Computing generating sets of lattice ideals

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Abstract

In this article, we present a new algorithm for computing generating sets and Gröbner bases of lattice ideals. In contrast to other existing methods, our algorithm starts computing in projected subspaces and then iteratively lifts the results back into higher dimensions, by using a completion procedure, until the original dimension is reached. We give a completely geometric presentation of our Project-and-Lift algorithm and describe also the two other existing main algorithms in this geometric framework. We then give more details on an efficient implementation of this algorithm, in particular on critical-pair criteria specific to lattice ideal computations. Finally, we conclude the paper with a computational comparison of our implementation of the Project-and-Lift algorithm in 4ti2 with algorithms for lattice ideal computations implemented in CoCoA and Singular. Our algorithm outperforms the other algorithms in every single instance we have tried.

Key words: Generating sets, Lattices, Lattice ideals, Markov bases, Integer Programming, Test Sets

1 Introduction

In this article, we present a new algorithm for computing a generating set of a lattice ideal

\[ I(\mathcal{L}) := \langle x^u^+ - x^u^- : u \in \mathcal{L} \rangle \subseteq k[x_1, \ldots, x_n], \]

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where $k$ is a field, $\mathcal{L}$ is a sub-lattice of $\mathbb{Z}^n$, and

$$x^{u^+} - x^{u^-} := x_1^{u_1^+} x_2^{u_2^+} \cdots x_n^{u_n^+} - x_1^{u_1^-} x_2^{u_2^-} \cdots x_n^{u_n^-}$$

where $u_i^+ = \max\{u_i, 0\}$ and $u_i^- = \max\{-u_i, 0\}$. We assume that $\mathcal{L} \cap \mathbb{N}^n = \{0\}$.

Generating set and Gröbner basis computations for general ideals are usually very time consuming. Fortunately, in the special setting of lattice ideals, many improvements are possible. Two interesting areas of application of lattice ideals are algebraic statistics and integer programming.

Generating sets of lattice ideals and Gröbner bases of lattice ideals have corresponding geometric concepts (see Thomas (1995); Urbaniak et al. (1997); Weismantel (1998)), which we call generating sets of lattices and Gröbner bases of lattices respectively (see Section 2). These concepts are related as follows: if a set $S \subseteq \mathcal{L}$ is a generating set of $\mathcal{L}$ or a Gröbner basis of $\mathcal{L}$ with respect to a term order $\succ$, then $G := \{x^{u^+} - x^{u^-} : u \in S\}$ is respectively a generating set of $I(\mathcal{L})$ or a Gröbner basis of $I(\mathcal{L})$ with respect to $\succ$; and also, if a set of monic binomials $G$ is a generating set of $I(\mathcal{L})$ or a Gröbner basis of $I(\mathcal{L})$ with respect to a term order $\succ$, then $S := \{\alpha - \beta : x^\alpha - x^\beta \in G\}$ is respectively a generating set of $\mathcal{L}$ or a Gröbner basis of $\mathcal{L}$ with respect to $\succ$. Note that we use the same order $\succ$ for monomials and vectors via the relation $x^\alpha \succ x^\beta$ if and only if $\alpha \succ \beta$ for $\alpha, \beta \in \mathbb{N}^n$. Also, note that any minimal reduced Gröbner basis of a lattice ideal is a set of monic binomials.

In this paper, we have chosen to present existing theory and the new algorithm only in a geometric framework following the approach in Thomas (1995); Urbaniak et al. (1997); Weismantel (1998), since for lattice ideals, we prefer the geometric approach to the algebraic one. However, note that for every geometric concept presented, there exists an equivalent algebraic notion, although we do not present it here. Also, we have tried to make this paper reasonably self contained, so for completeness, we present geometric proofs of existing results where pertinent.

Recently, there has been renewed interest in toric ideal computations because of applications in algebraic statistics. Here, we are interested in Markov bases, which are used in a Monte-Carlo Markov-Chain (MCMC) process to test validity of statistical models via sampling. Diaconis and Sturmfels (Diaconis and Sturmfels, 1999).
1998) showed that a (preferably minimal) generating set of a lattice

\[ L_A := \{ u \in \mathbb{Z}^n : Au = 0 \} \]

for some matrix \( A \in \mathbb{Z}^{d \times n} \) is a Markov basis. Note that \( I(L_A) \) is a toric ideal. However, at that time, no effective implementation of an algorithm to compute generating sets of toric ideals was available that could deal with moderate size problems in 50–100 variables. This situation has changed by now: several such implementations are available. Using 4ti2 (Hemmecke et al., 2005), Eriksson even reports, in Eriksson (2004), on successful computations of Gröbner bases and Markov bases of toric ideals in 2,048 variables. His problems arise from phylogenetic trees in computational biology.

In integer programming, test sets of integer programs correspond to Gröbner bases of lattices (or lattice ideals). Test sets were introduced in Graver (1975). Consider the general linear integer program

\[
\min \{ cz : Az = b, z_{\bar{\sigma}} \geq 0, z \in \mathbb{Z}^n \}
\]

where \( c \in \mathbb{Q}^{\sigma} \), \( A \in \mathbb{Z}^{d \times n} \), \( b \in \mathbb{Z}^d \), \( \sigma \subseteq \{1, \ldots, n\} \), \( \bar{\sigma} := \{1, \ldots, n\} \setminus \sigma \), and where \( z_{\bar{\sigma}} \) is the set of variables indexed by \( \bar{\sigma} \). Any integer program that has an optimal solution can be written in this form (Conti and Traverso, 1991). By projecting onto the \( \bar{\sigma} \) variables, we can rewrite these integer programs in the equivalent and more convenient form

\[
IP_{A,c,b}^{\sigma} = \min \{ cz : A\bar{\sigma}z \equiv b \pmod{A_{\sigma}Z}, z \in \mathbb{N}^{\bar{\sigma}} \},
\]

where \( A_{\sigma} \) and \( A_{\bar{\sigma}} \) are the sub-matrices of \( A \) whose columns are indexed by \( \sigma \) and \( \bar{\sigma} \) respectively, and \( A_{\sigma}Z := \{ A_{\sigma}z : z \in \mathbb{Z}^{\sigma} \} \). In the special case where \( \sigma = \emptyset \), we set \( A_{\sigma}Z := \{0\} \), and the problem \( IP_{A,c,b}^{\emptyset} \) simplifies to \( IP_{A,c,b} := \min \{ cz : Az = b, z \in \mathbb{N}^n \} \). Note that group relaxations and extended group relaxations of \( IP_{A,c,b}^{\sigma} \) are also of the form \( IP_{A,c,b}^{\sigma} \) for some cost vector \( \bar{c} \in \mathbb{Q}^{\bar{\sigma}} \) (Hosten and Thomas, 2002). Without loss of generality, we assume that \( c \) is generic meaning that \( IP_{A,c,b}^{\sigma} \) has a unique optimal solution for every feasible \( b \in \mathbb{Z}^d \). We can always easily perturb a given \( c \) so that it is generic.

A set \( T \subseteq \mathbb{Z}^{\bar{\sigma}} \) is called a test set for \( IP_{A,c,b}^{\sigma} \) if \( T \) contains an improving direction \( t \) for every non-optimal feasible solution \( z \in \mathbb{N}^{\bar{\sigma}} \) of \( IP_{A,c,b}^{\sigma} \) that is, \( z - t \) is also feasible and \( c(z - t) < cz \). Clearly, \( z - t \) being feasible implies that \( t \) is an element of the lattice

\[
\mathcal{L}_A^{\sigma} := \{ u \in \mathbb{Z}^{\bar{\sigma}} : A_{\sigma}u \equiv 0 \pmod{A_{\sigma}Z} \}.
\]

Moreover, a set \( T \subseteq \mathcal{L}_A^{\sigma} \) is called a test set for \( IP_{A,c,b}^{\sigma} \) the class of integer programs \( IP_{A,c,b}^{\sigma} \) for all \( b \in \mathbb{Z}^d \), if \( T \) is a test set for every integer program in \( IP_{A,c,b}^{\sigma} \). Graver showed that there exist finite sets \( T \) that are test sets for \( IP_{A,c}^{\sigma} \) (\( \sigma = \emptyset \)). In fact, his sets also constitute finite test sets for \( IP_{A,c}^{\sigma} \) for arbitrary
Having a finite test set available, an optimal solution of $IP_{A,c,b}^\sigma$ can be found by iteratively improving any given non-optimal solution of $IP_{A,c,b}^\sigma$.

In [Conti and Traverso (1991)] and [Sturmfels et al. (1995)], it was shown that given a generic cost vector $c$ and a term order $\succ$ where $c$ and $\succ$ are compatible, a set $S \subseteq L_A^\sigma$ is a Gröbner basis of $L_A^\sigma$ with respect to $\succ$ if and only if $S$ is a test set for $IP_{A,c}^\sigma$ where $c$ and $\succ$ are compatible if $c\alpha > c\beta$ implies $\alpha \succ \beta$ for all $\alpha, \beta \in \mathbb{N}^n$. A compatible term ordering $\succ$ exists for every generic $c$, and a compatible generic $c$ exists for every term ordering $\succ$. Additionally, any lattice $L$ can be written in the form $L_A^\sigma$ for some matrix $A \in \mathbb{Z}^{n \times d}$ and some index set $\sigma \subseteq \{1, \ldots, n\}$, and so, Gröbner bases of lattices and test sets of integer programs really are equivalent concepts.

We define generating sets and Gröbner bases of lattices, in Section 2, in a geometric context and present the completion procedure ([Buchberger, 1987, 1985; Cox et al., 1992]), which is the main building block for the algorithms for computing generating sets.

In Section 3, we present the two main existing algorithms for computing generating sets: the algorithm of Hosten and Sturmfels in [Hosten and Sturmfels (1995)], which we call the “Saturation” algorithm; and the algorithm of Bigatti, LaScala, and Robbiano in [Bigatti et al. (1999)], which we call the “Lift-and-Project” algorithm. We also describe our new algorithm for computing generating sets: the “Project-and-Lift” algorithm. The Saturation Algorithm is based upon the result that $I(L) = (\ldots ((J(S) : x_1^\infty) : x_2^\infty) \ldots) : x_n^\infty$ where $S$ is a lattice basis of $L$ and $J(S)$ is defined as above. Using this result, we can compute a generating set of $I(L)$ from $S$ via a sequence of saturation steps where each individual saturation step is performed via the completion procedure. The Lift-and-Project Algorithm is based upon the related result that $I(L) = J(S) : (x_1 \cdot x_2 \cdots \cdot x_n)^\infty$. Here, a generating set is computed via the completion procedure using an additional variable. The Project-and-Lift Algorithm is strongly related to the Saturation Algorithm; however, the computational speed-up is enormous as will be seen in Section 7. In contrast with the Saturation Algorithm, which performs saturation steps in the original space of the lattice $L$, the Project-and-Lift Algorithm performs saturation steps in projected subspaces of $L$ and then lifts the result back into the original space.

We are mainly interested in computing a generating set of $L$ where $L \cap \mathbb{N}^n = \{0\}$. However in Section 4, we address the question of how to compute a generating set $L$ if $L \cap \mathbb{N}^n \neq \{0\}$. We demonstrate that the above methods for the case where $L \cap \mathbb{N}^n = \{0\}$ can be extended to this more general case; it happens to be more straight-forward in some ways.

The completion procedure as it is presented in Section 2 is not very efficient. In
Section 5, we show how to increase the efficiency of the completion procedure. All the results in this section, which we present in a geometric framework, have corresponding results in an algebraic context. This section is rather technical and no other section depends upon it, so it may be skipped on first reading.

In Section 6, we give the solution of a computational challenge posed by Seth Sullivant to compute the Markov basis of $4 \times 4 \times 4$ tables with 2-marginals, a problem involving 64 variables. We solved this with the help of the new algorithm. Our computations led to 148,968 elements in the minimal generating set of $I(A)$ which fall into 15 equivalence classes with respect to the underlying symmetry group $S_4 \times S_4 \times S_4 \times S_3$.

In Section 7, we compare the performance of the implementation of the Project-and-Lift algorithm in 4ti2 v.1.2 (Hemmecke et al., 2005) with the implementation of the Saturation algorithm and the Lift-and-Project algorithm in Singular v3.0.0 (Greuel et al., 2005) and in CoCoA 4.2 (CoCoA Team, 2005). The Project-and-Lift algorithm is significantly faster than the other algorithms.

2 Generating sets and Gröbner bases

Given a lattice $\mathcal{L} \subseteq \mathbb{Z}^n$, and a vector $b \in \mathbb{Z}^n$, we define

$$\mathcal{F}_{\mathcal{L},b} := \{x : x \equiv b \pmod{\mathcal{L}}, x \in \mathbb{N}^n\}.$$ 

For $S \subseteq \mathcal{L}$, we define $\mathcal{G}(\mathcal{F}_{\mathcal{L},b}, S)$ to be the undirected graph with nodes $\mathcal{F}_{\mathcal{L},b}$ and edges $(x, y)$ if $x - y \in S$ or $y - x \in S$ for $x, y \in \mathcal{F}_{\mathcal{L},b}$.

**Definition 1** A set $S \subseteq \mathcal{L}$ a generating set of $\mathcal{L}$ if the graph $\mathcal{G}(\mathcal{F}_{\mathcal{L},b}, S)$ is connected for every $b \in \mathbb{Z}^n$.

We remind the reader that connectedness of $\mathcal{G}(\mathcal{F}_{\mathcal{L},b}, S)$ simply states that between each pair $x, y \in \mathcal{F}_{\mathcal{L},b}$, there exists a path from $x$ to $y$ in $\mathcal{G}(\mathcal{F}_{\mathcal{L},b}, S)$. Note the difference between a generating set of a lattice and a spanning set of a lattice: a spanning set of $\mathcal{L}$ is any set $S \subseteq \mathcal{L}$ such that any point in $\mathcal{L}$ can be represented as an linear integer combination of the vectors in $S$. A generating set of $\mathcal{L}$ is a spanning set of $\mathcal{L}$, but the converse is not necessarily true.

Recall that for any lattice $\mathcal{L}$, we have $\mathcal{L} = \mathcal{L}_A^\sigma$ for some some matrix $A \in \mathbb{Z}^{n \times d}$ and some index set $\sigma \subseteq \{1, \ldots, n\}$. Hence,

$$\mathcal{F}_{\mathcal{L},b} = \mathcal{F}_{A,\bar{b}} := \{x \in \mathbb{N}[\sigma] : A_\sigma x \equiv \bar{b} \pmod{A_\sigma \mathbb{Z}}\}.$$
for all $b \in \mathbb{Z}^n$ and all $\bar{b} \in \mathbb{Z}^d$ where $\bar{b} = A_b b$. So, $\mathcal{F}_{L,b}$ and $\mathcal{F}_{A,b}^\sigma$ are dual representations of feasible sets.

**Example 2** Let $S := \{(1,-1,-1,3,1,2),(1,0,2,-2,-2,1)\}$, and let $\mathcal{L} \subseteq \mathbb{Z}^6$ be the lattice spanned by $S$. So, by definition, $S$ is a spanning set of $\mathcal{L}$, but $S$ is not a generating set of $\mathcal{L}$. Observe that $\mathcal{L} = \mathcal{L}_{A}$ where

$$A = (\bar{A}, I), \quad \bar{A} = \begin{pmatrix} -2 & -1 \\ 2 & 1 \\ 2 & +1 \\ -1 & +1 \end{pmatrix}, \quad \text{and } I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

So, for every $b \in \mathbb{Z}^6$, $\mathcal{F}_{L,b} = \mathcal{F}_{A,b} = \{(x, s) : \bar{A} x + I s = \bar{b}, x \in \mathbb{N}^2, s \in \mathbb{N}^4\}$ where $\bar{b} = Ab \in \mathbb{Z}^4$. Hence, the projection of $\mathcal{F}_{L,b}$ onto the $(x_1, x_2)$-plane is the set of integer points in the polyhedron $\{x \in \mathbb{R}^2_+ : Ax \leq \bar{b}\}$, and the $s$ variables are the slack variables. Consider $b := (2, 2, 4, 2, 4, 1)$; then, $\mathcal{F}_{L,b} = \mathcal{F}_{A,b}$ where $\bar{b} = Ab = (-6, 4, 10, 1) \ (\text{see Figure 1a})$.

![Figure 1](image.png)

Fig. 1. The set $\mathcal{F}_{L,b}$ and the graphs $\mathcal{G}(\mathcal{F}_{L,b}, S)$ and $\mathcal{G}(\mathcal{F}_{L,b}, S')$ projected onto the $(x_1, x_2)$-plane.

The graph $\mathcal{G}(\mathcal{F}_{L,b}, S)$ is not connected because the point $(3, 4, 12, 2, 0, 0) \in \mathcal{F}_{L,b}$ is disconnected (see Figure 1b). Let $S' := S \cup \{(1,1,5,-1,-3,0)\}$. The graph of $\mathcal{G}(\mathcal{F}_{L,b}, S')$ is now connected (see Figure 1c); however, $S'$ is still not a generating set of $\mathcal{L}$ since we have $\mathcal{F}_{L,b'} = \{(0,0,0,0,1,1),(0,1,3,1,0,0)\}$ for $b' := (0,0,0,0,1,1)$, and the graph $\mathcal{G}(\mathcal{F}_{L,b'}, S')$ is disconnected; since there are only two feasible points in $\mathcal{F}_{L,b'}$, the vector between them $(0,1,3,1,-1,-1)$
must be in any generating of \( \mathcal{L} \). Finally, the set \( S'' := S' \cup \{(0,1,3,1,-1,-1)\} \) is a generating set of \( \mathcal{L} \).

For the definition of a Gröbner basis, we need a term ordering \( \succ \) for \( \mathcal{L} \). We call \( \succ \) a **term ordering** for \( \mathcal{L} \) if \( \succ \) is a total well-ordering on \( \mathcal{F}_{L,b} \) for every \( b \in \mathbb{Z}^n \) and \( \succ \) is an additive ordering meaning that for all \( b \in \mathbb{Z}^n \) and for all \( x,y \in \mathcal{F}_{L,b} \), if \( x \succ y \), then \( x + \gamma \succ y + \gamma \) for every \( \gamma \in \mathbb{N}^n \) (note that \( x + \gamma, y + \gamma \in \mathcal{F}_{L,b+\gamma} \)). We also need the notion of a decreasing path: a path \( (x^0, \ldots, x^k) \) in \( \mathcal{G}(\mathcal{F}_{L,b}, G) \) is **\( \succ \)-decreasing** if \( x^i \succ x^{i+1} \) for \( i = 0, \ldots, k-1 \).

We define \( \mathcal{L}_\succ := \{ u \in \mathcal{L} : u^+ \succ u^- \} \).

**Definition 3** A set \( G \subseteq \mathcal{L}_\succ \) is a **\( \succ \)-Gröbner basis** of \( \mathcal{L} \) if for every \( x \in \mathbb{N}^n \) there exists a \( \succ \)-decreasing path in \( \mathcal{G}(\mathcal{F}_{L,x}, G) \) from \( x \) to the unique \( \succ \)-minimal element in \( \mathcal{F}_{L,x} \).

If \( G \subseteq \mathcal{L}_\succ \) is a \( \succ \)-Gröbner basis, then \( G \) is a generating set of \( \mathcal{L} \) since given \( x,y \in \mathcal{G}(\mathcal{F}_{L,b}, G) \) for some \( b \in \mathbb{Z}^n \), there exists a \( \succ \)-decreasing path from \( x \) to the unique \( \succ \)-minimal element in \( \mathcal{F}_{L,b} \) and from \( y \) to the same element, and thus, \( x \) and \( y \) are connected in \( \mathcal{G}(\mathcal{F}_{L,b}, G) \). Also, \( G \subseteq \mathcal{L}_\succ \) is a Gröbner basis if and only if for every \( x \in \mathbb{N}^n \), \( x \) is either the unique \( \succ \)-minimal element in \( \mathcal{F}_{L,b} \) or there exists a vector \( u \in G \) such that \( x - u \in \mathcal{F}_{L,b} \) and \( x \succ x - u \); consequently, a Gröbner basis \( G \) is a test set for \( IP_{A,c}^\sigma \) where \( \mathcal{L}_A^\sigma = \mathcal{L} \) if \( c \) and \( \succ \) are compatible.

The defining property of a Gröbner basis is very strong, so we redefine it in terms of reduction paths. A path \( (x^0, \ldots, x^k) \) in \( \mathcal{G}(\mathcal{F}_{L,b}, G) \) is a **\( \succ \)-reduction path** if for no \( i \in \{1, \ldots, k-1\} \), we have \( x^i \succ x^0 \) and \( x^i \succ x^k \). For example, see Figure 2.

![Fig. 2. Reduction path between x and y.](attachment:image.png)

**Lemma 4** A set \( G \subseteq \mathcal{L}_\succ \) is a **\( \succ \)-Gröbner basis** of \( \mathcal{L} \) if and only if for each \( b \in \mathbb{Z}^n \) and for each pair \( x,y \in \mathcal{F}_{L,b} \), there exists a \( \succ \)-reduction path in \( \mathcal{G}(\mathcal{F}_{L,b}, G) \) between \( x \) and \( y \).

**Proof.** If \( \mathcal{G}(\mathcal{F}_{L,b}, G) \) contains \( \succ \)-decreasing paths from \( x,y \in \mathcal{F}_{L,b} \) to the unique \( \succ \)-minimal element in \( \mathcal{F}_{L,b} \), then joining the two paths (and removing cycles if necessary) forms a \( \succ \)-reduction path between \( x \) and \( y \).

For the other direction, we assume that there is a \( \succ \)-reduction path between each pair \( x,y \in \mathcal{F}_{L,b} \). Denote by \( x^* \) the unique \( \succ \)-minimal element in \( \mathcal{F}_{L,b} \); thus, every \( x \in \mathcal{F}_{L,b} \) is connected to \( x^* \) by a \( \succ \)-reduction path. In particular,
by the definition of a $\succ$-reduction path, if $x \neq x^*$, then the first node $x^1 \neq x$ in this path must satisfy $x \succ x^1$. Repeating this argument iteratively with $x^1$ instead of $x$, we get a $\succ$-decreasing path from $x$ to $x^*$. This follows from the fact that $\succ$ is a term ordering, which implies that every $\succ$-decreasing path must be finite. However, the only node from which the $\succ$-decreasing path cannot be lengthened is $x^*$. □

Checking for a given $G \subseteq L_{\succ}$ whether there exists a $\succ$-reduction path in $G(F_{L,b}, G)$ for every $b \in \mathbb{Z}^n$ and for each pair $x, y \in F_{L,b}$ involves infinitely many situations that need to be checked. In fact, far fewer checks are needed: we only need to check for a $\succ$-reduction path from $x$ to $y$ if there exists a $\succ$-critical path from $x$ to $y$.

**Definition 5** Given $G \subseteq L_{\succ}$ and $b \in \mathbb{Z}^n$, a path $(x, z, y)$ in $G(F_{L,b}, G)$ is a $\succ$-critical path if $z \succ x$ and $z \succ y$.

If $(x, z, y)$ is a $\succ$-critical path in $G(F_{L,b}, G)$, then $x + u = z = y + v$ for some pair $u, v \in G$, in which case, we call $(x, z, y)$ a $\succ$-critical path for $(u, v)$ (see Figure 3).

![Fig. 3. A critical path for $(u, v)$ between $x$, $z$, and $y$.](image)

The following lemma will be a crucial ingredient in the correctness proofs of the algorithms presented in Section 3. It will guarantee correctness of the algorithm under consideration, since the necessary reduction paths have been constructed during the run of the algorithm. In the next lemma, we cannot assume that $G$ is a generating set of $L$, since often this is what we are trying to construct.

**Lemma 6** Let $x, y \in F_{L,b}$ for some $b \in \mathbb{Z}^n$, and let $G \subseteq L_{\succ}$ where there is a path between $x$ and $y$ in $G(F_{L,b}, G)$. If there exists a $\succ$-reduction path between $x'$ and $y'$ for every $\succ$-critical path $(x', z', y')$ in $G(F_{L,b}, G)$, then there exists a $\succ$-reduction path between $x$ and $y$ in $G(F_{L,b}, G)$.

**Proof.** Assume on the contrary that no such $\succ$-reduction path exists from $x$ to $y$. Among all paths $(x = x^0, \ldots, x^k = y)$ in $G(F_{L,b}, G)$ choose one such that $\max\{x^0, \ldots, x^k\}$ is minimal. Such a minimal path exists since $\succ$ is a term ordering. Let $j \in \{0, \ldots, k\}$ where $x^j$ attains this maximum.

By assumption, $(x^0, \ldots, x^k)$ is not a $\succ$-reduction path, and thus, $x^j \succ x^0$ and $x^j \succ x^k$, and since $x^j$ is maximal, we have $x^j \succ x^{j-1}$ and $x^j \succ x^{j+1}$. Let $u = x^j - x^{j-1}$ and $v = x^j - x^{j+1}$. Then $(x^{j-1}, x^j, x^{j+1})$ forms a $\succ$-critical path.
Consequently, we can replace the path \((x^{j-1}, x^j, x^{j+1})\) with the \(\succ\)-reduction path \((x^{j-1} = \bar{x}^0, \ldots, \bar{x}^s = x^{j+1})\) in the path \((x^0, \ldots, x^k)\) and obtain a new path between \(x\) and \(y\) with the property that the \(\succ\)-maximum of the intermediate nodes is strictly less than \(x^j = \max\{x^1, \ldots, x^{k-1}\}\) (see Figure 4). This contradiction proves our claim. \(\Box\)

![Fig. 4. Replacing a critical path by a reduction path](image)

The following corollary is a straight-forward consequence of Lemma 6, but nonetheless, it is worthwhile stating explicitly.

**Corollary 7** Let \(G \subseteq L_\succ\). If for all \(b' \in \mathbb{Z}^n\) and for every \(\succ\)-critical path \((x', z', y')\) in \(G(F_{L,b}, G)\), there exists a \(\succ\)-reduction path between \(x'\) and \(y'\), then for all \(b \in \mathbb{Z}^n\) and for all \(x, y \in F_{L,b}\) where \(x\) and \(y\) are connected in \(G(F_{L,b}, G)\), there exists a \(\succ\)-reduction path between \(x\) and \(y\) in \(G(F_{L,b}, G)\).

Combining Corollary 7 with Lemma 4, we arrive at the following result for Gröbner bases.

**Corollary 8** A set \(G \subseteq L_\succ\) is a \(\succ\)-Gröbner basis of \(L\) if and only if \(G\) is a generating set of \(L\) and if for all \(b \in \mathbb{Z}^n\) and for every \(\succ\)-critical path \((x, z, y)\) in \(G(F_{L,b}, G)\), there exists a \(\succ\)-reduction path between \(x\) and \(y\) in \(G(F_{L,b}, G)\).

In Corollary 7 and Corollary 8, it is not necessary to check for a \(\succ\)-reduction path from \(x\) to \(y\) for every \(\succ\)-critical path \((x, y, z)\) in \(G(F_{L,b}, G)\) for all \(b \in \mathbb{Z}^n\). Consider the case where there exists another \(\succ\)-critical path \((x', y', z')\) in \(G(F_{L,b'}, G)\) for some \(b' \in \mathbb{Z}^n\) such that \((x, y, z) = (x' + \gamma, y' + \gamma, z' + \gamma)\) for some \(\gamma \in \mathbb{N}^n\). Then, a \(\succ\)-reduction path from \(x'\) to \(y'\) in \(G(F_{L,b'}, G)\) translates by \(\gamma\) to a \(\succ\)-reduction path from \(x\) to \(y\) in \(G(F_{L,b}, G)\). Thus, we only need to check for a \(\succ\)-reduction path from \(x'\) to \(y'\).

A \(\succ\)-critical path \((x, y, z)\) is **minimal** if there does not exist another \(\succ\)-critical path \((x', y', z')\) such that \((x, y, z) = (x' + \gamma, y' + \gamma, z' + \gamma)\) for some \(\gamma \in \mathbb{N}^n\) where \(\gamma \neq 0\), or equivalently, \(\min\{x_i, y_i, z_i\} = 0\) for all \(i = 1, \ldots, n\). Consequently, if there exists a \(\succ\)-reduction path between \(x\) and \(y\) for all minimal \(\succ\)-critical paths \((x, y, z)\), then there exists a \(\succ\)-reduction path between \(x'\) and \(y'\) for all \(\succ\)-critical paths \((x', y', z')\). Also, for each pair of vectors \(u, v \in L\), there exists a unique minimal \(\succ\)-critical path \((x^{(u,v)}, z^{(u,v)}, y^{(u,v)})\) determined by \(z^{(u,v)} := \max\{u^+, v^+\}\) component-wise, \(x^{(u,v)} := z^{(u,v)} - u\) and \(y^{(u,v)} := z^{(u,v)} - v\). So, any
other $\succ$-critical path for $(u, v)$ is of the form $(x^{(u,v)} + \gamma, z^{(u,v)}, y^{(u,v)} + \gamma)$ for some $\gamma \in \mathbb{N}^n$. Using minimal $\succ$-critical paths, we can rewrite Corollary 7 and Corollary 8, so that we only need to check for a finite number of $\succ$-reduction paths.

**Lemma 9** Let $G \subseteq L_{\succ}$. If there exists a $\succ$-reduction path between $x^{(u,v)}$ and $y^{(u,v)}$ for every pair $u, v \in G$, then for all $b \in \mathbb{Z}^n$ and for all $x, y \in F_{L,b}$ where $x$ and $y$ are connected in $G(F_{L,b}, G)$, there exists a $\succ$-reduction path between $x$ and $y$ in $G(F_{L,b}, G)$.

**Corollary 10** A set $G \subseteq L_{\succ}$ is a $\succ$-Gröbner basis of $L$ if and only if $G$ is a generating set of $L$ and for each pair $u, v \in G$, there exists a $\succ$-reduction path between $x^{(u,v)}$ and $y^{(u,v)}$ in $G(F_{L,z(u,v)}, G)$.

We now turn Lemma 9 into an algorithmic tool. The following algorithm, Algorithm 2 below, called a completion procedure (Buchberger, 1987), guarantees that if for a set $S \subseteq L$ the points $x$ and $y$ are connected in $G(F_{L,x}, S)$, then there exists a $\succ$-reduction path between $x$ and $y$ in $G(F_{L,x}, G)$, where $G$ denotes the set returned by the completion procedure. Thus, if $S$ is a generating set of $L$, then Algorithm 2 returns a set $G$ that is a $\succ$-Gröbner basis of $L$ by Corollary 10.

Given a set $S \subseteq L$, the completion procedure first sets $G := S$ and then directs all vectors in $G$ according to $\succ$ such that $G \subseteq L_{\succ}$. Note that at this point $G(F_{L,b}, S) = G(F_{L,b}, G)$ for all $b \in \mathbb{Z}^n$. The completion procedure then determines whether the set $G$ satisfies Lemma 9; in other words, it tries to find a reduction path from $x^{(u,v)}$ to $y^{(u,v)}$ for every pair $u, v \in G$. If $G$ satisfies Lemma 9, then we are done. Otherwise, no $\succ$-reduction path was found for some $(u, v)$, in which case, we add a vector to $G$ so that a $\succ$-reduction path exists, and then again, test whether $G$ satisfies Lemma 9, and so on.

To check for a $\succ$-reduction path, using the “Normal Form Algorithm”, Algorithm 1 below, we construct a maximal $\succ$-decreasing path in $G(F_{L,z(u,v)}, G)$ from $x^{(u,v)}$ to some $x'$, and a maximal $\succ$-decreasing path in $G(F_{L,z(u,v)}, G)$ from $y^{(u,v)}$ to some $y'$. If $x' = y'$, then we have found a $\succ$-reduction path from $x^{(u,v)}$ to $y^{(u,v)}$. Otherwise, we add the vector $r \in L_{\succ}$ to $G$ where $r := x' - y'$ if $x' > y'$, and $r := y' - x'$ otherwise, so therefore, there is now a $\succ$-reduction path from $x^{(u,v)}$ to $y^{(u,v)}$ in $G(F_{L,z(u,v), G})$. Note that before we add $r$ to $G$, since the paths from $x$ to $x'$ and from $y$ to $y'$ are maximal, there does not exist $u \in G$ such that $x' \geq u^+$ or $y' \geq u^+$ Therefore, there does not exist $u \in G$ such that $r^+ \geq u^+$. This condition is needed to ensure that the completion procedure terminates.

**Algorithm 1** Normal Form Algorithm

**Input**: a vector $x \in \mathbb{N}^n$ and a set $G \subseteq L_{\succ}$.
Output: a vector $x'$ where there is a maximal $\succ$-decreasing path from $x$ to $x'$ in $\mathcal{G}(\mathcal{F}_{L,x}, G)$.

$x' := x$

while there is some $u \in G$ such that $u^+ \leq x'$ do

$x' := x' - u$

return $x'$

We write $\mathcal{NF}(x, G)$ for the output of the Normal Form Algorithm.

Algorithm 2 Completion procedure

Input: a term ordering $\succ$ and a set $S \subseteq L$.

Output: a set $G \subseteq L_\succ$ such that if $x, y$ are connected in $\mathcal{G}(\mathcal{F}_{L,x}, S)$, then there exists a $\succ$-reduction path between $x$ and $y$ in $\mathcal{G}(\mathcal{F}_{L,x}, G)$.

$G := \{u : u^+ \succ u^-, u \in S\} \cup \{-u : u^- \succ u^+, u \in S\}$
$C := \{(u, v) : u, v \in G\}$

while $C \neq \emptyset$ do

Select $(u, v) \in C$
$C := C \setminus \{(u, v)\}$
$r := \mathcal{NF}(x^{(u,v)}, G) - \mathcal{NF}(y^{(u,v)}, G)$
if $r \neq 0$ then

if $r^- \succ r^+$ then $r := -r$
$C := C \cup \{(r, s) : s \in G\}$
$G := G \cup \{r\}$

return $G$.

We write $\mathcal{CP}(\succ, S)$ for the output of the Completion Procedure.

Lemma 11 Algorithm 2 terminates and satisfies its specifications.

Proof. Let $(r^1, r^2, \ldots)$ be the sequence of vectors $r$ that are added to the set $G$ during the Algorithm 2. Since before we add $r$ to $G$, there does not exist $u \in G$ such that $r^+ \geq u^+$, the sequence satisfies $r^{i+} \not\leq r^{j+}$ whenever $i < j$. By the Gordan-Dickson Lemma (see for example Cox et al. (1992)), such a sequence must be finite and thus, Algorithm 2 must terminate.

When the algorithm terminates, the set $G$ must satisfy the property that for each $u, v \in G$, there exists a $\succ$-reduction path from $x^{(u,v)}$ to $y^{(u,v)}$, and therefore, by Lemma 9, there exists a $\succ$-reduction path between $x$ and $y$ in $\mathcal{G}(\mathcal{F}_{L,b}, G)$ for all $x, y \in \mathcal{F}_{L,b}$ for all $b \in \mathbb{Z}^n$ where $x$ and $y$ are connected in $\mathcal{G}(\mathcal{F}_{L,b}, G)$. Moreover, by construction, $S \subseteq G \cup \overline{G}$, and therefore, if $x$ and $y$ are connected in $\mathcal{G}(\mathcal{F}_{L,b}, S)$, then $x$ and $y$ are connected in $\mathcal{G}(\mathcal{F}_{L,b}, G)$. \qed
Note that the completion procedure preserves connectivity: given $x, y \in \mathcal{F}_{L,b}$ for some $b \in \mathbb{Z}^n$, if $x$ and $y$ are connected in $\mathcal{G}(\mathcal{F}_{L,b}, S)$, then $x$ and $y$ are also connected in $\mathcal{G}(\mathcal{F}_{L,b}, G)$.

3 Computing a generating set

In this section, we finally present three algorithms to compute a generating set of $\mathcal{L}$: the “Saturation” algorithm (Hosten and Sturmfels [1995]), the “Lift-and-Project” algorithm (Bigatti, LaScala, and Robbiano [1999]), and our new “Project-and-Lift” algorithm. Each algorithm produces a generating set of $\mathcal{L}$ that is not necessarily minimal, and so, once a generating set of $\mathcal{L}$ is known, a minimal generating set of $\mathcal{L}$ can be computed by a single Gröbner basis computation (see Caboara et al. [2003] for more details). The fundamental idea behind all three algorithms is essentially the same, and the main algorithmic building block of the algorithms is the completion procedure.

3.1 The “Saturation” algorithm

Let $x, y \in \mathcal{F}_{L,b}$ for some $b \in \mathbb{Z}^n$, and let $S \subseteq \mathcal{L}$. Observe that if $x$ and $y$ are connected in $\mathcal{G}(\mathcal{F}_{L,b}, S)$, then $x + \gamma$ and $y + \gamma$ are connected in $\mathcal{G}(\mathcal{F}_{L,b}, S)$ for any $\gamma \in \mathbb{N}^n$ since we can just translate any path from $x$ to $y$ in $\mathcal{G}(\mathcal{F}_{L,b}, S)$ by $\gamma$ giving a path from $x + \gamma$ to $y + \gamma$ in $\mathcal{G}(\mathcal{F}_{L,b+\gamma}, S)$. However, it is not necessarily true that $x + \gamma$ and $y + \gamma$ are also connected for any $\gamma \in \mathbb{Z}^n$ ($\gamma$ may be negative) where $x + \gamma \geq 0$ and $y + \gamma \geq 0$.

Given a set $S \subseteq \mathcal{L}$, the Saturation algorithm constructs a set $T$ such that if $x$ and $y$ are connected in $\mathcal{G}(\mathcal{F}_{L,b}, S)$ for some $b \in \mathbb{Z}^n$, then $x + \gamma$ and $y + \gamma$ are connected in $\mathcal{G}(\mathcal{F}_{L,b}, T)$ for any $\gamma \in \mathbb{Z}^n$ where $x + \gamma \geq 0$ and $y + \gamma \geq 0$. Importantly then, if $S$ spans $\mathcal{L}$, then $T$ must be a generating set of $\mathcal{L}$. This follows since if $S$ spans $\mathcal{L}$, then for all $b \in \mathbb{Z}^n$ and for all $x, y \in \mathcal{F}_{L,b}$, there must exist a $\gamma \in \mathbb{N}^n$ such that $x + \gamma$ and $y + \gamma$ are connected in $\mathcal{G}(\mathcal{F}_{L,b}, T)$, and hence, $x$ and $y$ must also be connected in $\mathcal{G}(\mathcal{F}_{L,b}, T)$ from our assumption about $T$.

For convenience, we need some new notation. Given $x, y \in \mathbb{N}^n$, we define $x \wedge y$ as the component-wise minimum of $x$ and $y$ – that is, $(x \wedge y)_i = \min\{x_i, y_i\}$ for all $i = 1, \ldots, n$. Also, given $\sigma \subseteq \{1, \ldots, n\}$, we define $x \wedge \sigma y$ as the component-wise minimum of $x$ and $y$ for the $\sigma$ components and 0 otherwise – that is, $(x \wedge y)_i = \min\{x_i, y_i\}$ if $i \in \sigma$ and $(x \wedge y)_i = 0$ otherwise.
Definition 12  Let $\sigma \subseteq \{1, \ldots, n\}$, and let $S, T \subseteq \mathcal{L}$. The set $T$ is $\sigma$-saturated on $S$ if and only if for all $b \in \mathbb{Z}^n$ and for all $x, y \in \mathcal{F}_{L,b}$ where $x$ and $y$ are connected in $\mathcal{G}(\mathcal{F}_{L,b}, S)$, the points $x - \gamma$ and $y - \gamma$ are also connected in $\mathcal{G}(\mathcal{F}_{L,b-\gamma}, T)$ where $\gamma = x \land_\sigma y$.

So, when $T$ is $\sigma$-saturated on $S$, if $x$ and $y$ are connected in $\mathcal{G}(\mathcal{F}_{L,b}, S)$, then $x + \gamma$ and $y + \gamma$ are connected in $\mathcal{G}(\mathcal{F}_{L,b}, T)$ for any $\gamma \in \mathbb{Z}^n$ ($\gamma$ can be negative) where $x + \gamma \geq 0$, $y + \gamma \geq 0$, and $\text{supp}(\gamma) \subseteq \sigma$. Saturation is thus concerned with the connectivity of a set $T$ in relation to the connectivity of another set $S$. Note that, by definition, a set $S \subseteq \mathcal{L}$ is $\emptyset$-saturated on itself. Also observe that if $S$ spans $\mathcal{L}$, then $T \subseteq \mathcal{L}$ is $\{1, \ldots, n\}$-saturated on $S$ if and only if $T$ is a generating set of $\mathcal{L}$.

The fundamental idea behind the Saturation algorithm is given $S, T \subseteq \mathcal{L}$ where $T$ is $\sigma$-saturated on $S$ for some $\sigma \subseteq \{1, \ldots, n\}$, we can compute a set $T'$ that is a $(\sigma \cup \{i\})$-saturated on $S$ for any $i \in \bar{\sigma}$. Therefore, given a set $S \subseteq \mathcal{L}$ that spans $\mathcal{L}$, starting from a set $T = S$, which is $\emptyset$-saturated on $S$, if we do this repeatedly for each $i \in \{1, \ldots, n\}$, we arrive at a set $T' \subseteq \mathcal{L}$ that is $\{1, \ldots, n\}$-saturated on $S$ and, therefore, a generating set of $\mathcal{L}$.

The following two lemmas are fundamental to the saturation algorithm. First, we extend the definition of reduction paths. Given $\varphi \in \mathbb{Q}^n$, a path $(x^0, \ldots, x^k)$ in $\mathcal{G}(\mathcal{F}_{L,b}, G)$ is an $\varphi$-reduction path if for no $j \in \{1, \ldots, k - 1\}$, we have $\varphi x^j > \varphi x^0$ and $\varphi x^j > \varphi x^k$. Also, we define $e^i$ to be the $i$th unit vector and $\bar{e}^i = -e^i$. So, given $b \in \mathbb{Z}^n$, the path $(x^0, \ldots, x^k) \subseteq \mathcal{F}_{L,b}$ is a $\bar{e}^i$-reduction path if $x^j_i \geq x^0_i$ or $x^j_i \geq x^k_i$ for $j = 1, \ldots, k - 1$.

Lemma 13  Let $S, T \subseteq \mathcal{L}$ and $i \in \{1, \ldots, n\}$. The set $T$ is $\{i\}$-saturated on $S$ if and only if for all $b \in \mathbb{Z}^n$ and for all $x, y \in \mathcal{F}_{L,b}$ where $x$ and $y$ are connected in $\mathcal{G}(\mathcal{F}_{L,b}, S)$, there exists a $\bar{e}^i$-reduction path from $x$ to $y$ in $\mathcal{G}(\mathcal{F}_{L,b}, T)$.

Proof. Let $x, y \in \mathcal{F}_{L,b}$ for some $b \in \mathbb{Z}^n$ where $x$ and $y$ are connected in $\mathcal{G}(\mathcal{F}_{L,b}, S)$ and let $\gamma = x \land_{\{i\}} y$.

Assume $T$ is $\{i\}$-saturated on $S$, and so, $x - \gamma$ and $y - \gamma$ are connected in $\mathcal{G}(\mathcal{F}_{L,b-\gamma}, T)$. Let $(x - \gamma = x^0, \ldots, x^k = y - \gamma)$ be a path from $x - \gamma$ and $y - \gamma$ in $\mathcal{G}(\mathcal{F}_{L,b-\gamma}, T)$. The path $(x = x^0 + \gamma, \ldots, x^k + \gamma = y)$ is a $\bar{e}^i$-reduction path from $x$ to $y$ in $\mathcal{G}(\mathcal{F}_{L,b}, T)$.

Conversely, by assumption, there exists a $\bar{e}^i$-reduction path $(x = x^0, \ldots, x^k = y)$ in $\mathcal{G}(\mathcal{F}_{L,b}, T)$. The path $(x - \gamma = x^0 - \gamma, \ldots, x^k - \gamma = y - \gamma)$ is thus a feasible path from $x - \gamma$ to $y - \gamma$ in $\mathcal{G}(\mathcal{F}_{L,b-\gamma}, T)$. □

Given any vector $\varphi \in \mathbb{Q}^n$ and a term order $\prec$, we define the order $\prec_\varphi$ where $x \prec_\varphi y$ if $\varphi x < \varphi y$ or $\varphi x = \varphi y$ and $x \prec y$. Since we assume $\mathcal{L} \cap \mathbb{N}^n = \{0\}$, the order $\prec_{e^i}$ is thus a term ordering for $\mathcal{L}$. Importantly then, a $\prec_{e^i}$-reduction path
is also an $\bar{e}^i$-reduction path. Let $T = CP(\prec_{\bar{e}^i}, S)$. Then, by the properties of the completion procedure, for all $b \in \mathbb{Z}^n$ and $x, y \in \mathcal{F}_{L,b}$ where $x$ and $y$ are connected in $G(\mathcal{F}_{L,b}, S)$, there exists a $\prec_{\bar{e}^i}$-reduction path from $x$ to $y$ in $G(\mathcal{F}_{L,b}, T)$; $T$ is therefore \{i\}-saturated on $S$.

Let $S, T \subseteq L$ and $T$ is $\sigma$-saturated on $S$ for some $\sigma \subseteq \{1, \ldots, n\}$. Let $T' = CP(\prec_{\bar{e}^i}, T)$. So, $T'$ is therefore \{i\}-saturated on $T$. For the saturation algorithm to work, we need that $T'$ is also $(\sigma \cup \{i\})$-saturated on $S$, which follows from Lemma 14 below.

**Lemma 14** Let $\sigma, \tau \subseteq \{1, \ldots, n\}$ and $S, T, U \subseteq L$. If $U$ is $\sigma$-saturated on $S$, and $T$ is $\tau$-saturated on $U$, then $T$ is $(\sigma \cup \tau)$-saturated on $S$.

**Proof.** Let $b \in \mathbb{Z}^n$, and $x, y \in \mathcal{F}_{L,b}$ where $x$ and $y$ are connected in $G(\mathcal{F}_{L,b}, S)$. Let $\alpha = x \land_{\sigma} y$. Since $T$ is $\sigma$-saturated on $S$, $x - \alpha$ and $y - \alpha$ are connected in $G(\mathcal{F}_{L,b-\alpha}, T)$. Let $\beta = x - \alpha \land_{\tau} y - \alpha$. Then, since $U$ is $\tau$-saturated on $T$, $x - \alpha - \beta$ and $y - \alpha - \beta$ are connected in $G(\mathcal{F}_{L,b-\alpha-\beta}, U)$. Let $\gamma = \alpha + \beta$; then, $\gamma = x \land_{(\sigma \cup \tau)} y$. Therefore, there is a path from $x - \gamma$ to $y - \gamma$ in $G(\mathcal{F}_{L,b-\gamma}, U)$ as required. □

We now arrive at the Saturation algorithm below.

**Algorithm 3** Saturation algorithm

**Input:** a spanning set $S$ of $L$.

**Output:** a generating set $G$ of $L$.

\[
G := S \\
\sigma := \emptyset \\
\text{while } \sigma \neq \{1, \ldots, n\} \text{ do} \\
\quad \text{Select } i \in \bar{\sigma} \\
\quad G := CP(\prec_{\bar{e}^i}, G) \\
\quad \sigma := \sigma \cup \{i\} \\
\text{return } G.
\]

**Lemma 15** Algorithm 3 terminates and satisfies its specifications.

**Proof.** Algorithm 3 terminates, since Algorithm 2 always terminates. We show at the beginning of each iteration that $G$ is $\sigma$-saturated on $S$, and so, at the end of the algorithm $G$ is $\{1, \ldots, n\}$-saturated on $S$; therefore, $G$ is a generating set of $L$. At the beginning of the first iteration, $G$ is $\sigma$-saturated on $S$ since $\sigma = \emptyset$ and $G = S$. So, we can assume it is true for the current iteration, and now, we show it is true for the next iteration. Let $G' := CP(\prec_{\bar{e}^i}, G)$. Then, by Lemma 13, $G'$ is \{i\}-saturated on $G$, and so, by Lemma 14, $G'$ is $(\sigma \cup \{i\})$-saturated on $S$. So, $G$ is $\sigma$-saturated on $S$ at the beginning of the next iteration. □
During the Saturation algorithm, we saturate \( n \) times, once for each \( i \in \{1, \ldots, n\} \). However, as proven in Hosten and Shapiro (2000), it is in fact only necessary to perform at most \( \lfloor \frac{n}{2} \rfloor \) saturations. Given \( S, T \subseteq L \), we can show that there always exists a \( \sigma \subseteq \{1, \ldots, n\} \) where \( |\sigma| \leq \lfloor \frac{n}{2} \rfloor \) such that if \( T \) is \( \sigma \)-saturated on \( S \), then \( T \) is \( \{1, \ldots, n\} \)-saturated on \( S \). The following two lemmas prove the result.

**Lemma 16** Let \( \sigma \subseteq \{1, \ldots, n\} \), \( S, T \subseteq L \) where \( T \) is \( \sigma \)-saturated on \( S \), and \( u \in S \). If \( \text{supp}(u^-) \subseteq \sigma \) or \( \text{supp}(u^+) \subseteq \sigma \), then \( T \) is \( (\text{supp}(u) \cup \sigma) \)-saturated on \( S \).

**Proof.** Assume that \( \text{supp}(u^-) \subseteq \sigma \). Let \( x, y \in F_{L,b} \) for some \( b \in \mathbb{Z}^n \) where \( x \) and \( y \) are connected in \( G(F_{L,b}, S) \). Let \( \alpha = x \land_{\text{supp}(u^+)} y \) and \( \beta = x - \alpha \land_{\sigma} y - \alpha \). We must show that \( x - \alpha - \beta \) and \( y - \alpha - \beta \) are connected in \( G(F_{L,b - \alpha - \beta, T}) \) since \( \alpha + \beta = x \land_{(\text{supp}(u^+), \sigma)} y \). By translating the path from \( x \) to \( y \) by \( \alpha \), we get a path from \( x - \alpha \) to \( y - \alpha \) that is non-negative on all components except \( \text{supp}(u^+) \). This path can transformed into a path that is non-negative on all components except \( \text{supp}(u^-) \) by adding \( u \) to the start of the path as many times as necessary and subtracting \( u \) from the end of the path the same number of times. Therefore, \( x - \alpha + \gamma \) and \( y - \alpha + \gamma \) are connected in \( G(F_{L,b - \alpha - \gamma, T}) \) for some \( \gamma \in \mathbb{N}^n \) where \( \text{supp}(\gamma) \subseteq \text{supp}(u^-) \subseteq \sigma \). Observe that \( \text{supp}(\beta + \gamma) \subseteq \sigma \). Thus, since \( T \) is \( \sigma \)-saturated, \( x - \alpha - \beta \) and \( y - \alpha - \beta \) are connected in \( G(F_{L,b - \alpha - \beta, T}) \) as required.

The case where \( T \) is \( \text{supp}(u^+) \subseteq \sigma \) is essentially the same as above. \( \square \)

**Lemma 17** Let \( S, T \subseteq L \). There exists a \( \sigma \subseteq \{1, \ldots, n\} \) where \( |\sigma| \leq \lfloor \frac{n}{2} \rfloor \) such that if \( T \) is \( \sigma \)-saturated on \( S \), then \( T \) is \( \{1, \ldots, n\} \)-saturated on \( S \).

**Proof.** We show this by construction. Without loss of generality, we assume that \( L \) is not contained in any of the linear subspaces \( \{x_i : x_i = 0, x \in \mathbb{R}^n\} \) for \( i = 1, \ldots, n \); otherwise, we may simply delete this component.

Let \( \sigma = \emptyset \), \( \tau = \emptyset \), and \( U = \emptyset \). Repeat the following steps until \( \tau = \{1, \ldots, n\} \).

1. Select \( u \in S \) such that \( \text{supp}(u) \setminus \tau \neq \emptyset \).
2. If \( |\text{supp}(u^+) \setminus \tau| \geq |\text{supp}(u^-) \setminus \tau| \), then \( \sigma := \sigma \cup \text{supp}(u^-) \), else \( \sigma := \sigma \cup \text{supp}(u^+) \).
3. Set \( \tau := \tau \cup \text{supp}(u) \), and set \( U := U \cup \{u\} \).

The procedure must terminate since during each iteration we increase the size of \( \tau \). Note that, at termination, \( U \subseteq S \), \( \bigcup_{u \in U} \text{supp}(u) = \tau = \{1, \ldots, n\} \), and for all \( u \in U \) either \( \text{supp}(u^+) \subseteq \sigma \) or \( \text{supp}(u^-) \subseteq \sigma \). Therefore, by applying Lemma 16 recursively for each \( u \in U \), we have that if \( T \) is \( \sigma \)-saturated on \( S \), then \( T \) is \( \{1, \ldots, n\} \)-saturated on \( S \). Lastly, since in each iteration we add at least twice as many components to \( \tau \) as to \( \sigma \), we conclude that at termination
\(|\sigma| \leq \left\lfloor \frac{n}{2} \right\rfloor. \quad \square \)

**Example 18** Consider again the set \(S := \{(1,-1,-1,3,1,2),(1,0,2,-2,1,1)\}\). Let \(L\) be the lattice spanned by \(S\), and let \(\sigma = \{1,6\}\). Then, by Lemma 16, since supp\((1,-1,-1,3,1,2)^+\) = \(\{1,6\}\) and supp\((1,-1,-1,3,1,2)\) = \(\{1,2,3,4,5,6\}\), if a set \(T \subseteq L\) is \(\{1,6\}\)-saturated on \(S\), then \(T\) is \(\{1,2,3,4,5,6\}\)-saturated on \(S\). So, to compute a generating set of \(L\), we only need to saturate on \(\{1,6\}\). The following table gives the values of \(\sigma\), \(i\), and \(G\) at each stage of the Saturation algorithm when constructing a set that is \(\{1,6\}\)-saturated on \(S\), and hence, a generating set of \(L\).

<table>
<thead>
<tr>
<th>(\sigma)</th>
<th>(i)</th>
<th>(G := \mathcal{CP}(\prec_{{1,6}}, G))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\emptyset)</td>
<td>1</td>
<td>((-1,0,-2,2,2,-1),(1,0,2,-2,1,1),(0,1,3,1,-1,2),(1,0,2,-2,1,-1))</td>
</tr>
<tr>
<td>({1})</td>
<td>6</td>
<td>((-1,1,1,3,1,-2),(1,0,2,-2,1,-1),(1,-1,5,1,3,0),(1,2,8,0,-4,1))</td>
</tr>
</tbody>
</table>

Observe that after the first iteration, that \(G\) is not a generating set of \(L\). The set \(G\) does not contain the vector \((-1,1,-5,1,3,0)\), and so, the graph \(G(F_{1,6}, G)\) where \(b = (0,0,0,1,3,0)\) is disconnected. Note that the final set \(G\) is not a minimal generating set of \(L\); the vector \((1,2,8,0,-4,1)\) is not needed. See [Caboara et al. (2003)](Caboara et al. (2003)) for an algorithm to compute a minimal generating set.

We now introduce the concept of a \(\sigma\)-generating set of \(L\) for some \(\sigma \subseteq \{1,\ldots,n\}\) – a generalization of a generating set of \(L\). These new generating sets provide useful insights into saturation and the inspiration for the Project-and-Lift algorithm as well as a point of reference to compare the two algorithms.

Firstly, we define \(F_{1,6}^\sigma := \{z : z \equiv b \pmod{L}, z_\sigma \geq 0, z \in \mathbb{Z}^n\}\) where \(\sigma \subseteq \{1,\ldots,n\}\), so we now allow the \(\sigma\) components to be negative. Given \(S \subseteq L\), analogous to \(G(F_{1,6}, S)\), we define \(G(F_{1,6}^\sigma, S)\) to be the undirected graph with nodes \(F_{1,6}^\sigma\) and edges \((x,y)\) if \(x - y \in S\) or \(y - x \in S\). Observe that a path in \(G(F_{1,6}^\sigma, S)\) is non-negative on the \(\check{\sigma}\) components and may be negative on the \(\sigma\) components. Analogous to a generating set of \(L\), a set \(S \subseteq L\) is a \(\sigma\)-generating set of \(L\) if the graph \(G(F_{1,6}^\sigma, S)\) is connected for every \(b \in \mathbb{Z}^n\). Note that \(\emptyset\)-generating sets are equivalent to generating sets and \(\{1,\ldots,n\}\)-generating sets are equivalent to spanning sets.

**Lemma 19** Let \(\sigma \subseteq \{1,\ldots,n\}\) and \(S, T \subseteq L\) where \(S\) spans \(L\). If \(T\) is \(\sigma\)-saturated on \(S\), then \(T\) is a \(\check{\sigma}\)-generating set of \(L\).

**Proof.** Let \(x,y \in F_{1,6}^\sigma\) for some \(b \in \mathbb{Z}^n\). We must show that \(x\) and \(y\) are connected in \(G(F_{1,6}^\sigma, S)\). Since \(S\) spans \(L\), there must exist a \(\gamma \in \mathbb{N}^n\) such that \(x + \gamma\) and \(y + \gamma\) are connected in \(G(F_{1,6}, S)\). Let \(\alpha, \beta \in \mathbb{N}^n\) where \(\alpha + \beta = \gamma\), supp\((\alpha) \subseteq \sigma\), and supp\((\beta) \subseteq \check{\sigma}\). Since \(T\) is \(\sigma\)-saturated on \(S\) and supp\((\alpha) \subseteq \sigma\),
the points \( x + \beta = x + \gamma - \alpha \) and \( y + \beta = y + \gamma - \alpha \) are connected in \( \mathcal{G}(\mathcal{F}_{L, b + \beta}, S) \).

Therefore, since \( \text{supp}(\beta) \subseteq \bar{s} \), \( x \) and \( y \) are connected in \( \mathcal{G}(\mathcal{F}_{L, b}^2, S) \). \( \square \)

Interestingly, the converse of Lemma 19 is not true in general: a \( \bar{s} \)-generating set is not necessarily a \( \sigma \)-saturated set. Let \( S, T \subseteq \mathcal{L} \) where \( S \) spans \( \mathcal{L} \), \( \sigma \subseteq \{1, \ldots, n\} \), and let \( x, y \in \mathcal{F}_{L, b} \) for some \( b \in \mathbb{Z}^n \) where \( x \) and \( y \) are connected in \( \mathcal{G}(\mathcal{F}_{L, b}, S) \). If \( T \) is a \( \bar{s} \)-generating set of \( \mathcal{L} \), then \( x - \gamma \) and \( y - \gamma \) are connected in \( \mathcal{G}(\mathcal{F}_{L, \bar{s} - \gamma}, T) \) where \( \gamma = x \wedge y \). In other words, there is a path from \( x - \gamma \) and \( y - \gamma \) that remains non-negative on the \( \sigma \) components but may be negative on the \( \bar{s} \) components. On the other hand, if \( T \) is \( \sigma \)-saturated on \( S \), then \( x - \gamma \) and \( y - \gamma \) are connected in \( \mathcal{G}(\mathcal{F}_{L, \bar{s} - \gamma}, T) \) where again \( \gamma = x \wedge y \). In other words, there is a path from \( x - \gamma \) and \( y - \gamma \) that remains non-negative on all the components. So, while \( \bar{s} \)-generating sets, like \( \sigma \)-saturated sets, ensure path non-negativity on the \( \sigma \) components, they do not preserve existing path non-negativity on the other \( \bar{s} \) components like \( \sigma \)-saturated sets do. Indeed, \( \bar{s} \)-generating sets say nothing at all about the path non-negativity of the \( \bar{s} \) components. So, \( \sigma \)-saturation is a stronger concept than \( \bar{s} \)-generation.

In the Project-and-Lift algorithm, we compute \( \bar{s} \)-generating sets instead of \( \sigma \)-saturated sets. By doing so, we can effectively ignore the \( \bar{s} \) components, and therefore, we compute smaller intermediate sets, although we start and finish at the same point.

### 3.2 The “Project-and-Lift” algorithm

Given \( \sigma \subseteq \{1, \ldots, n\} \), we define the projective map \( \pi_\sigma : \mathbb{Z}^n \rightarrow \mathbb{Z}^{|\sigma|} \) that projects a vector in \( \mathbb{Z}^n \) onto the \( \bar{s} = \{1, \ldots, n\} \setminus \sigma \) components. For convenience, we write \( \mathcal{L}^\sigma \) where \( \sigma \subseteq \{1, \ldots, n\} \) as the projection of \( \mathcal{L} \) onto the \( \bar{s} \) components – that is, \( \mathcal{L}^\sigma = \pi_\sigma(\mathcal{L}) \). Note that \( \mathcal{L}^\sigma \) is also a lattice.

The fundamental idea behind the Project-and-Lift algorithm is that using a set \( S \subseteq \mathcal{L}^{(i)} \) that is a generating set of \( \mathcal{L}^{(i)} \) for some \( i \in \{1, \ldots, n\} \), we can compute a set \( S' \subseteq \mathcal{L}^{(i)} \) such that \( S' \) lifts to a generating set of \( \mathcal{L} \). So, for some \( \sigma \subseteq \{1, \ldots, n\} \), since \( \mathcal{L}^\sigma \) is also a lattice, starting with a generating set of \( \mathcal{L}^\sigma \), we can compute a generating of \( \mathcal{L}^{\sigma \setminus \{i\}} \) for some \( i \in \sigma \). So, by doing this repeatedly for every \( i \in \sigma \), we attain a generating set of \( \mathcal{L} \).

First, we extend the definition of Gröbner bases. Given \( \varphi \in \mathcal{Q}^n \), recall that a path \( (x^0, \ldots, x^k) \) in \( \mathcal{G}(\mathcal{F}_{L, b}, G) \) is a \( \varphi \)-reduction path if for no \( j \in \{1, \ldots, k-1\} \), we have \( \varphi x^j > \varphi x^0 \) and \( \varphi x^j > \varphi x^k \). A set \( G \subseteq \mathcal{L} \) is a \( \varphi \)-Gröbner basis of \( \mathcal{L} \) if for all \( b \in \mathbb{Z}^n \) and for every pair \( x, y \in \mathcal{F}_{L, b} \), there exists a \( \varphi \)-reduction path from \( x \) to \( y \) in \( \mathcal{G}(\mathcal{F}_{L, b}, G) \).

The following lemma is fundamental to the Project-and-Lift algorithm. Note
that the property that \( \ker(\pi_{\{i\}}) \cap \mathcal{L} = \{0\} \) for some \( i \in \{1, \ldots, n\} \) means that each vector in \( \mathcal{L}^{(i)} \) lifts to a unique vector in \( \mathcal{L} \), and thus, the inverse map \( \pi_{\{i\}}^{-1} : \mathcal{L}^{(i)} \to \mathcal{L} \) is well-defined. Moreover, by linear algebra, there must exist a vector \( \omega^i \in \mathbb{Q}^{n-1} \) such that for all \( u \in \mathcal{L}^{(i)} \), we have \( \omega^i u = (\pi_{\{i\}}^{-1}(u))_i \). We will always write such a vector as \( \omega^i \), and also, we define \( \bar{\omega}^i = -\omega^i \).

**Lemma 20** Let \( i \in \{1, \ldots, n\} \) where \( \ker(\pi_{\{i\}}) \cap \mathcal{L} = \{0\} \), and let \( S \subseteq \mathcal{L}^{(i)} \). The set \( S \) is a \( \bar{\omega}^i \)-Gröbner basis of \( \mathcal{L}^{(i)} \) if and only if \( \pi_{\{i\}}^{-1}(S) \) is a \( \bar{e}^i \)-Gröbner basis of \( \mathcal{L} \).

**Proof.** Assume \( S \) is a \( \bar{\omega}^i \)-Gröbner basis of \( \mathcal{L}^{(i)} \). Let \( x, y \in \mathcal{F}_{\mathcal{L}, b} \) for some \( b \in \mathbb{Z}^n \). We need to show that there is an \( \bar{e}^i \)-reduction path from \( x \) to \( y \) in \( \mathcal{G}(\mathcal{F}_{\mathcal{L}, b}, \pi_{\{i\}}^{-1}(S)) \). Let \( \bar{x} = \pi_{\{i\}}(x), \bar{y} = \pi_{\{i\}}(y) \), and \( \bar{b} = \pi_{\{i\}}(b) \). By assumption, there exists a \( \bar{\omega}^i \)-reduction path \( (\bar{x} = \bar{x}_0, \ldots, \bar{x}_k = \bar{y}) \) in \( \mathcal{G}(\mathcal{F}_{\mathcal{L}, b}, \mathcal{L}^{(i)}) \). So, we have \( \omega^i \bar{x}_j \geq \omega^i \bar{x} \) or \( \omega^i \bar{x}_j \geq \omega^i \bar{y} \) for all \( j \). We now lift this \( \bar{\omega}^i \)-reduction path in \( \mathcal{G}(\mathcal{F}_{\mathcal{L}, b}, \mathcal{L}^{(i)}) \) to an \( \bar{e}^i \)-reduction path in \( \mathcal{G}(\mathcal{F}_{\mathcal{L}, b}, \pi_{\{i\}}^{-1}(S)) \).

Let \( x^j = x + \pi_{\{i\}}^{-1}(\bar{x}_j - \bar{x}) = y + \pi_{\{i\}}^{-1}(\bar{x}_j - \bar{y}) \) for all \( j = 0, \ldots, k \). Hence, \( \pi_{\{i\}}(x^j) = \bar{x}_j \) and \( x^j_i = x_i + \omega^i \bar{x}_j - \omega^i \bar{x} = y_i + \omega^i \bar{x}_j - \omega^i \bar{y} \), and so, \( x^j_i \geq x_i \) or \( x^j_i \geq y_i \). Also, \( x^j - x^{j-1} = \pi_{\{i\}}^{-1}(\bar{x}_j - \bar{x}^{-1}) \in \pi_{\{i\}}^{-1}(S) \) for all \( j = 1, \ldots, k \). Therefore, \( (x = x^0, \ldots, x^k = y) \) is an \( \bar{e}^i \)-reduction path in \( \mathcal{G}(\mathcal{F}_{\mathcal{L}, b}, \pi_{\{i\}}^{-1}(S)) \) as required.

Assume \( \pi_{\{i\}}^{-1}(S) \) is a \( \bar{e}^i \)-Gröbner basis of \( \mathcal{L} \). Let \( x, y \in \mathcal{F}_{\mathcal{L}, b} \) for some \( b \in \mathbb{Z}^{n-1} \), and let \( \gamma = \omega^i(x - y) \). If \( \gamma > 0 \), then let \( \bar{x} = (x, 0) \) and \( \bar{y} = (y, -\gamma) \); hence, \( \bar{x}, \bar{y} \in \mathcal{F}_{\mathcal{L}, b} \) for some \( b \in \mathbb{Z}^n \), and \( \pi_{\{i\}}(b) = b \), \( \pi_{\{i\}}(\bar{x}) = x \), \( \pi_{\{i\}}(\bar{y}) = y \), and \( \min\{\bar{x}_i, \bar{y}_i\} = 0 \). By assumption, there exists a \( \bar{e}^i \)-reduction path \( (\bar{x} = \bar{x}_0, \ldots, \bar{x}_k = \bar{y}) \) in \( \mathcal{G}(\mathcal{F}_{\mathcal{L}, b}, \pi_{\{i\}}^{-1}(S)) \). Let \( x^j = \pi_{\{i\}}(\bar{x}^j) \). So, \( (x = x_0, \ldots, x^k = y) \) is a path in \( \mathcal{G}(\mathcal{F}_{\mathcal{L}, b}, \mathcal{L}^{(i)}) \). Moreover, since \( \bar{x}_j \geq \bar{x}_i \) or \( \bar{x}_j \geq \bar{y}_i \) for all \( j \), we have \( \omega^i x_j \geq \omega^i x \) or \( \omega^i x_j \geq \omega^i y \) for all \( j \). Therefore, the path is a \( \bar{\omega}^i \)-reduction path. \( \square \)

By definition, a \( \bar{e}^i \)-Gröbner basis of \( \mathcal{L} \) is also a generating set of \( \mathcal{L} \). On the other hand, a generating set of \( \mathcal{L} \) is also a \( \bar{e}^i \)-Gröbner basis of \( \mathcal{L} \). This follows since, given a generating set of \( \mathcal{L} \), for any \( x, y \in \mathcal{F}_{\mathcal{L}, b} \), there must exist a path from \( x - \gamma \) to \( y - \gamma \) where \( \gamma = x \wedge_{\{i\}} y \), and by translating such a path by \( \gamma \), we get a \( \bar{e}^i \)-reduction path from \( x \) to \( y \). This can also be shown using Lemma 13. So, we arrive at the following corollary.

**Corollary 21** Let \( i \in \{1, \ldots, n\} \) where \( \ker(\pi_{\{i\}}) \cap \mathcal{L} = \{0\} \), and let \( S \subseteq \mathcal{L}^{(i)} \). The set \( S \) is a \( \bar{\omega}^i \)-Gröbner basis of \( \mathcal{L}^{(i)} \) if and only if \( \pi_{\{i\}}^{-1}(S) \) is a generating set of \( \mathcal{L} \).

Given a vector \( \varphi \in \mathbb{Q}^n \), any \( \prec_{\varphi} \)-reduction path is also a \( \varphi \)-reduction path, and
so, any \( \prec \varphi \)-Gröbner basis is also a \( \varphi \)-Gröbner basis. So, given a set \( S \subseteq \mathcal{L}^{(i)} \) that generates \( \mathcal{L}^{(i)} \), we can compute a \( \prec \varphi \)-Gröbner basis \( S' \subseteq \mathcal{L}^{(i)} \) of \( \mathcal{L}^{(i)} \) by running the completion procedure with respect to \( \prec \varphi \) on \( S \). That is, \( S' = \mathcal{CP}(\prec \varphi, S) \). Hence, by Lemma 20, \( \pi_{(i)}^{-1}(S') \) is a generating set of \( \mathcal{L} \).

We can apply the above reasoning to compute a generating set of \( \mathcal{L}^{\sigma \setminus \{i\}} \) from a generating set of \( \mathcal{L}^{\sigma} \) for some \( \sigma \subseteq \{1, \ldots, n\} \) and \( i \in \sigma \). First, analogously to \( \pi_{(i)} \) and \( \prec \varphi \) in the context of \( \mathcal{L}^{(i)} \) and \( \mathcal{L} \), we define \( \pi_{(i)}^{\sigma} \) and \( \prec \varphi^{\sigma} \) in the same way except in the context of \( \mathcal{L}^{\sigma} \) and \( \mathcal{L}^{\sigma \setminus \{i\}} \) respectively.

We can now present our Project-and-Lift algorithm.

**Algorithm 4** Project-and-Lift algorithm

*Input:* a set \( S \subseteq \mathcal{L} \) that spans \( \mathcal{L} \).

*Output:* a generating \( G \) set of \( \mathcal{L} \)

Find a set \( \sigma \subseteq \{1, \ldots, n\} \) such that \( \ker(\pi_{\sigma}) \cap \mathcal{L} = \{0\} \) and \( \mathcal{L}^{\sigma} \cap \mathbb{N}^{[\sigma]} = \{0\} \). Compute a set \( G \subseteq \mathcal{L}^{\sigma} \) such that \( G \) is a generating set of \( \mathcal{L}^{\sigma} \) using \( S \).

while \( \sigma \neq \emptyset \) do
  Select \( i \in \sigma \)
  \( G := (\pi_{(i)}^{\sigma})^{-1}(\mathcal{CP}(\prec \varphi^{\sigma}, G)) \)
  \( \sigma := \sigma \setminus \{i\} \)
return \( G \).

**Lemma 22** Algorithm 4 terminates and satisfies its specifications.

**Proof.** Algorithm 4 terminates, since Algorithm 2 always terminates.

We claim that for each iteration of the algorithm, \( G \) is a generating set of \( \mathcal{L}^{\sigma} \), \( \ker(\pi_{\sigma}) \cap \mathcal{L} = \{0\} \), and \( \mathcal{L}^{\sigma} \cap \mathbb{N}^{[\sigma]} = \{0\} \); therefore, at termination, \( G \) is a generating set of \( \mathcal{L} \). This is true for the first iteration, so we assume it is true for the current iteration.

If \( \sigma = \emptyset \), then there is nothing left to do, so assume otherwise. Since by assumption, \( \mathcal{L}^{\sigma} \cap \mathbb{N}^{[\sigma]} = \{0\} \) and \( \ker(\pi_{\sigma}) \cap \mathcal{L}^{\sigma} = \{0\} \), we must have \( \ker(\pi_{(i)}^{\sigma}) \cap \mathcal{L}^{\sigma \setminus \{i\}} = \{0\} \), and so, the inverse map \( (\pi_{(i)}^{\sigma})^{-1} : \mathcal{L}^{\sigma} \to \mathcal{L}^{\sigma \setminus \{i\}} \) is well-defined. Let \( i \in \sigma \), \( G' := (\pi_{(i)}^{\sigma})^{-1}(\mathcal{CP}(\prec \varphi^{\sigma}, G)) \), and \( \sigma' := \sigma \setminus \{i\} \). Then, by Corollary 21, \( G' \) is a generating set of \( \mathcal{L}^{\sigma'} \). Also, since \( \sigma' \subseteq \sigma \), we must have \( \ker(\pi_{\sigma'}) \cap \mathcal{L} = \{0\} \) and \( \mathcal{L}^{\sigma'} \cap \mathbb{N}^{[\sigma']} = \{0\} \). Thus, the claim is true for the next iteration.

In our Project-and-Lift algorithm, we need to find a set \( \sigma \subseteq \{1, \ldots, n\} \) such that \( \ker(\pi_{\sigma}) \cap \mathcal{L} = \{0\} \) and \( \mathcal{L}^{\sigma} \cap \mathbb{N}^{[\sigma]} = \{0\} \), and then, we need to compute a generating set of \( \mathcal{L}^{\sigma} \).
For our purposes, the larger $\sigma$ the better. However, in general, finding the largest $\sigma$ is difficult; thus, we use the following method for finding a good $\sigma$. Let $B$ be a basis for the lattice $L$ ($L$ is spanned by the rows of $B$). Let $k := \text{rank}(B)$. Any $k$ linearly independent columns of $B$ then suffice to give a set $\bar{\sigma}$ such that every vector in $L^\sigma$ lifts to a unique vector in $L$; that is, $\ker(\pi_{\bar{\sigma}}) \cap L^\sigma = \{0\}$. Such a set $\bar{\sigma}$ can be found via Gaussian elimination. If $L^\sigma \cap N^{\bar{\sigma}} \neq \{0\}$, then remove some $i \in \sigma$ from $\sigma := \sigma \setminus \{i\}$ and recompute $L^\sigma \cap N^{\bar{\sigma}}$. Continue to do so until $L^\sigma \cap N^{\bar{\sigma}} = \{0\}$. This procedure must terminate since $L \cap N^n = \{0\}$ by assumption. To check if $L^\sigma \cap N^{\bar{\sigma}} = \{0\}$, we can either solve a linear programming problem or compute the extreme rays of $L^\sigma \cap N^{\bar{\sigma}}$ (see for example Avis and Fukuda (1996); Fukuda and Prodon (1996); Hemmecke (2002)). In practice, we compute extreme rays using the algorithm in Hemmecke (2002).

Once we have found such a $\sigma$, we can compute a generating set of $L^\sigma$ using either the Saturation algorithm, the Min-Max algorithm, or any other such algorithm. In practice and for this paper, we use the Saturation algorithm. It is possible that there does not exist such a $\sigma$ except the trivial case where $\sigma = \emptyset$, and so, the Project-and-Lift algorithm reduces to just the initial phase of computing a generating set of $L$ using some other algorithm. Though in practice, we usually found a non-trivial $\sigma$. We refer the reader to Section 4 for a description of a complete Project-and-Lift algorithm whereby we do not need another algorithm to start with.

**Example 23** Consider again the set $S := \{(1,-1,-1,-3,-1,2),(1,0,2,-2,-2,1)\}$. Let $L$ be the lattice spanned by $S$. Let $\sigma = \{3,4,5,6\}$. Then, $\ker(\pi_{\sigma}) \cap L^\sigma = \{0\}$. Note that $\pi_{\sigma}(S) = \{(1,-1),(1,0)\}$. However, $L^\sigma \cap N^{\bar{\sigma}} \neq \{0\}$. So, set $\sigma = \{3,4,6\}$. Now $\pi_{\sigma}(S) = \{(1,-1,-1),(1,0,-2)\}$, and $L^\sigma \cap N^{\bar{\sigma}} = \{0\}$.

The set $G = \{(0,-1,1),(-1,2,0)\}$ is a generating set of $L^\sigma$. We can compute this using the saturation algorithm. The following table gives the values of $\sigma$, $i$, $\omega^i$, and $G$ at each stage of the Project-and-Lift algorithm.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$i$</th>
<th>$\omega^i$</th>
<th>$\mathcal{CP}(\prec, G)$</th>
<th>$G := (\pi_{\sigma(i)}^{-1})(\mathcal{CP}(\prec, G))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${3,4,6}$</td>
<td>3</td>
<td>(2,3,0)</td>
<td>${(0,-1,1),(1,-2,0)}$</td>
<td>${(0,-1,3),(1,-2,-4)}$</td>
</tr>
<tr>
<td>${4,6}$</td>
<td>4</td>
<td>(-2,1,0,0)</td>
<td>${(0,-1,3),(1,-2,4)}$</td>
<td>${(0,-1,3,1),(1,-2,-4)}$</td>
</tr>
<tr>
<td>${6}$</td>
<td>6</td>
<td>(1,-1,0,0)</td>
<td>${(0,1,3,1,1), (-1,1,1,3,1), (-1,0,-2,2,2), (-1,1,5,1,3)}$</td>
<td>${(0,1,3,1,1,1), (-1,1,1,3,1,2), (-1,0,-2,2,2,1), (-1,1,5,1,3,0)}$</td>
</tr>
</tbody>
</table>

The final $G$ is a generating set of $L$. Again note that it is not minimal; the vector $(1,2,8,0,-4,-1)$ is not needed and can be removed from $G$, and $G$ will
still be a generating set of $\mathcal{L}$.

The concepts of $\sigma$-generating sets of $\mathcal{L}$ and generating sets of $\mathcal{L}^\sigma$ are, in fact, equivalent. So, as discussed before, unlike that Saturation algorithm, the Project-and-Lift algorithm computes $\sigma$-generating sets and thus does less work than the Saturation algorithm.

**Lemma 24** Let $\sigma \subseteq \{1, \ldots, n\}$ where $\ker(\pi_\sigma) \cap \mathcal{L} = \{0\}$ and $S \subseteq \mathcal{L}^\sigma$. The set $S$ is a generating set of $\mathcal{L}^\sigma$ if and only if $\pi_\sigma^{-1}(S)$ is a $\sigma$-generating set of $\mathcal{L}$.

**Proof.** Recall that a $\sigma$-generating set of $\mathcal{L}$ is a set where for all for all $b \in \mathbb{Z}^n$ and for all $x, y \in \mathcal{F}^\sigma_{\mathcal{L}, b}$ there exists a path from $x$ to $y$ in $\mathcal{G}(\mathcal{F}^\sigma_{\mathcal{L}, b}, S)$. Observe that $\pi_\sigma(\mathcal{F}^\sigma_{\mathcal{L}, b}) = \mathcal{F}^\sigma_{\mathcal{L}, \pi_\sigma(b)}$, and moreover, a path in $\mathcal{G}(\mathcal{F}^\sigma_{\mathcal{L}, b}, S)$ projects to a path in $\mathcal{G}(\mathcal{F}^\sigma_{\mathcal{L}, \pi_\sigma(b)}, S)$. Hence, a $\sigma$-generating set of $\mathcal{L}$ projects to a generating set of $\mathcal{L}^\sigma$. So, if $S$ is a $\sigma$-generating set of $\mathcal{L}$, then $\pi_\sigma(S)$ is a generating set of $\mathcal{L}^\sigma$. Also, assuming $\ker(\pi_\sigma) \cap \mathcal{L} = \{0\}$, if $S$ is a generating set of $\mathcal{L}^\sigma$, then $\pi_\sigma^{-1}(S)$ is a $\sigma$-generating set of $\mathcal{L}$. This follows since a path in $\mathcal{G}(\mathcal{F}^\sigma_{\mathcal{L}, b}, S)$ can be lifted to a path in $\mathcal{G}(\mathcal{F}^\sigma_{\mathcal{L}, b}, S)$ where $\pi_\sigma(b) = b$. □

Observe that if $\ker(\pi_\sigma) \cap \mathcal{L} \neq \{0\}$, then a path in $\mathcal{G}(\mathcal{F}^\sigma_{\mathcal{L}, b}, \pi_\sigma^{-1}(S))$ cannot necessarily be lifted to a path in $\mathcal{G}(\mathcal{F}^\sigma_{\mathcal{L}, b}, \pi_\sigma^{-1}(S))$ – the path may become disconnected – although, we can easily rectify this by adding a spanning set of the lattice $\ker(\pi_\sigma) \cap \mathcal{L}$ to $\pi_\sigma^{-1}(S)$.

The Project-and-Lift algorithm has some interesting properties. As we saw in Lemma 20, $\omega^i$-reduction paths lift to $\bar{e}^i$-reduction paths and $\bar{e}^i$-reduction paths project to $\omega^i$-reduction paths. The same holds true for $\prec_{\omega^i}$-reduction paths and $\prec_{\bar{e}^i}$-reduction paths, shown in exactly the same way, giving the following lemma.

**Lemma 25** Let $i \in \{1, \ldots, n\}$ where $\ker(\pi_{\{i\}}) \cap \mathcal{L} = \{0\}$, and let $S \subseteq \mathcal{L}^{(i)}$. Let $\prec$ be a term order. The set $\mathcal{L}_{\{i\}}^\sigma(S)$ is a $\prec_{\omega^i}$-Gröbner basis of $\mathcal{L}$ if and only if $S$ is a $\prec_{\omega^i}$-Gröbner basis of $\mathcal{L}^{(i)}$.

So, during the Project-and-Lift algorithm, we compute a $\prec_{\omega^i}$-Gröbner basis for some $i$ and then lift it to a $\prec_{\bar{e}^i}$-Gröbner basis. We then compute a $\prec_{\omega^j}$-Gröbner basis using some $j \neq i$ and again lift it to a $\prec_{\bar{e}^i}$-Gröbner basis, and repeat. So effectively, the Project-and-Lift algorithm just converts one Gröbner basis into another and lifts to another Gröbner basis. We therefore could use a Gröbner walk algorithm to move from one Gröbner basis to another (see [Collart et al. (1997); Fukuda et al. (2003)]). We have not yet implemented such an algorithm. It would be interesting to see its performance.

There are, in fact, two essentially equivalent ways to compute a $\prec_{\omega^i}$-Gröbner basis of $\mathcal{L}$ from a set $S \subseteq \mathcal{L}^{(i)}$ that generates $\mathcal{L}^{(i)}$ for some $i \in \{1, \ldots, n\}$ where
ker(\(\pi_{\{i\}}\)) \cap \mathcal{L} = \{0\}, as needed by the Project-and-Lift algorithm. Firstly, as we have already seen, the set \(T = \pi_{\{i\}}^{-1}(CP(\prec_{\omega^i}, S))\) is a \(\prec_{\omega^i}\)-Gröbner basis of \(\mathcal{L}\), but also, the set \(T' = CP(\prec_{\omega^i}, \pi_{\{i\}}^{-1}(S))\) is also a \(\prec_{\omega^i}\)-Gröbner basis of \(\mathcal{L}\). Essentially, to compute a \(\prec_{\omega^i}\)-Gröbner basis of \(\mathcal{L}\), we do not need a generating set of \(\mathcal{L}\), but instead, we only need a \{i\}-generating set of \(\mathcal{L}\). This follows from the following Lemmas, which are analogous to Lemmas 20 and 25 respectively.

**Lemma 26** Let \(i \in \{1, \ldots, n\}\) where \(ker(\pi_{\{i\}}) \cap \mathcal{L} = \{0\}\), and let \(S, T \subseteq \mathcal{L}\) where \(S\) is a \{i\}-generating set of \(\mathcal{L}\). The set \(T\) is a \(\prec_{\omega^i}\)-Gröbner basis of \(\mathcal{L}\) if and only if for all \(b \in \mathbb{Z}^n\) and for all \(x, y \in F_{\mathcal{L}, b}\) where \(x\) and \(y\) are connected in \(G(F_{\mathcal{L}, b}, S)\), there exists an \(\prec_{\omega^i}\)-reduction path from \(x\) to \(y\) in \(G(F_{\mathcal{L}, b}, T)\).

**Proof.** The forwards direction must hold by definition. Conversely, let \(x, y \in F_{\mathcal{L}, b}\) for some \(b \in \mathbb{Z}^n\). Since \(S\) is a \{i\}-generating set of \(\mathcal{L}\), there must exist \(\gamma \in \mathbb{N}^n\) where \(supp(\gamma) \subseteq \{i\}\), such that \(x + \gamma\) is connected to \(y + \gamma\) in \(G(F_{\mathcal{L}, b+\gamma}, S)\). So, by assumption, there exists an \(\prec_{\omega^i}\)-reduction path from \(x + \gamma\) to \(y + \gamma\) in \(G(F_{\mathcal{L}, b+\gamma}, T)\), which translates to a path from \(x\) to \(y\) in \(G(F_{\mathcal{L}, b}, T)\) since \(supp(\gamma) \subseteq \{i\}\). □

An analogous results holds for \(\prec_{\omega^i}\)-Gröbner bases for similar reasons.

**Lemma 27** Let \(i \in \{1, \ldots, n\}\) where \(ker(\pi_{\{i\}}) \cap \mathcal{L} = \{0\}\), and let \(S, T \subseteq \mathcal{L}\) where \(S\) is a \{i\}-generating set of \(\mathcal{L}\). The set \(T\) is a \(\prec_{\omega^i}\)-Gröbner basis of \(\mathcal{L}\) if and only if for all \(b \in \mathbb{Z}^n\) and for all \(x, y \in F_{\mathcal{L}, b}\) where \(x\) and \(y\) are connected in \(G(F_{\mathcal{L}, b}, S)\), there exists an \(\prec_{\omega^i}\)-reduction path from \(x\) to \(y\) in \(G(F_{\mathcal{L}, b}, T)\).

Therefore, if \(S \subseteq \mathcal{L}^{\{i\}}\) generates \(\mathcal{L}^{\{i\}}\), then \(\pi_{\{i\}}^{-1}(S)\) is a \{i\}-generating set of \(\mathcal{L}\) by Lemma 24. So, the set \(T' = CP(\prec_{\omega^i}, \pi_{\{i\}}^{-1}(S))\) is a \(\prec_{\omega^i}\)-Gröbner basis of \(\mathcal{L}\). Moreover, when computing \(CP(\prec_{\omega^i}, S)\) and computing \(CP(\prec_{\omega^i}, \pi_{\{i\}}^{-1}(S))\), the completion procedure performs essentially the same sequence of steps producing essentially the same output data and intermediate data with the exception that they perform the computation in different spaces. These two approaches are thus algorithmically equivalent.

In one iteration, the Saturation algorithm computes \(CP(\prec_{\omega^i}, T)\), in the space \(\mathcal{L}\), for some set \(T \subseteq \mathcal{L}\) that is \(\sigma\)-saturated on some spanning set for some \(\sigma \subseteq \{1, \ldots, n\}\) and \(i \in \sigma\). On the other hand, in one iteration, the Project-and-Lift effectively computes, in the space \(\mathcal{L}^{\sigma \setminus \{i\}}\), \(CP(\prec_{\omega^i}, T)\) for some set \(T \subseteq \mathcal{L}^{\sigma \setminus \{i\}}\) that is a \{i\}-generating set of \(\mathcal{L}^{\sigma}\) for some \(\sigma \subseteq \{1, \ldots, n\}\) and \(i \in \sigma\). So, the algorithms are very similar, but the Project-and-Lift algorithm performs intermediate steps in subspaces whereas the Saturation algorithm performs intermediate steps in the original space.
3.3 The “Lift-and-Project” algorithm

The idea behind this algorithm is to lift a spanning set $S$ of $\mathcal{L} \subseteq \mathbb{Z}^n$ to a spanning set $S' \subseteq \mathbb{Z}^{n+1}$ of $\mathcal{L}' \subseteq \mathbb{Z}^{n+1}$ in such a way that we can compute a set $G' \subseteq \mathcal{L}'$ that generates $\mathcal{L}'$ in only one saturation step. Then, we project $G'$ to $G \subseteq \mathcal{L}$, so that $G$ is a generating set of $\mathcal{L}$.

Let $S$ be a spanning set of $\mathcal{L} \subseteq \mathbb{Z}^n$. Let $S' := \{(u, 0) : u \in S\} \cup \{(1, \ldots, 1, -1)\}$, and let $\mathcal{L}' \subseteq \mathbb{Z}^{n+1}$ be the lattice spanned by $S'$. Since the vector $(1, \ldots, 1, -1)$ is in $S'$, it follows from Lemma 16, that if a set $G' \subseteq \mathcal{L}'$ is $\{n+1\}$-saturated on $S'$, then $G'$ is $\{1, \ldots, n+1\}$-saturated on $S$, and hence, $G'$ is a generating set of $\mathcal{L}'$. Also, since $\mathcal{L} \cap \mathbb{N}^n = \{0\}$, then $\mathcal{L}' \cap \mathbb{Z}^{n+1} = \{0\}$. Now, using exactly the same idea behind the Saturation algorithm, if we let $G' := CP(\prec_{\mathbb{Z}^{n+1}}, S')$, then $G'$ must be a generating set for the lattice $\mathcal{L}'$ by Lemma 13.

So, at the moment, we have a generating set $G'$ for $\mathcal{L}'$, and from this, we need to extract a generating set of $\mathcal{L}$. We define the linear map $\rho : \mathbb{Z}^{n+1} \mapsto \mathbb{Z}^n$ where

$$\rho(u') := (u'_1 + u'_{n+1}, u'_2 + u'_{n+2}, \ldots, u'_n + u'_{n+1}).$$

Observe that $\rho$ maps $\mathbb{Z}^{n+1}$ onto $\mathbb{Z}^n$, maps $\mathcal{L}'$ onto $\mathcal{L}$, and maps $\mathcal{F}_{\mathcal{L}', \mathcal{L}}$ onto $\mathcal{F}_{\mathcal{L}, \mathcal{L}}$ where $b = \rho(b')$. Let $G := \{\rho(u') : u' \in G'\} \setminus \{0\}$. So, $G \subseteq \mathcal{L}$, and we now show that in fact $G$ generates $\mathcal{L}$. Let $(x^0, \ldots, x^k)$ be a path in $\mathcal{G}(\mathcal{F}_{\mathcal{L}', \mathcal{L}}, G')$. Then, $(\rho(x^0), \ldots, \rho(x^k))$ is a walk from $\rho(x^0)$ to $\rho(x^k)$ in $\mathcal{G}(\mathcal{F}_{\mathcal{L}, \rho(b)}, G)$, so after removing cycles, we have a path from $\rho(x^0)$ to $\rho(x^k)$. Cycles may exist because the kernel of $\rho$ is non-trivial – $\ker(\rho) = \{(\gamma, \ldots, \gamma, -\gamma) : \gamma \in \mathbb{Z}\}$. Let $x, y \in \mathcal{F}_{\mathcal{L}, b}$ for some $b \in \mathbb{Z}^n$, and let $x' := (x, 0)$, $y' := (y, 0)$, and $b' := (b, 0)$; hence, $x = \rho(x')$, $y = \rho(y')$, and $\rho(b') = b$. Then, since $G'$ is a generating set of $\mathcal{L}'$ there must exist a path from $x'$ to $y'$ in $\mathcal{G}(\mathcal{F}_{\mathcal{L}', \mathcal{L}})$, and therefore, there exists a path from $x$ to $y$ in $\mathcal{G}(\mathcal{F}_{\mathcal{L}, b}, G)$. Hence, $G$ is a generating set of $\mathcal{L}$. We thus arrive at the Lift-and-Project algorithm.

**Algorithm 5** Lift-and-Project algorithm

**Input:** a set $S \subseteq \mathcal{L}$ that spans $\mathcal{L}$.

**Output:** a generating set $G$ of $\mathcal{L}$

$S' := \{(u, 0) : u \in S\} \cup \{(1, \ldots, 1, -1)\}$

$G' := CP(\prec_{\mathbb{Z}^{n+1}}, S')$

$G := \{\rho(u') : u' \in G'\} \setminus \{0\}$

**Return** $G$.

To make the algorithm more efficient, we can use a different additional vector to $(1, \ldots, 1, -1)$. By Lemma 17, we know that given a spanning set $S$, there exists a $\sigma$ where $|\sigma| \leq \left\lceil \frac{n}{2} \right\rceil$ such that if $T$ is $\sigma$-saturated on $S$, then $T$ is
a generating set of \( \mathcal{L} \). Then, instead of \((1, \ldots, 1, -1)\), it suffices to use the additional vector \( s_\sigma = \sum_{i \in \sigma} e^i - e^{n+1} \), which has the important property that \( \text{supp}(s_\sigma^+) = \sigma \) and \( \text{supp}(s_\sigma^-) = \{n+1\} \). Recall that \( e^i \) is the \( i \)-th unit vector. Set \( S' := \{(u,0) : u \in S\} \cup \{s_\sigma\} \), and let \( \mathcal{L}' \) be the lattice spanned by \( S' \).

Then, from Lemma 16, since \( s_\sigma \in S' \), if a set \( G' \subseteq \mathcal{L}' \) is \( \{n+1\} \)-saturated on \( S' \), then \( G' \) is \( (\sigma \cup \{n+1\}) \)-saturated on \( S' \). Also, since \( \{(u,0) : u \in S\} \subseteq S' \), from the proof of Lemma 17, it follows that if \( G' \) is \( \sigma \)-saturated on \( S' \), then \( G' \) is \( \{1, \ldots, n\} \)-saturated. Hence, \( G' \) is \( \{1, \ldots, n+1\} \)-saturated, and therefore, a generating set of \( \mathcal{L}' \). So again, we can compute a generating set \( G' \) of \( \mathcal{L}' \) in one saturation step. Also, we similarly define the linear map \( \rho_\sigma : \mathbb{Z}^{n+1} \to \mathbb{Z}^n \) where \( \rho_\sigma(x') := (x'_1, x'_2, \ldots, x'_n) + (\sum_{i \in \sigma} e_i) x_{n+1} \). Then, \( G := \{\rho_\sigma(x') : x' \in G'\} \) is a generating set of \( \mathcal{L} \). As a general rule, the smaller the size of \( \sigma \), the faster the algorithm.

### 4 What if \( \mathcal{L} \cap \mathbb{N}^n \neq \{0\} \)?

If \( \mathcal{L} \cap \mathbb{N}^n \neq \{0\} \), then computing a generating set of the lattice \( \mathcal{L} \) is actually more straight-forward than otherwise. The vectors in \( \mathcal{L} \cap \mathbb{N}^n \) are very useful when constructing generating sets.

We say that component \( i \in \{1, \ldots, n\} \) is **unbounded** if there exists a \( u \in \mathcal{L} \cap \mathbb{N}^n \) where \( i \in \text{supp}(u) \) and **bounded** otherwise. From Farkas’ lemma, \( i \) is unbounded if and only if the linear program \( \max \{x_i : x \equiv 0 \text{ (mod } \mathcal{L}) \} \) is unbounded. To find a \( u \in \mathcal{L} \) such that \( u \geq 0 \) and \( i \in \text{supp}(u) \), and so also, to check whether \( i \) is unbounded, we can solve a linear program or compute the extreme rays of \( \mathcal{L} \cap \mathbb{N}^n \) (see for example [Avis and Fukuda (1996); Fukuda and Prodon (1996); Hemmecke (2002)]). Given a term order \( \prec \) of \( \mathcal{L} \), the order \( \prec_{e_i} \) is a term order if and only if \( i \) is bounded.

Using the following lemma, we can extend the Saturation algorithm to the more general case where \( \mathcal{L} \cap \mathbb{N}^n \neq \{0\} \).

**Lemma 28** Let \( S \subseteq \mathcal{L} \). If there exists \( u \in S \) where \( u \in \mathcal{L} \cap \mathbb{N}^n \) and \( u \neq 0 \), then \( S \) is \( \text{supp}(u) \)-saturated (on \( S \)).

**Proof.** By definition, \( S \) is \( \emptyset \)-saturated (on \( S \)). Since \( u \geq 0 \), we have \( \text{supp}(u^-) = \emptyset \), and so it follows immediately from Lemma 16 that \( S \) is \( \text{supp}(u) \)-saturated (on \( S \)). \( \square \)

We can now extend the Saturation algorithm. Let \( \tau \subseteq \{1, \ldots, n\} \) be the set of unbounded components, and let \( S \) be a spanning set of \( \mathcal{L} \). Then, for each \( i \in \tau \), find a \( u \in \mathcal{L} \) such that \( u \geq 0 \) and \( u_i > 0 \) and add \( u \) to \( S \). Or equivalently, find a single \( u \geq 0 \) such that \( \text{supp}(u) = \tau \) and add it to \( S \). Now \( S \) is \( \tau \)-
saturated (on \( S \)) by Lemma 28. So, if a set \( T \) is \( \tau \)-saturated on \( S \), then \( T \) is \( \{1, \ldots, n\} \)-saturated on \( S \) by Lemma 14, and so, \( T \) is a generating set of \( \mathcal{L} \). So, we iteratively compute \( T := \mathcal{CP}(\langle \rho_i \rangle, T) \) for every \( i \in \tau \); then, \( T \) is \( \tau \)-saturated on \( S \) as required.

The Project-and-Lift algorithm can also be extended to the more general case where \( \mathcal{L} \cap \mathbb{N}^n \neq \{0\} \). First, we need the following lemma.

**Lemma 29** Let \( \sigma \subseteq \{1, \ldots, n\} \), \( S \subseteq \mathcal{L} \), and \( u \in \mathcal{L} \cap \mathbb{N}^n \) and \( u \neq 0 \). If \( S \) is a \( \sigma \)-generating set of \( \mathcal{L} \) and \( u \in S \), then \( S \) is a \( (\sigma \setminus \text{supp}(u)) \)-generating set of \( \mathcal{L} \).

**Proof.** Let \( x, y \in \mathcal{F}_{\mathcal{L},b} \) for some \( b \in \mathbb{Z}^n \). Since \( S \) is a \( \sigma \)-generating set of \( \mathcal{L} \), there exists a path from \( x \) to \( y \) in \( \mathcal{G}(\mathcal{F}_{\mathcal{L},b}, S) \). This path can transformed into a path in \( \mathcal{G}(\mathcal{F}_{\mathcal{L},b}^{\langle \sigma \setminus \text{supp}(u) \rangle}, S) \) by adding \( u \) to the start of the path as many times as necessary and subtracting \( u \) from the end of the path the same number of times. \( \Box \)

Let \( \sigma \subseteq \{1, \ldots, n\} \) where \( \ker(\pi_\sigma) \cap \mathcal{L}^\sigma = \{0\} \). Let \( S \subseteq \mathcal{L}^\sigma \) where \( S \) is a generating set of \( \mathcal{L}^\sigma \). Also, let \( i \in \sigma \). We now show how to construct a generating set of \( \mathcal{L}^{\sigma \setminus \{i\}} \), and thus by induction, a generating set of \( \mathcal{L} \). Firstly, since \( S \) is a generating set of \( \mathcal{L}^\sigma \), \( \pi^{-1}_{\{i\}}(S) \) is a \( \{i\} \)-generating set of \( \mathcal{L}^\sigma \setminus \{i\} \) from Lemma 24. If \( i \) is unbounded for \( \mathcal{L}^{\sigma \setminus \{i\}} \), then there exists a \( u \in \mathcal{L}^{\sigma \setminus \{i\}} \) such that \( u \geq 0 \) and \( u_i > 0 \). Thus, after adding \( u \) to \( (\pi^\sigma_{\{i\}})^{-1}(S) \), we then have a generating set of \( \mathcal{L}^{\sigma \setminus \{i\}} \). If \( i \) is bounded, then compute \( S := \mathcal{CP}(\langle \rho_i \rangle, S) \) and \( (\pi^\sigma_{\{i\}})^{-1}(S) \) is then a generating set of \( \mathcal{L}^{\sigma \setminus \{i\}} \).

We first need to find an initial \( \sigma \) and \( S \) such that \( \ker(\pi_\sigma) \cap \mathcal{L}^\sigma = \{0\} \) and \( S \) is a generating set of \( \mathcal{L}^\sigma \). Let \( B \) be a lattice basis of \( \mathcal{L} \) where the rows of \( B \) span \( \mathcal{L} \). Let \( \bar{\sigma} \) be any rank(\( B \)) linearly independent columns of \( B \). Let \( S = \pi_\sigma(B) \). Then, every vector in \( \mathcal{L}^\sigma \) lifts to a unique vector in \( \mathcal{L} \). Then, computing a \( u \in \mathcal{L}^\sigma \) such that \( u > 0 \) can be done by Gaussian elimination. After adding \( u \) to \( S \), \( S \) is a generating set of \( \mathcal{L}^\sigma \) as required.

We can also extend the Lift-and-Project algorithm in a similar way. As above, we can find a set \( S \) such that \( S \) is \( \tau \)-saturated (on \( S \)). The set

\[
S' := \{(u, 0) : u \in S \} \cup \{\sum_{i \in \bar{\tau}} e_i - e_{n+1}\}
\]

is also \( \tau \)-saturated (on \( S' \)) for the same reasons. If a set \( T' \subseteq \mathcal{L}' \) is \( \{n + 1\} \)-saturated on \( S' \), then \( T' = (\bar{\tau} \cup \{n + 1\}) \)-saturated on \( S' \) by Lemma 28, and so, by Lemma 14, \( T' \) is \( \{1, \ldots, n + 1\} \)-saturated on \( S' \) since \( S' \) is \( \tau \)-saturated (on \( S' \)); thus, \( T' \) is then a generating set of \( \mathcal{L}' \). Hence, in one saturation step, we can compute a generating set of \( \mathcal{L}' \). Note that the component \( n + 1 \) is bounded by construction. Then, the set \( T := \{\rho_\tau(u') : u' \in T' \} \) is a generating set of \( \mathcal{L} \).
5 Speeding-up the Completion Procedure

Finally, before presenting computational experience, we talk about ways in which the key algorithm, Algorithm 2, can be improved. This leads us to the critical pair criteria.

Algorithm 2 has to test for a reduction path between \( x^{(u,v)} \) and \( y^{(u,v)} \) for all critical pairs \( C := \{(u,v) : u, v \in G\} \). In the case of lattice ideals, computational profiling shows that this is the most time consuming part of the computation. So, we wish to reduce the number of critical pairs that we test, and avoid this expensive test as often as possible. We present three criteria that can reduce the number of critical pairs that need to be tested.

Criteria 1 and 3 (see [Buchberger, 1979, 1985]; [Gebauer and Möller, 1988]) are translated from the theory of Gröbner bases into a geometric context. Criterion 2 is specific to lattice ideals and corresponds to using the homogeneous Buchberger algorithm ([Caboara et al., 2003]; [Traverso, 1997]), but we give a slightly more general result. Note that all three criteria can be applied simultaneously.

**Criterion 1: The Disjoint-Positive-Support criterion**

For a pair \( u, v \in G \), the Disjoint-Positive-Support criterion is a simple and quick test for a \( \succ \)-reduction path from \( x^{(u,v)} \) to \( y^{(u,v)} \). So, using this quick test for a \( \succ \)-reduction path, we can sometimes avoid the more expensive test.

Given \( u, v \in G \), if \( \text{supp}(u^+) \cap \text{supp}(v^+) = \emptyset \), then there exists a simple \( \succ \)-reduction path from \( x^{(u,v)} \) to \( y^{(u,v)} \) using \( u \) and \( v \) in reverse order (see Figure 5).

![Fig. 5. Criterion 1.](image)

**Criterion 2: The Cancellation criterion**

Let \( G \) be a generating set of \( \mathcal{L} \). If \( \text{supp}(x^{(u,v)}) \cap \text{supp}(y^{(u,v)}) \neq \emptyset \) for some \( u, v \in G \) (or equivalently, \( \text{supp}(u^-) \cap \text{supp}(v^-) \neq \emptyset \)), then we do not need to check for a \( \succ \)-reduction path from \( x^{(u,v)} \) to \( y^{(u,v)} \) (we can remove the pair \( (u, v) \) from \( C \)).

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To show that this criteria holds, we need the concept of a grading. Let \( w \in \mathbb{Q}^n \). If \( wx = wy \) for all \( x, y \in \mathcal{F}_{\mathcal{L},b} \) for all \( b \in \mathbb{Z}^n \), then we call \( w \) a grading of \( \mathcal{L} \), and we define \( \text{deg}_w(\mathcal{F}_{\mathcal{L},b}) := wb \) called the \( w \)-degree of \( \mathcal{F}_{\mathcal{L},b} \). Importantly, if \( \mathcal{L} \cap \mathbb{N}^n = \{0\} \) (which we assume), then it follows from Farkas’ lemma that there exists a strictly positive grading \( w \in \mathbb{Q}_+^n \) of \( \mathcal{L} \).

First, we prove an analogous result to Corollary 8.

**Lemma 30** A set \( G \subseteq L_\succ \) is a \( \prec \)-Gröbner basis of \( \mathcal{L} \) if and only if \( G \) is a generating set of \( \mathcal{L} \) and if for every \( \succ \)-critical path \( (x, z, y) \) in \( G(\mathcal{F}_{\mathcal{L},b}, G) \) for all \( b \in \mathbb{Z}^n \) where \( \text{supp}(x) \cap \text{supp}(y) = \emptyset \), there exists a \( \succ \)-reduction path between \( x \) and \( y \) in \( G(\mathcal{F}_{\mathcal{L},b}, G) \).

**Proof.** The forwards implication follows from Corollary 8. For the backwards implication, we need to show that for every \( \succ \)-critical path \( (x, y, z) \) where \( \text{supp}(x) \cap \text{supp}(y) \neq \emptyset \), there exists a \( \succ \)-reduction path from \( x \) to \( y \) in \( G(\mathcal{F}_{\mathcal{L},b}, G) \), in which case, there is a \( \succ \)-reduction path for all \( \succ \)-critical paths, and so by Corollary 8, \( G \) is a Gröbner basis. Assume on the contrary that this is not the case. Let \( w \) be a strictly positive grading of \( \mathcal{L} \). Among all such \( \succ \)-critical paths \( (x, z, y) \) where \( \text{supp}(x) \cap \text{supp}(y) \neq \emptyset \) and there is no \( \succ \)-reduction path from \( x \) to \( y \), choose a \( \succ \)-critical path \( (x, z, y) \) such that \( \text{deg}_w(\mathcal{F}_{\mathcal{L},x}) \) is minimal. Let \( \gamma := x \land y, \bar{x} := x - \gamma, \) and \( \bar{y} := y - \gamma \). Note that \( \gamma \neq 0 \) since \( \text{supp}(x) \cap \text{supp}(y) \neq \emptyset \). Because \( G \) is a generating set of \( \mathcal{L} \), there must exist a path from \( \bar{x} \) to \( \bar{y} \) in \( G(\mathcal{F}_{\mathcal{L},\bar{x}}, G) \). Also, since \( w \) is strictly positive, \( \text{deg}_w(\mathcal{F}_{\mathcal{L},\bar{x}}) < \text{deg}_w(\mathcal{F}_{\mathcal{L},x}) \); therefore, by the minimality assumption on \( \text{deg}_w(\mathcal{F}_{\mathcal{L},x}) \), we can now conclude that for all \( \succ \)-critical paths in \( G(\mathcal{F}_{\mathcal{L},\bar{x}}, G) \) there exists a \( \succ \)-reduction path. Consequently, by Lemma 6, there exists a \( \succ \)-reduction path between \( \bar{x} \) and \( \bar{y} \) in \( G(\mathcal{F}_{\mathcal{L},\bar{x}}, G) \). This \( \succ \)-reduction path, however, can be translated by \( \gamma \) to a \( \succ \)-reduction path from \( x \) to \( y \) in \( G(\mathcal{F}_{\mathcal{L},x}, G) \) (see Figure 6a). But this contradicts our assumption that there is no such path between \( x \) and \( y \). □

![Fig. 6. Criterion 2.](image)

Now, for all \( u, v \in G \), if \( \text{supp}(x^{(u,v)}) \cap \text{supp}(y^{(u,v)}) \neq \emptyset \), then \( \text{supp}(x) \cap \text{supp}(y) \neq \emptyset \) for all \( \succ \)-critical paths \( (x, z, y) \) for \( (u, v) \). Using this observation, we arrive at an analogous result to Corollary 10.
Corollary 31 Let $G \subseteq \mathcal{L}$ be a generating set of $\mathcal{L}$; then, $G$ is a $\prec$-Gröbner basis of $\mathcal{L}$ if and only if for each pair $u, v \in G$ where $\text{supp}(x^{(u,v)}) \cap \text{supp}(y^{(u,v)}) = \emptyset$, there exists a $\succ$-reduction path between $x^{(u,v)}$ and $y^{(u,v)}$ in $\mathcal{G}(\mathcal{F}_{\mathcal{L},z^{(u,v)}}, G)$.

We can extend these results further leading to a more powerful elimination criterion. Let $u, v \in G$. We say the pair $(u, v)$ satisfies Criterion 2 if there exists $x', y' \in \mathcal{F}_{\mathcal{L},z^{(u,v)}}$ such that there exists a $\succ$-decreasing path in $\mathcal{G}(\mathcal{F}_{\mathcal{L},z^{(u,v)}}, G)$ from $x^{(u,v)}$ to $x'$ and from $y^{(u,v)}$ to $y'$, and $\text{supp}(x') \cap \text{supp}(y') \neq \emptyset$. Importantly, if $(u, v)$ satisfies Criterion 2, then we do not have to test for a $\succ$-reduction path from $x^{(u,v)}$ to $y^{(u,v)}$. Thus, we arrive at an extension of Corollary 31. Observe that the previous results are just a special case where $x' = x^{(u,v)}$ and $y' = y^{(u,v)}$.

Lemma 32 Let $G \subseteq \mathcal{L}$ be a generating set of $\mathcal{L}$; then, $G$ is a $\prec$-Gröbner basis of $\mathcal{L}$ if and only if for each pair $u, v \in G$ where $(u, v)$ does not satisfy Criterion 2, there exists a $\succ$-reduction path between $x^{(u,v)}$ and $y^{(u,v)}$ in $\mathcal{G}(\mathcal{F}_{\mathcal{L},z^{(u,v)}}, G)$.

If there is a $\succ$-reduction path from $x'$ to $y'$, then there exists a $\succ$-reduction path from $x^{(u,v)}$ to $y^{(u,v)}$. Since $\text{supp}(x') \cap \text{supp}(y') \neq \emptyset$, then $\gamma = x' \land y' \neq 0$. Let $\bar{x} = x' - \gamma$ and $\bar{y} = y' - \gamma$. So, if there exists a $\succ$-reduction path from $\bar{x}$ to $\bar{y}$, then there must exist a $\succ$-reduction path from $x'$ to $y'$, and therefore also, there must exist a $\succ$-reduction path from $x^{(u,v)}$ to $y^{(u,v)}$ (see Figure 6b). Again, we let $w$ be a strictly positive grading of $\mathcal{L}$, and so similarly to above, $\deg_w(\mathcal{F}_{\mathcal{L},\bar{x}}) < \deg_w(\mathcal{F}_{\mathcal{L},z^{(u,v)}})$. So, the proof of Lemma 32 is essentially as before.

For a pair $u, v \in G$, Criterion 2 can be checked not only before we search for a $\succ$-reduction path from $x^{(u,v)}$ to $y^{(u,v)}$ but also while searching for a $\succ$-reduction path. When searching for a $\succ$-reduction path, we construct a $\succ$-decreasing path from $x^{(u,v)}$ to $\mathcal{N}\mathcal{F}(x^{(u,v)}, G)$ and a $\succ$-decreasing path from $y^{(u,v)}$ to $\mathcal{N}\mathcal{F}(y^{(u,v)}, G)$. Therefore, we can take any point $x'$ on the $\succ$-decreasing path from $x^{(u,v)}$ to $\mathcal{N}\mathcal{F}(x^{(u,v)}, G)$ and any point $y'$ on the decreasing path from $y^{(u,v)}$ to $\mathcal{N}\mathcal{F}(y^{(u,v)}, G)$ and check Criterion 2, that is, we check if $\text{supp}(x') \cap \text{supp}(y') \neq \emptyset$. If this is true, then we can eliminate $(u, v)$.

We wish to point out explicitly here that Criterion 2 can be applied without choosing the vector pairs $u, v \in G$ in a particular order during Algorithm 2. In fact, when running Algorithm 2, if we apply Criterion 2 to eliminate a pair $u, v \in G$, it does not necessarily mean that there is a $\succ$-reduction path from $x^{(u,v)}$ to $y^{(u,v)}$ in $\mathcal{G}(\mathcal{F}_{\mathcal{L},z^{(u,v)}}, G)$ at that particular point in time in the algorithm but instead that a $\succ$-reduction path will exist when the algorithm terminates. This approach is in contrast to existing approaches that use the homogeneous Buchberger algorithm to compute a Gröbner basis whereby vector pairs $u, v \in G$ must be chosen in an order compatible with increasing $\deg_w(\mathcal{F}_{\mathcal{L},z^{(u,v)}})$ for some strictly positive grading $w$. This can be computationally costly.
we use these existing approaches, if a pair \((u, v)\) is eliminated by Criterion 2, then it is necessarily the case that there already exists a \(\succ\)-reduction path from \(x^{(u,v)}\) to \(y^{(u,v)}\).

Since we need a generating set of \(L\) for Criterion 2, we cannot apply Criterion 2 during the Saturation algorithm (Algorithm 3), and also, we cannot apply Criterion 2 when \(L \cap \mathbb{N}^n \neq \{0\}\). However, we can apply a less strict version.

Given \(u, v \in G\) and \(\tau \subseteq \{1, \ldots, n\}\), we say that \((u, v)\) satisfies Criterion 2 with respect to \(\tau\), if there exists \(x', y' \in F_{L,z(u,v)}\) such that there exists a decreasing path in \(G(F_{L,z(u,v), G})\) from \(x^{(u,v)}\) to \(x'\) and from \(y^{(u,v)}\) to \(y'\), and \(\text{supp}(x') \cap \text{supp}(y') \cap \tau \neq \emptyset\).

Let \(S, T \subseteq L\) where \(S\) spans \(L\) and \(T\) is a \(\sigma\)-saturated set on \(S\) for some \(\sigma \subseteq \{1, \ldots, n\}\). During an iteration of the Saturation algorithm 3, we compute a \((\sigma \cup \{i\})\)-saturated set of \(S\), by computing \(CP(\prec \bar{e}_i, T)\). While computing \(CP(\prec \bar{e}_i, T)\) here, we may apply Criterion 2 with respect to \(\sigma\). For an algebraic proof of this, see Bigatti et al. (1999).

Also, we can use Criterion 2 when \(L \cap \mathbb{N}^n \neq \{0\}\) if we have a generating set of \(L\). Let \(\tau\) be the set of bounded components. Then, we may apply Criterion 2 with respect to \(\tau\). Moreover, if we do not have a generating set and we are running the Saturation algorithm when \(L \cap \mathbb{N}^n \neq \{0\}\), we may apply Criterion 2 with respect to \(\sigma \cap \tau\).

**Criterion 3: The \((u, v, w)\) criterion**

Before presenting the \((u, v, w)\) criterion, we need a another result, Lemma 33, that is a less strict version of Lemma 6. First, we need to define a new type of path. A path \((x^0, \ldots, x^k)\) is \(z\)-bounded (with respect to \(\prec\)) if \(x^i \prec z\) for all \(i = 0, \ldots, k\). So, \(z\) is a strict upper bound on the path. Note that for a \(\succ\)-critical path \((x, z, y)\), a \(\succ\)-reduction path from \(x\) to \(y\) is a \(z\)-bounded path.

**Lemma 33** Let \(b \in \mathbb{Z}^n\), \(x, y \in F_{L,b}\), and let \(G \subseteq L_{\succ}\) where there is a path between \(x\) and \(y\) in \(G(F_{L,b}, G)\). If there exists a \(z'\)-bounded path between \(x'\) and \(y'\) for every \(\succ\)-critical path \((x', z', y')\) in \(G(F_{L,b}, G)\), then there exists a \(\succ\)-reduction path between \(x\) and \(y\) in \(G(F_{L,b}, G)\).

If we now re-examine the proof of Lemma 33, we find that we only need \(z'\)-bounded paths between \(x'\) and \(y'\) for every \(\succ\)-critical path \((x', z', y')\) in \(G(F_{L,b}, G)\), and that, a \(\succ\)-reduction path from \(x'\) and \(y'\) is more than we need. The proof proceeds in the same way as Lemma 6.

From Lemma 33, we arrive at an analogous result to Corollary 10.

**Corollary 34** A set \(G \subseteq L_{\succ}\) is a \(\prec\)-Gröbner basis of \(L\) if and only if \(G\) is a
generating set of $\mathcal{L}$ and if for each pair $u,v \in G$, there exists a $z^{(u,v)}$-bounded path between $x^{(u,v)}$ and $y^{(u,v)}$ in $\mathcal{G}(\mathcal{F}_{\mathcal{L},z^{(u,v)}}, G)$.

Corollary 34 does not fundamentally change Algorithm 2 since to test for a $z^{(u,v)}$-bounded path from $x^{(u,v)}$ to $y^{(u,v)}$, we still test for a $\succ$-reduction path from $x^{(u,v)}$ to $y^{(u,v)}$ which is a $z^{(u,v)}$-bounded path. However, we can use Corollary 34 to reduce the number of critical pairs $u,v \in G$ for which we need to compute a $\succ$-reduction path.

Now, we are able to present the $(u,v,w)$ criterion. Let $u,v,w \in G$ where $z^{(u,v)} \geq w^{+}$ (or equivalently, $z^{(u,v)} \geq z^{(u,w)}$ and $z^{(u,v)} \geq z^{(w,v)})$, and let $\bar{z} = z^{(u,v)} - w$. Then, a $z^{(u,v)}$-bounded path from $x^{(u,w)}$ to $\bar{z}$, and a $z^{(u,v)}$-bounded path from $\bar{z}$ to $y^{(w,v)}$ combine to form a $z^{(u,v)}$-bounded path from $x^{(u,v)}$ to $y^{(u,v)}$. Moreover, $(x^{(u,v)}, z^{(u,v)}, \bar{z})$ is a $\succ$-critical path for $(u,w)$ and $(\bar{z}, z^{(u,v)}, y^{(u,v)})$ is a $\succ$-critical path for $(w,v)$ (see Figure 7). Therefore, a $z^{(u,v)}$-bounded path from $x^{(u,v)}$ to $y^{(u,v)}$ and a $z^{(w,v)}$-bounded path from $x^{(w,v)}$ to $y^{(w,v)}$ combine to form a $z^{(u,v)}$-bounded path from $x^{(u,v)}$ to $y^{(u,v)}$, and so, we can remove $(u,v)$ from $C$.

![Fig. 7. Criterion 3.](image)

Note that in Figure 7a, a $\succ$-reduction path from $x^{(u,v)}$ to $\bar{z}$ and a $\succ$-reduction path from $\bar{z}$ to $y^{(u,v)}$ do combine to give a $\succ$-reduction path from $x^{(u,v)}$ to $y^{(u,v)}$; however, this is not the case in Figure 7b which is why we need the concept of bounded paths.

We can extend the previous result. Let $u,v \in G$, and $w^{1}, \ldots, w^{k} \in G$ where $z^{(u,v)} \geq (w^{i})^{+}$ for all $i = 1, \ldots, k$. If there exists a bounded path for the critical pairs $(u,w^{1})$, $(w^{k}, v)$, and $(w^{i}, w^{i+1})$ for all $i = 1, \ldots, k - 1$, then there is a bounded path for $(u,v)$. However, note that this can also be implied by a bounded path for $(u,w^{i})$ and $(w^{i}, v)$ for any $i = 1, \ldots, k$.

Unfortunately, we cannot just remove from $C$ all pairs $u,v \in G$ where there exists a $w \in G$ such that $z^{(u,v)} \geq w^{+}$. It may happen that in addition to $z^{(u,v)} \geq w^{+}$, we also have $z^{(u,w)} \geq w^{+}$, in which case, we would eliminate both the pairs $(u,v)$ and $(u,w)$ leaving only $(v,w)$ which is not sufficient. Moreover, at the same time, we may also have $z^{(w,v)} \geq w^{+}$, and we would eliminate all three pairs. To avoid these circular relationships, Gebauer and Möller [Gebauer and Möller (1988)] devised the following critical pair elimination criteria which we use in practice in 4ti2 v1.2.
Let $G = \{u^1, u^2, \ldots, u^{|G|}\}$, and let $u^i, u^j \in G$ where $i < j$. We define that the pair $(u^i, u^j)$ satisfies Criterion 3 if there exists $u^k \in G$ such that one of the following conditions hold:

1. $z(u^i, u^j) \preceq z(u^i, u^k)$ and $z(u^i, u^j) \preceq z(u^j, u^k)$;
2. $z(u^i, u^j) = z(u^i, u^k)$, $z(u^i, u^j) \preceq z(u^j, u^k)$, and $k < j$;
3. $z(u^i, u^j) \preceq z(u^i, u^k)$, $z(u^i, u^j) = z(u^j, u^k)$, and $k < i$; or
4. $z(u^i, u^j) = z(u^i, u^k) = z(u^j, u^k)$, and $k < i < j$.

So, if a pair $(u^i, u^j)$ satisfies Criterion 3, we can eliminate it. For example, if $G = \{u^1, u^2, u^3\}$ where $z(u^1, u^2) = z(u^1, u^3) \preceq z(u^2, u^3)$, then applying Criterion 3 to all three pairs $(u^1, u^2)$, $(u^1, u^3)$, and $(u^2, u^3)$ would eliminate only $(u^1, u^3)$.

After eliminating all pairs that satisfy Criterion 3, we are left with a set of critical pairs $C' \subseteq C = \{(u, v) : u, v \in G\}$ such that if there exists a $z(u', v')$-bounded path from $x(u', v')$ to $y(u', v')$ for all $(u', v') \in C'$, then there exists a $z(u, v)$-bounded path from $x(u, v)$ to $y(u, v)$ for all $(u, v) \in C$. However, this set of pairs may not be minimal. In Caboara et al. (2003), Caboara, Kreuzer, and Robbiano describe an algebraic algorithm for computing a minimal set of critical pairs with computational results. Their computational results show that the Gebauer and Möller criteria give a good approximation to the minimal set of critical pairs. We found that the Gebauer and Möller criteria were sufficient for our computations.

6 The $4 \times 4 \times 4$-challenge

The challenge posed by Seth Sullivant amounts to checking whether a given set of 145,512 integer vectors in $\mathbb{Z}^{64}$ is a Markov basis for the statistical model of $4 \times 4 \times 4$ contingency tables with 2-marginals. If $x = (x_{ijk})_{i,j,k=1,\ldots,4}$ denotes a $4 \times 4 \times 4$ array of integer numbers, the defining equations for the sampling moves are

$$
\begin{align*}
\sum_{i=1}^{4} x_{ijk} &= 0 \quad \text{for } j, k = 1, \ldots, 4, \\
\sum_{j=1}^{4} x_{ijk} &= 0 \quad \text{for } i, k = 1, \ldots, 4, \\
\sum_{k=1}^{4} x_{ijk} &= 0 \quad \text{for } i, j = 1, \ldots, 4.
\end{align*}
$$

This leads to a problem matrix $A_{444} \in \mathbb{Z}^{48 \times 64}$ of rank 37 and $\mathcal{L}_{A_{444}} = \{z : A_{444}z = 0, z \in \mathbb{Z}^{64}\}$. Note that the 145,512 vectors in the conjectured Markov
basis fall into 14 equivalence classes under the natural underlying symmetry group $S_4 \times S_4 \times S_4 \times S_3$.

In Aoki and Takemura (2003), Aoki and Takemura have computed these 14 symmetry classes via an analysis of sign patterns and under exploitation of symmetry. They claimed that the corresponding 145,512 vectors form the unique inclusion-minimal Markov basis of $A_{444}$.

Using our Project-and-Lift algorithm, however, we have computed the Markov basis from the problem matrix $A_{444}$ within less than 7 days on a Sun Fire V890 Ultra Sparc IV processor with 1200 MHz. Note that the symmetry of the problem was not used by the algorithm. This leaves room for a further significant speed-up. Our computation produced 148,968 vectors; that is, there are additionally 3,456 Markov basis elements. These vectors form a single equivalence class under $S_4 \times S_4 \times S_4 \times S_3$ of a norm 28 vector $z_{15}$ (or equivalently, of a degree 14 binomial).

A quick check via a Hilbert basis computation with 4ti2 shows that these Markov basis elements are indispensable, since $\{z \in \mathbb{Z}_4^4 : A_{444}z = A_{444}z_{15}^+\} = \{z_{15}^+, z_{15}^-\}$. As also all the other 145,512 Markov basis elements were indispensable, the Markov basis of $4 \times 4 \times 4$ contingency tables with 2-marginals is indeed unique. At least this claim can be saved from Aoki and Takemura (2003), although we have finally given a computational proof. Here is the list of the 15 orbit representatives, written as binomials:

1. $x_{111}x_{144}x_{414}x_{441} - x_{114}x_{141}x_{411}x_{441}$
2. $x_{111}x_{144}x_{334}x_{341}x_{414}x_{431} - x_{114}x_{141}x_{331}x_{344}x_{411}x_{434}$
3. $x_{111}x_{122}x_{134}x_{143}x_{414}x_{423}x_{432}x_{441} - x_{114}x_{123}x_{132}x_{141}x_{411}x_{422}x_{434}x_{443}$
4. $x_{111}x_{144}x_{324}x_{333}x_{341}x_{414}x_{423}x_{431} - x_{114}x_{141}x_{323}x_{331}x_{344}x_{411}x_{424}x_{433}$
5. $x_{111}x_{144}x_{234}x_{233}x_{323}x_{341}x_{414}x_{421}x_{433} - x_{114}x_{141}x_{233}x_{244}x_{321}x_{343}x_{411}x_{423}x_{434}$
6. $x_{111}x_{122}x_{133}x_{144}x_{324}x_{332}x_{341}x_{414}x_{423}x_{431} - x_{114}x_{123}x_{132}x_{141}x_{322}x_{331}x_{344}x_{411}x_{424}x_{433}$
7. $x_{111}x_{144}x_{222}x_{234}x_{243}x_{323}x_{341}x_{414}x_{421}x_{432} - x_{114}x_{141}x_{223}x_{232}x_{244}x_{321}x_{343}x_{411}x_{422}x_{434}$
8. $x_{111}x_{144}x_{222}x_{233}x_{324}x_{332}x_{341}x_{414}x_{423}x_{431} - x_{114}x_{141}x_{223}x_{232}x_{322}x_{331}x_{344}x_{411}x_{424}x_{433}$
9. $x_{111}x_{112}x_{133}x_{144}x_{223}x_{232}x_{241}x_{314}x_{322}x_{413}x_{421} - x_{113}x_{114}x_{132}x_{141}x_{221}x_{222}x_{233}x_{244}x_{312}x_{324}x_{411}x_{423}$
10. $x_{111}x_{112}x_{133}x_{144}x_{224}x_{232}x_{243}x_{313}x_{322}x_{414}x_{421} - x_{113}x_{114}x_{132}x_{141}x_{222}x_{233}x_{244}x_{312}x_{321}x_{343}x_{411}x_{424}$
We now compare the implementation of our new algorithm in 4ti2 v.1.2 (Hemmecke et al., 2005) with the implementation of the Saturation algorithm (Hosten and Sturmfels, 1995) and the Lift-and-Project algorithm (Bigatti et al., 1999) in Singular v3.0.0 (Greuel et al., 2005) (algorithmic options ‘hs’ and ‘blr’) and in CoCoA 4.2 (CoCoATeam, 2005) (functions ‘Toric’ and ‘Toric.Sequential’).

<table>
<thead>
<tr>
<th>Name</th>
<th>Software</th>
<th>Function</th>
<th>Algorithm</th>
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</thead>
<tbody>
<tr>
<td>Sing-blr</td>
<td>Singular v3.0.0</td>
<td>toric, option “blr”</td>
<td>Lift-and-Project</td>
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<td>Sing-hs</td>
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<td>Saturation</td>
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<td>CoCoA-t</td>
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<tr>
<td>P&amp;L</td>
<td>4ti2 v1.2</td>
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<td>Project-and-Lift</td>
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<tr>
<td>4ti2-gra</td>
<td>4ti2 v1.2</td>
<td>graver</td>
<td>Graver basis (Hemmecke, 2004)</td>
</tr>
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</table>

The first 4 problems correspond to three-way tables with 2-marginals, whereas $K_4$ and $K_5$ correspond to the binary models on the complete graphs $K_4$ and $K_5$, respectively. The problem s-magic333 is taken from an application in Ahmed et al. (2003) and computes the relations among the 66 elements of the Hilbert basis elements of $3 \times 3 \times 3$ semi-magic hypercubes. The example grin is taken from Hosten and Sturmfels (1995), while the examples hppi10-hppi14 correspond to the computation of homogeneous primitive partition identities, see for example Chapters 6 and 7 in Sturmfels (1996). Finally, the examples cuww1-cuww5 arise from knapsack problems presented in Cornuejols et al. (1997).
The computations were done on a Sun Fire V890 Ultra Sparc IV processor with 1200 MHz. Computation times are given in seconds, rounded up. See Figure 8. The running times give a clear ranking of the implementations: from left to write the speed increases and in all problems, the presented Project-and-Lift algorithm wins significantly.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Vars.</th>
<th>GB size</th>
<th>Sing-blr</th>
<th>Sing-hs</th>
<th>CoCoA-t</th>
<th>CoCoA-ts</th>
<th>P&amp;L</th>
<th>4ti2-gra</th>
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Fig. 8. Comparison of computing times.

The advantage of our Project-and-Lift algorithm is that it performs computations in projected subspaces of $L$. Thus, we obtain comparably small intermediate sets during the computation. Only the final iteration that deals with all variables reaches the true output size. In contrast to this, the Saturation algorithm usually comes close to the true output size already after the first saturation and then continues computing with as many vectors. See Figure 9, for a comparison of intermediate set sizes in each iteration for computing $3 \times 4 \times 4$ tables.
Moreover, the Project-and-Lift algorithm, performs Gröbner basis computations using a generating set, and thus can take full advantage of Criterion 2 which, as computational experience shows, is extremely effective. In fact, we only applied Criterion 2 and 1 (applied in that order) for the Project-and-Lift algorithm since Criterion 3 only slowed down the algorithm. However, for the Saturation algorithm where we cannot apply Criterion 2 fully, Criterion 3 was very effective. In this case, we applied Criterion 1, then 3, and then 2, in that order.

Note that in the knapsack problems cuww1-cuww5 the initial set \( \sigma \) chosen by the Project-and-Lift Algorithm 4 is empty. Thus, Algorithm 4 simplifies to the Saturation Algorithm 3. In fact, only a single saturation is necessary for each problem.

To us, the following observations were surprising.

- While Singular did not accept the inhomogeneous problems cuww1-cuww5 as input, CoCoA either could not solve them or produced incorrect answers.
- It is not clear why the CoCoA function Toric works well on problems hppi10-hppi14, but runs badly on the table problems 334, 335, 344.
- Problems hppi10-hppi14 are in fact Graver basis computations (see for example Chapter 14 in Sturmfels (1996)), for which 4ti2 has the state-of-the-art algorithm and implementation. Initially, it was a surprise to us that our Project-and-Lift Algorithm 4 comes so close to the speed of the state-of-the-art algorithm that computes Graver bases directly (Hemmecke, 2004). However, it turns out that our Project-and-Lift Algorithm 4 is an extension of the Project-and-Lift algorithm presented in Hemmecke (2004) to lattice ideal computations.

References

Aoki, S., Takemura, A., 2003. The list of indispensable moves of the unique minimal Markov basis for 3x4xk and 4x4x4 contingency tables with fixed two-dimensional marginals. METR Technical Report, 03–38.


