MODELLING HYBRID SYSTEMS WITH FLUID STOCHASTIC PETRI NETS

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Abstract

In this paper fluid stochastic Petri nets will be applied as a tool for performance analysis of hybrid systems. The general modelling formalism is introduced and a model of a chemical batch reactor is presented. The formalism allows to model flexibly the dependencies between the hybrid system and its discrete control mechanism.

Keywords: Stochastic Petri nets, second order stochastic fluid model, hybrid modelling.

1 Introduction

Specification as well as structural and quantitative analysis of hybrid systems have gained much attention over the last years. Among the formalisms that are used, Petri nets are the most popular. A number of different classes of hybrid Petri nets have been proposed. For the specification and qualitative analysis the autonomous non-timed and timed Petri nets (ACPNs) [4] have been used. Performance and dependability evaluation has been carried out with fluid stochastic Petri nets (FSPNs) [11, 6]. Various extensions of both have been developed recently.

Whereas the formalisms in [11, 6] are based on stochastic Petri nets, the continuous Petri nets (CPN) in [4] extend the timed and non-timed Petri nets. The latter are evaluated with simulation. For FSPNs numerical solution [6, 5, 12] as well as simulation [2] have been applied.

The discrete part of an FSPN taken on its own is a generalized stochastic Petri net (GSPN) [1] and the continuous part alone is equivalent to a fluid queueing system [9, 8]. In the original FSPNs in [11, 6] the continuous model part is represented by fluid places and fluid arcs due to deterministic in- and outflow. The level in the fluid places changes at deterministic rate. The stochastic behaviour of the model is determined by the discrete part and the dependencies between discrete and continuous submodel. The temporal behaviour of FSPNs is mathematically described by a system of first order partial differential equations (PDEs). In this paper a specific class of FSPNs is considered, the second order FSPNs. In second order FSPNs the fluid places are filled and emptied at normally distributed rate. Accordingly, the continuous model part has probabilistic behaviour too. The underlying equations are second order partial differential equations, hence the prefix *second order.* Second order FSPNs can, however, in some discrete states reduce to first order FSPNs, if in these states the variances of the fluid flow are zero. Then the second order derivative vanishes from the differential equation and a first order PDE corresponding to an ordinary (first order) FSPN is left [13]. If in all discrete states the variances of the fluid flow are zero, the whole model reduces to a first order FSPN. The idea of first and second order fluid models is known from queueing theory [7].

The example that is regarded in this paper is a model of a chemical reactor. The content of the reactor as well as its temperature are modelled with fluid places. The in- and outflow of the reactor happens at deterministic rate, whereas the temperature changes with some variation. The model is therefore a first order FSPN with respect to the first fluid place and a second order FSPN with respect to the second fluid place. For solving the equations underlying an FSPN a numerical solution method is used. The chosen discretization method is a flexible $\theta-$ scheme, that has the fully explicit, implicit and the Crank-Nicolson scheme as special cases.

In the next section the general second order FSPN formalism will be defined. In section 3 the stochastic process and the underlying differential equations are defined, section 4 holds the example and section 5 concludes.

2 The FSPN Formalism

For the definition of second order FSPNs the common notations for GSPNs [1] and for FSPNs [6] are taken over and extended as follows.

A second order FSPN is formally defined as a 8-tuple FSPN $= (\mathcal{P}, \mathcal{T}, \mathcal{A}, m_0, g, \lambda, R, w)$ where the set of places ($\mathcal{P} = \mathcal{P}_c \cup \mathcal{P}_d$) is divided into the *fluid (continuous)* and the *discrete* places. Discrete places (the elements of $\mathcal{P}_d$) are drawn as single lined circles and hold an integer number of *tokens*, whereas the continuous, or fluid, places (the elements of $\mathcal{P}_d$) are drawn as two concentrical circles and hold a real-valued amount of *fluid*. Fluid places have a limited capacity, that is bounded by $z^{\min}$ and $z^{\max}$. The set of transitions $\mathcal{T} = \mathcal{T}_E \cup \mathcal{T}_I$ is composed of the exponentially distributed, and the immediate transitions, respectively. Exponentially distributed transitions are drawn as empty rectangles and the immediate transitions as thin bars.

The set of arcs $\mathcal{A} = \mathcal{A}_d \cup \mathcal{A}_c$ is divided into two subsets, the discrete arcs (the elements of $\mathcal{A}_d$) and the continuous, or fluid, arcs ($\mathcal{A}_c$). The discrete arcs are drawn as single lined arrows, whereas the fluid double-lined arcs are drawn like pipes. The continuous arcs only connect transitions and fluid places, and vice versa. And the discrete arcs only connect transitions and discrete places. The generator matrix $\mathbf{Q}$ is defined as follows $\mathbf{Q} = (q_{ij})$ and

$$q_{ij} = \begin{cases} \lambda_t & \text{if} \quad i \neq j, \quad i,j \in \mathcal{S}, i \xrightarrow{t} j \\ -\sum\limits_{j,j\neq i} q_{ij} & \text{if} \quad i = j. \end{cases} \tag{1}$$

The *marking* $\mathbf{m} = (s, \mathbf{z}) \in \mathcal{M}$ consists of a discrete part $s = (\#p_i, i \in \mathcal{P}_d)$, where $\#p_i$ denotes the number of discrete tokens in the i-th discrete place p_i, and a continuous part, a vector representing the fluid level in each fluid place, $\mathbf{z} = (z_k, k \in \mathcal{P}_c) \in \mathcal{Z}$. The initial marking is $\mathbf{m}_0 = (s_0, \mathbf{z}_0)$. The total number of discrete states is S, the set of all discrete states is denoted $\mathcal{S}$.

$\mathrm{R}_{t,p}, \mathrm{R}_{p,t} : \mathcal{A}_c \times \mathcal{M} \to \mathbb{R}^2$ is the flow rate function along the fluid arc connecting the timed transition t and the fluid place p or vice versa. The flow rate $R = (ev, var)$ is a normally distributed random variable specified by expectation ev and variance var. Let Z_k be the fluid level in place $k \in \mathcal{P}_c$. Then the change rate of the fluid level is again a normally distributed random variable with

$$\frac{dZ_k}{dt} = \sum_{t \text{ enabled in } \mathbf{m}} R_{t,p}(\mathbf{m}) - \sum_{t \text{ enabled in } \mathbf{m}} R_{p,t}(\mathbf{m}).$$

Reflecting barriers assure, that the fluid level in each place stays within its range $[z^{\min}, z^{\max}]$.

$g_t : \mathcal{M} \to \mathbb{B}$ is the guard of transition t that can be a function of the discrete and the continuous state. The firing rate $\lambda_t : \mathcal{M} \to \mathbb{R} \cup \{\infty\}$ is a function of both the continuous and the discrete marking. Immediate transitions have firing rate $\lambda_t = \infty$.

The weight function $w_t : \mathcal{S} \to \mathbb{R}$ is defined for immediate transitions. The firing probability of each of the enabled immediate transitions in a vanishing state is $w_t(s) / \sum\limits_{t_i \text{enabled in } \mathbf{s}} w_{t_i}(s)$.

The enabling rules for the transitions are the same as in discrete Petri net models. The fluid arcs act only on the fluid places and do not influence the enabling conditions of the transitions, other than through a guard function.

3 Analysis

A second order FSPN gives rise to a stochastic process with mixed discrete and continuous state space that still is a Markov process. Therefore the PDEs describing the model's temporal behaviour can be derived by formulating the Kolmogorov forward equations. They result in a system of equations, one for each discrete state of the model. In states with probabilistic fluid flow they are diffusion-convection equations similar to the Fokker-Planck equation, whereas in states with deterministic flow they are first order PDEs. The single equations are coupled through the generator matrix $\mathbf{Q}$, describing the behaviour of the CTMC that underlies the discrete part of the model.

The change rate r_k, $k = 1, 2$ of the continuous variables is a normally distributed random variable that is defined by expectation $\mu_{s,k}$ and variance $\sigma^2_{s,k}$ in each discrete state s with

$$\mu_{s,k} = \sum_{t \text{ enabled in } \mathbf{m}} ev_{s,R_{t,p}} - \sum_{t \text{ enabled in } \mathbf{m}} ev_{s,R_{p,t}} \tag{2}$$

and

$$\sigma^2_{k,s} = \sum_{t \text{ enabled in } \mathbf{m}} var_{s,R_{t,p}} + \sum_{t \text{ enabled in } \mathbf{m}} var_{s,R_{p,t}}. \tag{3}$$

In matrix form expectation and variance are denoted $\mathbf{M}_k = \operatorname{diag}(\mu_{1,k}, \ldots, \mu_{S,k})$ and $\mathbf{\Sigma}^2_k = \operatorname{diag}(\sigma^2_{1,k}, \ldots, \sigma^2_{S,k})$, $k = 1, 2$. The stochastic process under consideration is a Markov process in continuous time with mixed discrete and continuous state space (see [3]).

$$\{(S(\tau), \mathbf{Z}(\tau)), t \in \mathbb{R}^+_0\}$$

where $S(\tau)$ is the discrete marking at time τ, and $\mathbf{Z}(\tau)$ is a vector of length m, if there are m fluid places, which represents the random variables of the fluid levels in each fluid place. We define $\mathbf{Z}(\tau) = (Z_1, Z_2)$, that is $m = 2$.

At time τ the transient probability of being in discrete state i with fluid levels in an infinitesimal environment around (z_1, z_1), is called the *fluid density* and is denoted by

$$\pi_i(\tau, z_1, z_2) = \partial/\partial z_1 \partial/\partial z_2 \;\left(P(S(\tau) = i, \mathbf{Z}(\tau) \leq (z_1, z_2)\right).$$

In vector notation for all discrete states i the fluid density is the solution of the following system of PDEs

$$\frac{\partial \boldsymbol{\pi}(\tau, z_1, z_2)}{\partial \tau} + \sum_{k=1}^{2} \frac{\partial}{\partial z_k}\Big(\boldsymbol{\pi}(\tau, z_1, z_2)\mathbf{M}_k(z_k)\Big)$$
$$= \sum_{k=1}^{2} \frac{1}{2}\frac{\partial^2}{\partial z_k^2}\Big(\boldsymbol{\pi}(\tau, z_1, z_2)\boldsymbol{\Sigma}_k^2(z_k)\Big) + \boldsymbol{\pi}(\tau, z_1, z_2)\mathbf{Q}(z_1, z_2)$$
$$\forall \tau, \forall z \in \overset{\circ}{\Omega}. \quad (4)$$

A reflecting barrier at the lower and at the upper bound define the boundary condition for the fluid flow. For its derivation see [3].

$$\Big[\frac{1}{2}\frac{\partial}{\partial z_1}\boldsymbol{\pi}(\tau, z_1, z_2)\boldsymbol{\Sigma}_1^2(z_1) + \frac{1}{2}\frac{\partial}{\partial z_2}\boldsymbol{\pi}(\tau, z_1, z_2)\boldsymbol{\Sigma}_2^2(z_2)$$
$$- \boldsymbol{\pi}(\tau, z_1, z_2)\mathbf{M}_1(z_1)$$
$$- \boldsymbol{\pi}(\tau, z_1, z_2)\mathbf{M}_2(z_2)\Big]_{\mathbf{z}=\{z_1^{\min}, z_2^{\min}\}, \{z_1^{\max}, z_2^{\max}\}} \quad (5)$$

Let $\boldsymbol{\pi}_0$ be the vector of the initial discrete state probabilities and let $\mathbf{z}_0 = (z_1^0, z_2^0)$ be the vector of initial fluid levels then the inital conditions are

$$\boldsymbol{\pi}(0, \mathbf{z}) = \boldsymbol{\pi}_0 \cdot \delta(\mathbf{z} - \mathbf{z}_0). \quad (6)$$

The transient behaviour of a second order FSPN is described by a system of partial differential equations (4), an initial condition (6) and a pair of boundary conditions (5).

4 Example

In this section an intrinsic continuous system will be regarded, a controlled chemical batch reactor, taken from [10]. It consists in a liquid-cooled reactor with measuring, stirring and feed supply devices, and a controller. The controller is designed to realize the following sequence of teps: first, a certain amount of highly concentrated, dissolved substance B is fed into the reactor by opening valve v_B for a specified period of time. Then the stirrer motor M is switched on and a solution of substance A is added (by opening v_A) until the total reactor content has reached an upper limit. The chemical reaction starts, when v_A is opened. The exothermic reaction leads to an increase of the liquid temperature. To avoid dangerous heat up, the cooling is switched on when the temperature reaches some limit. The reactor is drained by opening valve v_o and the stirrer is switched off when the reaction time T has elapsed.

If the cooling is switched on, the inflow of substance A is stopped and the reaction ends with a low concentration of substance A. Only if the process ends with the desired concentration, the reactor is emptied and a new process can start.

In figure 1 the system is sketched. The inflow of substance B takes on the average 30 minutes, the reaction time is on the average 25 minutes and the emptying of the reactor takes approximately 50 minutes. The cooling is only switched on if the temperature exceeds some limit, L_1. If the

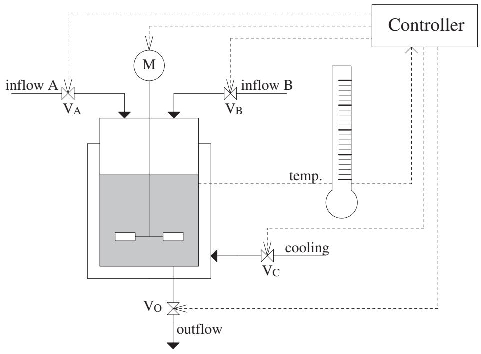

Figure 1: Sketch of the batch reactor

temperature exceeds the second threshold L_2, with $L_2 > L_1$, the state of critical temperature is entered. So, depending on the continuous state a discrete state transition of the controller takes place. The second order FSPN model is shown in figure 2.

The reachability graph is depicted in figure 3. It has six discrete states, the states of the controller, two of which are absorbing states. The states are

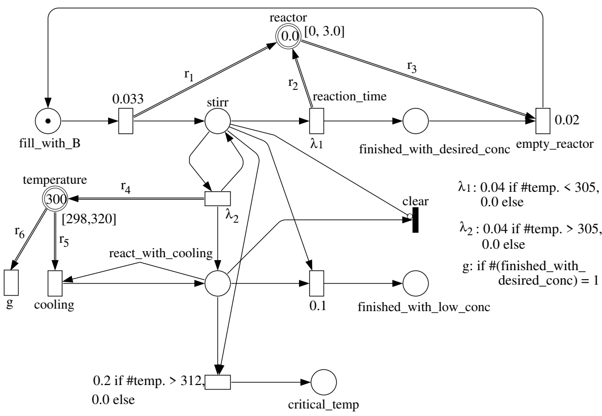

Figure 2: FSPN model of the batch reactor

labelled with the names of the controller states. Added to each discrete state are two continuous dimensions, the reactor content and the temperature. They are omitted in the reachability graph. The numbers in the small circles are used for referencing the solutions in the single states.

The fluid flow rates are defined through expectation and variance. For the reactor in- and outflow the rates are measured in cubic meter and are defined as deterministic rates ($\sigma_i^2 = 0$).

$$\mu_{r_1} = 0.05 \qquad \mu_{r_2} = 0.04 \qquad \mu_{r_3} = 0.06 \quad (7)$$

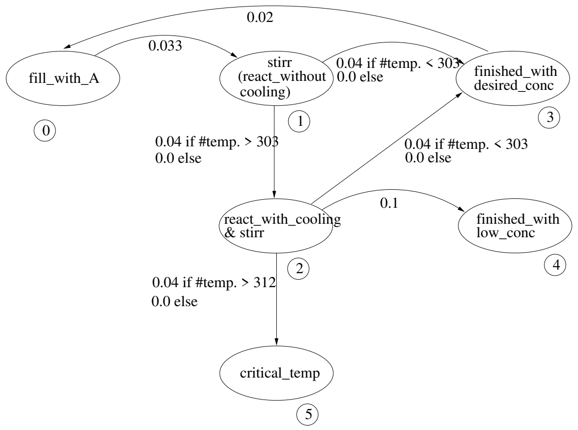

Figure 3: Reachability graph of the FSPN model

The temperature is measured in Kelvin and the rates, at which the temperature changes are

$$\begin{aligned} \mu_{r_4} &= 0.08 \cdot (\#reactor)^2 & \sigma^2_{r_4} &= 0.1 \\ \mu_{r_5} &= 0.04 \cdot (\#reactor)^2 & \sigma^2_{r_5} &= 0.1 \\ \mu_{r_6} &= -0.05 & \sigma^2_{r_6} &= 0.05. \end{aligned} \tag{8}$$

All the parameters in the model have been scaled by factor 10 for numerical convenience and it follows for the parameter matrices

$$\begin{aligned} \mathbf{M}_1 &= \mathrm{diag}(0.5,\ 0.4,\ 0.0,\ -0.6,\ 0.0,\ 0.0) \qquad \mathbf{\Sigma}_1^2 = \mathbf{0} \\ \mathbf{M}_2 &= \mathrm{diag}(0.0,\ 0.8z_1^2,\ 0.4z_1^2,\ -0.5,\ 0.0,\ 0.0) \\ \mathbf{\Sigma}_2^2 &= \mathrm{diag}(0.0,\ 1.0,\ 1.0,\ 0.5,\ 0.0,\ 0.0) \end{aligned} \tag{9}$$

The generator matrix is scaled analogously. The model has been solved over 20 time units, which corresponds to 200 minutes in the original model.

The solution of the differential equation gives a two-dimensional density, that changes over time. It cannot be represented in one plot, only in a series of plots for different time points, or in a film. In the following the transient evolution of the densities[1] in selected discrete states are shown. The full density over all states is omitted, since it is dominated by the behaviour in states 3 and 4, in which most probability mass is concentrated (see also figure 7).

In figure 4 the evolution of the density in the discrete state zero is shown. In state zero the reactor is filled with substance B and the temperature remains constant. The labelling refers to the time units of the solution procedure. The reactor fills up quickly and soon is full with high probability, whenever the controller is in state zero.

Two observations can be made for the system's evolution in state 2, which is shown in figure 6: this

[1]Strictly speaking this is a density only after normalization. Nonetheless, we will call the fractions of a density also density.

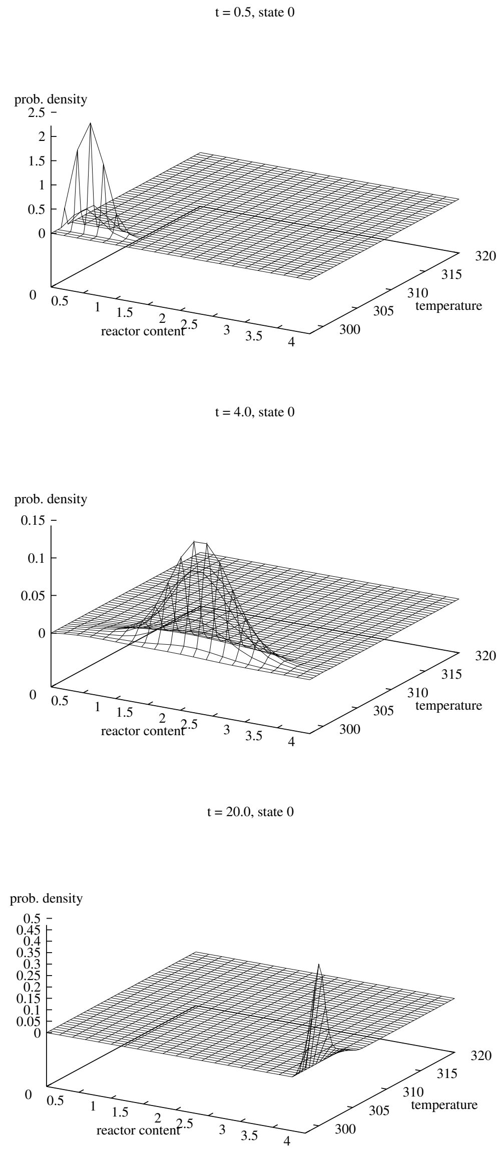

Figure 4: Probability density at selected time points in state 0

state is entered only, if the temperature is at least 305K (above L_1) and it is a transient state, that will after some time not be visited again (see also figure 7). The plots for state 3 and 1 are omitted.

State 4 is only entered if the temperature increases beyond the first threshold ($L_1 = 305$) and state 5 is reached only when the second threshold ($L_2 = 312$) is crossed. Both are shown in figure 5.

Integrating over one continuous dimension gives the marginal density with respect to the other continuous dimension. Figure 8 shows the transient curves for the marginal densities with respect to the temperature in the absorbing states 3, 4, and 5.

Integration over both continuous dimensions gives the discrete state probability. It is shown for all

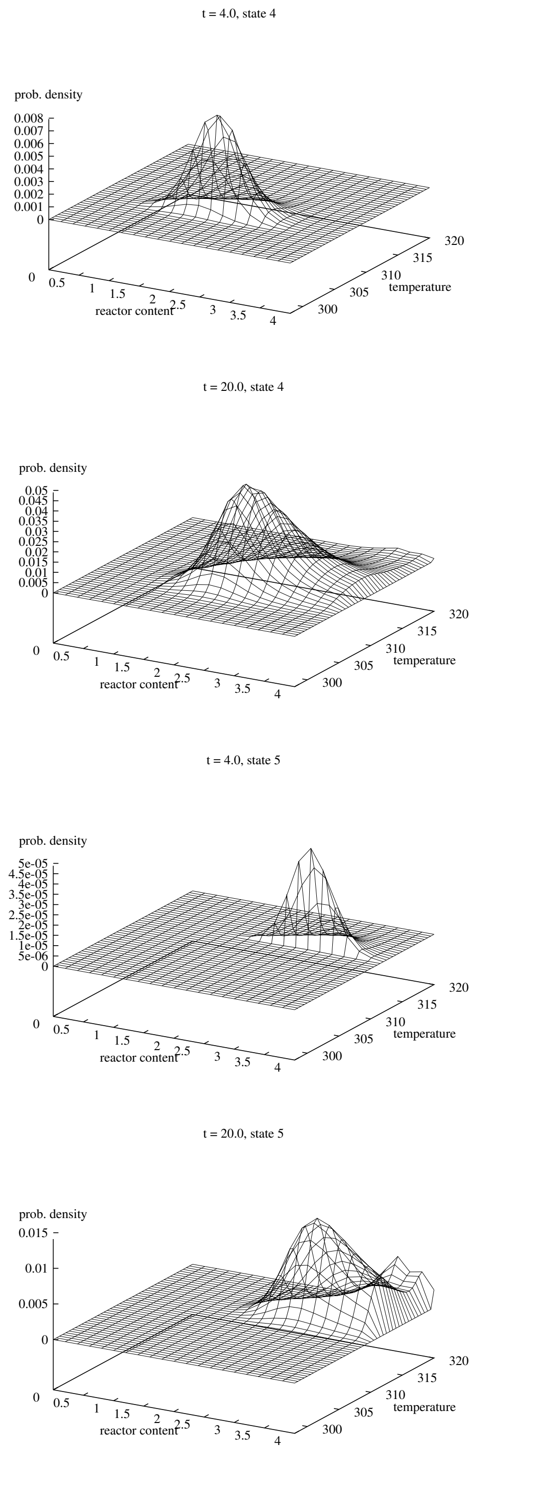

Figure 5: Probability density at selected time points in state 4 and 5

discrete states in figure 7. After 20 time units the probability of dangerous heat up is 8.8%. The probability that the cooling is needed is 53% (it is computed as the sum of the state probabilities of states zero, one and three).

The complexity of the solution depends on the

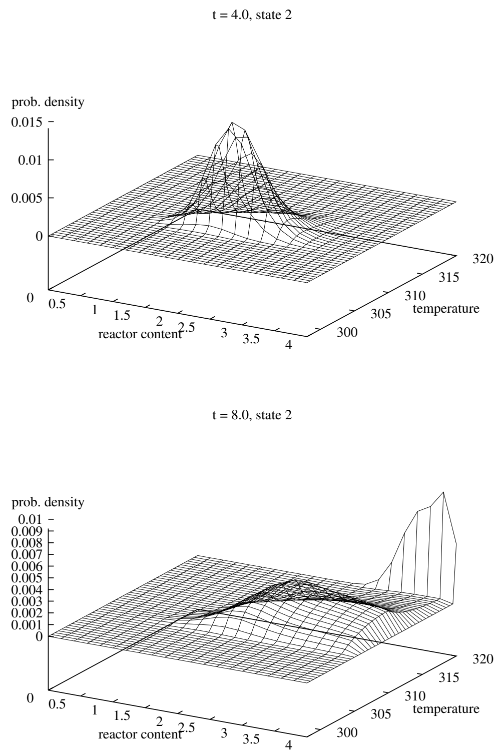

Figure 6: Probability density at selected time points in state 2

choice of the discretization stepsizes and the size of the state space. In each discretization step in time S systems of linear equations have to be solved, where S is the number of discrete states in the model. Each system of equations has $N * M$ unknowns and an equal number of equations, where N and M are the number of discretization steps in direction z_1 and z_2, respectively. Obviously, the computational effort depends on the size of the state space and good methods for state space reduction are worth while.

5 Conclusions

In this paper second order FSPNs have been presented as a formalism, that is well-suited for performance and dependability evaluation of hybrid systems. The continuous system components can have probabilistic or deterministic behaviour and their behaviour can depend on the continuous as well as the discrete state of the system. The modelling formalism has been defined and it has been shown how a second order FSPN can be mapped on a stochastic process. A model of a chemical batch reactor has been analyzed.

An extension of the numerical algorithm to three continuous dimensions would be desirable. In the existing algorithms a non-constant stepsize in time would be able to speed up the solution algorithm.

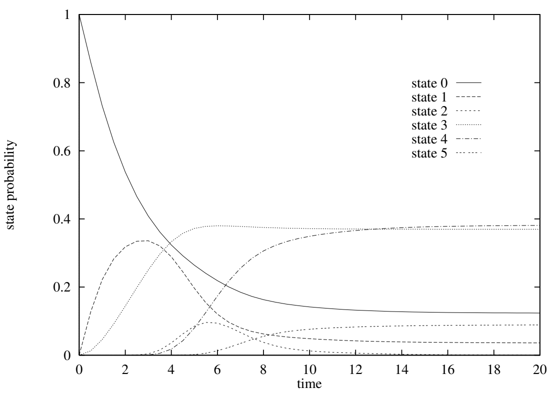

Figure 7: The discrete state probabilities

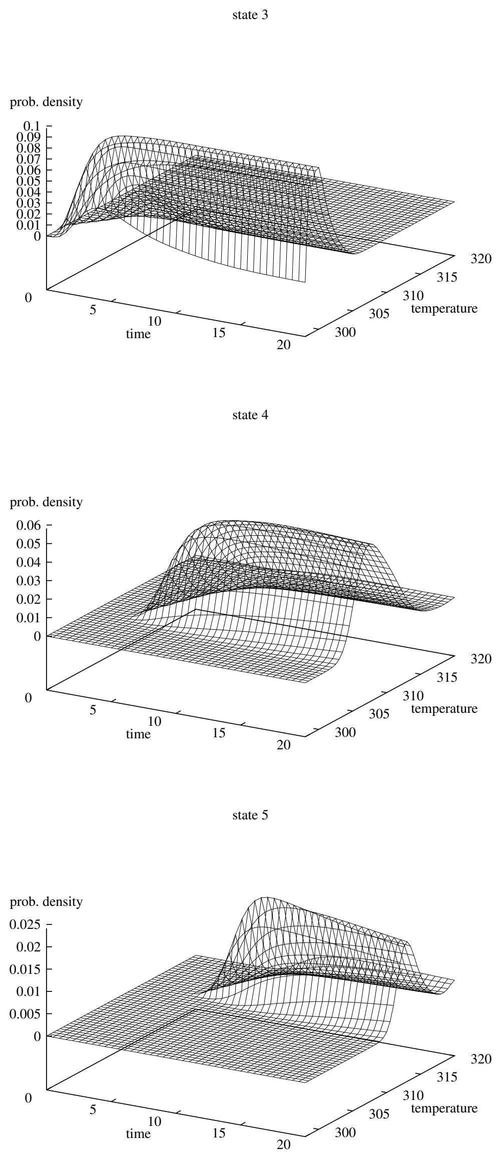

Figure 8: Marginal probability density

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