Combined Linear Local Passage at the 18-cycle in Bianchi $VI^*_{-1/9}$

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Abstract

In this paper, we find an example for a periodic heteroclinic chain in Bianchi $VI^*_{-1/9}$ that allows Takens Linearization at all base points. It turns out to be a "18-cycle", i.e. involving a heteroclinic chain of 18 different base points at the Kasner circle. We then show that the Combined Linear Local Passage at the 18-cycle is a contraction. This qualifies the 18-cycle as a candidate for proving the first rigorous convergence theorem in Bianchi $VI^*_{-1/9}$.

We proceed with an outlook on how to proceed further in studying Bianchi cosmologies, and also discuss directions for future research in inhomogeneous (PDE-) cosmological models. This puts our results in a broader perspective. The appendix contains symbolic and numerical computations done by Mathematica for examples discussed throughout the text.

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CHAPTER 1

Bianchi Spacetimes - Existing Results, Challenges and Techniques

1. The Equations of Wainwright and Hsu

In the paper [**59**] by Wainwright and Hsu, a formulation of the Einstein Equations for Bianchi models is presented that has several advantages, one of them beeing that it contains all models of Bianchi class A. Here are the equations, which are used throughout this dissertation:

$$
\begin{aligned}
N_1' &= (q - 4\Sigma_+)N_1, \\
N_2' &= (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-)N_2, \\
N_3' &= (q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-)N_3, \\
\Sigma_+' &= (q-2)\Sigma_+ - 3S_+, \\
\Sigma_-' &= (q-2)\Sigma_- - 3S_-.
\end{aligned} \tag{1}
$$

with constraint

$$
\Omega = 1 - \Sigma_+^2 - \Sigma_-^2 - K \tag{2}
$$

and abbreviations

$$
\begin{aligned}
q &= 2\left(\Sigma_+^2 + \Sigma_-^2\right) + \tfrac{1}{2}(3\gamma - 2)\Omega, \\
K &= \tfrac{3}{4}\left(N_1^2 + N_2^2 + N_3^2 - 2\left(N_1N_2 + N_2N_3 + N_3N_1\right)\right), \\
S_+ &= \tfrac{1}{2}\left((N_2 - N_3)^2 - N_1\left(2N_1 - N_2 - N_3\right)\right), \\
S_- &= \tfrac{1}{2}\sqrt{3}\left(N_3 - N_2\right)\left(N_1 - N_2 - N_3\right).
\end{aligned} \tag{3}
$$

The fixed parameter γ is related to the choice of matter model (e.g. $\gamma = 1$ represents dust, whereas $\gamma = 4/3$ represents radiation).

The properties of equations above have been studied intensively, for a recent survey see [**17**]. The main goal are rigorous results on the correspondence of iterations of the so-called "Kasner map" f to the dynamics of nearby trajectories to the Bianchi system (1) with reversed time, i.e. in the α-limit $t \to -\infty$. We will introduce the necessary background in the rest of this section.

Bianchi Class	N_1	N_2	N_3
I	0	0	0
II	$+$	0	0
VI_0	0	$+$	$-$
VII_0	0	$+$	$+$
VIII	$-$	$+$	$+$
IX	$+$	$+$	$+$

1.1. Vacuum Models of Bianchi Class A. From now on, we will restrict ourselves to the vacuum case $\Omega = 0$. This yields a 4-dimensional model, as we have five variables and one constraint:

DEFINITION 1.1. *(Phase Space of the Vacuum Wainwright-Hsu ODEs)*

$$\mathcal{W} = \{(N_1, N_2, N_3, \Sigma_+, \Sigma_-) \mid 0 = 1 - \Sigma_+^2 - \Sigma_-^2 - K\}$$

As we are interested in the dynamics of the Bianchi system (1) with reversed time, i.e. in the α-limit $t \to -\infty$, we will we denote by X^W the vector field corresponding to this time direction, for use in later chapters[1]. This means X^W stands for the vector field corresponding to the right side of (1), multiplied by -1.

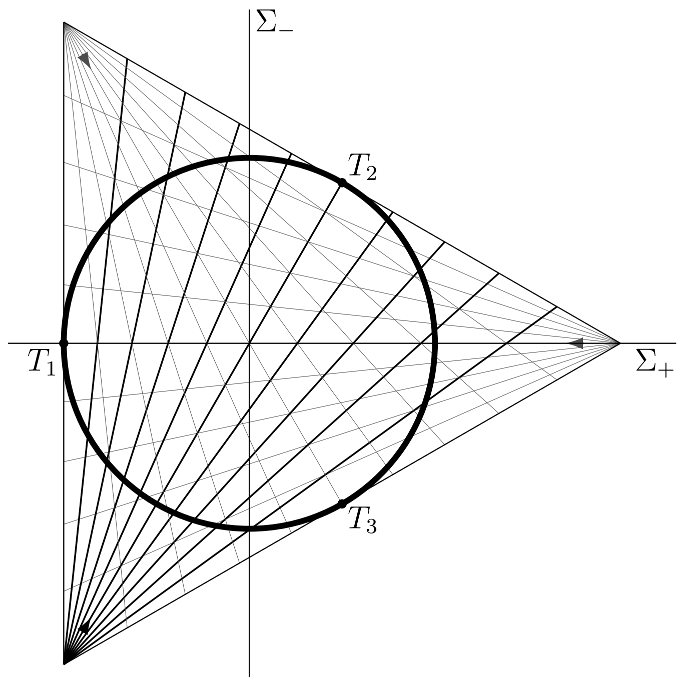
When we look at those equations, we observe that if $N_1 = N_2 = N_3 = 0$, the vector field is zero, as $K = 0$ and $q = 2$ in this case. We denote by $\mathcal{K} = \{N_1 = N_2 = N_3 = 0,\ \Omega = 0\}$ the resulting circle of equilibria: we obtain a circle because the constraint (2) reduces to $\Sigma_+^2 + \Sigma_-^2 = 1$. It is called the "Kasner circle", because the points $p \in \mathcal{K}$ represent the Kasner solution of the Einstein Equations discussed in section **??**.

In the classification of spatially homogenous models (based on the classification of 3-dimensional Lie-Algebras by Bianchi [**5**]), these are of Bianchi class I. One advantage of the Wainwright-Hsu equations is that they contain all models of Bianchi class A, with the signs of the N_i determining the type of Bianchi model, see the table above.

If we allow one of the N_i to be non-zero, the resulting half ellipsoids are called "Kasner caps": $C_k = \{N_k > 0,\ N_l = N_m = 0,\ \Omega = 0\}$ with $\{k, l, m\} = \{1, 2, 3\}$. They consist of heteroclinic orbits to equilibria on the Kasner circle and are of Bianchi class II. The projections of the trajectories of Bianchi class-II vacuum solutions onto the $\Sigma_\pm$-plane yield straight lines connecting two points of the Kasner circle.

[1] in the paper by Béguin [**2**], the equations are presented directly with time direction chosen towards the big bang, but we stick to the form of the equations used in the classic reference [**59**], and also in [**42, 27**].

These can be constructed geometrically in the following way: for a point $p \in \mathcal{K}$ of the Kasner circle, identify the nearest corner of the circumscribed triangle, and draw the resulting line as illustrated in the picture below:



Observe that this works for any $p \in \mathcal{K}$ except for the three points where the Kasner circle touches the circumscribed triangle, which we denote by T_1, T_2, T_3 and refer to them as Taub points.

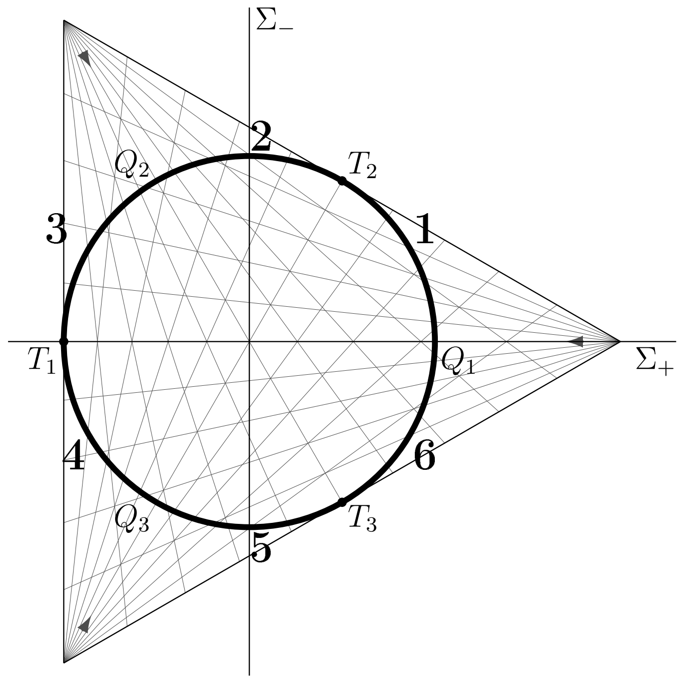
1.2. The Kasner Map. This leads to the definition of the so-called Kasner map $f : \mathcal{K} \to \mathcal{K}$. For each point $p_+ \in (\mathcal{K} \setminus \{T_1, T_2, T_3\})$ there exists a Bianchi class-II vacuum heteroclinic orbit $H(t)$ converging to p_+ as $t \to \infty$. This orbit is unique up to reflection $(N_1, N_2, N_3) \mapsto (-N_1, -N_2, -N_3)$. Its unique α-limit p_- defines the image of p_+ under the Kasner map

$$f(p_+) := p_- \tag{4}$$

Including the three fixed points, $f(T_k) := T_k$, this construction yields a continuous map, $f : \mathcal{K} \to \mathcal{K}$. In fact f is a non-uniformly expanding map and its image $f(\mathcal{K})$ is a double cover of $\mathcal{K}$, which can be seen directly from the geometric description given above.

For use in later chapters, let us denote by $H_{p,f(q)}$ the heteroclinic Bianchi-II-orbit from p to its image point under the Kasner map $f(q)$ and by $H_B = \bigcup_{p \in B} H_{p,f(q)}$ the set of all heterclinic Bianchi-II-orbits connecting two basepoints in the set $B \subset \mathcal{K}$.

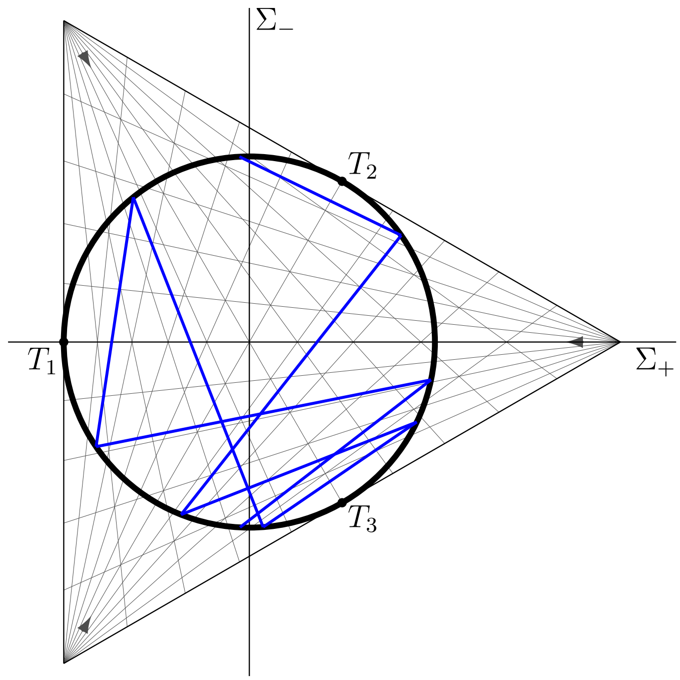
We will now introduce the so-called Kasner-parameter u, as it is a convenient way to parametrize the Kasner circle. Let us divide the Kasner circle into six sectors and label them as follows:



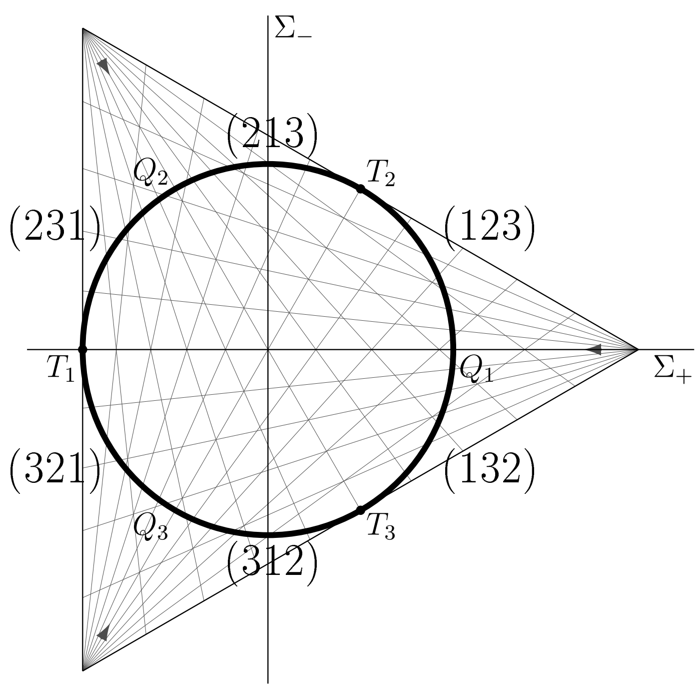
The Kasner parameter ranges in $[1, \infty]$, where it holds that $u = \infty$ at the Taub point T_i and $u = 1$ at the points Q_i shown in the picture. So for each value of u, we get an equivalence class of six points cooresponding to the $p \in \mathcal{K}$ in each sector of the Kasner circle (except for the values $u = \infty$ and $u = 1$, where the equivalence class consists only of three points). Expressed in u, the Kasner map has a very simple form:

$$f(u) = \begin{cases} u - 1 & u \in [2, \infty] \\ \frac{1}{u-1} & u \in [1, 2] \end{cases}$$

In the picture below, the dynamics of the Kasner-map is shown: If you start close to a Taub point (meaning that u is large compared to 1), there are first "bounces" around this Taub point as the value of u is decreased by 1 in each step. Then, after the value of u has fallen below 2, there is a "excursion" to a different part of the Kasner circle:



There is also an interesting connection to the Kasner solution described in section **??**. Each sector of the Kasner circle corresponds to a permutation of the Kasner expontents p_i, which can be expressed in the Kasner parameter u:



where sector (321) means e.g. that $p_3 < p_2 < p_1$ which fixes the formula for each of them. As an example, consider the sector 5 or (312) (for details see [**17**], p.8):

$$p_3 = \frac{-u}{1+u+u^2}$$
$$p_1 = \frac{(u+1)}{1+u+u^2}$$
$$p_2 = \frac{u(u+1)}{1+u+u^2}$$

We will need the formulas above, as they will allow us to express the eigenvalues of the linearized vector field at points of the Kasner circle in u, a key step for obtaining our results in the later chapters.

1.3. Eigenvalues in Terms of the Kasner Parameter u. When we linearize the vector field corresponding to equations (1) at points of the Kasner circle, we arrive at the following Matrix:

$$\begin{pmatrix} 2-4\Sigma_+ & 0 & 0 & 0 & 0 \\ 0 & 2+2\Sigma_+ + 2\sqrt{3}\Sigma_- & 0 & 0 & 0 \\ 0 & 0 & 2+2\Sigma_+ - 2\sqrt{3}\Sigma_- & 0 & 0 \\ 0 & 0 & 0 & 3(2-\gamma)\Sigma_+^2 & 3(2-\gamma)\Sigma_+\Sigma_- \\ 0 & 0 & 0 & 3(2-\gamma)\Sigma_+\Sigma_- & 3(2-\gamma)\Sigma_-^2 \end{pmatrix}.$$

and we can compute the following eigenvalues to eigenvectors ∂_{N_1}, ∂_{N_2}, ∂_{N_3} tangential to the Bianchi class-II vacuum heteroclinics:

$$\begin{aligned} \mu_1 &= 2-4\Sigma_+, \\ \mu_2 &= 2+2\Sigma_+ + 2\sqrt{3}\Sigma_-, \\ \mu_3 &= 2+2\Sigma_+ - 2\sqrt{3}\Sigma_- \end{aligned}$$

In addition we have the trivial eigenvalue zero to the eigenvector $-\Sigma_-\partial_{\Sigma_+} + \Sigma_+\partial_{\Sigma_-}$ tangential to the Kasner circle $\mathcal{K}$. The fifth eigenvalue $\mu_\Omega = 3(2-\gamma) > 0$ corresponds to the eigenvector $\Sigma_+\partial_{\Sigma_+} + \Sigma_-\partial_{\Sigma_-}$ transverse to the vacuum boundary $\{\Omega = 0\}$.

Now we use that it is possible to express the $\Sigma_{+/-}$-variables in terms of the Kasner exponents p_i (see [**17**], p.7):

$$\Sigma_+ = -\frac{3}{2}p_1 + \frac{1}{2}$$

$$\Sigma_- = -\frac{\sqrt{3}}{2}(p_1 + 2p_2 - 1)$$

Thus, we arrive at the following formulas for the eigenvalues expressed in u:

$$(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{-6u}{1+u+u^2}, \frac{6(1+u)}{1+u+u^2}, \frac{6u(1+u)}{1+u+u^2} \right) \tag{5}$$

As it holds that $u \in [1, \infty]$, we observe that at each point of the Kasner circle, there is one negative and two positive eigenvalues. But recall that we are interested in the time-direction $t \to -\infty$. The negative eigenvalue is unstable towards the past, while the two positive eigenvalues are stable in backwards-time. This means that for our vector field X^W (which has the time-direction already reversed, see beginning of section 1.1) there is one unstable eigenvalue λ_u and two stable eigenvalues λ_s, λ_{ss}, at a point of the Kasner circle. Away from the Taub points it also holds that:

$$|\lambda_u| < |\lambda_s| < |\lambda_{ss}|$$

Finally, also note that in Bianchi IX, the situation in different sectors of the Kasner-circle only differs by a permutation of those 3 formulas for the eigenvalues. This makes it easy to examine the question of resonances of the eigenvalues, which we are trying to exclude when linearizing the vector field.

However, we will also deal with Bianchi $VI^*_{-1/9}$ later, and there the situation is more complicated as the formulas for the eigenvalues at points on the Kasner circle do depend on the sector, so in order to check for resonances, a lot of cases have to be considered. This will be done in chapter 2. In the next section, we will introduce the equations for Bianchi $VI^*_{-1/9}$, which is of class B and not covered by the equations of Wainwright and Hsu.

2. Bianchi $VI^{*}_{-1/9}$

We now present the equations for Bianchi $VI^{*}_{-1/9}$, which is the most general model in Bianchi class B, and has a crucial importance for inhomogenous cosmologies (see e.g. [**21**]):

$$\Sigma_{+}^{'} = (q-2)\Sigma_{+} + 3\Sigma_{2}^{2} - 2N_{-}^{2} - 6A^{2} \tag{6}$$

$$\Sigma_{-}^{'} = (q-2)\Sigma_{-} - \sqrt{3}\Sigma_{2}^{2} + 2\sqrt{3}\Sigma_{\times}^{2} - 2\sqrt{3}N_{-}^{2} + 2\sqrt{3}A^{2} \tag{7}$$

$$\Sigma_{\times}^{'} = (q-2-2\sqrt{3}\Sigma_{-})\Sigma_{\times} - 8N_{-}A \tag{8}$$

$$\Sigma_{2}^{'} = (q-2-3\Sigma_{+} + \sqrt{3}\Sigma_{-})\Sigma_{2} \tag{9}$$

$$N_{-}^{'} = (q+2\Sigma_{+} + 2\sqrt{3}\Sigma_{-})N_{-} + 6\Sigma_{\times}A \tag{10}$$

$$A^{'} = (q+2\Sigma_{+})A \tag{11}$$

Abbreviations:

$$q = 2\Sigma^{2} + \frac{1}{2}(3\gamma-2)\Omega \tag{12}$$

$$\Sigma^{2} = \Sigma_{+}^{2} + \Sigma_{-}^{2} + \Sigma_{2}^{2} + \Sigma_{\times}^{2} \tag{13}$$

Constraints:

$$\Omega = 1 - \Sigma^{2} - N_{-}^{2} - 4A^{2} \tag{14}$$

$$g = (\Sigma_{+} + \sqrt{3}\Sigma_{-})A - \Sigma_{\times}N_{-} = 0 \tag{15}$$

Auxilliary Equations:

$$\Omega^{'} = [2q - (3\gamma-2)]\Omega \tag{16}$$

$$g^{'} = 2(q+\Sigma_{+}-1)g \tag{17}$$

Note that the auxilliary equations (21) and (22) follow from (11) – (16) and show the invariance of $\Omega = 0$ and $g = 0$, where $\Omega = 0$ results in the vacuum equations.

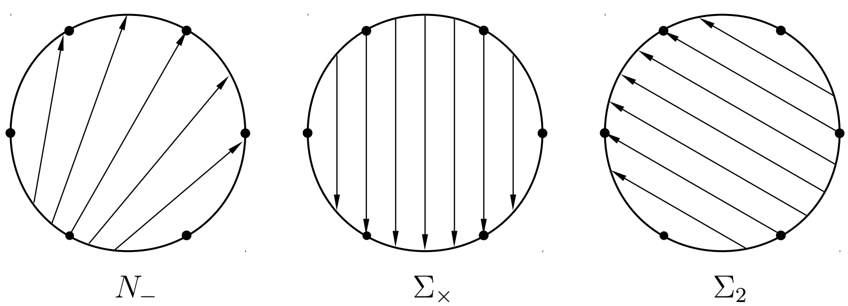
We define the phase space for vacuum Bianchi $VI^*_{-1/9}$ again by requiring that the constraints (14) and (15) are satisfied. This time, we have six variables and two constraints, yielding again a 4-dimensional state space for the Vacuum models, as in Bianchi IX before.

DEFINITION 1.2. *(Phase Space for Vacuum Bianchi $VI^*_{-1/9}$)*

$$\mathcal{B} = \{\Sigma_+, \Sigma_-, \Sigma_\times, \Sigma_2, N_-, A) \mid 0 = 1 - \Sigma^2 - N_-^2 - 4A^2 \text{ and } g = 0\}$$

The equations have been analysed in **[21]**, we give here only a very brief overview about the similarities and differences of Bianchi $VI^*_{-\frac{1}{9}}$ compared to Bianchi IX that are relevant for our own research (see chapter 2).

When we look at the equations, we observe there is also a Kasner circle of fixed points: $\mathcal{K} = \{\Sigma_\times = \Sigma_2 = N_- = A = 0\}$, leading again to $\Sigma_+^2 + \Sigma_-^2 = 1$. Similar to Bianchi IX, we can define caps of heteroclinic orbits connecting points of $\mathcal{K}$, but Bianchi $VI^*_{-\frac{1}{9}}$ is less symmetric than Bianchi IX. The transitions can be illustrated as follows:

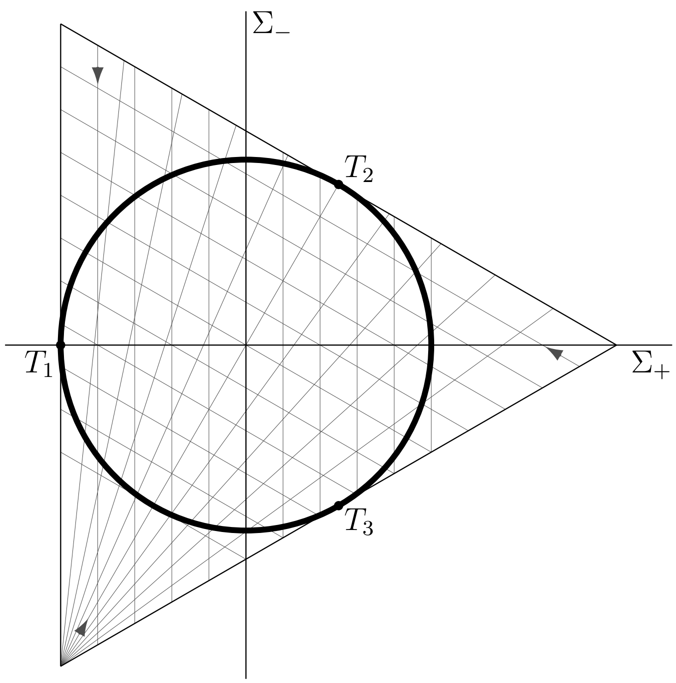


We have the following caps of heteroclinic orbits:

- $C_{N_-} = \{\Sigma_\times = \Sigma_2 = A = 0\}$, which represent transitions in the variable N_-, i.e. curvature transitions. This means the Kasner parameter u changes according to the Kasner map as explained in section 1.2 for Bianchi IX.
- $C_{\Sigma_\times} = \{\Sigma_2 = N_- = A = 0\}$, which represent transitions in the variable Σ_x, i.e. a frame transition. This means the Kasner parameter u is not changed by the transition, it rather connects two points of $\mathcal{K}$ that are in the same equivalence class with the same u in different sectors

- $C_{\Sigma_2} = \{\Sigma_\times = N_- = A = 0\}$ which represent transitions in the variable in the variable Σ_2, also a frame transition.

In addition to the traditional curvature transition in the variable N_- similar to those that also appear in Bianchi class A, we also observe frame transitions of two types, for the variables $\Sigma_\times$ and Σ_2. The fact that the frame transition do not change u can be seen from the fact that the projections of the heteroclinic orbits on the (Σ_+, Σ_-)-plane are parallel lines, and the (inverse) distance to the Taub points (which is one way to interpret u) stays the same after the transition.



Similar to what we did in Bianchi IX, we will also find expressions for the eigenvalues of the transition-variables in terms of the Kasner parameter u. This will be done in chapter 2.

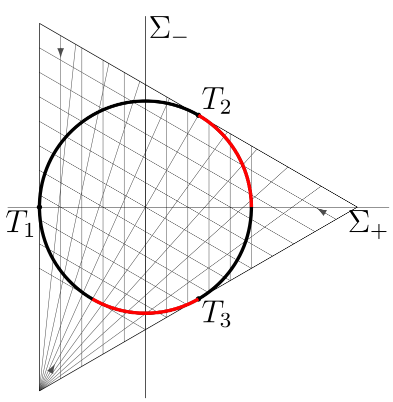
3. Existing Results in Bianchi IX and Difficulties in Bianchi B

Historically, the chaotic oscillations in Bianchi models have first been observed in the Bianchi IX model, see [**29, 3, 4**]. It is also known under the name "Mixmaster", a term coined by Misner [**34**]. The first rigorous theorem on the ancient dynamics in Bianchi IX was proved by Ringström [**42**], based on earlier results by Rendall [**39**]. For a recent survey on the "Facts and Beliefs" concerning the Mixmaster model, see [**17**].

One important research question is relating properties of the Kasner map (which is known to be chaotic, see e.g. the chapter 11 in the book [**58**]) to the real dynamics in Bianchi models. This can be seen as making the BKL-conjecture rigorous for spatially homogenous spacetimes, i.e. by proving the "oscillatory" (and possibly also the "vacuum") part in a case where the model is already "local".

In Bianchi IX, there exist such rigorous convergence results by Liebscher et al [**27**], Béguin [**2**], and Reiterer/Trubowitz [**38**]. For Bianchi VI_0 with a magnetic field as matter, there have been recent results by Liebscher, Rendall and Tschapa [**28**].

Until today, there exist no rigorous convergence results for Bianchi $VI^*_{-1/9}$, which is of class B (see chapter **??**, section **??**). The reason for this is that Bianchi $VI^*_{-1/9}$ is more difficult than Bianchi class A. One example is that in Bianchi IX, the normal hyperbolicity of the Kasner circle fails only at the three Taub points, while in Bianchi $VI^*_{-1/9}$ this is true for all of the six points that mark the borders of the sectors of the Kasner circle defined in section 1. Also there are non-unique heteroclinic chains because of multiple unstable eigenvalues for some sectors of the Kasner circle, marked in red in the picture below. We will discuss this matter further in chapter 2.



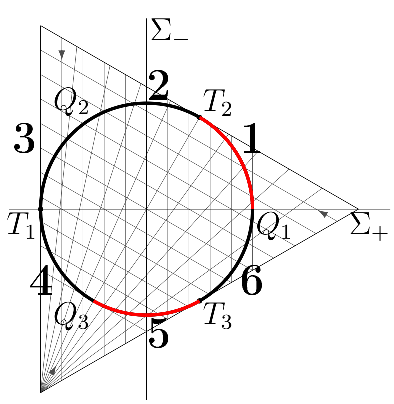
CHAPTER 2

Combined Linear Local Passage at the 18-cycle in Bianchi $VI^*_{-1/9}$

In this section, we show how some of the techniques developed in the chapter before can be applied to Bianchi $VI^*_{-1/9}$, where no rigorous convergence result exists to date. We construct an example for a periodic heteroclinic chain in Bianchi $VI^*_{-1/9}$ that allows Takens Linearization at all Base points. It will turn out to be a "18-cycle", i.e. involving a heteroclinic chain of 18 different base points at the Kasner circle. We then show that the combined linear local passage at the 18-cycle is a contraction. This qualifies it as a possible candidate for proving a rigorous convergence theorem in Bianchi $VI^*_{-1/9}$.

The situation in Bianchi $VI^*_{-1/9}$ is more involved than in Bianchi IX, as there are sectors of the Kasner circle with more than one unstable eigenvalue (towards the big bang, the time direction we are interested in). This means that even by starting with the same Kasner-parameter, one can have different realizations in terms of heteroclinic chains.

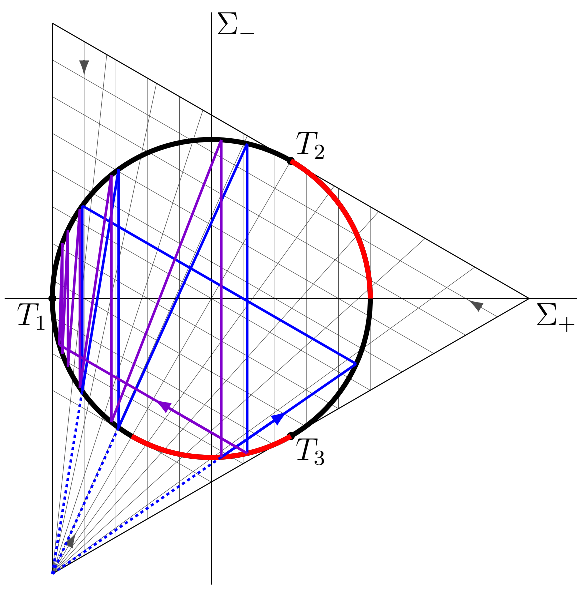
If we label the six sectors of the Kasner circle counter-clockwise as it is often done in Bianchi models (starting in the positive quadrant), this is true e.g. for sector 5, which is part of the 18-cycle. In the graphics below, the sectors with multiple unstable eigenvalues are marked in red color. The different families of heteroclinic orbits in Bianchi $VI^*_{-1/9}$ are illustrated by the lines in light gray in the background (see chapter 1, section 2 for details):



We will illustrate this ambiguity that can arise from the same continued fraction development $u = [3, 5, 3, 5, ...]$ by showing two possible "18-cycles" (details are discussed in section 3.1):

- the "classic" 18-cycle, with sector sequence 54343-425-43434343-425, where we go "left" to sector 4 both times in sector 5
- the "advanced" 18-cycle, with sequence 5634342-543434343425, where we go first "right" to sector 6, and the second time "left" to sector 4

The graphics below shows the "advanced 18 cycle" starting in sector 5. The bounces around the Taub-point can be seen clearly - the blue color shows the part with 3 bounces, while the purple color exhibits 5 bounces. The label "advanced 18-cylce" means that we first follow the blue part of the heteroclinic chain and then the purple one, reflecting our choice of u=[3,5,3,5,...].



Now we are in a position to state the two main results of this chapter:

Theorem 2.1. *Both the classic and the advanced 18-cycle allow Takens-Linearization at all base points.*

Lemma 2.2. *The Combined Linear Local Passage at the classic 18-cycle is a contraction.*

The proof of these results is the aim of this chapter, and we will now comment a little on how the rest of chapter is organized.

In order to illustrate our approach, we start by considering the 3-cycle in Bianchi $VI^*_{-1/9}$, as it is easier to handle and the method we develop applies also to longer cycles.

First we derive the formulas for the relevant eigenvalues of the linearized vector field at the base-points of the 3-cycle in Bianchi $VI^*_{-1/9}$ in terms of the Kasner Parameter u. We then consider the Combined Linear Local Passages and Takens Linearization at the 3-cycle. We are able to show that the Sternberg Non-Resonance Conditions are not satisfied for the 3-cycle, but they are satisfied for the 18-cycles.

This means that both the classic and the advanced 18-cycles are periodic heteroclinic chains in Bianchi $VI^*_{-1/9}$ that allow Takens Linearization at all of their Base points. In addition, we give evidence that the Combined Linear Local Passage at the classic 18-cycle is a contraction. This qualifies it as a candidate for proving a rigorous convergence theorem in Bianchi $VI^*_{-1/9}$.

In order to progress further and turn Lemma 2.2 into a rigorous convergence theorem, a better understanding of the global passages in Bianchi $VI^*_{-1/9}$ is necessary. We will comment on possible ideas how to do this in the section "Conclusion and Outlook".

As the formulas for the eigenvalues at points on the Kasner circle in Bianchi $VI^*_{-1/9}$ depend on the sector, many cases have to be checked. That's why we use Mathematica in order to do the necessary computations for the 18-cycles (see Appendix 1).

1. Eigenvalues in Terms of the Kasner Parameter u

1.1. General Formulas for Points on the Kasner Circle. At first, we recall that in Bianchi IX, the formula for the eigenvalues of the linearized vector field at points of the Kasner circle have an easy expression in terms of the Kasner parameter u, compare section 1, equation (5). In Bianchi IX, the situation in different sectors of the Kasner-circle only differs by a permutation of those 3 formulas for the eigenvalues (see [**17**], p.8), which does not matter for the question of resonances.

But in Bianchi $VI^*_{-1/9}$, the situation is more complicated. Here, the formulas for the eigenvalues at points on the Kasner circle do depend on the sector, so in order to check for resonances, a lot of cases have to be considered. Each sector corresponds to a permutation of the Kasner expontents p_i, where sector (321) means e.g. that $p_3 < p_2 < p_1$which fixes the formula for each of them. For this is it important to note that $u \in [1, \infty]$. As an example, consider the sector 5 or (312), compare section 1):

$$p_3 = \frac{-u}{1+u+u^2}$$

$$p_1 = \frac{(u+1)}{1+u+u^2}$$

$$p_2 = \frac{u(u+1)}{1+u+u^2}$$

The general formuals for the relevant eigenvalues (i.e. $\lambda_\times$, λ_2, λ_- corresponding to the variables involved in the heteroclinic chain: $\Sigma_\times$, Σ_2, $N-$, see section 2) in terms of Σ_+, Σ_-resp. the p_i are (see [**17**], p.7):

$$\Sigma_+ = -\frac{3}{2}p_1 + \frac{1}{2}$$

$$\Sigma_- = -\frac{\sqrt{3}}{2}(p_1 + 2p_2 - 1)$$

$$\lambda_\times = -2\sqrt{3}\Sigma_-$$

$$\lambda_2 = -3\Sigma_+ + \sqrt{3}\Sigma_-$$

$$\lambda_- = 2 + 2\Sigma_+ + 2\sqrt{3}\Sigma_-$$

1.2. Eigenvalues at the 3-Cycle. In the following, we present the formulas for the sectors that are involved in the 3-cycle with $u =$ golden mean $= \frac{1+\sqrt{5}}{2}$ and the sector-sequence "5-1-2-5". For the other sectors, similar formulas can be derived analogously (see Appendix 1.2).

Base Point B_1 in sector 5, i.e. (312).

$$\lambda_2 = \frac{3 - 3u^2}{1 + u + u^2}$$

$$\lambda_\times = \frac{6u + 3u^2}{1 + u + u^2}$$

$$\lambda_- = \frac{-6u}{1 + u + u^2}$$

Base Point B_2 in sector 1, i.e. (123).

$$\lambda_2 = \frac{-3 - 6u}{1 + u + u^2}$$

$$\lambda_\times = \frac{3 - 3u^2}{1 + u + u^2}$$

$$\lambda_- = \frac{6u + 6u^2}{1 + u + u^2}$$

Base Point B_3 in sector 2, i.e. (213).

$$\lambda_2 = \frac{3 + 6u}{1 + u + u^2}$$

$$\lambda_\times = \frac{-6u - 3u^2}{1 + u + u^2}$$

$$\lambda_- = \frac{6u + 6u^2}{1 + u + u^2}$$

2. The 3-Cycle in Bianchi $VI^*_{-1/9}$

2.1. (Non-)Resonance and Takens Linearization. In this section, we shortly check the Sternberg-Non-Resonance-Conditions (SNC) for the 3-cycle in Bianchi $VI^*_{-1/9}$. The procedure necessary to do this is described in detail in section **??**. The parameter-value at the 3-cylce is $u = g = \frac{1+\sqrt{5}}{2}$, which satisfies

$$1 + \frac{1}{u} = u \implies 1 + u - u^2 = 0$$

The equation for checking the (SNC) thus reads:

$$M * \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = z * \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

According to the formulas above, we observe that

$$M_{B_1} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & -6 \\ \text{-3} & 3 & 0 \end{pmatrix}, M_{B_2} = \begin{pmatrix} -3 & 3 & 0 \\ -6 & 0 & 6 \\ 0 & -3 & 6 \end{pmatrix}, M_{B_3} = \begin{pmatrix} 3 & 0 & 0 \\ 6 & -6 & 6 \\ 0 & -3 & 6 \end{pmatrix}$$

, which are all invertible, and give the following results (with $z = 6$ for the earliest possible resonance):

$$k_{B_1} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, k_{B_2} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}, k_{B_3} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

This means that (SNC) does hold at the Base points of the 3-cycle up to order 3, which is not enough to allow Takens-Linearization, as the the order necessary in the Takens-Theorem, named $\alpha(1)$, is bigger than 10 in all of the sectors involved (see Appendix 1.1). That's why Takens Linearization Theorem may not be employed at the base points of the 3-cycle, and we have to look for a different (longer) cycle. Nevertheless, we will stick to the 3-cycle in the following chapter in order to illustrate our method for calculating the Combined Linear Local Passage, as the method applies to longer cycles as well.

2.2. Combined Linear Local Passages. In this section, we use the linearized vectorfield at the base points of the 3-cycle to explicitly compute the local passages as was done for Bianchi IX before (see **??**-**??**). Be aware that Takens-Linearization is not allowed at the 3-cycle in Bianchi $VI^*_{-1/9}$, so this is only a formal calculation in this case to illustrate what we mean by "Combined Local Linear Passage". Later, for the 18-cycles, the calculation will be justified as Takens-Linearization is possible there.

The ratio of the relevant Eigenvalues at the Base-points B_1,B_2,B_3 of the 3-cycle (corresponding to the value $u = \frac{1+\sqrt{5}}{2}$ in the 3 sectors) is given by the following:

Near $B_1 = (-\frac{1}{4}, -\frac{\sqrt{15}}{4})$:

$$r_1 = -\frac{\lambda_\times}{\lambda_n} = \frac{u+2}{2} = 1.8090 > 0$$

$$r_2 = -\frac{\lambda_2}{\lambda_n} = -\frac{u^2-1}{2u} = -0.5000 < 0$$

Near $B_2 = (\frac{1+3\sqrt{5}}{8}, \frac{\sqrt{15}-\sqrt{3}}{8})$:

$$r_3 = -\frac{\lambda_n}{\lambda_2} = \frac{2u(u+1)}{2u+1} = 2 > 0$$

$$r_4 = -\frac{\lambda_u}{\lambda_{uu}} = -\frac{\lambda_\times}{\lambda_2} = -\frac{u^2-1}{2u+1} = -0.3820 < 0$$

Near $B_3 = (-\frac{1}{4}, \frac{\sqrt{15}}{4})$:

$$r_5 = -\frac{\lambda_2}{\lambda_\times} = \frac{2u+1}{u(u+2)} = 0.7236 > 0$$

$$r_6 = -\frac{\lambda_n}{\lambda_\times} = \frac{2(u+1)}{u+2} = 1.4472 > 0$$

We start our combined linear local passage at the In-Section of the local passage at B_1 in sector 5. Thus $\Sigma_\times$ is the incoming variable and we define $a := \Sigma_2^{in}$, $b := N_-^{in}$ (the two remaining relevant variables in the section).

Then we do the calculations for the 3 local passages involved at the 3-cycle near the B_1,B_2 and B_3 as in Bianchi IX before (see **??**-**??**). We arrive at the following formulas for the combined linear local passage near the 3-cycle:

$$\tilde{a} = ([b^{r_2}a]^{r_4}b^{r_1})^{r_5}$$

$$\tilde{b} = ([b^{r_2}a]^{r_4}b^{r_1})^{r_6}(b^{r_2}a)^{r_3}$$

Taking Logarithms on both sides yields the following:

$$\begin{pmatrix} \log \tilde{a} \\ \log \tilde{b} \end{pmatrix} = M_{3-cycle} * \begin{pmatrix} \log a \\ \log b \end{pmatrix}$$

with the following matrix $M_{3-cycle}$:

$$M_{3-cycle} = \begin{pmatrix} r_5 r_4 & r_5 r_4 r_2 + r_5 r_1 \\ r_6 r_4 + r_3 & r_6 r_4 r_2 + r_6 r_1 + r_3 r_2 \end{pmatrix}$$

As mentioned before, Takens-Linearization is not allowed at the 3-cycle in Bianchi $VI^*_{-1/9}$. However, for the 18-cycles, an analogous calculation will be justified as Takens-Linearization is possible there. We will discuss the properties $M_{18-cycle}$ in order to prove Theorem 2.2 in section 3.5.

3. The 18 Cycles in Bianchi $VI^*_{-1/9}$

3.1. Possible Passages in Bianchi $VI^*_{-1/9}$. When we look at the different transitions possible in Bianchi $VI^*_{-1/9}$ (see chapter 1, section 2), we are able to understand which sequence of sectors can arise when solutions converge to their corresponding heteroclinic chains. For the classification below, we have chosen to put our section always before the next "Curvature Transition", i.e. when we leave from sector 4 or 5, that's why all the passages start and end in one of these sectors:

- Passage A: Sectors 4-3-4
- Passage B1: Sectors 4-2-5
- Passage B2: Sectors 4-2-5-4
- Passage C1: Sectors 5-1-2-5
- Passage C2: Sectors 5-1-2-5-4
- Passage D: Sectors 5-6-3-4
- Passage E: Sectors 5-1-6-3-4

For the 18-cycles discussed below, only a few of the passages will occur, namely A, B1, B2 and D.

3.2. The Classic 18-Cycle. We now start with u=[3,5,3,5,...] in Sector 4 and prescribe the follwing dynamics:

u=	[3,5,...]	[2,5,...]	[2,5,...]	[1,5,...]	[1,5,...]	[5,3,...]	[5,3,...]
sector	4	3	4	3	4	2	5

u=	[5,3,...]	[4,3,...]	[4,3,...]	[3,3,...]	[3,3,...]	[2,3,...]	[2,3,...]
sector	4	3	4	3	4	3	4

u=	[1,3,...]	[1,3,...]	[3,5,...]	[3,5,...]	[3,5,...]	...	...
sector	3	4	2	5	4	...	...

This means we have the following sequence of Passages: A A B2 A A A A B2, and this pattern continues arbitrarily often. This involves 18 global passages, that's why I call it an "18-cycle".

Observe that both 18-cycles mentioned in the introduction to this chapter started in sector 5. This was done for illustrative purposes as there is an ambiguity how to continue in this sector. From now on, we refer to the "classic 18-cycle" with the sequence of sectors as illustrated in the table above, which means we have started in sector 4.

Note that we could also derive a different sequence of passages for the same u, as we have a choice in sector 5 either to go to sector 4 via a frame transition (as done above) or to go via curvature transition to sector 6, as done at the first transition for the advanced 18-cycle in the next section.

3.3. The Advanced 18-Cycle. Now we start with u=[3,5,3,5,...] in Sector 5 and prescribe the follwing dynamics:

u=	[3,5,...]	[2,5,...]	[2,5,...]	[2,5,...]	[1,5,...]	[1,5,...]	[5,3,...]
sector	5	6	3	4	3	4	2

u=	[5,3,...]	[5,3,...]	[4,3,...]	[4,3,...]	[3,3,...]	[3,3,...]	[2,3,...]
sector	5	4	3	4	3	4	3

u=	[2,3,...]	[1,3,...]	[1,3,...]	[3,5,...]	[3,5,...]	...	...
sector	4	3	4	2	5	...	...

This means we have the following sequence of passages: D A B2 A A A A B1, which defines the "advanced 18-cycle".

3.4. (Non-)Resonance and Takens Linearization at the 18-Cycle. We now show that both of the 18 cycles are infinite periodic heteroclinic chains in Bianchi $VI^*_{-1/9}$ that allows Takens Linearization at all of its base points. The Mathematica-output in Appendix 1.2 shows that for $u = [3,5,3,5,...]$ the Sternberg-Non-Resonance-Conditions (SNC) are satisfied, because the $\alpha(1)$ that is necessary for a C^1-linearization at each point is always smaller than the sum of the absolute value of the coefficients in the vector $k = \{k_1, k_2, k_3\}$. That's why we can employ the Takens Linearization Theorem for both of the 18-cycles mentioned above, and Theorem 2.1 is proven.

3.5. Combined Linear Local Passage at the 18-Cycle. Our Results from Mathematica (see Appendix 1.3) indicate that we get the following matrix for the combined linear local passage of the classic 18-cycle when we apply the same algorithm that we outlined for the 3-cycle in section 2.2:

$$M_{18-cycle} = \begin{pmatrix} 267.54 & 110.78 \\ 595.16 & 247.51 \end{pmatrix} =: \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}$$

$$\mu_1 = 514.49 \text{ with Eigenvector } v_1 = (0.45, 1)$$

$$\mu_2 = 0.5578 \text{ with Eigenvector } v_2 = (-0.41, 1)$$

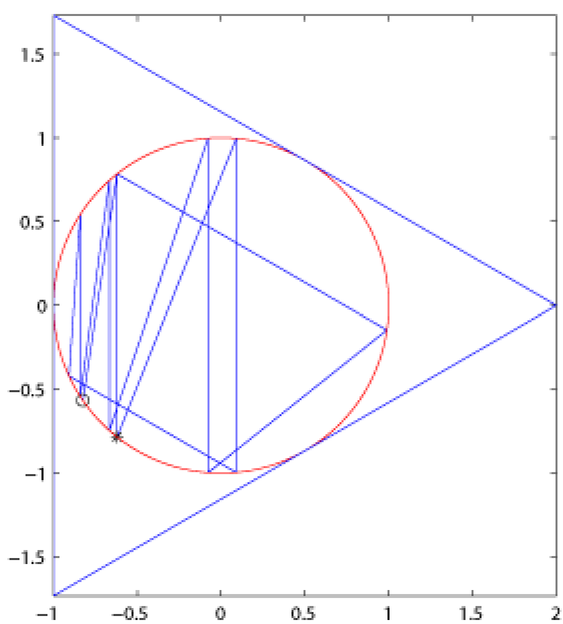
This implies that if we start with small positive a,b (i.e. log a, log b $<<$ 0), the combined linear local passage will bring us closer to the origin - this is what we mean by the term "contraction" in Lemma 2.2.

4. Numerical Simulation

The picture below shows a numerical simulation (with Matlab) of a periodic heteroclinic chain in Bianchi $VI^*_{-1/9}$, here a 13-cycle. According to the terminology developed above, it would be named the "advanced" 13 cycle, as both directions are taken from sector 5.

Although a much more detailed numerical analysis is necessary, our simulation shows that at least there are cases where both directions are taken from sector 5. Thus this possibility seems to really occur in the equations, at least numerically there are Bianchi $VI^*_{-1/9}$-solutions following the "advanced" 13 cycle towards the big bang.

Of course much more effort is needed in order to set up the numerics in an appropriate way instead of just using a built-in Matlab ODE solver[1]. One idea could be to use the explicit linear flow near the equilibria of the Kasner circle, where most of the time is spent, in order to achieve a higher precision.



[1]for producing the picture above, we have used the "ode113" solver, which is a variable order Adams-Bashforth-Moulton PECE solver (according to the Matlab documentation [**33**]). We thank Woei Chet Lim for providing us with some Matlab code that we used in order to carry out our numerical simulations.

Summary

We have constructed, for the fist time, the 18-cycle as a relatively simple example for a periodic heteroclinic chain in Bianchi $VI^*_{-1/9}$ that allows Takens Linearization at all Base points. In addition, we were able to show that the Combined Linear Local Passage at the classic 18-cycle is a contraction. This could be seen as a first step for proving a rigorous convergence theorem in Bianchi $VI^*_{-1/9}$.

In order to progress further, a better understanding of the global passages in Bianchi $VI^*_{-1/9}$ is necessary. This is not an easy task, as there are less invariant subspaces than in Bianchi IX that restrict the signs of the heteroclinic orbits, so much more complicated transitions are possible. A first step could be to check in detail which sequence of signs for the different transitions occurs numerically, leading to a classification of possible cases. Speculatively, one could think about the possibility to prove a theorem that for heteroclinic chains with periodic continued fraction developments and with a clearly defined seqence of signs for the transitions there exist solutions of the Bianchi $VI^*_{-1/9}$ equations that show this behaviour. But this matter requires further investigation.

Until now, the application of Dynamical Systems Techniques to spatially homogeneous cosmological models yielding Ordinary Differential Equations has been discussed. However, there has also been the attempt to apply such techniques to inhomogeneous cosmologies, yielding Partial Differential Equations. The reason is that in a way the main point[2] of the BKL-picture is the question of "locality", asking if

[2] let us again quote the recent survey paper by Uggla on this issue ([**54**],p.2): "However, arguably the most central, and controversial, assumption of BKL is their 'locality' conjecture. According to BKL, asymptotic dynamics toward a generic spacelike singularity in inhomogeneous cosmologies is 'local,' in the sense that each spatial point is assumed to evolve toward the singularity individually and independently of its neighbors as a spatially homogeneous model"

the "complicated" Einstein Equations that are PDEs[3] can be approximated by "simpler/less complicated" ODEs towards the big bang. Until today, mostly numerical and heuristic results exist in inhomogeneous cosmologies, but very few rigorous mathematical theorems.

In the paper "The past attractor in inhomogeneous cosmologies" ([**56**]), it is outlined how it could be achieved to make the "local" part of the BKL-picture more rigorous, compare also [**18**]. After some results have been proven in the oscillatory spatially homogeneous setting of Bianchi IX and in an inhomogeneous, but non-oscillatory setting of Gowdy-spacetimes (see e.g. [**41**]), the logical next step seems to be to consider inhomogeneous oscillatory cosmological models. Arguably the simplest case is given by the G_2-cosmologies, that's why it has received rising attention in recent years ([**11, 30, 32, 7**]). However, there is not a single rigorous convergence theorem comparable to the results that could be achieved in spatially homogeneous models.

A particular complication in inhomogeneous models is the occurrence of spikes, i.e. the formation of spatial structure. Numerical experiments support the conjecture that spikes form the non-local part of the generalized Mixmaster attractor ([**32**]). Lim has also found explicit spike solutions that are compatible with the usual Bianchi II - transitions, giving rise to a "non-local" version of the mimaster dynamics involving "spike transitions" ([**31**]). Recent progress has been achieved by Heinzle and Uggla, who report more in detail about the role of the spike solutions as building blocks of such an extended non-local mixmaster-dynamics ([**19**]). In addition, they have done a statistical analysis on the spikes in G_2-models ([**20**]).

A first step towards achieving rigorous results in inhomogeneous oscillatory models could be to investigate the process of spike formation in G_2-models. A good understanding of the underlying spatially homogeneous model (which is Bianchi $VI^*_{-1/9}$) is probably necessary for this project, but as Uggla writes in his recent survey: "Unfortunately, there exist no rigorous mathematical results concerning their past asymptotic dynamics", refering to Bianchi $VI^*_{-1/9}$ ([**54**], p.11).

[3] for General Relativity from the viewpoint of Partial Differential Equations see [**41, 43**]

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1. Results on Non-Resonance-Conditions and CLLP for Heteroclinic Cycles in Bianchi $VI^*_{-1/9}$

1.1. Takens Linearization at the Base Points of the 3-Cycle. For u=[m,a,a,...] and m=1...a with the following PARAMETERS:
a= 1, k= 1 (k=1 means smoothness is C^1)

m= 1
Sektor 1 alpha= 22 beta= 3 k= 2,0,1
Sektor 2 alpha= 14 beta= 5 k= -2,0,1
Sektor 3 alpha= 9 beta= 3 k= 2,4,1
Sektor 4 alpha= 15 beta= 6 k= 2,0,1
Sektor 5 alpha= 16 beta= 3 k= -2,0,1
Sektor 6 alpha= 17 beta= 8 k= 2,4,1

1.2. Takens Linearization at the Base Points of the 18-Cycle. We first give the coefficient matricies for the eigenvalues in the other sectors not part of the 3-cycle (see chapter 2, section 1.2):

$$M_{S_3} = \begin{pmatrix} 0 & -3 & 6 \\ 6 & -6 & 6 \\ 3 & 0 & 0 \end{pmatrix}, M_{S_4} = \begin{pmatrix} -3 & 3 & 0 \\ 0 & 6 & -6 \\ 3 & 0 & 0 \end{pmatrix}, M_{S_6} = \begin{pmatrix} 0 & -3 & 6 \\ -6 & 0 & 6 \\ -3 & 3 & 0 \end{pmatrix}$$

For u=[m,a,b,a,b,...] and m=1...b with the following PARAMETERS:
a= 3 b= 5 k= 1 (k=1 means smoothness is C^1)

m= 1
Sektor 1 alpha= 34 beta= 3 k= {-46,-80,-37}
Sektor 2 alpha= 11 beta= 4 k= {-34,-80,-37}
Sektor 3 alpha= 8 beta= 3 k= {6,-40,-37}
Sektor 4 alpha= 25 beta= 10 k= {6,-28,-37}
Sektor 5 alpha= 25 beta= 3 k= {-34,-28,-37}
Sektor 6 alpha= 30 beta= 14 k= {-46,-40,-37}
m= 2
Sektor 1 alpha= 15 beta= 3 k= {-46,-92,-43}
Sektor 2 alpha= 17 beta= 6 k= {-46,-92,-43}
Sektor 3 alpha= 13 beta= 4 k= {6,-40,-43}
Sektor 4 alpha= 10 beta= 4 k= {6,-40,-43}
Sektor 5 alpha= 11 beta= 3 k= {-46,-40,-43}
Sektor 6 alpha= 10 beta= 5 k= {-46,-40,-43}
m= 3
Sektor 1 alpha= 16 beta= 3 k= {-34,-80,-37}
Sektor 2 alpha= 22 beta= 8 k= {-46,-80,-37}
Sektor 3 alpha= 15 beta= 4 k= {6,-28,-37}

Sektor 4 alpha= 12 beta= 4 k= {6,-40,-37}
Sektor 5 alpha= 13 beta= 3 k= {-46,-40,-37}
Sektor 6 alpha= 9 beta= 5 k= {-34,-28,-37}
m= 4
Sektor 1 alpha= 20 beta= 3 k= {-10,-44,-19}
Sektor 2 alpha= 26 beta= 9 k= {-34,-44,-19}
Sektor 3 alpha= 21 beta= 5 k= {6,-4,-19}
Sektor 4 alpha= 14 beta= 4 k= {6,-28,-19}
Sektor 5 alpha= 15 beta= 3 k= {-34,-28,-19}
Sektor 6 alpha= 11 beta= 6 k= {-10,-4,-19}
m= 5
Sektor 1 alpha= 23 beta= 3 k= {26,16,11}
Sektor 2 alpha= 31 beta= 11 k= {-10,16,11}
Sektor 3 alpha= 23 beta= 5 k= {6,32,11}
Sektor 4 alpha= 19 beta= 5 k= {6,-4,11}
Sektor 5 alpha= 17 beta= 3 k= {-10,-4,11}
Sektor 6 alpha= 13 beta= 7 k= {26,32,11}

For u=[m,b,a,b,a,...] and m=1...a with the following
PARAMETERS: a= 3 b= 5 k= 1 (k=1 means smoothness is C^1)

m= 1
Sektor 1 alpha= 50 beta= 3 k= {-26,-52,-21}
Sektor 2 alpha= 11 beta= 4 k= {-26,-52,-21}
Sektor 3 alpha= 11 beta= 4 k= {10,-16,-21}
Sektor 4 alpha= 38 beta= 15 k= {10,-16,-21}
Sektor 5 alpha= 37 beta= 3 k= {-26,-16,-21}
Sektor 6 alpha= 47 beta= 21 k= {-26,-16,-21}
m= 2
Sektor 1 alpha= 16 beta= 3 k= {-6,-32,-11}
Sektor 2 alpha= 17 beta= 6 k= {-26,-32,-11}
Sektor 3 alpha= 12 beta= 4 k= {10,4,-11}
Sektor 4 alpha= 10 beta= 4 k= {10,-16,-11}
Sektor 5 alpha= 12 beta= 3 k= {-26,-16,-11}
Sektor 6 alpha= 12 beta= 6 k= {-6,4,-11}
m= 3
Sektor 1 alpha= 16 beta= 3 k= {34,28,19}
Sektor 2 alpha= 20 beta= 7 k= {-6,28,19}
Sektor 3 alpha= 14 beta= 4 k= {10,44,19}
Sektor 4 alpha= 11 beta= 4 k= {10,4,19}
Sektor 5 alpha= 13 beta= 3 k= {-6,4,19}
Sektor 6 alpha= 9 beta= 5 k= {34,44,19}

1.3. CLLP for the Classic 18-Cycle. The classic 18-cycle has the sector sequence 4343-425-43434343-425, i.e. we start in sector 4 with u=[3,5,3,5,...] which is around 3.18819:

Sector 4, u= 3.18819
Passage A: GP from Sector 4 to 3. In sector 4, the eigenvalues are:
lc= 1.5418 lt= 1.91557 ln= -1.33278 la= 0.209019
Passage A: GP from Sector 3 to 4. In sector 3, the eigenvalues are:
lc= -2.02211 lt= 3.44689 ln= 2.39822 la= 0.37611
The Eigenvalues are {1.4570,1.0000}
The Eigenvectors are {{0.167325,1.0000},{0,1.0000}}

Sector 4, u= 2.18819
Passage A: GP from Sector 4 to 3. In sector 4, the eigenvalues are:
lc= 2.02211 lt= 1.42478 ln= -1.646 la= 0.37611
Passage A: GP from Sector 3 to 4. In sector 3, the eigenvalues are:
lc= -2.81366 lt= 3.15683 ln= 3.64699 la= 0.833333
The Eigenvalues are {1.8416,1.0000}
The Eigenvectors are {{0.62500,1.0000},{0,1.0000}}

Sector 4, u= 1.18819
Passage B2: GP from Sector 4 to 2.In sector 4,the eigenvalues are:
lc= 2.81366 lt= 0.343171 ln= -1.98032 la= 0.833333
Passage B2: GP from Sector 2 to 5.In sector 2,the eigenvalues are:
lc= -3.37457 lt= 1.00965 ln= 5.82633 la= 2.45176
Passage B2: tGP from Sector 5 to 4.In sector 5,the eigenvalues are:
lc= 3.37457 lt= -2.36492 ln= -0.922814 la= 2.45176
The Eigenvalues are {3.1761,-1.9879}
The Eigenvectors are {{1.01227,1.0000},{-0.239459,1.0000}}

Sector 4, u= 5.31366
Passage A: GP from Sector 4 to 3. In sector 4, the eigenvalues are:
lc= 1.00965 lt= 2.36492 ln= -0.922814 la= 0.0868342
Passage A: GP from Sector 3 to 4. In sector 3, the eigenvalues are:
lc= -1.20737 lt= 3.41557 ln= 1.33278 la= 0.125411
The Eigenvalues are {1.2318,1.0000}
The Eigenvectors are {{0.045618,1.0000},{0,1.0000}}

Sector 4, u= 4.31366
Passage A: GP from Sector 4 to 3. In sector 4, the eigenvalues are:
lc= 1.20737 lt= 2.2082 ln= -1.08196 la= 0.125411
Passage A: GP from Sector 3 to 4. In sector 3, the eigenvalues are:

lc= -1.49614 lt= 3.45384 ln= 1.6923 la= 0.196156
The Eigenvalues are {1.3018,1.0000}
The Eigenvectors are {{0.075222,1.0000},{0,1.0000}}

Sector 4, u= 3.31366
Passage A: GP from Sector 4 to 3. In sector 4, the eigenvalues are:
lc= 1.49614 lt= 1.9577 ln= -1.29998 la= 0.196156
Passage A: GP from Sector 3 to 4. In sector 3, the eigenvalues are:
lc= -1.94792 lt= 3.45473 ln= 2.29407 la= 0.346154
The Eigenvalues are {1.4322,1.0000}
The Eigenvectors are {{0.150000,1.0000},{0,1.0000}}

Sector 4, u= 2.31366
Passage A: GP from Sector 4 to 3. In sector 4, the eigenvalues are:
lc= 1.94792 lt= 1.50681 ln= -1.60176 la= 0.346154
Passage A: GP from Sector 3 to 4. In sector 3, the eigenvalues are:
lc= -2.69398 lt= 3.23295 ln= 3.43668 la= 0.742693
The Eigenvalues are {1.7612,1.0000}
The Eigenvectors are {{0.49035,1.0000},{0,1.0000}}

Sector 4, u= 1.31366
Passage B2: GP from Sector 4 to 2.In sector 4, the eigenvalues are:
lc= 2.69398 lt= 0.538969 ln= -1.95129 la= 0.742693
Passage B2: GP from Sector 2 to 5.In sector 2, the eigenvalues are:
lc= -3.45737 lt= 1.5418 ln= 5.58196 la= 2.12459
Passage B2: GP from Sector 5 to 4.In sector 5, the eigenvalues are:
lc= 3.45737 lt= -1.91557 ln= -1.33278 la= 2.12459
The Eigenvalues are {2.8062,-1.4925}
The Eigenvectors are {{2.03045,1.0000},{-0.262732,1.0000}}

Sector 4, u= 3.18819
The Eigenvalues are {514.49,0.55783}
The Eigenvectors are {{0.448591,1.0000},{-0.414926,1.0000}}