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Non-Markovian Fluid Stochastic Petri Nets

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Abstract

This paper combines two different extensions of the modeling power of stochastic Petri nets in one single framework. First, the firing times of transitions may be non-exponentially distributed. Second, certain places can hold a continuous (fluid), rather than discrete, number of tokens. Both extensions lead to a state space which is partially discrete and partially continuous and the model dynamics can be described by similar equations: systems of first-order hyperbolic partial differential and integral equations. These extensions are therefore closely related. We show that the method of supplementary variables provides a unified mathematical framework for the derivation of the state equations. Earlier restrictions on the coupling of fluid places and immediate transitions are removed by the introduction of new boundary conditions. It is now possible for the enabling of immediate transitions to depend on fluid levels and for their firing to transport a volume of fluid in zero time. Examples are included together with their numerical solutions.

Key words: stochastic Petri nets, fluid flow approximation, method of supplementary variables.

1 Introduction

Stochastic Petri nets (SPNs) [1] are well suited for the model-based performance and dependability evaluation of complex systems. However, some difficulties restrict their applicability: state-space explosion, the inadequacy of the exponential distribution in describing the system's behaviour, and the inability to model continuous quantities that may be present in the system.

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Regarding non-exponential distributions, several possibilities have been investigated in recent years. Besides the use of phase-type distributions [13], two approaches have been considered: Markov regenerative theory and the method of supplementary variables. Markov regenerative theory can be used under the restriction that transitions with non-exponentially distributed firing times (referred to as general transitions) may not be enabled concurrently. Stationary and transient analysis of this class of nets based on Markov regenerative theory are presented in [2, 7, 9, 5, 21]. In principle, the method of supplementary variables can be used for the analysis of nets without this restriction [14]. However, in order to bound algorithmic complexity, the same restriction (general transitions not concurrently enabled) was assumed for the stationary and transient analysis in [16]. A comparison of both approaches is given in [15].

The idea of the supplementary variable method is to include age variables into the state description which represent the time since enabling of the general transitions. Thus, a mixed discrete and continuous state-space is defined. The inclusion of the age variables makes the stochastic process Markovian and it is thus possible to derive a set of state equations which describe the model dynamics. The equations consist of a system of first order partial differential equations (PDE), each component of which represents a probability density function for one discrete state of the net and which depends on the age variables of the general transitions enabled in this state and, in the transient case, on time. The differential equations are combined with initial conditions (in the transient case), and boundary conditions which represent the dynamics at the instants of time at which a general transition fires. If in each state at most one age variable is included, efficient solution algorithms are known [14, 15, 16], otherwise one has to resort to discretizing the entire PDE, which leads to more expensive and more complex solution algorithms.

Regarding the problem of state space explosion, various methods can be used. Examples for approaches are: truncation of the state space, lumping, decomposition [8], structured model representation using tensor algebra [6], or fluid flow approximation. A fluid model contains continuous variables which change with deterministic flow rates [12, 19]. The objective of this approach is to approximate large buffer levels by continuous fluid levels and thus help to decrease the state-space complexity. Fluid models have been used and investigated in queueing theory [3, 20], and an application to buffer sizing in ATM systems is given in [23]. Recently, the concept of fluid models was put in the context of SPNs [22, 17], referred to as *fluid stochastic Petri nets* (FSPNs). An extension of fluid flow approximation is diffusion approximation where the flow rate fluctuates probabilistically [19, 18, 4]. A second advantage of the FSPN approach is that continuous quantities in the system being modelled may be represented directly by the fluid levels in the net.

In FSPNs, the fluid variables are represented by fluid places which can hold fluid rather than discrete tokens. FSPNs allow a very flexible definition of fluid models. The dynamics of an FSPN are described by a system of first order hyperbolic partial differential equations combined with initial (transient case) and boundary equations. The system of differential equations is similar to that arising in the method of supplementary variables. However, the boundary conditions are less complex. For certain structures, specialized numerical methods can be used for the solution of the equations. The general system of PDEs may be solved via a standard discretization approach. In [17] certain restrictions are imposed on the immediate transitions: the enabling of the immediate transitions may not depend on the fluid levels and the immediate transitions may not transport fluid in zero time. The former restriction was due to a resulting ambiguity in the definition of a state being tangible or vanishing and the latter would have necessitated the use of infinite rates in the stochastic process. These restrictions were somewhat limiting, as they for example prevented the modeling of state changes with deterministic firing times and the modeling of impulse rewards.

In this paper we consider non-Markovian FSPNs, an extension of GSPNs to allow general transitions and fluid places. This appears natural, since both extensions lead to a similar class of stochastic processes with a mixed discrete and continuous state space and with a similar underlying system of equations. The method of supplementary variables is taken and extended for fluid places: the fluid levels represent additional supplementary space variables. It is then possible to follow [16, 17] in order to derive a system of state equations. In doing so, the problem of the immediate transitions in relation to the fluid levels can also be solved: the transportation of fluid in zero time is described by appropriately chosen boundary conditions. In order to allow the enabling of immediate transitions to depend on fluid levels, the vanishing markings have to be taken into account.

In this paper we concentrate on the definition and derivation of the state equations. We always assume that numerical solution of the discretized differential equations can be performed. The development of efficient discretization schemes (preferably with an adaptive stepsize) is left open for future research.

The rest of the paper is organized as follows. Section 2 defines the class of non-Markovian FSPNs considered in this paper. An example which is used throughout the paper is also presented: a generally modulated D/D/1/K queueing system. Section 3 contains the main contribution of the paper: the derivation of state equations which describe the model dynamics. Special attention is given to the definition of boundary conditions which represent the behaviour of the immediate transitions in relation to the fluid places. In Section 4 the equations of Section 3 are specialized to Markovian FSPNs and are illustrated by an FSPN model of an M/D/1/K queueing system. Conclusions are given in Section 5.

2 Non-Markovian Fluid Stochastic Petri Nets

We define a non-Markovian FSPN following the common notation for stochastic Petri nets [8, 2] as in [17, 15] and recall briefly the main definitions. FSPNs consist of *places, transitions* and *arcs*. The set of places is divided into *fluid* (continuous) and *discrete* places ($\mathcal{P} = \mathcal{P}_c \cup \mathcal{P}_d$), the former containing a continuous amount of fluid and the latter a number of discrete *tokens*. The set of transitions $\mathcal{T} = \mathcal{T}_E \cup \mathcal{T}_I \cup \mathcal{T}_G$ is composed of the exponentially distributed, immediate and generally distributed transitions, respectively. Places and transitions are connected by *input*, *output* or *inhibitor* arcs $\mathcal{A} = \mathcal{A}_d \cup \mathcal{A}_c$. The subscript c refers to continuous places and the subscript d to the discrete ones. Note that we have extended the definition in [17] by allowing fluid places to be connected to immediate transitions.

Arcs connecting transitions and fluid places have the same graphical representation as arcs connecting transitions and discrete places but different semantics. The cardinality of an arc connecting a fluid place to a timed transition determines the flow rate of the fluid through that arc while the timed transition is enabled. The firing time distribution of the timed transition is of no importance for the fluid flow. When the arc connects to an immediate transition, the firing of that transition instantaneously removes or deposits an amount of fluid denoted by the cardinality of the arc.

The *marking* $\boldsymbol{\mu} = (\boldsymbol{\mu}_d, \mathbf{Z})$ consists of a discrete marking vector $\boldsymbol{\mu}_d = (\#p_i, i \in \mathcal{P}_d)$, where $\#p_i$ denotes the number of discrete tokens in the i-th discrete place p_i, and a continuous part, a vector representing the fluid level in each fluid place $\mathbf{Z} = (Z_k, k \in \mathcal{P}_c)$. The initial marking is $\boldsymbol{\mu}_0 = (\boldsymbol{\mu}_{d0}, \mathbf{Z}_0)$. Let $\mathcal{M}$ be the set of all markings and $\mathcal{S}$ be the set of all discrete markings. The markings in which one or more immediate transitions are enabled with reference to the discrete marking alone are called vanishing markings, and may be eliminated as outlined in [8]. Those markings in which the enabling of immediate transitions depends on fluid levels cannot be eliminated, since they are of mixed tangible/vanishing type.

We allow at most one general transition to be enabled at once. The firing times of exponential transitions and the enabling of both timed and immediate transitions may depend on fluid levels. A deterministic firing time equal to zero is attached to immediate transitions. For every general transition $g \in \mathcal{T}^G$ the firing time is specified by a *probability distribution function* (PDF) $F^g(x)$, the corresponding pdf $f^g(x)$, or by the *hazard rate* $\lambda^g(x) = \frac{f^g(x)}{1-F^g(x)}$, which we do not allow to depend on the marking. As firing policy *race with enabling memory* is used.

The flow rate function $\mathcal{R} : \mathcal{A}_c \times \mathcal{M} \to \mathbb{R}$ is defined as in [17] and is represented as the cardinality of the arc connecting a timed transition and a fluid place in the net. We define the instantaneous net fluid inflow rate at the k-th fluid place as:

$$r_k(\mathbf{z}, \mu) = \begin{cases} \sum\limits_{\tau \text{ enabled in } \mu} \mathcal{R}((\tau, k), \mu) - \sum\limits_{\tau \text{ enabled in } \mu} \mathcal{R}((k, \tau), \mu) & 0 < Z_k < B_k \\ \sum\limits_{\tau \text{ enabled in } \mu} \mathcal{R}((\tau, k), \mu) & Z_k = 0 \\ -\sum\limits_{\tau \text{ enabled in } \mu} \mathcal{R}((k, \tau), \mu) & Z_k = B_k \end{cases},$$

where τ is a timed transition and B_k denotes an upper bound for the amount of fluid in the k-th fluid place. If the capacity of place k is infinite, then $B_k = \infty$. Thus we prohibit both an overflow and a negative level in a fluid place.

For every place $k \in \mathcal{P}_c$ there is an instantaneous flow rate in every marking $\mu \in \mathcal{S}$ (if there is none the entry is equal to zero) which we require to be continuous functions of $\mathbf{z}$. These are collected into a diagonal matrix

$$\mathbf{R}_k(\mathbf{z}) = \operatorname{diag}(r_k(\mathbf{z}, \mu_i)) \qquad i = 1, \ldots, |\mathcal{S}|$$

where $|\mathcal{S}|$ denotes the number of discrete markings.

Figure 1 shows a non-Markovian FSPN model of a queueing system with deterministic arrivals and departures. Transition t_1 has a generally distributed firing time, so the modulated arrival process is not Markovian. We will therefore refer to this example as a GMD/D/1/K queueing system, where the abbreviation GM stands for "generally modulated". The arrival rate to the queue alternates between b customers per time unit during a period of average length $1/\lambda$ and a arrivals during a period of length $\mathrm{E}(X)$, where $\mathrm{E}(.)$ is the expectation of the firing time distribution. The service time is deterministic, because fluid leaves place P_4 at fixed rate c as long as t_5 is enabled, which is the case whenever P_4 is not empty. The rates a, b and c may depend on the marking.

A steady state solution for a two-state Markovian model with active and silent periods has been investigated in [23].

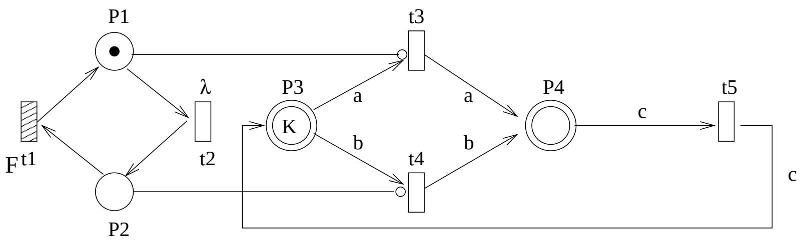

Figure 1: Non-Markovian FSPN model of a GMD/D/1/K queueing system

The discrete reachability graph of the Petri net is shown in figure 2. It contains two states. The first component of each state description is the number of tokens in P_1. If no token is in P_1 the general transition t_1 is enabled and the state description is supplemented by a variable x. Two variables representing the fluid levels are added to each state. In this example the fluid levels are uniquely specified by the level in only one place, as both always sum to K units.

3 Derivation of State Equations

In this section we will derive the systems of partial differential equations which describe the transient behaviour of the non-Markovian FSPN defined in the previous section. The nodes of the reachability graph consist of all discrete

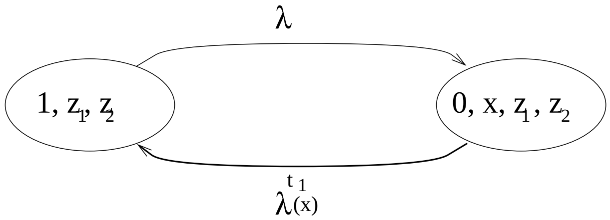

Figure 2: Reachability graph of the Petri net in figure 1

markings supplemented by the vector of fluid levels, plus an age variable if a general transition is enabled in that state. It gives rise to a stochastic process in continuous time with continuous state space. Due to the supplemented variables, which provide a full description of each state, the stochastic process is memoryless and can be analyzed by using Markov theory.

The stochastic process under consideration is

$$\{(N(t), X(t), \mathbf{Z}(t)), t \in \mathbb{R}_0^+\} \tag{1}$$

where $N(t)$ is the discrete marking at time t, $X(t)$ is the time elapsed since the enabling of the general transition g and $\mathbf{Z}(t)$ is a vector of length m, which represents the fluid level in each fluid place. As there is never more than one general transition enabled, the supplementary variable X is a scalar. If no general transition is enabled, $X(t)$ is not defined.

Let $\mathcal{S}^E$ denote the subset of the discrete state space where only exponential transitions are enabled and let $\mathcal{S}^g$ be the subset of the state space where the general transition g is enabled.

For each fluid place $k \in \mathcal{P}_c$ we define a set of discrete *fluent* states $\mathcal{S}_k$, in which the fluid level of that place might change. In states of the complementary set of *non-fluent* states $\mathcal{S}_{\overline{k}}$ the fluid level of place k remains fixed, but others can change. The sets are defined as

$$\mathcal{S}_k = \{i \mid i \in \mathcal{S} \text{ and } \mathbf{R}_{ii}(\mathbf{z}) \neq 0 \text{ for the fluid place } k\} \tag{2}$$

$$\mathcal{S}_{\overline{k}} = \{i \mid i \in \mathcal{S} \text{ and } \mathbf{R}_{ii}(\mathbf{z}) = 0 \text{ for the fluid place } k.\} \tag{3}$$

For fluid places k and $j \in \mathcal{P}_c$ the sets $\mathcal{S}_k$ and $\mathcal{S}_j$ might be identical, as in the example of the GMD/D/1/K queueing system, they might be overlapping or disjoint. Figure 3 depicts a possible situation for general transitions $g, h \in \mathcal{T}^G$ and fluid places $k_1, k_2, k_3 \in \mathcal{P}_c$.

At time t the transient probability of being in discrete state i with fluid levels in an infinitesimal environment around z_k, for all fluid places $k \in \mathcal{P}_c$ is called the *volume density* and is denoted by $\pi_i(t, \mathbf{z}) = \partial/\partial\mathbf{z}\ P(N(t) = i, \mathbf{Z}(t) \leq \mathbf{z})$. If a general transition is enabled the joint probability of being in state i, observing the fluid levels $\mathbf{z}$ and the remaining firing time x is called the *volume age density* and is defined as

$$\pi_i(t, x, \mathbf{z}) = \frac{\partial}{\partial \mathbf{z}} \frac{\partial}{\partial x} P\left(N(t) = i, X(t) \leq x, \mathbf{Z}(t) \leq \mathbf{z}\right). \tag{4}$$

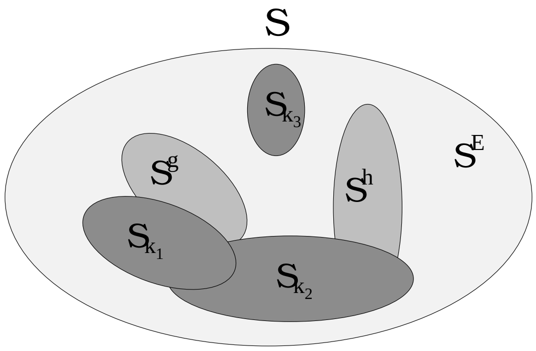

Figure 3: A possible composition of the state space

Let $\boldsymbol{\pi}(t,\mathbf{z}) = (\pi_1(t,\mathbf{z}),\ldots,\pi_n(t,\mathbf{z}))$ denote the vector of probabilities with respect to the discrete states, and of densities with respect to $\mathbf{z}$. Let there be m fluid places so that $\mathbf{z} = (z_1,\ldots,z_m)$. Furthermore let $\boldsymbol{\pi}(z_1+\mathbf{R}_1(\mathbf{z})\Delta t,\ldots,z_m+\mathbf{R}_m(\mathbf{z})\Delta t)$ be an abbreviation for $(\pi_1(z_1+r_1(\mathbf{z},1)\Delta t,\ldots,z_m+r_m(\mathbf{z},1)\Delta t),\ldots,\pi_n(z_1+r_1(\mathbf{z},n)\Delta t,\ldots,z_m+r_m(\mathbf{z},n)\Delta t))$.

We will now define the required vectors and matrices and derive the differential equations. The reader who is interested in the results only may jump to equations 20 - 31. Analogous to [16] we define the following vectors

$$\boldsymbol{\pi}^E(t,\mathbf{z}), \quad \boldsymbol{\pi}^g(t,\mathbf{z}) \quad \text{and} \quad \boldsymbol{\pi}^g(t,x,\mathbf{z}).$$

All vectors have dimension $|\mathcal{S}|$. $\boldsymbol{\pi}^E(t,\mathbf{z})$ contains all volume densities for states in $\mathcal{S}^E$. $\boldsymbol{\pi}^g(t,\mathbf{z})$ and $\boldsymbol{\pi}^g(t,x,\mathbf{z})$ denote the volume and the age volume densities of states in $\mathcal{S}^g$ for each $g \in \mathcal{T}^G$. The elements of every vector that do not correspond to the superscripts are set to zero.

For each fluid place $k \in \mathcal{P}_c$ the vectors of volume ($\boldsymbol{\pi}^E$) and age volume ($\boldsymbol{\pi}^g$) densities are split into a sum of two vectors.

$$\begin{aligned}
\boldsymbol{\pi}^g(t,x,\mathbf{z}) &= \boldsymbol{\pi}^g_k(t,x,\mathbf{z}) + \boldsymbol{\pi}^g_{\overline{k}}(t,x,\mathbf{z}) \\
\boldsymbol{\pi}^g(t,\mathbf{z}) &= \boldsymbol{\pi}^g_k(t,\mathbf{z}) + \boldsymbol{\pi}^g_{\overline{k}}(t,\mathbf{z}) \\
\boldsymbol{\pi}^E(t,\mathbf{z}) &= \boldsymbol{\pi}^E_k(t,\mathbf{z}) + \boldsymbol{\pi}^E_{\overline{k}}(t,\mathbf{z}).
\end{aligned}$$

The subscript k refers to the vectors containing densities of states in which place k is fluent and the subscript $\overline{k}$ to those, where place k is non-fluent. More precisely,

$$\pi^g_{i,k}(t,x,\mathbf{z}) = \begin{cases} \pi_i(t,x,\mathbf{z}) & \text{if } i \in \mathcal{S}^g \text{ and } i \in \mathcal{S}_k \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$$\pi^g_{i,k}(t,\mathbf{z}) = \begin{cases} \pi_i(t,\mathbf{z}) & \text{if } i \in \mathcal{S}^g \cap \mathcal{S}_k \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

$$\pi^E_{i,k}(t,\mathbf{z}) = \begin{cases} \pi_i(t,\mathbf{z}) & \text{if } i \in \mathcal{S}^E \cap \mathcal{S}_k \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

For states in which no changes in the fluid levels take place, the vectors of densities are

$$\pi^g_{i,\overline{k}}(t,x,\mathbf{z}) = \begin{cases} \pi_i(t,x,\mathbf{z}) & \text{if } i \in \mathcal{S}^g \cap \mathcal{S}_{\overline{k}} \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

$$\pi^g_{i,\overline{k}}(t,\mathbf{z}) = \begin{cases} \pi_i(t,\mathbf{z}) & \text{if } i \in \mathcal{S}^g \cap \mathcal{S}_{\overline{k}} \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

$$\pi^E_{i,\overline{k}}(t,\mathbf{z}) = \begin{cases} \pi_i(t,\mathbf{z}) & \text{if } i \in \mathcal{S}^E \cap \mathcal{S}_{\overline{k}} \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Furthermore matrices of dimension $|\mathcal{S}| \times |\mathcal{S}|$ are defined

$$\mathbf{Q}^{E,E}, \quad \mathbf{Q}^{E,g}, \quad \mathbf{Q}^{g,E}, \quad \mathbf{Q}^{g,h} \qquad \mathbf{\Delta}^{g,E}, \quad \mathbf{\Delta}^{g,h}, \quad g,h \in \mathcal{T}^G \quad \text{and}$$

$$\mathbf{Q}_k, \quad \mathbf{Q}_{\overline{k}}, \quad \mathbf{Q}_{\overline{k},k}, \quad \mathbf{Q}_{k,\overline{k}}, \qquad \mathbf{\Delta}_k, \quad \mathbf{\Delta}_{\overline{k}}, \quad \mathbf{\Delta}_{\overline{k},k} \quad \mathbf{\Delta}_{k,\overline{k}} \quad \forall k \in \mathcal{P}_c.$$

The $\mathbf{Q}$- matrices contain transition rates of exponential transitions causing the state changes denoted by the sub- and the superscripts. All combinations of sub- and superscripts are possible. If the type of the state does not change after the firing of a transition, one sub- or superscript can be omitted. The $\mathbf{\Delta}$-matrices describe the branching probabilities after the firing of a general transition and possibly several immediate transitions. Similar to the vectors, in the matrices all entries that do not correspond to the sets denoted by the superscripts contain zeros.

So $\mathbf{Q}_k$ contains all exponential transition rates causing state changes, while place k is fluent, $\mathbf{Q}_{\overline{k}}$ those while place k is non-fluent. Place k becomes fluent, being non-fluent before, by the state changes corresponding to the rates in $\mathbf{Q}_{\overline{k},k}$ and it becomes non-fluent with state changes denoted by $\mathbf{Q}_{k,\overline{k}}$. The switching matrices $\mathbf{\Delta}$ are defined analogously.

We define matrices $\widetilde{\mathbf{Q}}$ and $\widetilde{\mathbf{\Delta}}$ that consist of rates or probabilities for state changes, where no fluid place becomes fluent, being non-fluent before.

$$\widetilde{\mathbf{Q}} = \mathbf{Q} - \mathbf{Q}_{\overline{k},k} \qquad \forall k \in \mathcal{P}_c \tag{11}$$

$$\widetilde{\mathbf{\Delta}} = \mathbf{\Delta} - \mathbf{\Delta}_{\overline{k},k} \qquad \forall k \in \mathcal{P}_c. \tag{12}$$

The value of the supplementary variable increases linearly with time, whereas the change in the fluid levels Z_k takes place at the rate specified by the matrix $\mathbf{R}_k(\mathbf{z})$. Thus in states with a supplementary age variable the Kolmogorov forward equations are given by

$$\boldsymbol{\pi}^g(t+\Delta t, x+\Delta x, z_1+\mathbf{R}_1(\mathbf{z})\Delta t, \ldots, z_m+\mathbf{R}_m(\mathbf{z})\Delta t) = \\ \boldsymbol{\pi}^g(t,x,\mathbf{z}) \cdot \left(\mathbf{I} + \widetilde{\mathbf{Q}}^g \Delta t - \lambda^g(x)\Delta t\right) + o(\Delta t). \quad (13)$$

for $0 \le x < x^g_{\max}$. $\mathbf{I}$ is the identity matrix and $o(\Delta t)$ a function that tends to zero more rapidly than its argument. We let x^g_{max} denote the range over which the firing time is defined (i.e. $x^g_{max} = \min\{x \ge 0 : F^g(x) = 1\}$. The age volume densities after time Δt are determined by the possible changes of state. Exponential transitions may fire without deactivating the general transition and the general transition can also fire. The matrices of flow rate functions $\mathbf{R}_k$ also contribute to the volume densities.

The Kolmogorov forward equation does not include state transitions from non-fluent to fluent states. These have to be handled seperately. They are expressed by the boundary conditions (eqns. 26, 27).

For states without a supplementary age variable ($i \in \mathcal{S}^E$) we have:

$$\boldsymbol{\pi}^E(t+\Delta t, z_1+\mathbf{R}_1(\mathbf{z})\Delta t, \ldots, z_m+\mathbf{R}_m(\mathbf{z})\Delta t) = \boldsymbol{\pi}^E(t,\mathbf{z}) \left(\mathbf{I} + \widetilde{\mathbf{Q}}^{E,E}(\mathbf{z})\Delta t\right) \\ + \sum_{g\in\mathcal{T}^G} \int_0^{x^g_{\max}} \boldsymbol{\pi}^g(t,x,\mathbf{z}) \cdot \lambda^g(x) \widetilde{\boldsymbol{\Delta}}^{g,E} \cdot dx\ \Delta t \\ + \sum_{g\in\mathcal{T}^G} \int_0^{x^g_{\max}} \boldsymbol{\pi}^g(t,x,\mathbf{z}) \cdot \widetilde{\mathbf{Q}}^{g,E}(\mathbf{z}) dx\ \Delta t + o(\Delta t). \quad (14)$$

If no general transition is enabled after time Δt, the volume densities are determined by the firing of exponential transitions or a general transition, analogous to models without fluid places [16] and by the changes of fluid levels as specified by the rate functions $\mathbf{R}_k$.

Equation 13 can be transformed into a differential equation. This is achieved by adding to the left side of eqn. 13 the following terms, that sum to zero

$$-\boldsymbol{\pi}^g(t, x+\Delta x, z_1+\mathbf{R}_1\Delta t, \ldots, z_m+\mathbf{R}_m\Delta t) + \boldsymbol{\pi}^g(t, x+\Delta x, z_1+\mathbf{R}_1\Delta t, \ldots, z_m+\mathbf{R}_m\Delta t) \\ - \boldsymbol{\pi}^g(t, x, z_1+\mathbf{R}_1\Delta t, \ldots, z_m+\mathbf{R}_m\Delta t) + \boldsymbol{\pi}^g(t, x, z_1+\mathbf{R}_1\Delta t, \ldots, z_m+\mathbf{R}_m\Delta t) \\ - \boldsymbol{\pi}^g(t, x, z_1, z_2+\mathbf{R}_2\Delta t, \ldots, z_m+\mathbf{R}_m\Delta t) + \boldsymbol{\pi}^g(t, x, z_1, z_2+\mathbf{R}_2\Delta t, \ldots, z_m+\mathbf{R}_m\Delta t) \\ \vdots \\ - \boldsymbol{\pi}^g(t, x, z_1, \ldots, z_{m-1}, z_m+\mathbf{R}_m\Delta t) + \boldsymbol{\pi}^g(t, x, z_1, \ldots, z_{m-1}, z_m+\mathbf{R}_m\Delta t),$$

subtracting $\boldsymbol{\pi}^g(t,x,\mathbf{z})$ from both sides, dividing by Δt and taking the limit as Δt approaches zero leads to

$$\frac{\partial}{\partial t}\boldsymbol{\pi}^g(t,x,\mathbf{z}) + \frac{\partial}{\partial x}\boldsymbol{\pi}^g(t,x,\mathbf{z}) + \sum_{k=1}^{m} \frac{\partial}{\partial z_k}\boldsymbol{\pi}^g(t,x,\mathbf{z})\mathbf{R}_k(\mathbf{z}) = \\ \boldsymbol{\pi}^g(t,x,\mathbf{z}) \cdot \widetilde{\mathbf{Q}}^g(\mathbf{z}) - \boldsymbol{\pi}^g(t,x,\mathbf{z})\lambda^g(x) \qquad 0 < x < x^g_{\max} \quad (15)$$

A differential equation can be derived from eqn. 14 by carrying out the equivalent transformations to those applied to eqn. 13.

$$\begin{aligned}\frac{\partial}{\partial t}\boldsymbol{\pi}^E(t,\mathbf{z}) + \sum_{k=1}^{m}\frac{\partial}{\partial z_k}\boldsymbol{\pi}^E(t,\mathbf{z})\mathbf{R}_k(\mathbf{z}) &= \boldsymbol{\pi}^E(t,\mathbf{z})\widetilde{\mathbf{Q}}^{E,E}(\mathbf{z}) \\ &+ \sum_{g\in\mathcal{T}^G}\int_0^{x^g_{\max}}\boldsymbol{\pi}^g(t,x,\mathbf{z})\lambda^g(x)\,dx\,\widetilde{\boldsymbol{\Delta}}^{g,E} + \\ &\sum_{g\in\mathcal{T}^G}\int_0^{x^g_{\max}}\boldsymbol{\pi}^g(t,x,\mathbf{z})\,dx\,\widetilde{\mathbf{Q}}^{g,E}(\mathbf{z}). \quad (16)\end{aligned}$$

If $\boldsymbol{\pi}_0^E$ and $\boldsymbol{\pi}_0^g$ are the initial state probabilities of the discrete markings, $\mathbf{z}_0$ the vector of initial fluid levels in the fluid places and we assume that no transition $g\in\mathcal{T}^G$ was enabled before $t=0$ then the initial conditions are

$$\boldsymbol{\pi}^E(0,\mathbf{z}) = \boldsymbol{\pi}_0^E\cdot\delta(\mathbf{z}_0), \quad \boldsymbol{\pi}^g(0,\mathbf{z}) = \boldsymbol{\pi}_0^g\cdot\delta(\mathbf{z}_0), \quad \boldsymbol{\pi}^g(0,x,\mathbf{z}) = \boldsymbol{\pi}_0^g\cdot\delta(0,\mathbf{z}_0) \quad (17)$$

where $\delta(\mathbf{z}_0)$ and $\delta(0,\mathbf{z}_0)$ denote multivariate Dirac impulses. The supplementary variable X is set to zero in the instant of enabling of a general transition g so no change of fluid levels takes place.

$$\begin{aligned}\boldsymbol{\pi}^g(t,0,\mathbf{z}) = \boldsymbol{\pi}^E(t,\mathbf{z})\cdot\mathbf{Q}^{E,g}(\mathbf{z}) + \sum_{h\in\mathcal{T}^G}\int_0^{x^h_{\max}}\boldsymbol{\pi}^h(t,x,\mathbf{z})\cdot\lambda^h(x)\cdot\Delta^{h,g}\,dx \\ + \sum_{h\in\mathcal{T}^G}\int_0^{x^h_{\max}}\boldsymbol{\pi}^h(t,x,\mathbf{z})\cdot\mathbf{Q}^{h,g}(\mathbf{z})\,dx. \quad (18)\end{aligned}$$

Integrating the volume age densities with respect to x yields the state probabilities

$$\boldsymbol{\pi}^g(t,\mathbf{z}) = \int_0^\infty \boldsymbol{\pi}^g(t,x,\mathbf{z})\,dx. \quad (19)$$

In order to simplify the above equations further we need

$$q_i(t,x,\mathbf{z}) = \frac{\pi_i(t,x,\mathbf{z})}{1-F^g(x)} \qquad x < x^g_{\max},\ i\in\mathcal{S}^g, g\in\mathcal{T}^G. \quad (20)$$

Substituting the vectors of densities in eqn. 15 by using eqn. 20 and applying the chain rule for differentiation to the left side of eqn. 15 leads to

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{q}^g(t,x,\mathbf{z})\cdot\Big(1-F^g(x)\Big) + \left(\frac{\partial}{\partial x}\mathbf{q}^g(t,x,\mathbf{z})\right)\cdot\Big(1-F^g(x)\Big) + \\ f^g(x)\cdot\mathbf{q}^g(t,x,\mathbf{z}) + \sum_{k=1}^{m}\frac{\partial}{\partial z_k}\mathbf{q}^g(t,x,\mathbf{z})\mathbf{R}_k(\mathbf{z})\cdot\Big(1-F^g(x)\Big) = \\ \mathbf{q}^g(t,x,\mathbf{z})\Big(1-F^g(x)\Big)\cdot\widetilde{\mathbf{Q}}^g(\mathbf{z}) - \boldsymbol{\pi}^g(t,x,\mathbf{z})\lambda^g(x).\end{aligned}$$

Since $f^g(x) \cdot \mathbf{q}^g(t,x,\mathbf{z}) = \boldsymbol{\pi}^g(t,x,\mathbf{z})\lambda^g(x)$ we have

$$\frac{\partial}{\partial t}\mathbf{q}^g(t,x,\mathbf{z}) + \frac{\partial}{\partial x}\mathbf{q}^g(t,x,\mathbf{z}) + \sum_{k=1}^{m} \frac{\partial}{\partial z_k}\mathbf{q}^g(t,x,\mathbf{z})\mathbf{R}_k(\mathbf{z}) = \mathbf{q}^g(t,x,\mathbf{z}) \cdot \widetilde{\mathbf{Q}}^g(\mathbf{z}). \tag{21}$$

The same transformations were carried out in [16].

Using eqns. 20 and 19, equation 16 simplifies to

$$\begin{aligned}\frac{\partial}{\partial t}\boldsymbol{\pi}^E(t,\mathbf{z}) + \sum_{k=1}^{m} \frac{\partial}{\partial z_k}\boldsymbol{\pi}^E(t,\mathbf{z})\mathbf{R}_k(\mathbf{z}) = \boldsymbol{\pi}^E(t,\mathbf{z}) \cdot \widetilde{\mathbf{Q}}^{E,E}(\mathbf{z}) + \\ \sum_{g\in\mathcal{T}^G} \int_0^{x^g_{\max}} \mathbf{q}^g(t,x,\mathbf{z})\, dF^g(x) \cdot \widetilde{\boldsymbol{\Delta}}^{g,E} + \sum_{g\in\mathcal{T}^G} \boldsymbol{\pi}^g(t,\mathbf{z})\widetilde{\mathbf{Q}}^{g,E}.\end{aligned} \tag{22}$$

The initial conditions now are

$$\boldsymbol{\pi}^E(0,\mathbf{z}) = \boldsymbol{\pi}^E_0 \cdot \delta(\mathbf{z}_0), \quad \boldsymbol{\pi}^g(0,\mathbf{z}) = \boldsymbol{\pi}^g_0 \cdot \delta(\mathbf{z}_0), \quad \mathbf{q}^g(0,x,\mathbf{z}) = \boldsymbol{\pi}^g_0 \cdot \delta(0,\mathbf{z}_0) \tag{23}$$

Transforming eqns. 18 and 19 leads to

$$\begin{aligned}\mathbf{q}^g(t,0,\mathbf{z}) = \boldsymbol{\pi}^E(t,\mathbf{z}) \cdot \mathbf{Q}^{E,g}(\mathbf{z}) + \sum_{h\in\mathcal{T}^G} \int_0^{\infty} \mathbf{q}^h(t,x,\mathbf{z})\, dF^h(x)\boldsymbol{\Delta}^{h,g} \\ + \sum_{h\in\mathcal{T}^G} \boldsymbol{\pi}^h(t,\mathbf{z}) \cdot \mathbf{Q}^{h,g}(\mathbf{z})\end{aligned} \tag{24}$$

and

$$\boldsymbol{\pi}^g(t,\mathbf{z}) = \int_0^{\infty} \mathbf{q}^g(t,x,\mathbf{z})\left(1 - F^g(x)\right)\, dx. \tag{25}$$

For each fluid place $k \in \mathcal{P}_c$ the switching from non-fluent to fluent states at any fluid level in the fluid places is expressed by the following boundary equations

$$\begin{aligned}\boldsymbol{\pi}^E_k(t,\mathbf{z}) &= \boldsymbol{\pi}^E_{\overline{k}}(t,\mathbf{z})\mathbf{Q}^{E,E}_{\overline{k},k}(\mathbf{z}) + \sum_{g\in\mathcal{T}^G} \int_0^{x^g_{\max}} \boldsymbol{\pi}^g_{\overline{k}}(t,x,\mathbf{z})\lambda^g(x)\, dx\, \boldsymbol{\Delta}^{g,E}_{\overline{k},k} \\ &\quad + \sum_{g\in\mathcal{T}^G} \int_0^{x^g_{\max}} \boldsymbol{\pi}^g_{\overline{k}}(t,x,\mathbf{z})\, dx\, \mathbf{Q}^{g,E}_{\overline{k},k}(\mathbf{z}),\end{aligned} \tag{26}$$

$$\boldsymbol{\pi}^g_k(t,x,\mathbf{z}) = \boldsymbol{\pi}^g_{\overline{k}}(t,x,\mathbf{z})\mathbf{Q}^g_{\overline{k},k}(\mathbf{z}). \tag{27}$$

Since the boundary conditions for some fluid places can be identical, at most $2m$ boundary conditions arise from these state changes.

For states in which no immediate transition is enabled by fluid levels the boundary conditions determined by the rate functions with respect to $\mathbf{z}$ are

$$\pi_i(t, x, z_1, \ldots, z_{k-1}, 0, z_{k+1}, \ldots, z_m) = 0 \qquad \text{if } r_k(\mathbf{z}, i) > 0 \\ \text{for } k = 1, \ldots, m \text{ and } i \in \mathcal{S} \quad (28)$$

where $\boldsymbol{\pi} = \boldsymbol{\pi}^E \cup \boldsymbol{\pi}^g$.

For each state in which an inflow to a fluid place k is defined by the flow rate function r_k, the probability of finding place k empty is equal to zero.

Immediate transitions and fluid places may interact in two different ways. In both cases the transition is enabled by a guard function that is a function of the fluid levels. Either the firing of an immediate transition causes a switching from one discrete state to another, or fluid is removed from or filled into fluid places in no time, which can be seen as a jump to another location in the m-dimensional hypercube defined by the components of the fluid vector $\mathbf{z}$.

We first define the boundary condition arising from the discrete change of states due to the firing of an immediate transition depending on the fluid level of place k. Let $g_\rho : \mathcal{T} \times \mathcal{M} \to \{0, 1\}$ be a guard function for the immediate transition ρ. Let $\boldsymbol{\Delta}_\rho$ be the matrix of switching probabilities after the firing of the immediate transition ρ, similar to the branching probabilities after the firing of a general transition mentioned at the beginning of this section, then the boundary condition

$$\boldsymbol{\pi}(t, \mathbf{z}) = \boldsymbol{\pi}_k(t, \mathbf{z}) \cdot \boldsymbol{\Delta}_\rho \qquad \text{if} g_\rho(\mathbf{z}) = 1 \quad (29)$$

holds.

For the transportation of fluid due to the firing of an immediate transition we have to define another kind of matrix, $\mathbf{R}_{k,\tau}$. These matrices have different semantics to the rate matrices used until now. The entries of $\mathbf{R}_{k,\tau}$ denote the amount of fluid that is moved immediately in each discrete marking, whereby the fluid place k is filled or emptied by the firing of the immediate transition τ. This will be illustrated by the example in section 4.

Let $g_\tau : \mathcal{T} \times \mathcal{M} \to \{0, 1\}$ be another guard function and $\boldsymbol{\Delta}_\tau$ another matrix of switching probabilities, then the immediate change in the fluid levels due to the firing of τ is expressed by

$$\boldsymbol{\pi}_k(t, z_1 + \mathbf{R}_{1,\tau}(\mathbf{z}), \ldots, z_m + \mathbf{R}_{m,\tau}(\mathbf{z})) = \boldsymbol{\pi}_k(t, \mathbf{z}) \cdot \boldsymbol{\Delta}_\tau \quad \text{if } g_\tau(\mathbf{z}) = 1 \\ \forall k \in \mathcal{P}_c \text{ that are filled or emptied by the firing of } \tau. \quad (30)$$

If the guard functions of transition ρ and τ are identical, the boundary conditions can be unified to

$$\boldsymbol{\pi}(t, z_1 + \mathbf{R}_{1,\tau}(\mathbf{z}), \ldots, z_m + \mathbf{R}_{m,\tau}(\mathbf{z})) = \boldsymbol{\pi}_k(t, \mathbf{z}) \cdot \boldsymbol{\Delta}_\tau \cdot \boldsymbol{\Delta}_\rho \\ \forall k \in \mathcal{P}_c \text{ where } \mathbf{R}_{k,\tau} \neq \mathbf{0} \text{if } \rho \text{ is prior to } \tau, g_{\rho,\tau}(\mathbf{z}) = 1. \quad (31)$$

The dynamic behaviour of a non-Markovian FSPN is completely described by the differential equations (21 and 22), the initial condition (23), and the boundary conditions (24, 26, 27 28,29, 30).

Example. We illustrate the above equations by presenting the formulas for the model in figure 1. We will refer to the states of the reachability graph in figure 2 by the number of tokens in P_1. State 1 is the only state in $\mathcal{S}^E_{P_3} = \mathcal{S}^E_{P_4}$, state 0 is the only one in $\mathcal{S}^{t_1}_{P_3} = \mathcal{S}^{t_1}_{P_4}$, $\mathcal{S}_{\overline{P}_3} = \mathcal{S}_{\overline{P}_4} = \emptyset$ so in the following we leave out the subscripts denoting fluent or non-fluent states. Thus, $\boldsymbol{\pi}^E(t,\mathbf{z}) = (0, \pi^E_1(t,\mathbf{z}))$, $\boldsymbol{\pi}^{t_1}(t,\mathbf{z}) = (\pi^{t_1}_0(t,\mathbf{z}), 0)$, $\boldsymbol{\pi}^{t_1}(t,x,\mathbf{z}) = (\pi^{t_1}_0(t,x,\mathbf{z}), 0)$ and $\mathbf{q}^{t_1}(t,x,\mathbf{z}) = (q^{t_1}_0(t,x,\mathbf{z}), 0)$. The matrices defined are

$$\mathbf{Q}^{E,E} = \begin{pmatrix} 0 & 0 \\ 0 & -\lambda \end{pmatrix} \qquad \mathbf{Q}^{E,t_1} = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \qquad \boldsymbol{\Delta}^{t_1,E} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{R}_1 = \begin{pmatrix} c-a & 0 \\ 0 & c-b \end{pmatrix} \qquad \mathbf{R}_2 = \begin{pmatrix} a-c & 0 \\ 0 & b-c \end{pmatrix}$$

The fluid place P_3 is abbreviated as $k=1$ and P_4 as $k=2$. The differential equations for the GMD queueing system are derived from equation 21

$$\frac{\partial}{\partial t}\mathbf{q}^{t_1}(t,x,\mathbf{z}) + \frac{\partial}{\partial x}\mathbf{q}^{t_1}(t,x,\mathbf{z}) + \sum_{k=1}^{2}\frac{\partial}{\partial z_k}\mathbf{q}^{t_1}(t,x,\mathbf{z})\mathbf{R}_k(\mathbf{z}) = 0 \tag{32}$$

and from eqn. 22

$$\frac{\partial}{\partial t}(0, \pi^E_1(t,\mathbf{z})) + \frac{\partial}{\partial z_1}(0, \pi^E_1(t,\mathbf{z})\cdot(c-b)) + \frac{\partial}{\partial z_2}(0, \pi^E_1(t,\mathbf{z})\cdot(b-c)) = \int_0^{x^{t_1}_{\max}} \mathbf{q}^{t_1}(t,x,\mathbf{z})\, dF^{t_1}(x)\cdot\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{33}$$

The boundary conditions are

$$\begin{array}{lcll}
\mathbf{q}^{t_1}(t,0,\mathbf{z}) & = & (\lambda\cdot\pi^E_1(t,\mathbf{z}), 0) & \\
\pi_0(t,(0,z_2)) & = & 0 & \text{if } c-a>0 \Leftrightarrow c>a \\
\pi_0(t,(z_1,0)) & = & 0 & \text{if } a-c>0 \Leftrightarrow a>c \\
\pi_1(t,(0,z_2)) & = & 0 & \text{if } c>b \\
\pi_1(t,(z_1,0)) & = & 0 & \text{if } b>c
\end{array}$$

In this example we have only two boundary conditions because either $c>a$ or $a>c$ and accordingly either $b>c$ or $c>b$.

4 Markovian FSPNs

The FSPN, which is a GSPN with fluid places, forms a special case of the non-Markovian FSPN that was considered in the previous section. As the GSPN contains no general transitions, the Kolmogorov forward equation is the following difference equation

$$\boldsymbol{\pi}(t+\Delta t, z_1+\Delta t\mathbf{R}_1(\mathbf{z}),\ldots,z_m+\Delta t\mathbf{R}_m(\mathbf{z})) = \boldsymbol{\pi}(t,z_1,\ldots,z_m)\left(\mathbf{I}+\widetilde{\mathbf{Q}}(\mathbf{z})\cdot\Delta t)\right)+o(\Delta t).$$

The corresponding differential equation is

$$\frac{\partial}{\partial t}\boldsymbol{\pi}(t,\mathbf{z})+\sum_{k=1}^{m}\frac{\partial}{\partial z_k}\boldsymbol{\pi}(t,\mathbf{z})\mathbf{R}_k(\mathbf{z})=\boldsymbol{\pi}(t,\mathbf{z})\cdot\widetilde{\mathbf{Q}}(\mathbf{z}). \tag{34}$$

which can be seen to be a special case of eqn. 22 on page 11. If $\boldsymbol{\pi}_0$ is the vector of initial state probabilities and $\mathbf{z}_0$ the vector of initial fluid levels, then the initial conditions are

$$\begin{aligned}\boldsymbol{\pi}(0,\mathbf{z}) &= \boldsymbol{\pi}_0\cdot\delta(\mathbf{z}_0) \quad \text{where } \mathbf{z}=(z_1,\ldots,z_m) \\ \boldsymbol{\pi}(0,\mathbf{z}) &= 0 \quad \text{for all } \boldsymbol{\pi}\neq\boldsymbol{\pi}_0.\end{aligned} \tag{35}$$

For each fluid place $k\in cP_c$ the boundary conditions are

$$\boldsymbol{\pi}_k(t,\mathbf{z})=\boldsymbol{\pi}_{\overline{k}}(t,\mathbf{z})\cdot\mathbf{Q}_{\overline{k},k}(\mathbf{z}), \tag{36}$$

eqn. 28, 29 and eqn. 30 on page 12.

In the remainder of this section we present an example in which immediate transitions are enabled by the level in a fluid place using a guard function. Figure 4 shows a M/D/1/K queueing system, where the fluid place has the function of an hourglass that models the deterministic service time. Transition t_4 is enabled only while a token is in P_3. Then the fluid place fills at rate 1 up to the level τ, which corresponds to the service time. As soon as this level is reached, t_3 fires, the service is complete and the fluid place is emptied at once by the firing of the immediate transition t_5 at rate τ. Hence, in this example the fluid place is used to model a non-exponentially distributed firing time. Of course, the use of a transition with deterministic firing time, being a special case of general firing time distributions, would simplify the model.

Figure 5 shows the reachability graph of the model in figure 4. The state description is (N,z), where N is the number of tokens in P_2 and P_3. State $(0,z)$ is the only tangible marking, all other markings (N,z) where $N>0$ are vanishing.

In this example the vector of state probabilities is

$$\boldsymbol{\pi}(t,z)=(\pi_0(t,z),\ldots,\pi_K(t,z)) \text{ where}$$

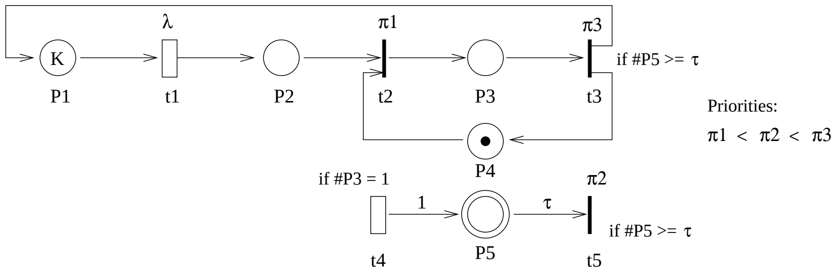

Figure 4: FSPN model of a M/D/1/K queueing system

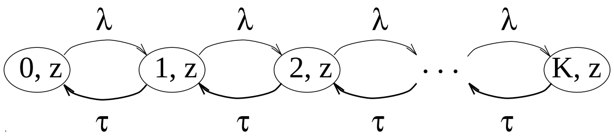

Figure 5: Reachability graph of the Petri net in figure 4

$$\boldsymbol{\pi}_{\overline{P}_5}(t,z) = (\pi_0(t,z), 0, \ldots, 0) \text{ and } \boldsymbol{\pi}_{P_5}(t,z) = (0, \pi_1(t,z), \ldots, \pi_n(t,z))$$

and the following matrices are of dimension $(K+1) \times (K+1)$

$$\mathbf{Q}_{(D+C+C,D)} = \begin{pmatrix} -\lambda & 0 & 0 & \ldots & 0 \\ 0 & -\lambda & \lambda & & \\ \vdots & & \ddots & & \vdots \\ 0 & & 0 & -\lambda & \lambda \\ 0 & & \cdots & & 0 \end{pmatrix} \qquad \mathbf{Q}_{D,C} = \begin{pmatrix} 0 & \lambda & 0 & \ldots & 0 \\ & 0 & 0 & & \\ \vdots & & \ddots & & \vdots \\ & & 0 & \ddots & \\ 0 & & \cdots & & 0 \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} 0 & & \cdots & & 0 \\ 0 & 1 & 0 & & \\ \vdots & & \ddots & & \vdots \\ 0 & & 0 & \ddots & 0 \\ 0 & & \cdots & 0 & 1 \end{pmatrix} \qquad \boldsymbol{\Delta}_{t_3} = \begin{pmatrix} 0 & & \cdots & & 0 \\ 1 & 0 & & & \\ 0 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

$$\boldsymbol{\Delta}_{t_5} = \mathbf{I} \qquad \mathbf{R}_{t_5} = \begin{pmatrix} 0 & & \cdots & & 0 \\ 0 & -\tau & 0 & & \\ \vdots & & \ddots & & \vdots \\ 0 & & 0 & \ddots & 0 \\ 0 & & \cdots & 0 & -\tau \end{pmatrix}$$

We assume that the queue – represented by the fluid place – is initially empty, yielding the initial conditions

$$\begin{aligned} \pi_0(0,z) &= \delta(0) \\ \pi_n(0,z) &= 0 \text{ for } n > 0,\ z \in \mathbb{R}. \end{aligned}$$

The boundary condition is

$$\boldsymbol{\pi}_{P_5}(t, \mathbf{z}) = \boldsymbol{\pi}_{\overline{P}_5}(t, \mathbf{z}) \cdot \mathbf{Q}_{\overline{P}_5, P_5}(\mathbf{z}),$$

and (30)

$$\boldsymbol{\pi}(t, z + \mathbf{R}_{t_5}) = \boldsymbol{\pi}_{P_5}(t, z) \cdot \boldsymbol{\Delta}_{t_5} \cdot \boldsymbol{\Delta}_{t_3} \qquad \text{if } z \geq \tau.$$

More precisely this is

$$\pi_n(t, 0) = \pi_{n+1}(t, z) \qquad \text{if } z \geq \tau, \quad 0 < n \leq K - 1.$$

We have a linear first order partial differential equation in two variables with one initial and $(K-1)$ boundary equations. In this example the PDE can be reduced to an ordinary differential equation (ODE) as shown in [15]. In general this is not the case.

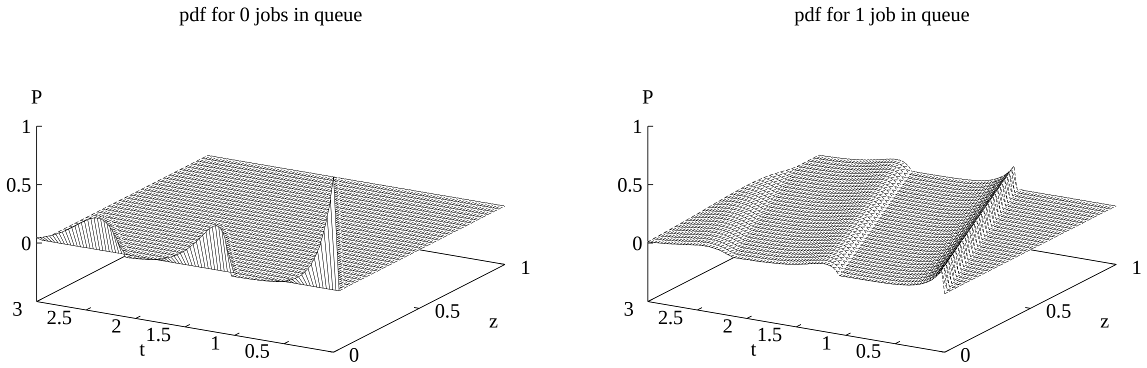

Figure 6: Numerical Solution of M/D/1/K model (pdf). Left: #P3=0, Right: #P3=1

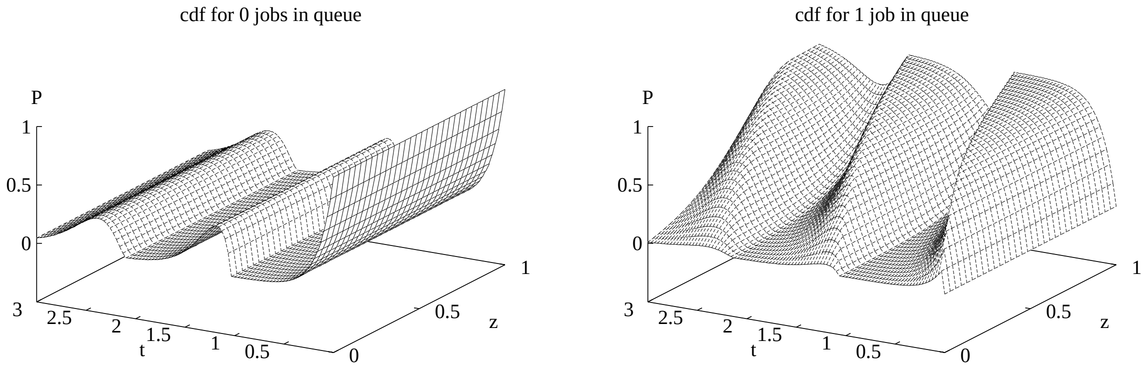

Figure 7: Numerical Solution of M/D/1/K model (cdf). Left: #P3=0, Right: #P3=1

Figure 7 shows a solution to the M/D/1/K model obtained by solving the discretized equations 34. These are the cumulative distribution functions, i.e. the integrals of the densities with respect to z. We chose parameter values of $K = 1$, $\lambda = 6$, $\tau = 1$. The left figure shows the case that no jobs are in the queue or in service, and the right figure shows the case for one job in service. Time advances from right to left and into the page, and z advances from left to right into the page. Note that the left curve has no structure parallel to the z-axis. This is because the fluid level is necessarily zero when the queue is empty and the curve is determined entirely by the probabilities at $z = 0$. The right hand curve clearly shows the fluid level increasing linearly with time and the probability. We have chosen $K = 1$ for reasons of brevity only: there is one such three-dimensional curve for each discrete marking. Also, the choice of a larger arrival rate than service rates is unrealistic. Here, too, we are just interested in showing the form of the solution.

5 Conclusions

In this paper non-Markovian fluid stochastic Petri nets have been presented. This class of nets unifies those with non-exponentially distributed firing times and those containing fluid places in one framework. The unification appears to be natural, since both extensions lead to mixed discrete and continuous state spaces and to very similar state equations. The proposed formalism provides a high modeling flexibility which allows both the use of a larger variety of distributions for the changes of state and the use of fluid places for the reduction of the state space complexity and the modeling of inherently continuous quantities. Since the fluid levels can be regarded as additional supplementary variables, we may consider this as an enhancement of the method of supplementary variables to deal with both extensions. The use of appropriate boundary conditions gives the user full modeling flexibility concerning the dependencies between immediate transitions and fluid places.

In this work we have concentrated on the derivation of state equations. The development of efficient numerical solution methods is a subject of future research. A further restriction of the current presentation is that general transitions may not concurrently be enabled (on the level of the stochastic process this corresponds to the restriction that at most one supplementary age variable is allowed in each state). This restriction is not inherent in the method of supplementary variables, but merely allows an algorithmic simplification. In the case of several concurrently enabled general transitions the derivation of state equations becomes much more tedious. In addition we will address the question of alternative firing policies.

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