Polyhedral Combinatorics of Coxeter Groups

Dissertation’s Defense

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A few motivations

Open Problem (Dyer (1993))

Is there, for each infinite Coxeter group, a complete ortholattice that contains the weak order?

Open Problem (Jonsson (2003))

Is there a polytopal realization of the multi-associahedron?
A few motivations

Open Problem (Dyer (1993))

*Is there, for each infinite Coxeter group, a complete ortholattice that contains the weak order?*

⇒ Introduction of the limit roots of an infinite Coxeter group.

Open Problem (Jonsson (2003))

*Is there a polytopal realization of the multi-associahedron?*

⇒ Introduction of the multi-cluster complex of a finite Coxeter group.
PART I: Ortholattice for infinite Coxeter groups?
Fixed-gear drivetrain
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Algebra

Geometry
PART I: Ortholattice for infinite Coxeter groups?  
Fixed-gear drivetrain

Algebra  Combinatorics  Geometry
The weak order on Coxeter groups

\((W, S) – \text{infinite Coxeter group } (\langle s, t \in S | e = s^2 = (st)^{m_s, t} \rangle)\)

**Definition (Weak order)**

Let \(u, v \in W\). Then \(u \leq v \iff u\) is a prefix of \(v\).

**Example**

\(tus\) and \(tustut\).
The weak order on Coxeter groups

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**Example**

\(tus\) and \(tustut\).

\((W, \leq)\) is a meet-semilattice. There is no maximal element.

**Question**

*How to tell when the join of two elements exists?*
Asymptotical behaviour of roots

Study the directions of the roots.

\[ \beta = \rho_1 \quad \hat{\rho}_2 \quad \ldots \quad \hat{\rho}'_2 \quad \alpha = \rho'_1 \]

\[ \hat{Q} \]

The infinite dihedral group \( l_2(\infty) \).

The set $E(\Phi)$ of accumulation points of normalized roots $\hat{\Phi}$ is contained in the isotropic cone of $(V, B)$. 

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A first step using biconvexity

**Theorem (L. 2012)**

Let $(W, S)$ be a Coxeter group of rank $n \leq 3$, there is a complete ortholattice containing the weak order.

- The join of two elements can be computed using convex hulls.

\[ \rightsquigarrow \text{The join of two elements can be computed using convex hulls.} \]
A first step using biconvexity

**Theorem (L. 2012)**

Let $(W, S)$ be a Coxeter group of rank $n \leq 3$, there is a complete ortholattice containing the weak order.

$\leadsto$ The join of two elements can be computed using convex hulls.

**Fact (L. 2012)**

For Coxeter groups of rank $n \geq 4$ the join can not be computed using convex hulls.

The notion of convexity is too restrictive.

$\leadsto$ biclosedness seems to be the right geometry to look at.
PART II: Subword Complexes in Discrete Geometry

Algebra

Combinatorics

Geometry
Multi-cluster complexes

triangulations

Finite Coxeter Groups (Algebraic)
Multi-cluster complexes

Finite Coxeter Groups (Algebraic)

triangulations

Integer $k > 1$
(Geometric)
Multi-cluster complexes

Finite Coxeter Groups (Algebraic)

Integer $k > 1$ (Geometric)
Multi-cluster complexes

Finite Coxeter Groups (Algebraic)

Integer $k > 1$ (Geometric)
Fix a convex $m$-gon.

**Multi-triangulation**: Maximal set of diagonals not containing $k + 1$ pairwise crossing diagonals.
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$\Delta_{m,k}$: the simplicial complex with

$$\text{faces } \leftrightarrow \text{ sets of (relevant) diagonals not containing } k + 1 \text{ pairwise crossing diagonals}$$

Conjecture (Jonsson, 2003) $\Delta_{m,k}$ is isomorphic to the boundary complex of a simplicial polytope.
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**Conjecture (Jonsson, 2003)**

$\Delta_{m,k}$ is isomorphic to the boundary complex of a simplicial polytope.
Let $m = 6$ and $k = 2$

When $m = 2k + 2$, $\Delta_{m,k}$ is a $k$-simplex.
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Subword complexes

\((W, S)\) finite Coxeter system of rank \(n\)

**Definition (Knutson-Miller, 2004)**

Let \(Q = (q_1, \ldots, q_r)\) be a word in \(S\) and \(\pi \in W\).

**subword complex** \(\Delta(Q, \pi)\) := facets \(\leftrightarrow\) complements (in \(Q\)) of reduced expressions of \(\pi\)
(\(W, S\)) finite Coxeter system of rank \(n\)

**Definition (Knutson-Miller, 2004)**

Let \(Q = (q_1, \ldots, q_r)\) be a word in \(S\) and \(\pi \in W\).

A *subword complex* \(\Delta(Q, \pi)\) is defined as:

\[
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\]

**Theorem (Knutson-Miller, 2004)**

Subword complexes are topological spheres or balls.
Definition (Ceballos-L.-Stump, J. of Alg. Comb. 2013)

The multi-cluster complex $\Delta_k^c(W)$ is the subword complex $\Delta(c^k w_\circ(c), w_\circ)$ of type $W$. 
Definition (Ceballos-L.-Stump, *J. of Alg. Comb. 2013*)

The multi-cluster complex $\Delta^k_c(W)$ is the subword complex $\Delta(c^k w_\circ(c), w_\circ)$ of type $W$.

Theorem (CLS, 2013)

The subword complex $\Delta(cw_\circ(c), w_\circ)$ is isomorphic to the $c$-cluster complex of type $W$. 
Multi-cluster complexes of type $A$ and $B$

Theorem (Pilaud-Pocchiola 2012, Stump 2011)

The multi-cluster complex $\Delta^k_c(A_n) \cong$ simplicial complex of $k$-triangulations of a convex $m$-gon where $m = n + 2k + 1$. 

Corollary $\Delta_{sym}^m, k$ is a vertex-decomposable simplicial sphere.
Multi-cluster complexes of type $A$ and $B$

**Theorem (Pilaud-Pocchiola 2012, Stump 2011)**

The multi-cluster complex $\Delta^k_c(A_n) \cong \text{simplicial complex of }\text{k-triangulations of a convex }m\text{-gon}$

where $m = n + 2k + 1$.

**Theorem (CLS, 2013)**

The multi-cluster complex $\Delta^k_c(B_{m-k}) \cong \text{simplicial complex of centrally symmetric }k\text{-triangulations of a regular convex }2m\text{-gon}$

**Corollary**

$\Delta_{m,k}^{\text{sym}}$ is a vertex-decomposable simplicial sphere.
Question (Knutson-Miller, 2004)

*Charaterize all simplicial spheres that can be realized as a subword complex.*
Universality and polytopality of $\Delta^k_c(W)$

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Characterize all simplicial spheres that can be realized as a subword complex.

Theorem (CLS, 2013)

A simplicial sphere is realized as $\iff$ it is the link of a face of a multi-cluster subword complex $\Delta^k_c(W)$. 

Corollary

The following two statements are equivalent.

(i) Every spherical subword complex is polytopal.

(ii) Every multi-cluster complex is polytopal.
Universality and polytopality of $\Delta^k_c(W)$

Question (Knutson-Miller, 2004)

Characterize all simplicial spheres that can be realized as a subword complex.

Theorem (CLS, 2013)

A simplicial sphere is realized as it is the link of a face of a multi-cluster complex $\Delta^k_c(W)$ if and only if it is the link of a face of a multi-cluster complex $\Delta^k_c(W)$.

Corollary

The following two statements are equivalent.

(i) Every spherical subword complex is polytopal.

(ii) Every multi-cluster complex is polytopal.
Conjecture (⇒ Knutson-Miller’04, Jonsson’05, Soll-Welker’09)

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The multi-cluster complex is the boundary complex of a simplicial polytope.

- True for $k = 1$: Chapoton-Fomin-Zelevinsky, Hohlweg-Lange-Thomas, Pilaud-Stump, Stella
- True for $l_2(m)$, $k \geq 1$: cyclic polytope, Ceballos-L.-Stump
- True for $A_3$, $k = 2$: Bokowski-Pilaud, Ceballos-L.
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Recent work of Bergeron-Ceballos-L.:

- New combinatorial construction giving $60000+$ realizations of $A_3$, $k = 2$
- Fan realizations of all subword complexes of type $A_3$
Thanks!

Merci! Thank you! Grazie! Danke! Gracias!
Recent developments on limit roots

Relation with the Tits cone [Dyer-Hohlweg-Ripoll, (arxiv:2013)]

Limit set of Kleinian groups [Hohlweg-Préaux-Ripoll, (arxiv:2013)]

Sphere packings and geometric invariants for Coxeter groups [Chen-L., (in preparation)]
Recent developments on multi-cluster complexes

〜 Generalized brick polytope, spanning trees [Pilaud-Stump, arxiv:2011-12]

〜 Denominator vectors of cluster algebras of finite types [Ceballos-Pilaud, (arxiv:2013)]

Geometric computation of the join