in: Proc. World Congress on Systems Simulation ,pp. 328–332 *Singapore, Sept. 1-3, 1997.*

Second Order Fluid Stochastic Petri Nets: an Extension of GSPNs for Approximate and Continuous Modelling

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Abstract

A Petri net formalism is presented that allows for mixed discrete and continuous stochastic models. The continuous part of the models consists of fluid places that are filled and emptied at random (normally distributed) rate. Fluid places can be used for continuous approximation of heavily loaded discrete places in order to avoid state space explosion as well as for the modelling of continuous system components. The dynamics of a second order FSPN are described by second order partial differential equations. They can be solved by using an implicit Crank-Nicolson scheme. As an example a processing system with modulated arrival stream and service break downs is used. QoS measures such as waiting time, loss rate and throughput are calculated for both a discrete and a fluid model.

1 Introduction

Stochastic Petri nets (SPNs) [1] are well suited for the model-based performance and dependability evaluation of computer and communication systems. As complexity of the systems increases the state space of the models tends to be extremely large so that often no analytical solution can be obtained. Furthermore, certain properties of SPNs do not always seem appropriate. In the context of fluid models the inadequacy of the exponential distribution in describing the system's behaviour and the inability to model continuous quantities that may be present in the system come into consideration.

To reduce state space complexity various methods are known such as lumping, decomposition [2], or continuous approximation [11]. Continuous Petri net models where first constructed from non-stochastic Petri nets by splitting tokens into an infinitely large number of particles [5].

First and second order (fluid and diffusion) approximation to heavily loaded queueing systems are well known from queueing theory [11, 9]. Recently, the concept of fluid models was put in the context of SPNs [10, 8], referred to as *Fluid Stochastic Petri nets* (FSPNs). FSPNs have been invented to extend SPNs for the modelling of continuous quantities and are meant to be clearly divided into a discrete and a continuous submodel that can affect each other. FSPNs allow a very flexible definition of fluid models, but no probabilistic variation of the fluid flow is possible, since the fluid flow is described by a first order (fluid flow) process, leading to a system of first order partial differential equations (PDEs).

We will define a semantics of a fluid place that serves for the approximation of large contents of discrete places as well as for the modelling of continuous system components.

Second order (diffusion) approximation will be used in order to preserve the random nature of a continuously approximated discrete place and to allow non-deterministic modelling of continuous quantities. Using diffusion approximation the stochastic process is described by second order PDEs in the continuous variables representing the fluid levels.

The formalism presented here will be restricted to GSPNs [1] in the discrete part of the models, whereas in the continuous part any probability distribution function is possible. Derivation of the differential equations for models containing transitions with deterministic or generally distributed firing times in the discrete part of the model is carried out by using the method of supplementary variables [3] following [7].

In order to give an impression of the quality of a second order fluid model a queueing system will be solved both as a continuous and as a discrete model, the latter is analyzed with `TimeNET` [6].

This paper is organized as follows: in the next Section second order FSPNs are defined and their temporal behaviour is described, in Section 3 transient analysis is carried out and in Section 4 the stationary analysis. Section 5 contains an example and Section 6 concludes.

[1] The author's research is supported by a PhD scholarship from the German Research Council (DFG) under grant GRK 86.

2 Model Definition and Behaviour

We define a second order FSPN following the common notation for stochastic Petri nets [1]. As in [8] FSPNs consist of *places, transitions* and *arcs*. The set of places is divided into *fluid (continuous)* and *discrete* places ($\mathcal{P} = \mathcal{P}_c \cup \mathcal{P}_d$), the former containing a continuous amount of fluid and the latter discrete *tokens*. The set of transitions $\mathcal{T} = \mathcal{T}_E \cup \mathcal{T}_I$ is composed of the exponentially distributed, and the immediate transitions, respectively. Places and transitions are connected by *input*, *output*, or *inhibitor* arcs $\mathcal{A} = \mathcal{A}_d \cup \mathcal{A}_c$, that may be discrete or continuous, as denoted by the subscripts d and c.

The *marking* $\mathbf{m} = (s, \mathbf{z})$ consists of a discrete part $s = (\#p_i, i \in \mathcal{P}_d)$, where $\#p_i$ denotes the number of discrete tokens in the i-th discrete place p_i, and a continuous part, a vector representing the fluid level in each fluid place $\mathbf{z} = (z_k, k \in \mathcal{P}_c)$. The initial marking is $\mathbf{m}_0 = (s_0, \mathbf{z}_0)$.

Arcs connecting fluid places and transitions are drawn like pipes and are labelled with the flow rate function $R : \mathcal{A}_c \times \mathcal{M} \to \mathbb{R}^2$ a normally distributed random variable, that is specified by the distribution's expectation μ and variance σ^2. Fluid flows along an arc connecting a timed transition and a fluid place as long as the transition is enabled. A timed transition is enabled only by the required number of tokens in all its discrete input places, or by a guard.

We use immediate transitions for removing and depositing a certain amount of fluid at once, that means for jumps in the continuous part of the state space. But fluid immediate transitions do not fire at once without a delay as do their discrete counterparts. Their firing occurs in a Poisson process at rate λ. The rates in all discrete states are collected into a matrix $\mathbf{\Lambda}$. When firing, they remove or deposit a random amount sampled from the distribution $b(.)$. The matrix $\mathbf{\Delta}_k$ contains the switching probabilities after the firing of a fluid immediate transition to other discrete states.

Each single fluid place can be regarded as a fluid queueing system. The arrival process is determined by the sum of all input processes from timed transitions and the departure process by the output arcs to all timed transitions. Thus, the results of queueing theory can be applied [11].

We now define the parameters of a single fluid inflow or outflow process. Let $\alpha(t)$ be the total number of arrivals up to time t. Then

$$P(\alpha(t) \geq n) = P(\tau_n \leq t)$$

where τ_n is the arrival time of the n-th customer. This is equal to the sum of n interarrival times, since $\tau_n = t_1 + t_2 + \ldots t_n$. The interarrival times are assumed to be independent and identically distributed with distribution function $A(t)$. So if the time t and the number n get large, τ_n is the sum of a large number of iid random variables. Thus the central limit theorem holds and we assume $\alpha(t)$ (and analogously the departure process $\delta(t)$) to be normally distributed. This holds for each incoming and outgoing flow to a fluid place.

Let $A(t)$ be the distribution of the interarrival times (with known expectation μ_a and variance σ_a^2) and let $D(t)$ be the distribution of the service times (with known expectation μ_d and variance σ_d^2). Both may be any probability distribution functions. Then for the arrival process we have

$$\mu_{\alpha(t)} = \frac{t}{\mu_a(t)}, \qquad \sigma^2_{\alpha(t)} = \sigma_a^2(t) \cdot \frac{t}{(\mu_a(t))^3}$$

and for the departure process

$$\mu_{\delta(t)} = \frac{t}{\mu_b(t)}, \qquad \sigma^2_{\delta(t)} = \sigma_d^2(t) \cdot \frac{t}{(\mu_d(t))^3}.$$

For deriving the differential equations the infinitesimal mean and variance, which are derivatives with respect to time of the conditional mean and variance (see [9]) are required. They evaluate for all time homogenous arrival and departure processes to

$$\mu_{(\alpha)} = \frac{1}{\mu_a}, \qquad \sigma_\alpha^2 = \frac{\sigma_a^2}{(\mu_a)^3}.$$

and μ_δ analogously.

In the underlying stochastic process each fluid place is represented by an axis in the mixed continuous and discrete state space. All the incoming and outgoing streams in one marking add and we get for the random instantaneous flow rate $r_k(s, \mathbf{z})$ in every marking $s \in \mathcal{S}$ the following parameters

$$\mu_{k,s} = \sum_{arrivals} \frac{1}{\mu_a} - \sum_{departures} \frac{1}{\mu_b} \quad \forall k \in \mathcal{P}_c, s \in \mathcal{S}, \quad (1)$$

$$\sigma^2_{k,s} = \sum_{arrivals} \frac{\sigma_a^2}{(\mu_a)^3} + \sum_{departures} \frac{\sigma_b^2}{(\mu_b)^3} \quad \forall k \in \mathcal{P}_c, s \in \mathcal{S}. \quad (2)$$

because variances must add. The expectations and variances for each fluid place are collected into diagonal matrices (if there is none in a discrete state, the entry is equal to zero):

$$\boldsymbol{\mu}_k = \mathbf{diag}(\mu_{k,1}, \ldots, \mu_{k,|\mathcal{S}|}), \quad (3)$$

$$\boldsymbol{\sigma}_k^2 = \mathbf{diag}(\sigma^2_{k,1}, \ldots, \sigma^2_{k,|\mathcal{S}|}). \quad (4)$$

3 Transient Analysis

The stochastic process under consideration is a Markov process in continuous time with mixed discrete and continuous state space (see [4]).

$$\{(S(t), \mathbf{Z}(t)), t \in \mathbb{R}_0^+\}$$

where $S(t)$ is the discrete marking at time t, and $\mathbf{Z}(t)$ is a vector of length m, if there are m fluid places, which represents the fluid level in each fluid place.

At time t the transient probability of being in discrete state i with fluid levels in an infinitesimal environment around z_k, for all fluid places $k \in \mathcal{P}_c$ is called the *fluid density* and is denoted by $\pi_i(t, \mathbf{z}) = \partial/\partial \mathbf{z} \ P(S(t) = i, \mathbf{Z}(t) \leq \mathbf{z})$.

The Kolmogorov forward equation is

$$\boldsymbol{\pi}(t+\Delta, \mathbf{z}) = \sum_k \int \boldsymbol{\pi}(t+\Delta, \mathbf{z}|t, \mathbf{z}')\, \mathbf{diag}\,(\boldsymbol{\pi}(t, \mathbf{z}'))\, dz'_k + \boldsymbol{\pi}(t, \mathbf{z})\Big(\mathbf{Q}(\mathbf{z}) - \sum_k \boldsymbol{\Lambda}_k(\mathbf{z})\Big)\Delta + \sum_k \Big(\int_0^{z_k} \boldsymbol{\pi}(t, \mathbf{z} - \mathbf{y_k}) b(y) dy + \int_{z_k}^{z_k^{max}} \boldsymbol{\pi}(t, \mathbf{z_y}) b(z_k - y) dy \Big)\, \boldsymbol{\Lambda}_k(\mathbf{z}) \boldsymbol{\Delta}_k(\mathbf{z}) \Delta + o(\Delta) \quad (5)$$

where $\mathbf{y_k} = (y_{k,i})_{i=1,\dots,m}$ with $y_{k,i} = y$ if $i = k, y_{k,i} = 0$ otherwise and $\mathbf{z_y} = (z_{y,i})_{i=1,\dots,m}$ with $z_{y,i} = y$ if $i = k, z_{y,i} = z_i$ otherwise.. Substituting as in [11] we get

$$\boldsymbol{\pi}(t+\Delta, \mathbf{z}) = \boldsymbol{\pi}(t, \mathbf{z}) - \Delta \sum_k \frac{\partial \boldsymbol{\pi}(t, \mathbf{z})}{\partial z_k} \boldsymbol{\mu}_k + \boldsymbol{\pi}(t, \mathbf{z})\Big(\mathbf{Q}(\mathbf{z}) - \sum_k \boldsymbol{\Lambda}_k(\mathbf{z})\Big)\Delta + \sum_k \frac{\partial^2 \boldsymbol{\pi}(t, \mathbf{z})}{2\partial z_k^2}(\Delta^2 \boldsymbol{\mu}_k^2 + \boldsymbol{\sigma}_k^2 \cdot \Delta) + \sum_k \Big(\int_0^{z_k} \boldsymbol{\pi}(t, \mathbf{z} - \mathbf{y_k}) b(y) dy + \int_{z_k}^{z_k^{max}} \boldsymbol{\pi}(t, \mathbf{z_y}) b(z_k - y) dy \Big) \boldsymbol{\Lambda}_k(\mathbf{z}) \boldsymbol{\Delta}_k(\mathbf{z}) \Delta + o(\Delta)$$

$$\Leftrightarrow \frac{\boldsymbol{\pi}(t+\Delta, \mathbf{z}) - \boldsymbol{\pi}(t, \mathbf{z})}{\Delta} = -\sum_k \frac{\partial \boldsymbol{\pi}(t, \mathbf{z})}{\partial z_k} \boldsymbol{\mu}_k + \boldsymbol{\pi}(t, \mathbf{z})\Big(\mathbf{Q}(\mathbf{z}) - \sum_k \boldsymbol{\Lambda}_k(\mathbf{z})\Big) + \sum_k \frac{\partial^2 \boldsymbol{\pi}(t, \mathbf{z})}{2\partial z_k^2}(\Delta \boldsymbol{\mu}_k^2 + \boldsymbol{\sigma}_k^2) + \sum_k \Big(\int_0^{z_k} \boldsymbol{\pi}(t, \mathbf{z} - \mathbf{y_k}) b(y) dy + \int_{z_k}^{z_k^{max}} \boldsymbol{\pi}(t, \mathbf{z_y}) b(z_k - y) dy \Big) \boldsymbol{\Lambda}_k(\mathbf{z}) \boldsymbol{\Delta}_k(\mathbf{z}) + o(\Delta)$$

$$\Delta \to 0: \quad \frac{\partial \boldsymbol{\pi}(t, \mathbf{z})}{\partial t} + \sum_k \frac{\partial \boldsymbol{\pi}(t, \mathbf{z})}{\partial z_k} \boldsymbol{\mu}_k - \sum_k \frac{1}{2} \frac{\partial^2 \boldsymbol{\pi}(t, \mathbf{z})}{\partial z_k^2} \boldsymbol{\sigma}_k^2 - \boldsymbol{\pi}(t, \mathbf{z})\Big(\mathbf{Q}(\mathbf{z}) - \sum_k \boldsymbol{\Lambda}_k(\mathbf{z})\Big) - \sum_k \Big(\int_0^{z_k} \boldsymbol{\pi}(t, \mathbf{z} - \mathbf{y_k}) b(y) dy + \int_{z_k}^{z_k^{max}} \boldsymbol{\pi}(t, \mathbf{z_y}) b(z_k - y) dy \Big) \boldsymbol{\Delta}_k = 0. \quad (6)$$

Note, that the discrete transition rates may depend on the fluid levels.

A reflecting barrier at the origin (and at an upper bound, if there is one) define the boundary condition for the fluid flow [4].

$$\left[\frac{1}{2} \frac{\partial}{\partial z_k} \boldsymbol{\pi}(t, \mathbf{z}) \boldsymbol{\sigma}_k^2 - \boldsymbol{\pi}(t, \mathbf{z}) \boldsymbol{\mu}_k \right]_{z_k = 0} = 0 \quad (7)$$

Let $\boldsymbol{\pi}_0$ be the vector of initial discrete state probabilities and let $\mathbf{z_0}$ be the vector of initial fluid levels then the inital conditions are

$$\boldsymbol{\pi}(0, \mathbf{z}) = \boldsymbol{\pi}_0 \cdot \delta(\mathbf{z} - \mathbf{z}_0). \quad (8)$$

4 Steady State Analysis

In the following only models without immediate transitions in the continuous part will be considered. For the steady state analysis, the rate functions are required to be constant over the fluid levels. We will regard models with only one fluid place and with integrating each component separately we denote

$$\boldsymbol{\pi} = \int \boldsymbol{\pi}(z)\ dz.$$

Eqn. 6 then reduces to:

$$\frac{\partial \boldsymbol{\pi}(z)}{\partial z} \boldsymbol{\mu} - \frac{1}{2} \frac{\partial^2 \boldsymbol{\pi}(z)}{\partial z^2} \boldsymbol{\sigma}^2 - \boldsymbol{\pi} \mathbf{Q} = \mathbf{0}. \quad (9)$$

In state of equilibrium it is known that $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$. It is thus possible to compute the vector of state probabilities $\boldsymbol{\pi}$ by solving $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$. The fluid densities then are determined by the solutions of

$$\frac{\partial \boldsymbol{\pi}(z)}{\partial z} \boldsymbol{\mu} - \frac{1}{2} \frac{\partial^2 \boldsymbol{\pi}(z)}{\partial z^2} \boldsymbol{\sigma}^2 = 0. \quad (10)$$

Integrating twice and the use of boundary condition 7 (see [11]) yields for every discrete state $s \in \mathcal{S}$

$$\pi_s(z) = C \cdot \exp^{\frac{2\mu_s}{\sigma_s^2} z} \quad (11)$$

where C is a constant of integration. Since

$$\int C \cdot \exp^{\frac{2\mu_s}{\sigma_s^2} z}\ dz = \pi_s \quad (12)$$

the following three cases can be distinguished

$$
\begin{aligned}
1)\quad \mu_s < 0 \quad : \quad \int_0^\infty C \cdot \exp^{\frac{2\mu_s}{\sigma_s^2} z}\, dz &= \pi_s \\
\Rightarrow \quad C &= -\frac{2\mu_s}{\sigma_s^2} \cdot \pi_s \\
\Rightarrow \quad \pi_s(z) &= -\pi_s \cdot \frac{2\mu_s}{\sigma_s^2} \exp^{\frac{2\mu_s}{\sigma_s^2} z} \qquad (13)
\end{aligned}
$$

If $\pi_s(z)$ is computed on a bounded interval $[0, z_{max}]$ eqn. 13 has the more general form

$$\pi_s(z) = \pi_s \cdot \frac{2\mu_s}{\sigma_s^2 \cdot (\exp^{\frac{2\mu_s}{\sigma_s^2} z_{max}} - 1)} \cdot \exp^{\frac{2\mu_s}{\sigma_s^2} z}. \qquad (14)$$

Note, that since $\mu_s < 0$ if $z_{max} \to \infty$ eqn. 14 approaches eqn. 13.

$$
\begin{aligned}
2)\quad \mu_s > 0 \quad : \quad C &= \pi_s \cdot \frac{2\mu_s}{\sigma_s^2 \cdot \exp^{\frac{2\mu_s}{\sigma_s^2} z_{max}} - 1} \\
\Rightarrow \quad \pi_s(z) &= \pi_s \cdot \frac{2\mu_s}{\sigma_s^2 \cdot (\exp^{\frac{2\mu_s}{\sigma_s^2} z_{max}} - 1)} \cdot \exp^{\frac{2\mu_s}{\sigma_s^2} z} \qquad (15)
\end{aligned}
$$

3) $\mu_s = 0$. In this case eqn. 10 on the preceding page reduces to

$$\frac{\sigma_s^2}{2} \frac{\partial^2 \pi_s(z)}{\partial z^2} = 0 \qquad (16)$$

and eqn. 11 becomes

$$\pi_s(z) = C \cdot \frac{2}{\sigma_s^2}. \qquad (17)$$

The constant C is determined by

$$
\begin{aligned}
\int_0^{z_{max}} C \cdot \frac{2}{\sigma_s^2}\, dz &= \pi_s \\
\Rightarrow \quad \pi_s(z) &= \pi_s \cdot \frac{1}{z_{max}}. \qquad (18)
\end{aligned}
$$

A steady state distribution on an infinite interval only exists if $\mu_s < 0$, so in both other cases only finite intervals are considered.

5 Example

Figure 1 shows a second order FSPN model of a queueing system which can be regarded as a model of a node in a communication network. The buffer is represented by the fluid place. The arrivals form a (normally) modulated stream. The values of the parameters arr1, arr2 and serve denote the firing rates of transitions $t3, t4$ and $t5$ of the corresponding discrete model. The infinitesimal flow rate parameters evaluate according to eqn. 1 and

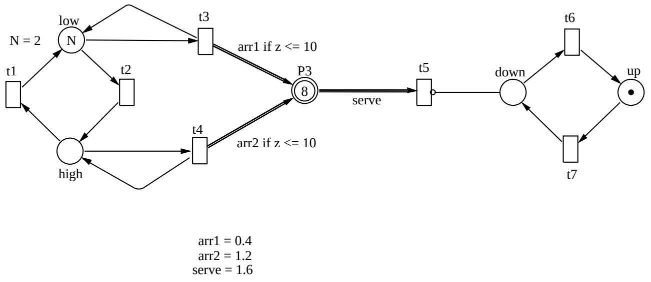

Figure 1: Second order FSPN model of a ΣNM/N/1/K queueing system

eqn. 2 (when counting the discrete states of the reachability graph in figure 2 from left to right, top to bottom) to

$$
\begin{aligned}
\boldsymbol{\mu} &= \mathbf{diag}(-1.2, 0.4, 0, 1.6, -0.4, 1.2) \\
\boldsymbol{\sigma^2} &= \mathbf{diag}(2.0, 0.4, 3.2, 1.6, 2.8, 1.2).
\end{aligned}
$$

The underlying Markov process has six discrete states each of which is supplemented by a variable representing the fluid level. Figure 3 and the left plot in figure 4 show

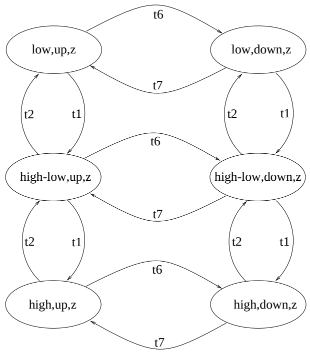

Figure 2: Reachability graph of the model in fig. 1

the probability density for the fluid level in a discrete state each. For the right side of figure 4 the fluid densities of all discrete states have been added, so it shows the overall density for z.

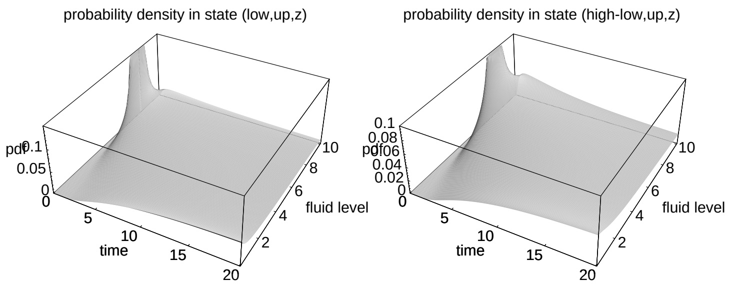

Figure 3: Fluid density in state one and three

At the end of the transient epoch analyzed, no steady state distribution is yet reached, so the curves in figure 4 differ from the cut through the transient densities at time 20 for the discrete states. If we had taken first

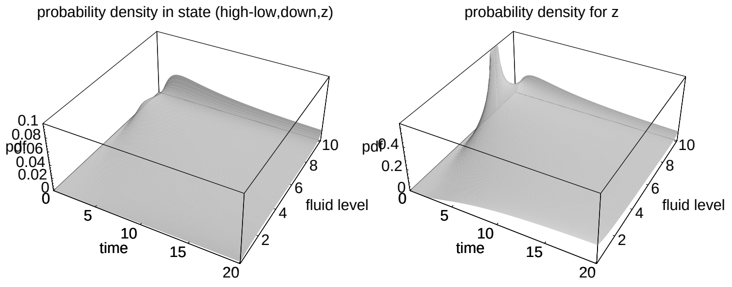

Figure 4: Probability density in state five and for all states

order approximation all steady state probability mass in state (low,up,z) would have been at $z = 0$. So, here the advantage of second order continuous processes becomes apparent. If there is no drift, (as in state (high-low,up,z)) probability mass will spread out homogeneously over the z-axis.

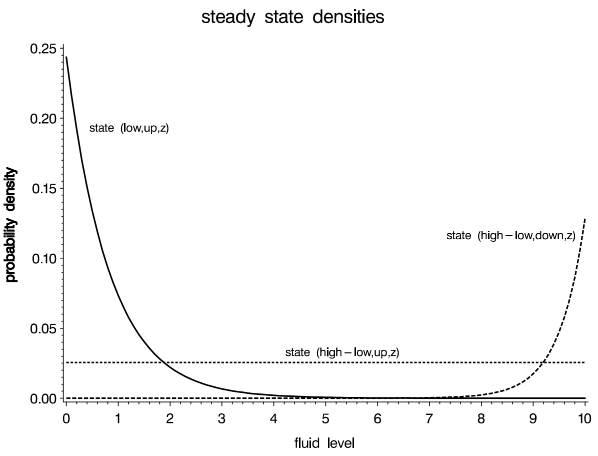

Figure 5: Steady state fluid densities for the discrete states

In figure 5 the waiting time of a customer in the queue, the loss rate, and the throughput of the server are plotted for the fluid and the discrete model. The loss rate

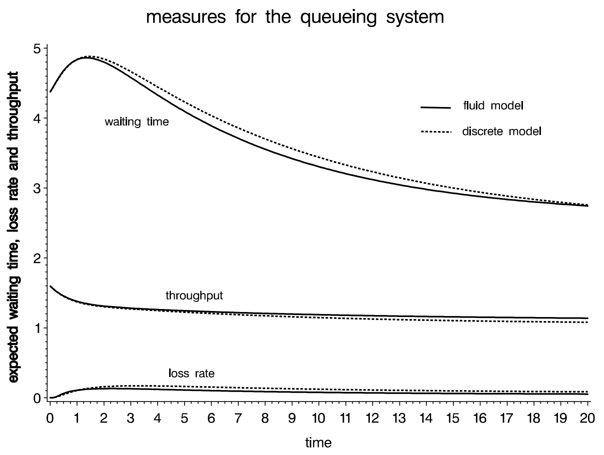

Figure 6: QoS measures for the processing system

is computed as $P(\text{buffer full}) * \sum P(\text{arrival stream on}) *$ arrival rate. In the discrete model $P(\text{buffer full}) = P(\text{buffer content} = 10)$, whereas in the continuous model $P(\text{buffer full}) = 1 - P(\text{buffer content} < 10)$ which is not the same. So, for the continuous queueing system we have defined $P(\text{buffer full}) = P(\text{buffer content} > 9.5)$. Obviously, we can not expect the results to be identical. The same considerations hold for the throughput. The expected waiting time decreases in the fluid model more rapidly then in the discrete one because of the smoothing property of the diffusion equation.

6 Conclusions

An extension of SPNs has been presented, that allows for the modelling of continuous quantities arising in manufacturing systems as well as for a continuous approximation to heavily loaded discrete places. By using a second order approximation the probabilistic nature of discrete models is preserved. The model fit seems reasonable, when comparing the measures from the continuous and the discrete model in figure 5 as it was expected because of known results from queueing theory.

Much work is still left for the future, such as an efficient, maybe parallel, implementation of the solution algorithm, including immediate transitions in the fluid part of the models, an extension to several fluid dimensions and the inclusion of general firing times in the discrete part of the models.

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