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Second Order Non-Markovian Fluid Stochastic Petri Nets

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Abstract

We present a Petri net formalism that allows for mixed discrete and continuous stochastic models. The continuous part of the models consists of fluid places that are filled and emptied at random (normally distributed) rate. Fluid places can be used for the modelling of continuous system components as well as for continuous approximation of heavily loaded discrete places in order to avoid state space explosion. The dynamics of a second order FSPN are described by second order partial differential equations.

Introduction

Stochastic Petri nets (SPNs) [1] are well suited for the model-based performance and dependability evaluation of complex systems. However, some difficulties restrict their applicability: state-space explosion, the inadequacy of the exponential distribution in describing the system's behaviour, and the inability to model continuous quantities that may be present in the system.

Regarding non-exponential distributions phase-type distributions, Markov regenerative theory [3, 4], and the method of supplementary variables [6] have been considered. The idea of the supplementary variable method [5] is to include age variables into the state description which represent the time since enabling of the transitions with generally distributed firing times. Thus, a mixed discrete and continuous state-space is defined.

To reduce state space complexity various methods such as lumping, decomposition, structured model representation using tensor algebra or fluid flow approximation [8] are known. The concept of first and second order (fluid and diffusion) approximation to heavily loaded queueing systems is well known from queueing theory [8]. Recently, the concept of fluid models was put in the context of SPNs [9, 7], referred to as *fluid stochastic Petri nets* (FSPNs). FSPNs have been invented to extend SPNs for the modelling of continuous quantities and are meant to be clearly divided into a discrete and a continuous submodel that can affect each other. In FSPNs, the fluid variables are represented by fluid places which can hold fluid rather than discrete tokens. FSPNs allow a very flexible definition of fluid models, but no probabilistic variation of the fluid flow is possible.

We will define a semantics of a fluid place that serves for the approximation of large contents of discrete places as well as for the modelling of continuous system components. Of course both may in practice lead to different models, e.g. immediate transitions being enabled by fluid places will only occur when the place is used as an approximated discrete one.

Supplementary variables will be used as age variables of transitions with generally distributed firing times, which we refer to as *general* transitions, and to represent the fluid level in fluid places. Although the method does not exclude concurrently enabled general transitions we do impose this restriction in order to bound algorithmic complexity. The inclusion of supplementary variables makes the stochastic process Markovian and it is thus possible to derive a set of state equations which describe the model dynamics.

When using first order (fluid flow)[2] approximation, the equations consist of a system of first order partial differential equations (PDEs). Each component represents a probability density function for one discrete state of the net and depends on the age variables of the general transitions enabled in this state, the fluid variables, and, in the transient case, on time.

We will use second order (diffusion) approximation to allow a non-deterministic modelling of continuous quantities and to preserve the random nature of a continuously approximated discrete place. Using diffusion approximation the stochastic process is described by second order PDEs in the supplementary variables representing the fluid levels.

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Model Definition and Behaviour

We define a non-Markovian FSPN following the common notation for stochastic Petri nets [1]. As in [7] FSPNs consist of *places, transitions* and *arcs*. The set of places is divided into *fluid (continuous)* and *discrete* places ($\mathcal{P} = \mathcal{P}_c \cup \mathcal{P}_d$), the former containing a continuous amount of fluid and the latter discrete *tokens*. The set of transitions $\mathcal{T} = \mathcal{T}_E \cup \mathcal{T}_I \cup \mathcal{T}_G$ is composed of the exponentially distributed, immediate and generally distributed transitions, respectively. Places and transitions are connected by *input*, *output*, or *inhibitor* arcs $\mathcal{A} = \mathcal{A}_d \cup \mathcal{A}_c$, that may be discrete or continuous, as denoted by the subscripts d and c. Note that we have extended the definition in [7] by allowing fluid places to be connected to immediate transitions.

The *marking* $\mathbf{m} = (\boldsymbol{\mu}, \mathbf{Z})$ consists of a discrete part $\boldsymbol{\mu} = (\#p_i, i \in \mathcal{P}_d)$, where $\#p_i$ denotes the number of discrete tokens in the i-th discrete place p_i, and a continuous part, a vector representing the fluid level in each fluid place $\mathbf{Z} = (Z_k, k \in \mathcal{P}_c)$ which is a random variable. The initial marking is $\mathbf{m}_0 = (\boldsymbol{\mu}_0, \mathbf{Z}_0)$.

Arcs connecting fluid places and transitions are drawn like pipes and are labelled by a normally distributed random variable, that is specified by the distribution's expectation and variance. Fluid flows along arcs connecting a timed transition and a fluid place as long as the transition is enabled. A timed transition is enabled only by the required number of tokens in all its discrete input places, or by a guard, which may depend on the discrete state only. The firing times of exponential transitions may depend on the discrete and on the continuous marking, whereas the firing times of general transitions may depend on the discrete marking only. In general, the behaviour of a FSPN may depend on the discrete and, in a limited way, the continuous state.

Immediate transitions are enabled if the fluid input places contain the amount of fluid, that is removed by the firing of the immediate transition or by a guard that is a function of the complete state description. When firing, they remove or deposit a random amount sampled from the normal distribution with the parameters of the random variable the arc is labelled with. If more than one immediate transition is enabled at once and there is a conflict, they fire according to their weights. This is meant to be a definition as close as possible to the definition of the corresponding discrete model and is only meaningful when using a fluid place as a continuously approximated discrete one.

We allow at most one general transition to be enabled at once. For each general transition $g \in \mathcal{T}^G$ the firing time is specified by a *probability distribution function* (PDF) $F^g(x)$ which we do not allow to depend on the marking. As firing policy *race with enabling memory* is used.

Each single fluid place can be regarded as a fluid queueing system. The arrival process is determined by the sum of all input processes from timed transitions and the departure process by the output arcs to all timed transitions. Thus, the results from queueing theory can be applied [8]. Each single arrival and departure process is normally distributed and specified by its expectation, the flow rate, and its variance. The fluid flow's distribution may be piecewise defined, depending on the complete marking. The flow rate function $R : \mathcal{A}_c \times \mathcal{M} \to \mathbb{R}^2$ is a normal distribution with mean and variance.

For each fluid place $k \in \mathcal{P}_c$ there is a random instantaneous flow rate $r_k(\mu_i, \mathbf{z})$ in every marking $\mu \in \mathcal{S}$ (if there is none, the entry is equal to zero). Expectation and variance of the instantaneous flow rate are determined by the sum of the parameters of the incoming flow minus the sum of the outgoing flow, as a linear combination of independent identically distributed (iid) random variables is distributed with the linear combination of the expectations and the sum of the variances. The rates are independent, as long as no upper or lower bound, which are to avoid overflow or negative content of fluid places, is reached. Boundary conditions preserve the independence in the PDEs. The random instantaneous rate functions for each fluid place and for each discrete marking are collected into a diagonal matrix

$$\mathbf{R}_k(\mathbf{z}) = \mathrm{diag}(r_k(\mu_i, \mathbf{z})) \qquad i = 1, \ldots, |\mathcal{S}|$$

where $|\mathcal{S}|$ denotes the number of discrete markings.

For the analysis of a FSPN the underlying stochastic process must be defined. The nodes of the reachability graph consist of all discrete markings supplemented by a vector of random variables for the fluid levels, plus an age variable if a general transition is enabled in that marking. It gives rise to a stochastic process in continuous time with continuous state space. Due to the supplemented variables, which provide a full description of each state, the stochastic process is memoryless and can be analyzed by using Markov theory.

If we did allow the enabling of exponential transitions to depend on the continuous marking the definition of the underlying Markov process would be more difficult. It could happen then, that a transition was enabled in a discrete state and by the firing of that transition the model would switch to an other state, whereas in the same discrete marking the mentioned transition could as well be not enabled, depending on the continuous marking. But on the level of the reachability graph we can not distinguish states that are identical in their discrete markings but differ in the continuous state description, so that different timed transitions are enabled, leading to different possible state changes. If transitions can be enabled by the continuous marking it is not possible to construct a Markov process by simply adding variables to the state description, because the behaviour of the Markov process can depend only in a very limited way on the value of the supplemented variables.

The stochastic process under consideration is

$$\{(N(t), X(t), \mathbf{Z}(t)), t \in \mathbb{R}_0^+\}$$

where $N(t)$ is the discrete marking at time t, $X(t)$ is the time elapsed since the enabling of the general transition g and $\mathbf{Z}(t)$ is a vector of length m, if there are m fluid places, which represents the fluid level in each fluid place. Since there is never more than one general transition enabled at once, the supplementary variable $X(t)$ is a scalar. If no general transition is enabled, $X(t)$ is not defined.

At time t the transient probability of being in discrete state i with fluid levels in an infinitesimal environment around z_k, for all fluid places $k \in \mathcal{P}_c$ is called the *volume density* and is denoted by $\pi_i(t, \mathbf{z}) = \partial/\partial\mathbf{z}\ P(N(t) = i, \mathbf{Z}(t) \leq \mathbf{z})$. If a general transition is enabled, the joint probability of being in state i, observing the fluid levels $\mathbf{z}$ and the remaining firing time x, is called the *volume age density* and is defined as

$$\pi_i(t, x, \mathbf{z}) = \frac{\partial}{\partial \mathbf{z}} \frac{\partial}{\partial x} P\left(N(t) = i, X(t) \leq x, \mathbf{Z}(t) \leq \mathbf{z}\right).$$

The probabilistic description of the dynamic behaviour of a second order non-Markovian FSPN will consist of a stochastic process in continuous time with continuous state space that is Markovian due to the supplemented variables. The movement of fluid will be described by a diffusion process. This leads to a system of second order partial differential equations with respect to the fluid levels and first order partial equations with respect to the age variables of the general transitions. The reflecting barrier at the origin and an upper bound of the fluid variables determine boundary conditions of the PDEs. If the diffusion process has variance zero it reduces to the special case of fluid flow, for which the PDEs have been derived in [10]. Due to limited space, formal derivation of the second order PDEs is ommited here.

Examples

We present two examples, one from each field of application for the proposed formalism. In the first example the fluid place is used as an approximation to a discrete place, and in the second for the modelling of temperature, which is considered to be a continuous system component. In figure 1 a continuously approx-

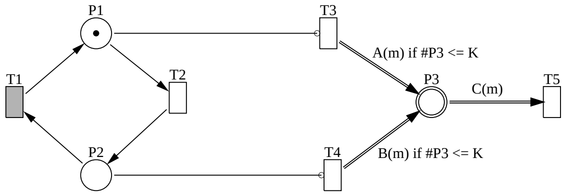

Figure 1: Non-Markovian FSPN model of a GMN/N/1/K queueing system

imated queueing system is shown. The arrival process is **G**enerally **M**odulated **N**ormal because transition t_1 has generally distributed firing time. The fluid place represents the buffer. The arrival processes $A(m)$ and $B(m)$ and the departure process $C(m)$ are normally distributed and may depend on the marking. Overflow can be avoided either by an appropriately chosen definition of the rate processes or by an upper bound of place P_3. The reachability graph of the model is

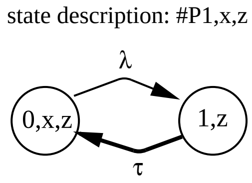

Figure 2: Reachability graph of a GMN/N/1/K queueing system

shown in figure 2. The underlying Markov process has only two discrete states, one of which is supplemented by two variables, the other by one. A measure of interest would be e.g. the probability of overflow of place P_3.

Figure 3 shows a model of a manufacturing system with a continuous system component. Work pieces are burnt and the naturally continuous quantity is the temperature of the kiln. Place P_5 represents the temperature of the kiln, the heating of which (at rate $A(m)$) is modelled by transition t_4 and cooling down (at rate $B(m)$) by t_5. Pieces enter the kiln one by one and queue up in front of it (P_2). The burning consumes heat (at rate $C(m)$) and time and is modelled with the general transition t_3. Transition t_3 is enabled only when a token is in P_3, which means there

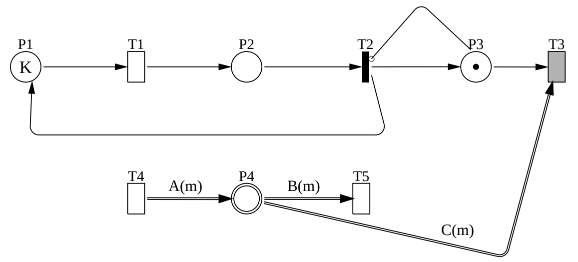

Figure 3: Non-Markovian FSPN model of a kiln

is a work piece in the oven. The system's capacity is limited to $(K+1)$ pieces.

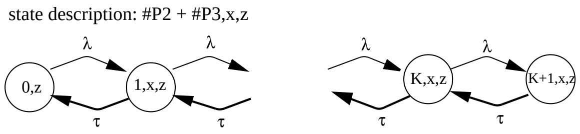

Figure 4: Reachability graph of the model of a kiln

One can ask for the average number of pieces being burnt in a certain period of time, or the expected temperature of the kiln.

Conclusions

An extension of SPNs has been presented, that allows for the modelling of continuous quantities such as temperatures arising in manufacturing systems as well as for a continuous approximation to heavily loaded discrete places. By using a second order approximation the probabilistic nature of discrete models is preserved and we expect a better model fit, than would be achieved with a first order model, as it is known from queueing theory. Deterministic fluid flow is still included as a special case. General transitions provide flexibility in the model's behaviour in time. The method of supplementary variables makes the underlying complex stochastic process Markovian and permits the derivation of PDEs by formulating the Kolmogorov forward equations in a direct and straight forward way.

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