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Hybrid Modelling with Second Order Fluid Stochastic Petri Nets

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Abstract

Stochastic Petri nets have been extended for the modelling of hybrid systems by adding fluid places, that are filled and emptied continuously at random (normally distributed) rate. The flow rates as well as the discrete transition rates may depend on the content a fluid place holds. The dynamics of the model are described by second order partial differential equations, that can be solved by using an implicit Crank-Nicolson scheme. Reward measures such as the fluid throughput can be computed from the continuous part of the solution. As an example we present a model of a chemical reactor.

1 Introduction

Stochastic Petri nets (SPNs) [1] are well suited for the model-based performance and dependability evaluation of complex systems. So far they are used for systems that take on discrete states only. Continuous Petri nets where first constructed by splitting tokens into a large number of particles [4], but they serve only poorly for performance evaluation since they do not include stochastic elements.

Purely continuous systems are usually modelled by using differential equations rather than graphical paradigms. The integration of both leads to *second order fluid Petri nets* for hybrid modelling. The continuous part of a model is then described by a diffusion process as it has been used before for the approximation of discrete quantities.

First and second order (fluid and diffusion) approximation to heavily loaded queueing systems are well known from queueing theory [10, 8]. Recently, the concept of fluid models was put in the context of SPNs [9, 7], referred to as *Fluid Stochastic Petri nets* (FSPNs). FSPNs have been developed to extend SPNs for the modelling of continuous quantities and are meant to be clearly divided into a discrete and a continuous submodel that can affect each other. FSPNs allow a very flexible definition of fluid models, but no probabilistic variation of the fluid flow is possible, since the fluid flow is described by a first order (fluid flow) process, leading to a system of first order partial differential equations (PDEs). Each equation represents a probability density function for one discrete state of the net and depends on the fluid variables, and, in the transient case, on time.

We will define a semantics of a fluid place that serves for the approximation of large contents of discrete places as well as for the modelling of continuous system components. Second order (diffusion) approximation will be used in order to allow non-deterministic modelling of continuous quantities. Using diffusion approximation the stochastic process is described by second order PDEs in the continuous variables representing the fluid levels.

The formalism presented here will be restricted to GSPNs [1] in the discrete part of the models, whereas in the continuous part any probability distribution function is possible. Derivation of the differential equations for models containing transitions with deterministic or generally distributed firing times in the discrete part of the model is performed by using the method of supplementary variables [2] following [6]. This will not be done here because no numerical solutions can be presented yet for non-Markovian FSPNs.

2 Model Definition

We define a second order FSPN following the common notation for stochastic Petri nets [1]. As in [7] FSPNs consist of *places, transitions* and *arcs*. The set of places is divided into *fluid (continuous)* and *discrete* places ($\mathcal{P} = \mathcal{P}_c \cup \mathcal{P}_d$), the former containing a continuous amount of fluid and the latter discrete *tokens*. The set of transitions $\mathcal{T} = \mathcal{T}_E \cup \mathcal{T}_I$ is composed of exponen-

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tially distributed, and immediate transitions, respectively. Places and transitions are connected by *input*, *output*, or *inhibitor* arcs $\mathcal{A} = \mathcal{A}_d \cup \mathcal{A}_c$, that may be discrete or continuous, as denoted by the subscripts d and c.

The *marking* $\mathbf{m} = (s, \mathbf{z})$ consists of a discrete part $s = (\#p_i, i \in \mathcal{P}_d)$, where $\#p_i$ denotes the number of discrete tokens in the i-th discrete place p_i, and a continuous part, a vector representing the fluid level in each fluid place $\mathbf{z} = (z_k, k \in \mathcal{P}_c)$. The initial marking is $\mathbf{m}_0 = (s_0, \mathbf{z}_0)$.

Arcs connecting fluid places and transitions are drawn like pipes and are labelled with a normally distributed random variable, that is specified by the distribution's expectation and variance. Fluid flows along arcs connecting a timed transition and a fluid place as long as the transition is enabled. A timed transition is enabled by the required number of tokens in all its discrete input places, or by a guard, which may depend on the complete state description. The firing times of exponential transitions may depend on the discrete and on the continuous marking. In general, the behaviour of an FSPN may depend on the discrete as well as on the continuous state.

For each fluid place $k \in \mathcal{P}_c$ there is a random instantaneous flow rate $r_k(s, \mathbf{z})$ in every marking $s \in \mathcal{S}$ (if there is none, the entry is equal to zero). Expectation μ_k and variance σ_k^2 of the instantaneous flow rate are determined by the sum of the parameters of the incoming flow minus the sum of the outgoing flow, since a linear combination of independent identically distributed (iid) random variables is distributed with the linear combination of the expectations and the sum of the variances. The expectations and variances for each fluid place are collected into diagonal matrices:

$$\boldsymbol{\mu}_k = \mathbf{diag}\,(\mu_{k,1}, \dots, \mu_{k,|\mathcal{S}|}), \tag{1}$$

$$\boldsymbol{\sigma}_k^2 = \mathbf{diag}\,(\sigma_{k,1}^2, \dots, \sigma_{k,|\mathcal{S}|}^2). \tag{2}$$

The rates are independent, as long as no upper or lower bound, which are used to avoid overflow or negative content of fluid places is reached. Boundary conditions preserve the independence in the PDEs.

For the analysis of an FSPN the underlying stochastic process must be defined. The nodes of the reachability graph consist of all discrete markings supplemented by a vector of random variables for the fluid levels. It gives rise to a stochastic process in continuous time with continuous state space.

The departure process is independent of the arrival process only as long as the fluid place is not empty or full. For the derivation of the diffusion equations we assumes this to be the case. Later on a boundary condition will replace the differential equation in case of an empty (or full) fluid place.

To fully specify the diffusion process the expectations μ_α, μ_δ and the variances $\sigma_\alpha^2, \sigma_\delta^2$ of the arrival stream (α) and the departure stream (δ) have to be known.

Let $A(t)$ be the distribution of the arrival periods for a unit of fluid (with known expectation μ_a and variance σ_a^2) and let $B(t)$ be the distribution of the departure periods for a unit of fluid (with known expectation μ_b and variance σ_b^2). Both may be arbitrary probability distribution functions. Then for the arrival process we have

$$\mu_{\alpha(t)} = \frac{t}{\mu_a(t)}, \qquad \sigma_{\alpha(t)}^2 = \sigma_a^2(t) \cdot \frac{t}{(\mu_a(t))^3}$$

and for the departure process

$$\mu_{\delta(t)} = \frac{t}{\mu_b(t)}, \qquad \sigma_{\delta(t)}^2 = \sigma_b^2(t) \cdot \frac{t}{(\mu_b(t))^3}.$$

For deriving the differential equations the infinitesimal mean and variance, which are derivatives with respect to time of the conditional mean and variance (see Kleinrock [8]) are required. They evaluate for all time homogenous arrival and departure processes to

$$\mu_{(\alpha)} = \frac{1}{\mu_a}, \qquad \sigma_\alpha^2 = \frac{\sigma_a^2}{(\mu_a)^3}.$$

and μ_δ and σ_δ^2 analogously.

In most cases we are interested in the actual fluid change in a place. All the incoming and outgoing streams in one marking add and we get the following parameters

$$\mu = \sum_{arrivals} \frac{1}{\mu_a} - \sum_{departures} \frac{1}{\mu_b} \tag{3}$$

and

$$\sigma^2 = \sum_{arrivals} \frac{\sigma_a^2}{(\mu_a)^3} + \sum_{departures} \frac{\sigma_b^2}{(\mu_b)^3} \tag{4}$$

because variances must add.

3 Transient Analysis

The stochastic process under consideration is a Markov process in continuous time with mixed discrete and continuous state space (see [3]).

$$\{(S(t), \mathbf{Z}(t)), t \in \mathbb{R}_0^+\}$$

where $S(t)$ is the discrete marking at time t, and $\mathbf{Z}(t)$ is a vector of length m, if there are m fluid places, which represents the random variables of the fluid levels in each fluid place.

At time t the transient probability of being in discrete state i with fluid levels in an infinitesimal environment around z_k, for all fluid places $k \in \mathcal{P}_c$ is called the *fluid density* and is denoted by $\pi_i(t, \mathbf{z}) = \partial / \partial \mathbf{z} \; P(S(t) = i, \mathbf{Z}(t) \leq \mathbf{z})$.

The Kolmogorov forward equation is

$$\boldsymbol{\pi}(t+\Delta, \mathbf{z}) = \sum_k \int \boldsymbol{\pi}(t+\Delta, z_k | t, z_{k0}) \boldsymbol{\pi}(t, z_{k0}) dz_{k0} + \boldsymbol{\pi}(t, z)\mathbf{Q}(\mathbf{z})\Delta + o(\Delta) \quad (5)$$

The differential equation obtained from eqn. 5 is as shown in [11]

$$\frac{\partial \boldsymbol{\pi}(t,\mathbf{z})}{\partial t} + \sum_k \frac{\partial \boldsymbol{\pi}(t,\mathbf{z})}{\partial z_k} \boldsymbol{\mu}(\mathbf{z}) - \sum_k \frac{\partial^2 \boldsymbol{\pi}(t,\mathbf{z})}{\partial z_k \, z_k'} \, \frac{1}{2} \, \boldsymbol{\sigma}^2(\mathbf{z}) - \boldsymbol{\pi}(t,\mathbf{z})\mathbf{Q}(\mathbf{z}) = 0. \quad (6)$$

Note, that the generator matrix $\mathbf{Q}$ may depend on the value of the the vector of fluid levels $\mathbf{z}$. A reflecting barrier at the origin (and at an upper bound, if there is one) define the boundary condition for the fluid flow [3].

$$\left[\frac{1}{2} \frac{\partial}{\partial z_k} \boldsymbol{\pi}(t,z) \boldsymbol{\sigma}_k^2(\mathbf{z}) - \boldsymbol{\pi}(t,z) \boldsymbol{\mu}_k(\mathbf{z}) \right]_{z_k = 0} = 0. \quad (7)$$

Let $\boldsymbol{\pi}_0$ be the vector of the initial discrete state probabilities and let $\mathbf{z}_0$ be the vector of initial fluid levels then the inital conditions are

$$\boldsymbol{\pi}(0, \mathbf{z}) = \boldsymbol{\pi}_0 \cdot \delta(\mathbf{z} - \mathbf{z}_0). \quad (8)$$

Eqn. 6 is solved by using an implicit discretization technique, the Crank-Nicolson scheme, combined with an upwind descretization for the first derivatives in space. The method is absolutely stable and the truncation error is $O(h + k^2)$, where k is the stepsize in time and h in space.

4 Example

As an example of a hybrid system we model a chemical reactor that produces a liquid from two different substances (see figure 1). We will study the transient behaviour of the system with respect to the reactor's average content and throughput. The two substances enter the reactor simultaneously at different rates (fill1 and fill2) but in constant relation. When the reactor is full, the inflow is stopped, the content is stirred and susequently drained out. We assume that opening and closing of valves may cause random delays. This is modelled with the transitions $t3$ (closing the inflow valves) and $t5$ (closing the outgoing and opening the inflow valves). They are enabled only when the reactor is almost full (contains at least 4.8 units of fluid), or almost empty (contains at most 0.2 units of fluid), respectively. So, the two transitions are enabled not only in certain discrete states but also only in a subset of the continuous state space.

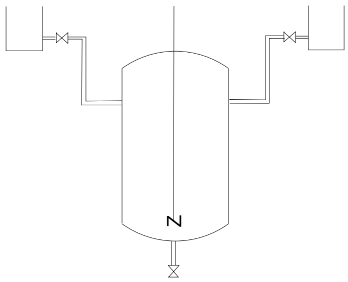

Figure 1. Model of the hybrid system

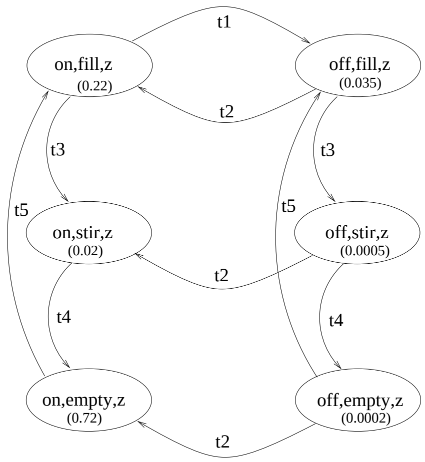

Figure 2. Reachability graph of the model

After exponentially distributed time intervals an inflow container is empty and needs to be replaced by a new one. In the meantime the inflow is switched off completely, in order to keep the relation of the two substances constant. Figure 2 shows the reachability graph of the Petri net model in figure 3. In brackets are the steady state probabilities. The parameters for the fluid flow are set to

$$\boldsymbol{\mu} = \mathbf{diag}\,(fill1 + fill2, 0, 0, 0, -empty, -empty)$$
$$\boldsymbol{\sigma^2} = \mathbf{diag}\,(fill1 + fill2, 0, 0, 0, empty, empty).$$

where

$$fill1(z) = \begin{cases} 0.8 & \text{if} z \leq 4 \\ 0.4 & \text{else} \end{cases}$$

$$fill2(z) = \begin{cases} 1.6 & \text{if} z \leq 4 \\ 0.8 & \text{else} \end{cases}$$

$$empty(z) = \frac{z^2}{4}$$

(see eqn. 1 and eqn. 2 on page 2). They are now content dependent functions. The fluid place is filled at constant rate. When it contains four units of fluid the inflow rates are reduced to half. The speed, at which the place empties is proportional to the fluid level. The underlying Markov process has one continuous axis and

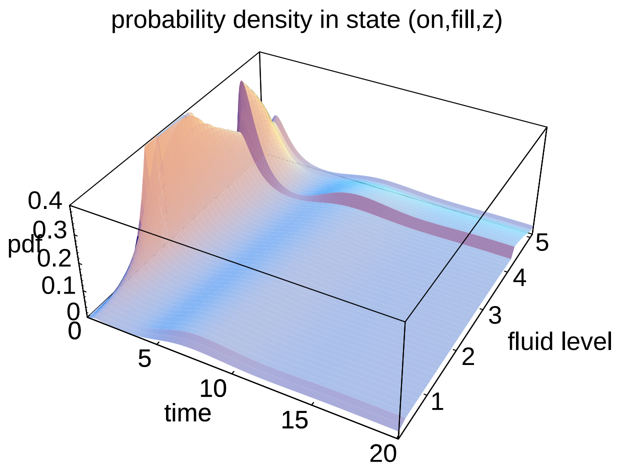

Figure 4. Fluid density in state (on,fill,z)

six discrete states each of which is supplemented by a variable representing the fluid level.

Figures 4, 5 and 6 show the transient fluid densities in the states (on,fill,z), (on,stir,z) and (on,empty,z). At $z = 4$ the step in the inflow function is visible. In figure 5 a state is presented, that is only entered when the fluid place holds at least 4.8 units of fluid. Figure 7 shows the overall fluid density, which is computed by adding the densities in all discrete states. Here both steps, at 4 and at 4.8 units of fluid become apparent.

We compute as a measure of interest the expected content of the reactor and it's throughput over a transient period of 20 units of time. Both are shown in figure 8.

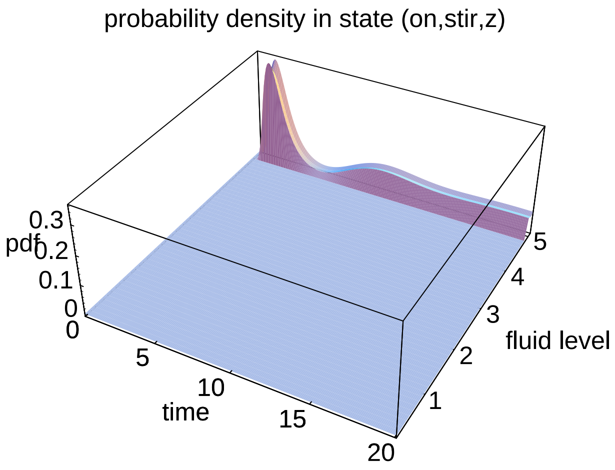

Figure 5. Fluid density in state (on,stir,z)

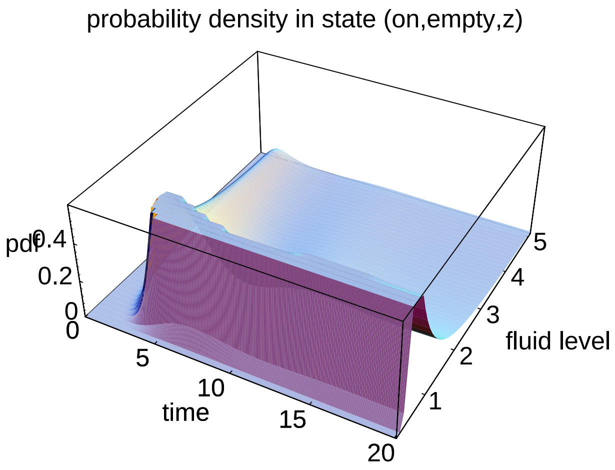

Figure 6. Fluid density in state (on,empty,z)

5 Conclusions

An extension of SPNs has been presented, that allows for the modelling of continuous quantities arising in many technical processes as well as for a continuous approximation to heavily loaded discrete places. By using a second order approximation the probabilistic nature of the systems is preserved.

Dependencies between discrete and continuous part of a system can now be modelled very flexibly. Both, the continuous (fluid flow rates) and the discrete (dis-

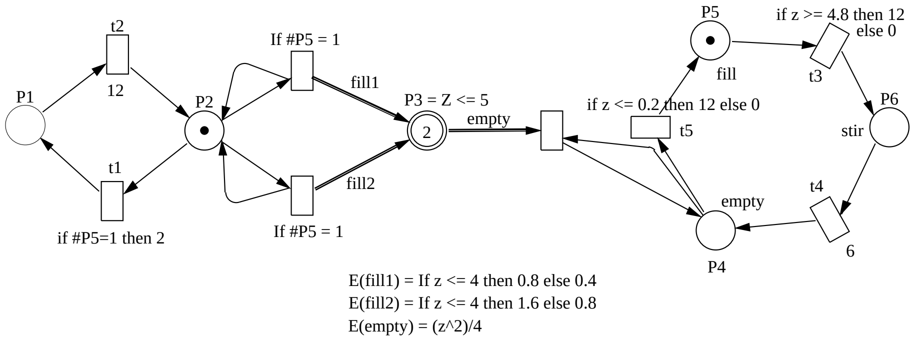

Figure 3. Second order FSPN model of the hybrid system

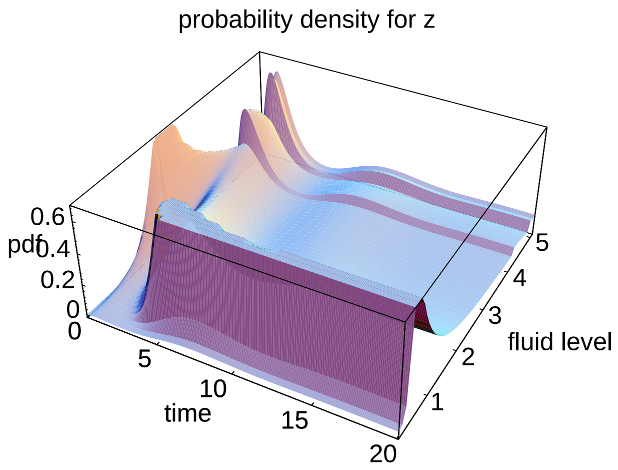

Figure 7. Probability density for the fluid level

crete state transitions) actions in the model may depend on the complete state description (the continuous as well as the discrete marking).

We have modelled a hybrid system where the discrete part of the model represents the control over the continuous part and is at the same time regulated by the continuous model description. We have shown how to calculate performance measures such as the throughput from the continuous part of the model.

But still, the analysis component needs to be extended in different ways: for fluid modelling of more than one continuous system component (the inflow containers and the reactor, for instance) and for solving models in which actions in the discrete part are other than exponentially distributed. For models with purely discrete state space the latter is possible under certain restrictions and the analysis is implemented in the software tool TimeNET [5].

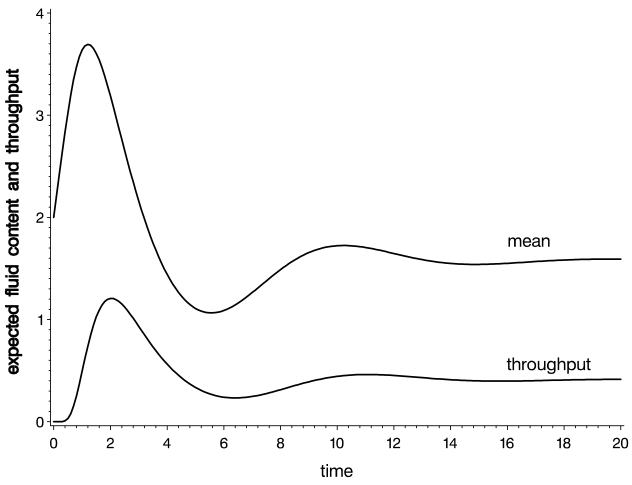

Figure 8. Measures for the hybrid system

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