq B[-u].$$

Since $u^{i+1}$ is a lattice point of $B[-u]$ there is a direct summand of $F_0(B[-u])$ containing $u^{i+1}$. Thus we may choose $\Phi_0 : F_0(A^i) \to F_0(B[-u])$ to map everything into this direct summand. But this implies for the composition

$$[\ker(d_0^{A^i}) \hookrightarrow F_0(A^i)] \circ \Phi_0 = 0.$$

Thus the map $h_0$ is zero as well as the resulting map $h$. Therefore neither $\mathrm{Ext}^1(A^i, B)$, nor $\mathrm{Ext}^1(A^{i+1}, B)$ are supported in degree $u + w$.

$\mathrm{Ext}^n \times \mathrm{Ext}^n$

**Theorem 8.21.** *Let*

$$(u, A, B) \in \mathrm{Ext}^1(A, B),$$
$$(v, B^{n-1} \to \ldots \to B, C) \in \mathrm{Ext}^n(B, C).$$

*If there is an element*

$$(u + v, A^n \to \ldots \to A, C) \in \mathrm{Ext}^{n+1}(A, C)$$

*such that*

$$A^i \subseteq B^{i-1}[-u] \quad \textit{for all } i = 1, \ldots, n,$$

*where we set* $B^0 := B$*. Then*

$$(v, B^{n-1} \to \ldots \to B, C) \cdot (u, A, B) = (u + v, A^n \to \ldots \to A, C).$$

*Otherwise the product is zero.*

*Proof.* We need to expand the diagram of Figure 8.13 for the sequence

$$0 \to \ker_{d^A_{n-1}} \hookrightarrow F_{n-1}(A) \to \ldots \to F_0(A) \twoheadrightarrow A \to 0.$$

Thus assume $E$ to be a direct summand of $\ker d_0^A$ yielding the element $(u, A, B) \in \mathrm{Ext}^1(B, A)$. By the containment condition we know that $A^1$ is such a summand, hence we pick $E := A^1$.

The containment condition ensures that we may lift the map $\varphi^u : A^1 \hookrightarrow B[-u]$ along the following chain:

$$\begin{tikzcd}
\cdots \arrow[rr] & & F_i(A) \arrow[rr] \arrow[ddd, "\varphi^{i+1}"] & & F_{i-1}(A) \arrow[r] \arrow[ddd, "\varphi^i"] & \cdots \\
& A^{i+1} \arrow[ur, hook] \arrow[d, hook] & & A^i \arrow[ur, hook] \arrow[d, hook] & & \\
& B^i[-u] \arrow[dr, hook] & & B^{i-1}[-u] \arrow[dr, hook] & & \\
\cdots \arrow[rr] & & F_{i-1}(B[-u]) \arrow[rr] & & F_{i-2}(B[-v]) \arrow[r] & \cdots
\end{tikzcd},$$

where we assume $\varphi^i$ to be zero on all components of $F_{i-1}(A)$ that do not contribute to $A^i$. On the two components $R[u^k]$ and $R[u^{k+1}]$ of $F_{i-1}(A)$, whose intersection is $A^i$, $\varphi^i$ may be any homogeneous map that makes the diagram commute.

Next we may construct $\varphi^{i+1}$ inductively: By the construction of $\varphi^i$ we only need to worry about $F_0(A^i)$ inside of $F_{i-1}(A)$. We map this to $F_0(B^{i-1}[-u])$ by again mapping every direct summand that does not belong to $A^{i+1}$ to zero. We can now map the two direct summands generating $A^{i+1}$ into the direct summands generating $B^i[-u]$ via the containment condition.

At the end of the chain we may use the same arguments as in the proof of Theorem 8.11. For the beginning of the chain take the inclusion of $A^1 \subseteq B[-u]$. Since there are no commutativity conditions at this position of the chain, we may choose the map $\varphi^1 : F_0(A) \to B[-u]$ in the desired way, i.e. zero on all direct summands that do not contribute to $A^1$. The two direct summands with intersection $A^1$ are mapped in such a way that they restrict to the inclusion $A^1 \hookrightarrow B[-u]$. $\square$

This formula explains the multiplication with an $\mathrm{Ext}^1$-element from the right. As a last step we may lift this theorem quite canonically to the general case via the direct sum decomposition of higher $\mathrm{Ext}^n$ explained in Theorem 5.16.

**Corollary 8.22.** Let

$$(u, A^{m-1} \to \ldots \to A, B) \in \mathrm{Ext}^m(A, B),$$
$$(v, B^{n-1} \to \ldots \to B, C) \in \mathrm{Ext}^n(B, C)$$

and assume that there exists

$$(u+v, A^{n+m-1} \to \ldots \to A, C) \in \mathrm{Ext}^{n+m}(A, C)$$

such that

$$A^i \subseteq B^{i-m}[-u] \quad \text{for all } i = m, \ldots, n+m-1,$$

where we set $B^0 := B$. Then

$$(v, B^{n-1} \to \ldots \to B, C) \cdot (u, A^{m-1} \to \ldots \to A, B) = (u+v, A^{n+m-1} \to \ldots \to A, C).$$

Otherwise the product is zero.

*Remark* 8.23. From an algorithmic perspective it is very interesting that the product monomial must be unique. This means given the the path

$$B^{n-1} \to \ldots \to B$$

of the first factor we may construct the product starting at $A^{m-1}$, following the unique path yielding the product, i.e. satisfying the containment condition. If we have a choice at any point, the product must be zero. This extends Remark 8.20.

**Example 8.24.** The first and simplest example is of course $n = 2$, $q = 1$. Here we only have one non-trivial divisor to consider, namely $D = E^1$. The only monomial in $\operatorname{Ext}^1(E^1, E^1)$ is

$$([-1, -1], E^1, E^1) \in \operatorname{Ext}^1(E^1, E^1).$$

To determine the quiver $\mathcal{R}$ we just need the short exact sequence

$$0 \to E^1[1, 1] \hookrightarrow R^2 \twoheadrightarrow E^1 \to 0$$

and thus we compute

$$\operatorname{Ext}^n(E^1, E^1) = \operatorname{Ext}^{n-1}(E^1[1, 1], E^1) = \ldots = \mathbb{C} \cdot \bar{x}^{[-n,-n]}$$

for $n \geq 1$. Hence we have monomials

$$([-n, -n], E^1[n, n] \to E^1[n - 1, n - 1] \to \ldots \to E^1, E^1).$$

The only thing left to do is to compute the product

$$([-n, -n], E^1[n, n] \to E^1[n - 1, n - 1] \to \ldots \to E^1, E^1) \cdot ([-1, -1], E^1, E^1).$$

By Theorem 8.21 this is exactly

$$([-(n + 1), -(n + 1)], E^1[n + 1, n + 1] \to E^1[n, n] \to \ldots \to E^1, E^1).$$

We even have equality for all inclusions $A^i \subseteq B^{i-1}[-v]$ of the theorem. Thus the algebra $\operatorname{Ext}(E^1, E^1)$ is generated in degree 0 and 1 and we can describe it as

$$\operatorname{Ext}(E^1, E^1) = \mathbb{C}[y, xy, x^2y, a] \Big/ (ya,\ ay),$$

with $a$ having degree $[-1, -1]$. In this case the ring is even commutative. This is not true in general, as we will see in Section 8.7.

Using Corollary 8.22 we can formulate Algorithm 8.25 to compute the product of two monomials. Applying the appropriate shifts as discussed in Definition 8.3 is important here.

**Algorithm 8.25:** Computing the product of two monomials of $\mathrm{Ext}(D)$

**Input**: $a := (u, A^{m-1} \to \ldots \to A, B) \in \mathrm{Ext}^m(A, B)$,
$b := (v, B^{n-1} \to \ldots \to B, C) \in \mathrm{Ext}^n(B, C)$ two elements of $\mathrm{Ext}(D)$, and the quiver $\mathcal{R}$.
**Output**: The product $b \cdot a \in \mathrm{Ext}^{n+m}(A, C)$.

```
begin
  resultPath := A^{m-1} → ... → A;
  last := A^{m-1};
  for i = 0, ..., n-1 do
    S := {D ∈ V R | There is a path D → last and D ⊆ B^i[-u]};
    if #S > 1 then
      return 0
    else
      Take D the only element of S;
      resultPath = D → resultPath;
      last = D;
  if u + v ∈ Ext^1(last, C) then
    return (u + v, resultPath, C)
  else
    return 0
```

## 8.6. Generators of the Ext algebra

Based on Theorem 8.21 and the recursion of Theorem 5.16 we can now give a sufficient criterion for finite generation of the algebra $\mathrm{Ext}(D)$ as a $\mathbb{C}$-algebra.

Fix a Weil divisor $D$ then we can define the subquiver $\mathcal{R}(D)$:

**Definition 8.26.** The subquiver $\mathcal{R}(D)$ of $\mathcal{R}$ is the quiver with vertices

$$\mathrm{V}(\mathcal{R}(D)) := \{E \in \mathrm{V}(\mathcal{R}) \mid \text{There is a path from } E \text{ to } D\}$$

and all labelled edges of $\mathcal{R}$ in between those vertices.

The vertices are exactly the divisors that may appear while resolving $D$ recursively. For our purposes these are still too many. We only need those that appear repeatedly. Hence we define $\mathcal{R}(D)^\infty$:

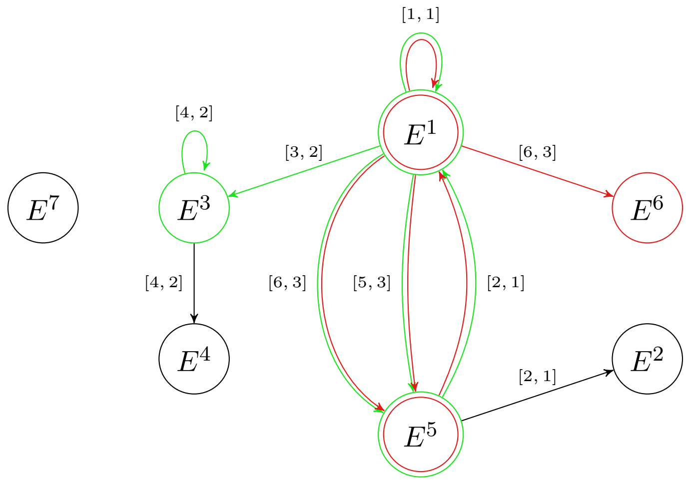
**Definition 8.27.** The subquiver $\mathcal{R}(D)^\infty$ is the quiver with vertices

$$\mathrm{V}(\mathcal{R}(D)^\infty) := \{E \in \mathrm{V}(\mathcal{R}(D)) \mid E \text{ is part of a cycle of } \mathcal{R}(D)\}$$

and all labelled edges of $\mathcal{R}(D)$ in between those vertices.

Basically both $\mathcal{R}(D)$ and $\mathcal{R}(D)^\infty$ continue the ideas of Conjecture 3.10. The vertices $\mathrm{V}(\mathcal{R}(D))$ are exactly the same as the elements of $\bar{P}(D)$.

**Example 8.28.** In the running example with $n = 7$ and $q = 3$, let us draw $\mathcal{R}(E^3)$ and $\mathcal{R}(E^6)$:




As one can easily see we have

$$\mathrm{V}(\mathcal{R}(E^3)^\infty) = \{E^3, E^1, E^5\} \text{ and } \mathrm{V}(\mathcal{R}(E^6)^\infty) = \{E^1, E^5\}.$$

Furthermore we have

$$\mathrm{V}(\mathcal{R}(E^3)^\infty) = \mathrm{V}(\mathcal{R}(E^4)^\infty),\ \mathrm{V}(\mathcal{R}(E^6)^\infty) = \mathrm{V}(\mathcal{R}(E^2)^\infty) = \mathrm{V}(\mathcal{R}(E^5)^\infty) = \mathrm{V}(\mathcal{R}(E^1)^\infty).$$

Thus it is not true that $\mathcal{R}(D)^\infty$ is the same independently of $D$. This becomes especially clear for the two extremal cases:

*Remark* 8.29. For the extremal cases $q = n - 1$ and $q = 1$ we have

1. If $q = n - 1$, then $\mathrm{V}(\mathcal{R}(E^i)) = \mathrm{V}(\mathcal{R}(E^{n-i})) = \mathrm{V}(\mathcal{R}(E^i))^\infty = \{E^i, E^{n-i}\}$ for all $i \in \{1, \dots, n-1\}$;
2. If $q = 1$, then $\mathrm{V}(\mathcal{R}(E^i)) = \{E^i, E^1\}$ and $\mathrm{V}(\mathcal{R}(E^i))^\infty = \{E^1\}$ for all $i \in \{1, \dots, n-1\}$.

*Remark* 8.30. An useful observation is that $\mathcal{R}(E)$ is a subquiver of $\mathcal{R}(D)$ if there is a path from $E$ to $D$. This is also true for $\mathcal{R}(\bullet)^\infty$.

**Corollary 8.31.** Assume there exists $n \in \mathbb{Z}_{>0}$ such that

$$\mathrm{Ext}^n(D, D) \times \mathrm{Ext}^1(E, D) \twoheadrightarrow \mathrm{Ext}^{n+1}(E, D)$$

is surjective for all $E \in \mathcal{R}(D)^\infty$. Then $\mathrm{Ext}(D)$ is finitely generated and the generators have degree $\leq n$.

*Proof.* Assume that there exist $n_0$ and $m_0$ in $\mathbb{Z}_{>0}$ such that the multiplication map

$$\mathrm{Ext}^n(D, D) \times \mathrm{Ext}^m(D, D) \twoheadrightarrow \mathrm{Ext}^{n+m}(D, D)$$

is surjective for all $n \geq n_0$ and $m \geq m_0$. Then $\mathrm{Ext}(D)$ is finitely generated, and the generators are among the elements of $\mathrm{Ext}^i(D, D)$ for $0 \leq i \leq \max(m_0, n_0)$. Now we apply the recursion of Theorem 5.16 to the second factor and display it as a direct sums of $\mathrm{Ext}^1(E, D)$ with $E \in \mathcal{R}(D)$. If we choose $m_0$ big enough, we get the desired $E \in \mathcal{R}(D)^\infty$ by Remark 8.30. $\square$

## 8.7. Example: $n = 7$, $q = 3$

Let us consider the running example with $n = 7$, $q = 3$. We want to compute the algebra $\mathrm{Ext}(E^3, E^3)$. We already discussed the quiver $\mathcal{R}$ in Example 8.28 and thus we observe that we only need to compute the three Ext-modules

$$\begin{aligned}
\mathrm{Ext}^1(E^3, E^3) &= \mathbb{C} \cdot \{[-1,-1], [-4,-2], [-2,-1]\},\\
\mathrm{Ext}^1(E^1, E^3) &= \mathbb{C} \cdot \{[1,0], [-2,-1]\}, \text{ and}\\
\mathrm{Ext}^1(E^5, E^3) &= \mathbb{C} \cdot \{[-3,-2], [-4,-2]\}.
\end{aligned}$$

Now for $\mathrm{Ext}^2(E^3, E^3)$ we have

$$\begin{aligned}
\mathrm{Ext}^2(E^3, E^3) &= \mathrm{Ext}^1(E^3[4,2], E^3) \oplus \mathrm{Ext}^1(E^1[3,2], E^3)\\
&= \mathrm{Ext}^1(E^3, E^3)[-4,-2] \oplus \mathrm{Ext}^1(E^1, E^3)[-3,-2]\\
&= \mathbb{C} \cdot \{[-5,-3], [-8,-4], [-6,-3]\} \oplus \mathbb{C} \cdot \{[-2,-2], [-5,-3]\}.
\end{aligned}$$

Next we describe the basis of the 5-dimensional $\mathbb{C}$-vector space $\mathrm{Ext}^2(E^3, E^3)$ in terms of Section 8.2:

$$\mathrm{Ext}^2(E^3, E^3) = \left\langle \begin{matrix} ([-5,-3], E^3[4,2] \to E^3, E^3),\\ ([-8,-4], E^3[4,2] \to E^3, E^3),\\ ([-6,-3], E^3[4,2] \to E^3, E^3),\\ ([-2,-2], E^1[3,2] \to E^3, E^3),\\ ([-5,-3], E^1[3,2] \to E^3, E^3) \end{matrix} \right\rangle_{\mathbb{C}}.$$

We want to compute the products of the three basis elements

$$a := ([-1,-1], E^3, E^3),\ b := ([-4,-2], E^3, E^3),\ \text{and } c := ([-2,-1], E^3, E^3)$$

of $\mathrm{Ext}^1(E^3, E^3)$ with each other. Just considering the degrees by Theorem 8.19 we already know

$$c^2 = ac = ca = 0.$$

Let us consider the remaining squares first. For $a^2$ we have to check the following condition according to Theorem 8.11:

$$E^1[3,2] \subseteq E^3[1,1].$$

This is true, as well as $E^3[4,2] \subseteq E^3[4,2]$ for $b^2$. This yields

$$\begin{aligned}
a^2 &= ([-2,-2], E^1[3,2] \to E^3, E^3), \text{ and}\\
b^2 &= ([-8,-4], E^3[4,2] \to E^3, E^3).
\end{aligned}$$

Now let us check the products $ab$ and $ba$. We check the following two conditions, respectively:

$$E^3[4,2] \subseteq E^3[4,2] \text{ and } E^1[3,2] \subseteq E^3[1,1].$$

Again these are true giving us the formulas

$$\begin{aligned}
ab &= ([-5,-3], E^3[4,2] \to E^3, E^3), \text{ and}\\
ba &= ([-5,-3], E^1[3,2] \to E^3, E^3).
\end{aligned}$$

Thus $a$ and $b$ do not commute with each other. As expected, we see that there really was no choice to be made, since

$$E^3[4,2] \not\subseteq E^3[1,1] \text{ and } E^1[3,2] \not\subseteq E^3[4,2].$$

Otherwise the product would be zero. For the remaining products we get

$$bc = ([-6,-3], E^3[4,2] \to E^3, E^3) \text{ and } cb = 0.$$

Thus we see that the multiplication map

$$\mathrm{Ext}^1(E^3,E^3) \times \mathrm{Ext}^1(E^3,E^3) \to \mathrm{Ext}^2(E^3,E^3)$$

is surjective. Using the criterion above and the algorithm for the multiplication on a computer, one finds that $\mathrm{Ext}(E^3)$ is generated by the elements $a, b, c$ as an $R$-algebra. Unfortunately there is no software able to compute the relations of the generators of a non-commutative ring of our format. One problem is that Groebner bases do not have to be finite in the non-commutative case.

Please do also have a look at Section A.1 to see how to use `polymake` to compute the product in $\mathrm{Ext}(D)$.

# Code

For the implementation we use `polymake` ([GJ00]) for the combinatorial aspects and **Singular** ([Dec+15]) for algebro-geometric tasks. Many experiments were also done using Macaulay2 ([GS]). For the concrete implementation we settled for `polymake`, since our algorithms are purely combinatorial in nature.

Most of the code on toric varieties, interfacing **Singular**, and cyclic quotients has already been incorporated in the `polymake` core. The code provided at the end of this chapter is especially for experiments within the topics of this thesis. It is experimental and possibly still unstable.

## A.1. Running example with $n = 7$ and $q = 3$

The following code shows how to reproduce the results of the running example $n = 7$, $q = 3$.

```
polytope > application "fulton";

fulton > $cqs = new CyclicQuotient(N=>7, Q=>3);

fulton > print $cqs->WEIGHT_CONE->HILBERT_BASIS;
7 3
0 1
2 1
1 1

fulton > print $cqs->CONTINUED_FRACTION;
3 2 2
fulton > print $cqs->DUAL_CONTINUED_FRACTION;
2 4
fulton > $e1 = $cqs->add("DIVISOR",COEFFICIENTS=>new Vector
    (-1,0));

fulton > $e3 = $cqs->add("DIVISOR",COEFFICIENTS=>new Vector
    (-3,0));

fulton > $canonical = $cqs->add("DIVISOR",COEFFICIENTS=>new
    Vector(-6,0));
```

```
fulton > print $cqs->EXT1_MATRIX;
2 1 2 1 2 1 0
1 1 1 1 1 0 0
2 1 3 2 2 1 0
1 1 2 1 1 0 0
2 1 2 1 2 1 0
1 0 1 0 1 0 0
0 0 0 0 0 0 0

fulton > print $e1->MODULE_GENERATORS;
7 3
1 1
2 1

fulton > print $e3->MODULE_GENERATORS;
7 3
3 2
4 2

fulton > print $canonical->MODULE_GENERATORS;
6 3
7 3

fulton > print $cqs->ext1($e1,$canonical);

fulton > print $cqs->ext1($e3,$canonical);

fulton > print $cqs->ext1($e1,$e1);
-1 -1
-2 -1

fulton > print $cqs->ext1($e3,$e3);
-1 -1
-4 -2
-2 -1

fulton > $ext1e3e3 = $cqs->ext1($e3,$e3);

fulton > $e3c = $e3->COEFFICIENTS;

fulton > $a = new ExtMonomial(DEGREE=>$ext1e3e3->[0], PATH=>new
    Matrix([$e3c]), SOURCE=>$e3c);

fulton > $b = new ExtMonomial(DEGREE=>$ext1e3e3->[1], PATH=>new
    Matrix([$e3c]), SOURCE=>$e3c);
```

```
fulton > $c = new ExtMonomial(DEGREE=>$ext1e3e3->[2], PATH=>new
    Matrix([$e3c]), SOURCE=>$e3c);

fulton > $aa = $cqs->multiply($a, $a);

fulton > $ab = $cqs->multiply($a, $b);

fulton > $ac = $cqs->multiply($a, $c);

fulton > $ba = $cqs->multiply($b, $a);

fulton > $bb = $cqs->multiply($b, $b);

fulton > $bc = $cqs->multiply($b, $c);

fulton > $ca = $cqs->multiply($c, $a);

fulton > $cb = $cqs->multiply($c, $b);

fulton > $cc = $cqs->multiply($c, $c);

fulton > $cqs->print_monomial_nicely($aa);
([-2,-2], E1[3,2]->E3[0,0], E3[0,0])

fulton > $cqs->print_monomial_nicely($ab);
([-5,-3], E3[4,2]->E3[0,0], E3[0,0])

fulton > $cqs->print_monomial_nicely($ac);
0

fulton > $cqs->print_monomial_nicely($ba);
([-5,-3], E1[3,2]->E3[0,0], E3[0,0])

fulton > $cqs->print_monomial_nicely($bb);
([-8,-4], E3[4,2]->E3[0,0], E3[0,0])

fulton > $cqs->print_monomial_nicely($bc);
([-6,-3], E3[4,2]->E3[0,0], E3[0,0])

fulton > $cqs->print_monomial_nicely($ca);
0

fulton > $cqs->print_monomial_nicely($cb);
([-6,-3], E3[4,2]->E3[0,0], E3[0,0])

fulton > $cqs->print_monomial_nicely($cc);
0
```

## A.2. Finiteness of $\bar{P}(D)$

In this section we will compute the set $\bar{P}(D)$ defined in Section 3.2. As stated in Conjecture 3.10, we believe it to be finite for all $\sigma$ and $D$. This is trivial for simplicial $\sigma$, since the class group is finite in this case. Thus this is not interesting for CQS. We have prepared two examples, both are given as the cone over a polytope at height one. The first example even appears on the titlepage and is given by a rectangle of length $2 \times 1$ at height one. The second is given as the cone over a hexagon.

Unfortunately, the computation of $\bar{P}(D)$ becomes very slow for more complicated $\sigma$. Additionally the code we provide is not suitable for checking the finiteness of $\bar{P}(D)$, it only compute $\bar{P}(D)$ and hence, it only terminates if $\bar{P}(D)$ is finite.

### A.2.1. Rectangle example

In this example we will take $\sigma^\vee$ to be generated by a rectangle at height one. We will identify divisors up to linear equivalence by the shape of the convex hull of their generators.

```
fulton > $rectangle = new Cone(INPUT_RAYS
    =>[[1,0,0],[1,1,0],[1,1,2],[1,0,2]]);

fulton > $tv = new AffineNormalToricVariety(new Cone(RAYS=>
    $rectangle->FACETS));

fulton > print $tv->WEIGHT_CONE->HILBERT_BASIS;

1 0 0
1 1 0
1 1 2
1 0 1
1 1 1
1 0 2

fulton > $div1 = $tv->add("DIVISOR", COEFFICIENTS=>new Vector
    (-1,0,0,0));
```

The method `check_finiteness` takes two arguments. The first are the facets of $\sigma^\vee$. The second are the coefficients of the divisor. We proceed by calculating $\bar{P}(D)$ for $D$ having the square at height one as generators. We receive a lot of output, namely the generators of every new divisors, and the divisors we do not need to add to the list, since their class was already found.

```
fulton > print join("\n",check_finiteness($rectangle->FACETS,
    new Vector(-1,0,0,0)));
1: -1 0 0 0 Remembering.
Gens for: -1 0 0 0
1 0 1
1 1 1
1 0 2
```

```
1 1 2

11: −2 −1 −1 −1 Remembering.
Gens for: −2 −1 −1 −1
2 1 2
2 1 3

11: −3 −1 −2 −1 Dropping. −1 0 0 0
10: −2 −1 −1 −1 Dropping. −2 −1 −1 −1
9: −2 −1 −1 −1 Dropping. −2 −1 −1 −1
8: −2 −1 −1 −1 Dropping. −2 −1 −1 −1
7: −2 −1 −1 −1 Dropping. −2 −1 −1 −1
6: −2 −1 0 −1 Remembering.
Gens for: −2 −1 0 −1
2 1 2
2 1 3
2 1 4

9: −4 −1 −2 −1 Remembering.
Gens for: −4 −1 −2 −1
3 1 4
3 2 4

9: −4 −2 −2 −2 Dropping. −2 −1 0 −1
8: −4 −1 −1 −1 Dropping. −1 0 0 0
7: −4 −1 −2 −1 Dropping. −4 −1 −2 −1
6: −3 −1 −2 −1 Dropping. −1 0 0 0
5: −2 −1 −1 0 Dropping. −1 0 0 0
4: −2 −1 −1 −1 Dropping. −2 −1 −1 −1
3: −2 −1 −1 −1 Dropping. −2 −1 −1 −1
2: −2 0 −1 −1 Dropping. −1 0 0 0
1: −1 −1 −1 −1 Dropping. −2 −1 0 −1
−1 0 0 0
−2 −1 −1 −1
−2 −1 0 −1
−4 −1 −2 −1
```

As one can see, there are four elements in the resulting $\bar{P}(D)$. These are the vertical lines of length one and two, the horizontal line of length one, and the square itself.

Let us choose a bigger divisor, namely take the rectangle of (horizontal) length two:

```
fulton > print join("\n",check_finiteness($rectangle->FACETS,
    new Vector(-3,0,0,0)));
1: −3 0 0 0 Remembering.
Gens for: −3 0 0 0
2 0 3
2 1 3
2 2 3
```

```
2 0 4
2 1 4
2 2 4

57: -4 -2 -1 -2 Remembering.
Gens for: -4 -2 -1 -2
4 2 4
4 2 7
4 2 5
4 2 6

67: -7 -2 -4 -2 Dropping. -3 0 0 0
66: -7 -2 -3 -2 Remembering.
Gens for: -7 -2 -3 -2
5 2 7
5 3 7

66: -7 -3 -3 -3 Remembering.
Gens for: -7 -3 -3 -3
6 3 7
6 3 8
6 3 9

69: -9 -3 -5 -3 Dropping. -7 -2 -3 -2
68: -9 -3 -4 -3 Remembering.
Gens for: -9 -3 -4 -3
7 3 9
7 4 9
7 3 10
7 4 10

78: -10 -4 -5 -4 Remembering.
Gens for: -10 -4 -5 -4
8 4 10
8 4 11

78: -11 -4 -6 -4 Dropping. -9 -3 -4 -3
# ...
35: -4 -2 0 -2 Remembering.
Gens for: -4 -2 0 -2
4 2 4
4 2 8
4 2 5
4 2 6
4 2 7

60: -8 -2 -4 -2 Remembering.
```

```
Gens for: -8 -2 -4 -2
6 4 8
6 3 8
6 2 8

63: -8 -4 -4 -4 Dropping. -4 -2 0 -2
# ...
1: -3 -1 -1 -2 Dropping. -7 -3 -3 -3
-3 0 0 0
-4 -2 -1 -2
-7 -2 -3 -2
-7 -3 -3 -3
-9 -3 -4 -3
-10 -4 -5 -4
-4 -2 0 -2
-8 -2 -4 -2
fulton >
```

In this case there are eight elements in $\bar{P}(D)$. Namely the vertical lines of length one to four, the horizontal lines of length one to two, the square and the rectangle itself.

### A.2.2. Hexagon example

We proceed just as in the previous case of a rectangle at height one. Looking at the generators in between, we see that even though $\sigma$ was generated at height one, just like $D$, this does not have to be true for the other elements of $\bar{P}(D)$.

```
polytope > application "fulton";

fulton > $c = new Cone(INPUT_RAYS
    =>[[1,0,0],[1,1,0],[1,2,1],[1,2,2],[1,1,2],[1,0,1]]);

fulton > $m = $c->FACETS;

fulton > $d = -dense(unit_vector(6,1));

fulton > @pd = check_finiteness($m, $d);
1: 0 -1 0 0 0 0 Remembering.
Gens for: 0 -1 0 0 0 0
1 0 1
1 2 2
1 1 1
1 1 2
1 2 1

26: -2 -2 -2 -2 -1 -2 Remembering.
Gens for: -2 -2 -2 -2 -1 -2
2 2 2
```

```
3 4 5

26: -4 -5 -4 -2 -2 -2 Remembering.
Gens for: -4 -5 -4 -2 -2 -2
4 4 5
4 5 5
4 4 6
4 5 6
4 6 6

51: -6 -6 -6 -4 -3 -4 Remembering.
Gens for: -6 -6 -6 -4 -3 -4
6 6 6
5 6 7

51: -6 -7 -6 -6 -6 -6 Dropping. 0 -1 0 0 0 0
50: -6 -6 -6 -4 -3 -4 Dropping. -6 -6 -6 -4 -3 -4
49: -6 -6 -6 -4 -3 -4 Dropping. -6 -6 -6 -4 -3 -4
48: -5 -6 -6 -4 -3 -4 Remembering.
Gens for: -5 -6 -6 -4 -3 -4
5 6 6
5 6 7

48: -6 -7 -6 -5 -4 -4 Remembering.
Gens for: -6 -7 -6 -5 -4 -4
6 7 7
6 6 7
6 8 8
6 7 8

58: -7 -8 -8 -6 -5 -6 Dropping. -5 -6 -6 -4 -3 -4
57: -7 -8 -8 -6 -5 -6 Dropping. -5 -6 -6 -4 -3 -4
56: -7 -8 -8 -6 -5 -5 Remembering.
Gens for: -7 -8 -8 -6 -5 -5
7 8 8
7 8 9
7 9 9

59: -8 -9 -9 -7 -6 -6 Dropping. -7 -8 -8 -6 -5 -5
58: -8 -9 -9 -7 -5 -6 Remembering.
Gens for: -8 -9 -9 -7 -5 -6
8 9 9
8 9 10
8 10 10
8 10 11

68: -9 -11 -10 -8 -7 -7 Remembering.
```

```
Gens for: -9 -11 -10 -8 -7 -7
9 10 11
9 11 11

68: -10 -11 -11 -9 -7 -8 Dropping. -8 -9 -9 -7 -5 -6
67: -9 -11 -10 -8 -6 -7 Remembering.
Gens for: -9 -11 -10 -8 -6 -7
9 10 11
9 11 11
9 11 12

70: -10 -12 -11 -9 -7 -8 Dropping. -9 -11 -10 -8 -6 -7
69: -10 -12 -11 -9 -7 -7 Dropping. -6 -7 -6 -5 -4 -4
68: -10 -12 -11 -8 -7 -8 Remembering.
Gens for: -10 -12 -11 -8 -7 -8
10 11 12
10 12 12
10 11 13
10 12 13

78: -12 -13 -12 -10 -8 -9 Remembering.
Gens for: -12 -13 -12 -10 -8 -9
11 12 13
11 13 14

78: -12 -14 -13 -10 -9 -10 Dropping. -10 -12 -11 -8 -7 -8
# ...
45: -6 -6 -6 -4 -2 -4 Remembering.
Gens for: -6 -6 -6 -4 -2 -4
6 6 6
6 8 10
5 6 7

48: -8 -10 -8 -6 -6 -6 Remembering.
Gens for: -8 -10 -8 -6 -6 -6
8 8 10
8 10 10
8 9 10

51: -10 -10 -10 -8 -6 -8 Dropping. -6 -6 -6 -4 -2 -4
# ...
1: -2 -2 -2 -1 -1 -2 Dropping. -5 -6 -6 -4 -3 -4

fulton > $n = @pd;

fulton > print $n;
14
```

```
fulton >
```

The calculation takes about 30 Minutes for this example.

### A.2.3. polymake code

#### CQS code

```
use List::MoreUtils;

declare object ExtMonomial;

object ExtMonomial{

   property DEGREE : Vector;

   property PATH : Matrix;

   property SOURCE : Vector;

}


declare object ResolutionData;

object ResolutionData {

   property WEIGHTED_EDGES : Map<Pair<Vector, Vector>, Matrix>;

   property LABELS : String;

   property QUIVER : Graph<Directed>;

   property INCIDENCE_MATRIX : Matrix;

}



object CyclicQuotient {

   property CLASS_GROUP_REPRESENTATIVES : Matrix;

   rule CLASS_GROUP_REPRESENTATIVES : N, Q{
      my $n = $this->N;
      my $q = $this->Q;
      my @class_group = map((new Vector(-$_, 0)),1..(new Int($n
         )));
```

```
    $this->CLASS_GROUP_REPRESENTATIVES = new Matrix<Rational
        >(@class_group);
}

property RESOLUTION : ResolutionData;

property EXT1_MATRIX : Matrix;

property TOR1_MATRIX : Matrix;

rule TOR1_MATRIX : EXT1_MATRIX, RESOLUTION.INCIDENCE_MATRIX{
    my $ext1 = $this->EXT1_MATRIX;
    my $incidence_matrix = $this->RESOLUTION->
        INCIDENCE_MATRIX;
    $this->TOR1_MATRIX = $incidence_matrix *
        $incidence_matrix * $ext1;
}

rule EXT1_MATRIX : DUAL_CONTINUED_FRACTION{
    my $dcf = $this->DUAL_CONTINUED_FRACTION;
    $this->EXT1_MATRIX = ext1_mat_from_dcf($dcf);
}

rule RESOLUTION.INCIDENCE_MATRIX : N, RESOLUTION.
    WEIGHTED_EDGES{
    my $n = $this->N;
    my $weighted_edges = $this->RESOLUTION->WEIGHTED_EDGES;
    my $incidence_matrix = new Matrix(new Int($n), new Int($n
        ));
    foreach my $edge (keys %$weighted_edges){
        my $source = $edge->[0];
        my $target = $edge->[1];
        #print $source," ",$target,"\n";
        my $row = -$target->[0]-1;
        my $col = -$source->[0]-1;
        $incidence_matrix->($row, $col) = $weighted_edges->{
            $edge}->rows;
    }
    $this->RESOLUTION->INCIDENCE_MATRIX = $incidence_matrix;
}

rule RESOLUTION.WEIGHTED_EDGES : CLASS_GROUP_REPRESENTATIVES
    , N, Q {
    my $G = $this->CLASS_GROUP_REPRESENTATIVES;
    my $n = $this->N;
    my $weighted_edges = new Map<Pair<Vector, Vector>, Matrix
        >();
```

```
    foreach my $coefficients (@$G){
        my $divisor = $this->DIVISOR(COEFFICIENTS=>
            $coefficients, temporary);
        my $generators = sort_matrix_rows_by_first_coordinate(
            $divisor->MODULE_GENERATORS);
        # print $generators,"\n";
        my $target = $coefficients;
        for(my $j = 1; $j<$generators->rows; $j++){
            my $inbetween = $generators->[$j]->[0] -
                $generators->[$j-1]->[0];
            my $source = new Vector(-$inbetween, 0);
            my $edge = new Pair<Vector,Vector>($source, $target
                );
            my $shift = $generators->[$j-1];
            if(!defined $weighted_edges->{$edge}){
                $weighted_edges->{$edge} = new Matrix($shift);
            } else {
                my $oldshift = $weighted_edges->{$edge};
                $weighted_edges->{$edge} = $oldshift / $shift;
            }
        }
    }
    $this->RESOLUTION->WEIGHTED_EDGES = $weighted_edges;
}

rule RESOLUTION.QUIVER.ADJACENCY, RESOLUTION.QUIVER.
    NODE_LABELS : CLASS_GROUP_REPRESENTATIVES, N, RESOLUTION.
    WEIGHTED_EDGES{
    my $class_group = $this->CLASS_GROUP_REPRESENTATIVES;
    my $graph = new common::Graph<Directed>($class_group->
        rows);
    my $weighted_edges = $this->RESOLUTION->WEIGHTED_EDGES;
    my $numbering = new Map<Vector, Int>();
    my $k = 0;
    foreach my $g (@$class_group){
        $numbering->{$g} = $k;
        $k++;
    }
    foreach my $edge (keys %$weighted_edges){
        my $source = $edge->[0];
        my $target = $edge->[1];
        $graph->edge($numbering->{$source}, $numbering->{
            $target});
    }
    $this->RESOLUTION->QUIVER->NODE_LABELS = @$class_group;
    $this->RESOLUTION->QUIVER->ADJACENCY = $graph;
}
```

```
user_method ext1(TDivisor, TDivisor){
   my($cqs, $d1, $d2) = @_;
   my $rho1 = new Vector<Rational>(1, new Rational($cqs->Q,
      $cqs->N));
   my $d1Gens = new Matrix<Rational>(
      sort_matrix_rows_by_first_coordinate($d1->
      MODULE_GENERATORS));
   my $annVertex = -$d1Gens->[$d1Gens->rows-1] + ($d1Gens->[
      $d1Gens->rows-1]->[0] - $d1Gens->[0]->[0])*$rho1;
   $annVertex = $annVertex - $rho1 - (new Vector<Rational
      >(0,new Rational(1,$cqs->N)));
   # print "Ann: ",$annVertex,"\n";
   my $d2Vertex = $d2->SECTION_POLYTOPE->VERTICES->[0]->
      slice(1);
   $annVertex += $d2Vertex;
   my $ann = new Polytope(POINTS=>[[1,@$annVertex
      ],[0,0,-1],[0,-$cqs->N,-$cqs->Q]]);
   # print $d1Gens,"\n",$d2Vertex,"\n";
   my @result = ();
   for(my $i=1; $i<$d1Gens->rows; $i++){
      my $intersectionVertex = $d1Gens->[$i-1] + ($d1Gens->[
         $i]->[0] - $d1Gens->[$i-1]->[0])*$rho1;
      # print "int: ",$intersectionVertex,"\n";
      $intersectionVertex *= (-1);
      $intersectionVertex += $d2Vertex;
      my $intersector = new Polytope(POINTS=>[[1,
         @$intersectionVertex],[0,0,1],[0,$cqs->N,$cqs->Q]])
         ;
      my $intersection = intersection($ann, $intersector);
      # print "LP:\n",$intersection->LATTICE_POINTS,"\n";
      @result = (@result, @{$intersection->LATTICE_POINTS->
         minor(All,~[0])});
   }
   return new Matrix(uniq(@result));
}

user_method multiply(ExtMonomial, ExtMonomial){
   my($cqs, $b, $a) = @_;
   my $weightedEdges = $cqs->RESOLUTION->WEIGHTED_EDGES;
   my $facets = new Matrix<Rational>(primitive($cqs->RAYS));
   my $resultPath = $a->PATH;
   # print "Path is:\n",$resultPath,"\n";
   my $last = $resultPath->[$resultPath->rows - 1];
   for(my $i = 0; $i<$b->PATH->rows; $i++){
      my $bi = vertex_from_ineq($b->PATH->[$i], $facets);
      $bi = $bi - $a->DEGREE;
```

```
        $bi = ineq_from_vertex($bi, $facets);
        # print "bi: ",$bi,"\n";
        my @S = find_incoming_arrows($last, $weightedEdges,
            $facets);
        # print "S:\n",join("\n",@S),"\n\n";
        @S = grep(($_->[0] <= $bi->[0]) && ($_->[1] <= $bi
            ->[1]),@S);
        # print "S:\n",join("\n",@S),"\n\n";
        if(@S == 1){
            $last = pop @S;
            $resultPath = $resultPath/$last;
        } else {
            return 0;
        }
    }
    my $d1 = $cqs->add('DIVISOR',COEFFICIENTS=>$last);
    my $d2 = $cqs->add('DIVISOR',COEFFICIENTS=>$b->SOURCE);
    my $possibleDegrees = $cqs->ext1($d1, $d2);
    # print "Possible Degrees:\n",$possibleDegrees,"\n";
    my $resultDegree = $a->DEGREE + $b->DEGREE;
    if(grep($_ == $resultDegree, @$possibleDegrees) == 1){
        return new ExtMonomial(DEGREE=>$resultDegree, PATH=>
            $resultPath, SOURCE=>$b->SOURCE);
    } else {
        return 0;
    }
}

user_method print_monomial_nicely(ExtMonomial){
    my($cqs, $monomial) = @_;
    my $facets = new Matrix<Rational>(primitive($cqs->RAYS));
    my $n = $cqs->N;
    print "([",join(",",@{$monomial->DEGREE}),"], ";
    for(my $i=$monomial->PATH->rows-1; $i>=0; $i--){
        my $current = $monomial->PATH->[$i];
        my($ei, $shift) = find_lineq_ei($current, $facets, $n)
            ;
        my $index = -$ei->[0];
        print "E",$index,"[",join(",",@$shift),"]";
        if($i>0){
            print "->";
        }
    }
    print ", ";
    my($sourceEi, $sourceShift) = find_lineq_ei($monomial->
        SOURCE, $facets, $n);
```

```
      print "E",-$sourceEi->[0],"[",join(",",@$sourceShift),"])
          \n";
   }

}



user_function ext1_mat_from_dcf(Vector<Integer>){
   my($dcf) = @_;
   $dcf = new Vector<Integer>($dcf);
   my $length = $dcf->dim;
   if(($length == 0) || ($dcf == ones_vector<Integer>(1))){
      return zero_matrix(1,1);
   }
   if($dcf->[$length-1] == 1){
      $dcf = $dcf->slice(0, $length-1);
      $dcf->[$dcf->dim - 1]--;
      return ext1_mat_from_dcf(new Vector<Integer>($dcf));
   }
   $dcf->[$length-1]--;
   my @dcf = @$dcf;
   my $upper_left_dcf = new Vector<Integer>(@dcf[0..($length-2)
      ]);
   my $lower_right_dcf = new Vector<Integer>(\@dcf);
   my $upper_left = ext1_mat_from_dcf($upper_left_dcf);
   my $lower_right = ext1_mat_from_dcf($lower_right_dcf);
   my $A;
   if($upper_left->rows > $lower_right->rows){
      my $start = $upper_left->rows - $lower_right->rows;
      $A = $upper_left->minor(All, [$start..($upper_left->rows
         -1)]);
   } else {
      my $start = $lower_right->rows - $upper_left->rows;
      $A = $lower_right->minor([$start..($lower_right->rows-1)
         ], All);
   }
   my $result = ($upper_left | $A) / (transpose($A) |
      $lower_right);
   $result = $result + upper_triangular_ones_matrix($result->
      rows);
   return new Matrix($result);
}

sub vertex_from_ineq{
   my($val, $facets) = @_;
   my $p = new Polytope(INEQUALITIES=>($val | $facets));
```

```
    return $p->VERTICES->[0]->slice(1);
}

sub ineq_from_vertex{
    my($vertex, $facets) = @_;
    return -$facets * $vertex;
}

sub are_linearly_equivalent{
    my($div1, $div2, $facets) = @_;
    my $v1 = vertex_from_ineq($div1, $facets);
    my $v2 = vertex_from_ineq($div2, $facets);
    return is_integral($v1-$v2);
}

sub find_lineq_ei{
    my($div, $facets, $n) = @_;
    for(my $i=0; $i<$n; $i++){
        my $candidate = new Vector(-$i, 0);
        if(are_linearly_equivalent($div, $candidate, $facets)){
            my $shift = vertex_from_ineq($div, $facets) -
                vertex_from_ineq($candidate, $facets);
            return($candidate, $shift);
        }
    }
    die "No fitting candidate.";
}

sub find_incoming_arrows{
    my($div, $weightedEdges, $facets) = @_;
    my @selected = grep($_->[1] == $div, keys %$weightedEdges);
    # print "Selection done.\n";
    my @result = map{
        my $sourceVertex = vertex_from_ineq($_->[0], $facets);
        # print "Have source vertex.\n";
        my $degrees = $weightedEdges->{$_};
        map($_+$sourceVertex, @$degrees);
    }@selected;
    return map(ineq_from_vertex($_, $facets), @result);
}

sub upper_triangular_ones_matrix{
    my($n) = @_;
    return new Matrix(map(ones_vector($n-$_) | zero_vector($_),
        1..$n));
}
```

```
sub sort_matrix_rows_by_first_coordinate{
   my($matrix) = @_;
   if($matrix->rows == 1){
      return $matrix;
   }
   my @rows = @$matrix;
   my @sorted_rows = sort{$a->[0] <=> $b->[0]} @rows;
   return new Matrix(@sorted_rows);
}
```

#### Interfacing Singular

```
object TDivisor {

   property SINGULAR_IDEAL : String;

   property SINGULAR_SYZYGIES : String;

}


object AffineNormalToricVariety {

   property SINGULAR_TORIC_RING : String;

   rule SINGULAR_TORIC_RING : WEIGHT_CONE {
      my $monoid = $this->WEIGHT_CONE;
      my $ringname = get_random_string();
      my @variables = map("x(".$_.")",1..$monoid->HILBERT_BASIS
         ->rows);
      my $toric_ideal = toric_ideal_as_string($monoid,
         @variables);
      singular_eval("ring R".$ringname." = 0,(".join(",",
         @variables)."),dp;");
      singular_eval("ideal toric_ideal = ".$toric_ideal.";");
      singular_eval("qring r_".$ringname." = std(toric_ideal);"
         );
      $this->SINGULAR_TORIC_RING = $ringname;
   }
   precondition : AFFINE;

   rule DIVISOR.SINGULAR_SYZYGIES : SINGULAR_TORIC_RING,
      DIVISOR.SINGULAR_IDEAL{
      my $ringname = $this->SINGULAR_TORIC_RING;
      my $idealname = $this->DIVISOR->SINGULAR_IDEAL;
      singular_eval("setring r_".$ringname.";");
```

```
    singular_eval("module␣syz_".$idealname."␣=␣syz(div_".
        $idealname.");");
    $this->DIVISOR->SINGULAR_SYZYGIES = $idealname;
}
precondition : AFFINE;

rule DIVISOR.SINGULAR_IDEAL : SINGULAR_TORIC_RING,
    WEIGHT_CONE, DIVISOR.MODULE_GENERATORS{
    my $ringname = $this->SINGULAR_TORIC_RING;
    my $generators = $this->DIVISOR->MODULE_GENERATORS;
    # print "Gens: ",$generators,"\n";
    my $monoid = $this->WEIGHT_CONE;
    my $mod_vector = find_vector_moving_points_inside_cone(
        $generators, $monoid);
    # print "Mod vector: ",$mod_vector,"\n";
    my @variables = map("x(".$_.")",1..$monoid->HILBERT_BASIS
        ->rows);
    my $mod_gens = new Matrix(map($mod_vector + $_,
        @$generators));
    my $mod_gens_monomial_exponents =
        represent_vectors_in_Hilbert_basis($mod_gens, $monoid
        ->HILBERT_BASIS, $monoid->FACETS);
    my @mod_gens_monomials = map(vector_to_monomial_string($_
        ,@variables), @$mod_gens_monomial_exponents);
    my $idealname = get_random_string();
    singular_eval("setring␣r_".$ringname.";");
    singular_eval("ideal␣div_".$idealname."␣=␣".join(",",
        @mod_gens_monomials).";");
    singular_eval("div_".$idealname."␣=␣std(div_".$idealname.
        ");");
    $this->DIVISOR->SINGULAR_IDEAL = $idealname;
}
precondition : AFFINE;

user_method singular_exti_dimension( $ , TDivisor, TDivisor)
    {
    my $toric_variety = $_[0];
    my $i = $_[1];
    my $divisor1 = $_[2];
    my $divisor2 = $_[3];
    my $ringname = $toric_variety->SINGULAR_TORIC_RING;
    my $syzygies1 = $divisor1->SINGULAR_SYZYGIES;
    my $syzygies2 = $divisor2->SINGULAR_SYZYGIES;
    singular_eval("setring␣r_".$ringname.";");
    load_singular_library("homolog.lib");
    singular_eval("module␣M␣=␣Ext(".$i.",␣syz_".$syzygies1.",
        ␣syz_".$syzygies2.");");
```

```
      singular_eval("M␣=␣std(M);");
      singular_eval("int␣d␣=␣vdim(M);");
      singular_eval("int␣vd␣=␣vdim(M);");
      return new Vector(singular_get_var("d"), singular_get_var
         ("vd"));
   }

   user_method singular_tori_dimension( $ , TDivisor, TDivisor)
      {
      my $toric_variety = $_[0];
      my $i = $_[1];
      my $divisor1 = $_[2];
      my $divisor2 = $_[3];
      my $ringname = $toric_variety->SINGULAR_TORIC_RING;
      my $syzygies1 = $divisor1->SINGULAR_SYZYGIES;
      my $syzygies2 = $divisor2->SINGULAR_SYZYGIES;
      singular_eval("setring␣r_".$ringname.";");
      load_singular_library("homolog.lib");
      singular_eval("module␣M␣=␣Tor(".$i.",␣syz_".$syzygies1.",
         ␣syz_".$syzygies2.");");
      singular_eval("M␣=␣std(M);");
      singular_eval("int␣d␣=␣vdim(M);");
      singular_eval("int␣vd␣=␣vdim(M);");
      return new Vector(singular_get_var("d"), singular_get_var
         ("vd"));
   }

}

object CyclicQuotient{

   user_method singular_ext1_matrix(){
      my $cqs = $_[0];
      my $canonical_divisor = new Vector(-1, -1);
      my @divisors= ();
      my @canonical_minus_divisors= ();
      for(my $i=1; $i<=$cqs->N; $i++){
         my $divisor_coefficients = new Vector(-$i, 0);
         push @divisors, $cqs->add("DIVISOR",COEFFICIENTS=>
            $divisor_coefficients);
         push @canonical_minus_divisors, $cqs->add("DIVISOR",
            COEFFICIENTS=>$canonical_divisor -
            $divisor_coefficients);
      }
      my @result = ();
      foreach my $divisor (@divisors){
```

```
        my @ext_vector = map($cqs->singular_exti_dimension(1,
            $divisor, $_),@canonical_minus_divisors);
        my $ext_vector = new Vector(map($_->[1], @ext_vector))
            ;
        push @result, $ext_vector;
    }
    return new Matrix(@result);
}

user_method singular_ext3_matrix(){
    my $cqs = $_[0];
    my $canonical_divisor = new Vector(-1, -1);
    my @divisors= ();
    my @canonical_minus_divisors= ();
    for(my $i=1; $i<=$cqs->N; $i++){
        my $divisor_coefficients = new Vector(-$i, 0);
        push @divisors, $cqs->add("DIVISOR",COEFFICIENTS=>
            $divisor_coefficients);
        push @canonical_minus_divisors, $cqs->add("DIVISOR",
            COEFFICIENTS=>$canonical_divisor -
            $divisor_coefficients);
    }
    my @result = ();
    foreach my $divisor (@divisors){
        my @ext_vector = map($cqs->singular_exti_dimension(3,
            $divisor, $_),@canonical_minus_divisors);
        my $ext_vector = new Vector(map($_->[1], @ext_vector))
            ;
        push @result, $ext_vector;
    }
    return new Matrix(@result);
}

user_method singular_tor1_matrix(){
    my $cqs = $_[0];
    my @divisors= ();
    for(my $i=1; $i<=$cqs->N; $i++){
        my $divisor_coefficients = new Vector(-$i, 0);
        push @divisors, $cqs->add("DIVISOR",COEFFICIENTS=>
            $divisor_coefficients);
    }
    my @result = ();
    foreach my $divisor (@divisors){
        my @tor_vector = map($cqs->singular_tori_dimension(1,
            $divisor, $_),@divisors);
        my $tor_vector = new Vector(map($_->[1], @tor_vector))
            ;
```

```
            push @result, $tor_vector;
        }
        return new Matrix(@result);
    }

}
```

**Checking finiteness of** $P(D)$

```
user_function check_finiteness(Matrix, Vector){
    my($facets, $d) = @_;
    my @toAdd = ($d);
    my @result = ();
    while(@toAdd > 0){
        my $m = @toAdd;
        my $current = pop @toAdd;
        print $m,": ", $current," ";
        my @equivalents = grep(are_equivalent($facets, $current,
            $_), @result);
        my $equivalents = @equivalents;
        if($equivalents == 0){
            print "Remembering.\n";
            @toAdd = (@toAdd, all_intersections($facets, $current)
                );
            push @result, $current;
        }
        else { print "Dropping. ",join(" ",@equivalents),"\n";}
    }
    return @result;
}

sub all_intersections{
    my($facets, $d) = @_;
    my @result;
    my $ineq = $d | $facets;
    my $p = new Polytope(INEQUALITIES=>$ineq);
    my $gens = lower_lattice_points($p);
    $gens = $gens->minor(All, ~[0]);
    print "Gens for: ",$d," \n";
    print $gens,"\n";
    $gens = -$gens * transpose($facets);
    my $n = $gens->rows;
    for(my $i=2; $i<=$n; $i++){
        my @indices = all_subsets_of_k($i, 0..($n-1));
        foreach my $index (@indices){
            my $subgens = $gens->minor($index, All);
            push @result, componentwise_minimum($subgens);
```

```
        }
    }
    return @result;
}

sub componentwise_minimum{
    my($vectors) = @_;
    my @columns = @{transpose($vectors)};
    my @maxValues = map(minimum($_), @columns);
    return new Vector(@maxValues);
}

sub are_equivalent{
    my($facets, $d1, $d2) = @_;
    my $eq = ($d1-$d2 | $facets);
    my $ineq = new Matrix(unit_vector($facets->cols+1,0));
    my $p = new Polytope(EQUATIONS=>$eq, INEQUALITIES=>$ineq);
    return $p->N_LATTICE_POINTS > 0;
}
```

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# Acknowledgements

This thesis would not have been possible without the support of many people. First of all I would like to thank my advisor, Klaus Altmann, for constant encouragement, motivation and many helpful discussions. Many mathematicians helped me through discussions and answering my many questions. Thus, I would like to thank Winfried Bruns, Jan Christophersen, Lutz Hille, Andreas Hochenegger, David Ploog, Karl Schwede and Michael Wemyss.

I am very grateful to Nikolai Beck, Alexandru Constantinescu, Marianne Merz, Irem Portakal and Anna-Lena Winz for proofreading this thesis. I also wish to thank my workgroup for their constant interest in my work, their mathematical input, and their talent to cheer me up when I felt lost. This thesis would not have been possible without the supportive and encouraging atmosphere created by the members of both AG Altmann and Schmitt. Many thanks also to Mary Metzler-Kliegl for helping me with many non-mathematical tasks. Benjamin Lorenz helped me a lot with programming in `polymake`, many experiments would have been impossible without his expertise and support.

I wish to thank my friends and family in Berlin for their support. Finally, I thank Anna-Lena for her patience and love.

# Summary

The topic of this thesis are the Ext modules of $T$-invariant Weil divisors on a normal affine toric variety. Such divisors can be described by a polyhedron with the same facet vectors as the cone describing the variety. Hence, using these combinatorial descriptions of the involved objects, we want to derive a combinatorial description of the Ext-modules. In particular, we want to give a criterion for $D$ being maximal Cohen-Macaulay.

Denote by $X$ a normal affine toric variety and let $D$ and $D'$ be two $T$-invariant Weil divisors. We want to compute $\operatorname{Ext}^i_X(D, D')$. Since $X = \operatorname{Spec} R$ is affine this is the same as computing $\operatorname{Ext}^i_R(M, M')$ for $M$ and $M'$ being the divisorial ideals over $R$, given by the global sections of $D$, $D'$ respectively.

The structure of this thesis is the following:

First we give a short introduction into the toric geometry used, and describe the relationship of Ext and maximal Cohen-Macaulayness.

The modules $M$ and $M'$ are isomorphic to certain monomial ideals in $R$. Hence, we want to resolve $M$ freely. Thus, we use a generalization of the Ansatz of Taylor for resolving monomial ideals in polynomial rings. This results in a spectral sequence, providing a sufficient criterion for the vanishing of higher Ext. Furthermore we give a superset of the combinatorial support of $\operatorname{Ext}^1(D, D')$.

Next we restrict to the case of $X$ being a cyclic quotient singularity (CQS). In this case the class group is finite. Furthermore, the generalized Taylor resolution becomes a short exact sequence. Hence, we can encode all free resolutions in a quiver. Higher $\operatorname{Ext}^i(D, D')$ become direct sums of $\operatorname{Ext}^1(G, D')$, where we read off the necessary $G$ from the quiver.

Thus, we are interested in understanding $\operatorname{Ext}^1(D, D')$. The combinatorial data of $D$ and $D'$ results in the desired combinatorial description of $\operatorname{Ext}^1(D, D')$. If one is interested in the dimension of $\operatorname{Ext}^1(D, D')$ as a vector space, the key is the relationship of CQS with continued fractions. These have already proven useful in the deformation theory of CQS, as shown by Stevens and Christophersen. Here we can construct the matrix with entries $\dim \operatorname{Ext}^1(D, D')$, $[D], [D'] \in \operatorname{Cl} X$, recursively from the continued fraction.

The last part concerns the algebra $\operatorname{Ext}(D)$. We construct a homogeneous basis of $\operatorname{Ext}(D)$ as a vector space, which then allows us to formulate a combinatorial description of the multiplication.

# Zusammenfassung

Diese Arbeit beschäftigt sich mit Ext-Moduln $T$-invarianter Weil-Divisoren auf normalen affinen torischen Varietäten. Solche Weil-Divisoren lassen sich durch Polyeder beschreiben, die dieselben Facetten-Vektoren haben, durch die auch der Kegel gegeben ist, der die torische Varietät beschreibt. Das Ziel ist es daher, diese kombinatorische Beschreibung auf die Ext-Moduln zu übertragen, um damit zu bestimmen, ob ein Weil-Divisor maximal Cohen-Macaulay ist.

Ziel ist es $\operatorname{Ext}^i_X(D, D')$ zu gegeben $T$-invarianten Weil-Divisoren $D$ und $D'$ auf der normalen affinenen torischen Varietät $X$ zu berechnen. Da $X = \operatorname{Spec} R$ affin ist, ist das äquivalent zur Berechnung von $\operatorname{Ext}^i_R(M, M')$, wobei $M$ und $M'$ die $R$-Moduln der globalen Schnitte von $D$, bzw. $D'$, bezeichnen.

Die Arbeit gliedert sich nun wie folgt:

Zuerst geben wir eine kurze Einführung in die torische Geometrie, die wir verwenden, und beschreiben den Zusammenhang von Ext und maximal Cohen-Macaulay.

Die Moduln $M$ und $M'$ sind sogenannte divisorielle Ideale und sind in unserem Fall isomorph zu bestimmten Monomidealen in $R$. Um Ext zu berechnen, konstruieren wir eine freie Auflösung von $M$. Dazu verallgemeinern wir den Ansatz von Taylor zur Auflösung von Monomidealen in Polynomringen auf Halbgruppenringe. Dies mündet in eine Spektralsequenz, die es uns erlaubt, ein hinreichendes Kriterium für das Verschwinden aller höherer Ext-Moduln anzugeben. Außerdem können wir eine Obermenge des kombinatorischen Trägers von $\operatorname{Ext}^1(D, D')$ angeben.

Danach schränken wir uns auf den Fall ein, dass $X$ eine zyklische Quotienten-Singularität (ZQS) ist. In diesem Fall ist die Klassengruppe endlich und die Verallgemeinerung der Taylor-Auflösung mündet in eine kurze exakte Sequenz für jeden Divisor. Dies erlaubt es uns alle freien Auflösungen als einen Köcher darzustellen. Höhere $\operatorname{Ext}^i(D, D')$ sind nun direkte Summen von $\operatorname{Ext}^1(G, D')$, wobei die benötigten $G$ aus dem Köcher gewonnen werden.

Als nächstes interessieren wir uns daher für die Berechnung von $\operatorname{Ext}^1(D, D')$. Wir können aus den kombinatorischen Daten von $D$ und $D'$ eine kombinatorische Beschreibung von $\operatorname{Ext}^1(D, D')$ gewinnen. Falls wir nur an der Dimension der $\operatorname{Ext}^1(D, D')$ für alle Äquivalenzklassen aus der Klassengruppe interessiert sind, gibt es ein weiteres hilfreiches Datum auf ZQS. ZQS sind eng verbunden mit Kettenbrüchen, z.B. beschrieben Stevens und Christophersen die Komponenten der versellen Deformation einer ZQS durch bestimmte Kettenbrüche. Die Matrix mit den Dimensionen der $\operatorname{Ext}^1(D, D')$ lässt sich nun rekursiv aus dem Kettenbruch gewinnen, der auch die ZQS beschreibt.

Zuletzt widmen wir uns der Algebra $\operatorname{Ext}(D)$. Anhand der bisherigen Überlegung entwickeln wir eine Basis von $\operatorname{Ext}(D)$ als Vektorraum, die eine kombinatorische Beschreibung der Multiplikation erlaubt.