Plan of the Seminar

Crystalline cohomology is an attempt to get a “nice” $p$-adic Weil cohomology theory for schemes of positive characteristic $p$. This was developed as a cohomological theory to contain the information about the $p$-torsion of a scheme in characteristic $p$.

The aim of the seminar is to understand the definition and main properties of crystalline cohomology with special attention to comparison isomorphisms with the de Rham Cohomology. We will also follow the theory of crystals along and we end the seminar with a crystalline version of Gieseker’s conjecture. As examples we will consider the crystalline cohomology groups of K3 surfaces and Abelian varieties.

In the first part of the seminar we will mainly follow the notes [2] of Berthelot and Ogus on crystalline cohomology. It is available here. Very useful shorter survey papers on this subject are [8], a bit more up to date [6] and [11]. In this part we will do the construction of P.D. structures and define the crystalline cohomology. In the second part of our seminar we continue by studying some specific examples of computing crystalline cohomology groups, prove comparison isomorphisms with de Rham cohomology, mention in passing its relations with $p$-adic Hodge theory and conclude the seminar by more recent results, as in [3].

For any questions about the seminar, or help with finding the references needed, please contact us at: katsief@zedat.fu-berlin.de or at tanya.k.srivastava@fu-berlin.de.

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1 When referring to [2], we denote the chapter numbers by roman numbers to avoid confusion with the numbering of the propositions. The page numbers have then the form, eg. II.9
Schedule

Note: The reference for the talks 2-7 will be [2], unless otherwise stated.

17.10 Introduction to crystalline cohomology and Weil cohomology theories

(Tanya)

In this talk, we begin by giving a brief introduction to Weil cohomology theories and Witt vectors. We sketch the various cohomology theories for algebraic varieties in characteristic $p$, try to see why we need crystalline cohomology and to motivate the technical definition of a crystalline site following [2].

For more details about the examples mentioned in the talk, see [5],[4],[12],[13].

24.10 Review of connections and stratifications

In this talk we are going to review the notions of stratifications, connections and differential operators, following Chapter 2 of [2]. As we will see later, these notions are closely related with crystals. As a matter of fact, these notions, under their preliminary form, are relevant to crystalline cohomology only in characteristic zero. We will then explain how they have to be modified to yield a good formalism in characteristic $p$.

In more detail, start this talk by summarizing the discussion in pages II.1 and II.2., state definition 2.1 and propositions 2.2. and 2.4 without giving their proofs. As in page II.6, define the sheaf of germs of differential operators $\text{Diff}^n_{X/S}$ and state propositions 2.6 and 2.7, which give an explicit local description of differential operators and their composition.

The next part of the talk focuses on connections: give definition 2.8 and state proposition 2.9 . Following the discussion in page II.9, try to explain the motivation behind Grothendieck’s description of a connection and in the following explain definition 2.10 (summarizing the discussion in pages II.11-II.13) and state proposition 2.11 and remark 2.13. Finally sketch the proof of theorem 2.15.

31.10 Divided powers

In this talk we study the divided powers and introduce the notion of a divided power envelope, which will be crucial to what follows.

Start by explaining the definition 3.1 that introduces the notions of “divided powers on an ideal $I$”, and “divided power rings and morphisms".
Explain the examples in 3.2 without giving the proof of example 3. Next, give the definition 3.14 and state propositions 3.15 and 3.16. Define "compatible P.D. structures" as in definition 3.17 and sketch the construction of a "P.D. envelope" given in theorem 3.19. Discuss the remarks 1,3-8 in pages III.13-14 and state the result 3.21-3.23 without proof. Finally, give the definition of a "P.D. nilpotent ideal" as in definitions 3.24, 3.27.

07.11 Calculus with divided powers
Start this talk by presenting the discussion on page III.18, stating proposition 3.30 and covering the discussion in page III.9. Define the $n$-th order power neighborhood as on page III.20 and state proposition 3.32.

In the second part of the talk, we continue by modifying the objects presented in the second talk of the seminar, so that we can work in positive characteristic.

First, present definition 4.1, remark 4.2 and define (H)P.D. stratifications as in definitions 4.3, 4.3H. discuss definition 4.4 and example 4.5. After stating proposition 4.6 and corollary 4.7, state the important theorem 4.8. Then state the definition 4.10, theorem 4.12 and corollary 4.13.

14.11 The crystalline topos
This week we start using all the material that was introduced before to introduce the crystalline site and the crystalline topos, as in the fifth chapter of [2].

Explain in detail the definition of the crystalline site, following the discussion on pages V.1 to V.3, and explain proposition 5.1 and the examples 5.2. Continue with the remark 5.3 and explaining the material of page V.5 and subsequently give definition 5.6 and propositions 5.7 to 5.9. Also explain the natural ringed topoi morphism after remark 5.14 and the remark itself as well.

21.11 Crystalline cohomology
We continue with definition 5.15 and follow the discussion after it to define crystalline cohomology. State the results 5.16 and 5.17 and explain proposition 5.18 and the discussion that follows. Sketch the proofs of propositions 5.25, 5.26 about the description of localization in crystalline topos and prove corollary 5.27. Finally, explain the paragraph 5.28 and if time permits the paragraph 5.29.
28.11 Crystals

This will be the first of two talks about sheaves on the crystalline site; crystals. Define crystals as in definition 6.1, state proposition 6.2 and prove corollary 6.3. You can do exercise 6.5 and then prove theorem 6.6 and its corollaries 6.7, 6.8.

For what follows, first go back to chapter 2 and explain briefly the construction of Grothendieck’s linearization functor on pages II.17-II.19. Then, do the P.D. analogue of this, as on pages VI.10 and VI.11 (up to corollary 6.9).

05.12 Crystals II

Continuing from the last talk, recall corollary 6.9 and explain the discussion following it. Then state the results 6.10, 6.10.1 and 6.11 and prove theorem 6.12. Last, state theorems 6.13 and 6.14.

12.12 Cohomology of a crystal

Wouter

The goal of this talk is to establish a basis property of crystalline cohomology; its relation to de Rham cohomology.

We follow chapter VII in [2]. First prove theorems 7.1, 7.2 and corollaries 7.3, 7.4 and state remark 7.5. Recall if necessary theorem 6.14. In what follows we define a quasi-coherent crystal and state some properties of the cohomology of such crystals; state theorem 7.6 and proposition 7.7 about the adjunction formula. Then summarize the discussion on page VII.11 and state 7.8, 7.9 and remark 7.10, corollary 7.11 and 7.12.

09.01 Properties of crystalline cohomology

In this talk we follow [6] to study the main properties of crystalline cohomology. This paper gives a survey on results presented mainly in [1].

Start with section 2 of [6] and recall briefly if necessary, the notion of a divided power envelope, that is summarized there. Then state proposition 2.3, lemma 2.3.1 and the discussion after. Then explain paragraphs 2.4 and 2.5. Following the third section of [6] explain in detail the properties of crystalline cohomology up to 3.10.
16.01 De Rham-Witt complex

In this talk, we define the de Rham-Witt complex and see its relation with crystalline cohomology. We mainly follow [11] sections 3 and 4, which present a short introduction of the notions, but all the results and a more detailed survey of these can be found in [7].

In detail, explain the entire sections 3.1, 3.2 and 3.3 of [11]. finally, state theorem 4.1. Summarize the construction of the maps in question, with as much detail as possible, which can be found in [7], section II 1.1-1.4, without giving the proof that they are actually isomorphisms as in theorem 1.4.

23.01 Crystalline cohomology of K3 and abelian surfaces

The goal of this talk is to give two examples, where we can see crystalline cohomology in use.

We follow [7], sections II 7.1 and 7.2. Start by defining Newton and Hodge polygons, as in section 3.2 of [9]. (More details about this talk, will be added.)

30.01 Comparison Isomorphisms and $p$-adic Hodge Theory

(Tanya)

In this talk we follow [10] and prove the comparison isomorphism at the isocrystal (i.e., after tensoring the crystal with $\mathbb{Q}$) level between crystalline and de Rham cohomology. Then we state a few more comparison isomorphisms of crystalline cohomology, like with $l$-adic cohomology, arising in $p$-adic Hodge theory following [6].

06.02 The crystalline analogue of Gieseker’s conjecture-Gauss-Manin isocrystal

(Efstathia)

In this talk we talk about some results in [3], about crystals on simply connected varieties. This gives us an analogue of the conjecture of Gieseker, which we studied in the previous semester.

13.02

Open
References


