Numerics of Stochastic Processes III (21.11.2008)

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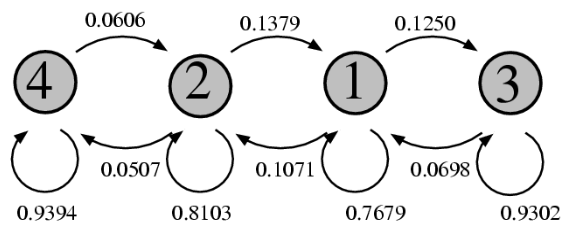
Short Reminder

Discrete State Space,
Discrete Time:
Markov Chain



Discrete State Space,
Continuous Time:
Markov Process

Stochastic Processes

Continuous State Space,
Discrete Time:
Markov Process

Continuous State Space,
Continuous Time:
Partial Differential Equation

Last Time

$$X_{t+\tau} = X_t + \tau f(X_t) + \sigma\sqrt{\tau}\epsilon_t \qquad \boxed{\tau \rightarrow 0} \qquad \partial_t \pi(X,t) = \mathbf{G}\pi(X,t)$$

Gaussian and Wiener Processes

process is Gaussian if and only if for every finite set of indices $t_1, \ldots, t_k$ in the index set T

$$\vec{\mathbf{X}}_{t_1,\ldots,t_k} = (\mathbf{X}_{t_1}, \ldots, \mathbf{X}_{t_k})$$

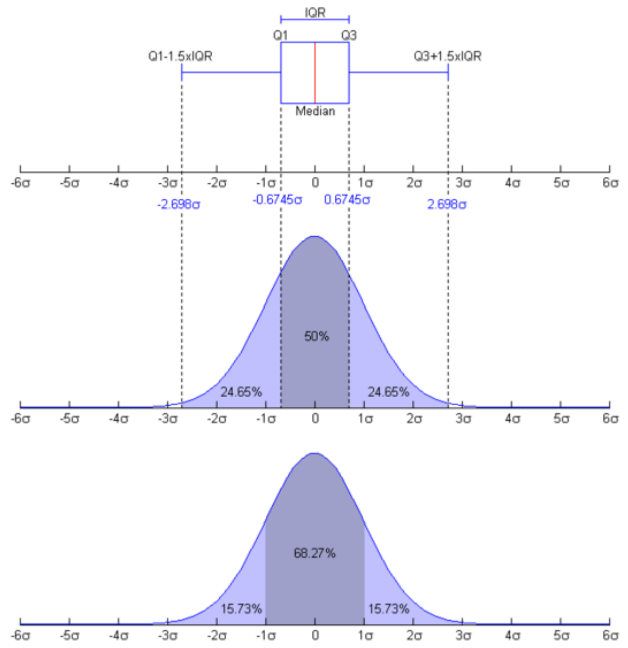
is a vector-valued Gaussian random variable

Wiener Process (also called ***Brownian Motion***):

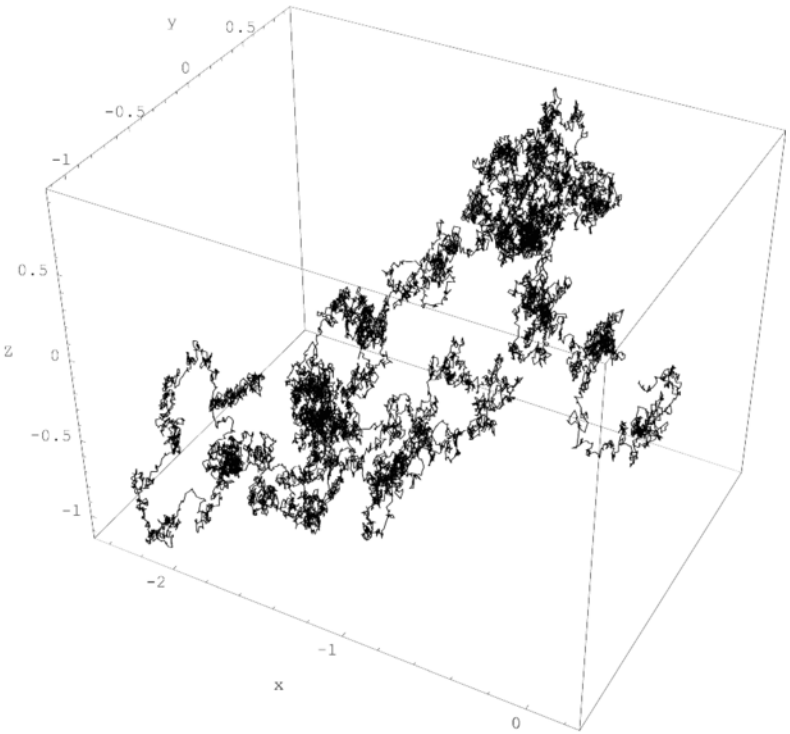
1. $W_0 = 0$
2. W_t is almost surely continuous
3. W_t has independent increments with distribution

$$W_t - W_s \sim \mathcal{N}(0, t-s) \text{ (for } 0 \leq s < t\text{).}$$

White Noise: $\epsilon_t \sim \mathcal{N}(0,1)$



Wiener Processes: Properties I



From: wikipedia.org

Properties of a one-dimensional Wiener process

The unconditional probability density function at a fixed time t:

$$f_{W_t}(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}.$$

The expectation is zero:

$$E(W_t) = 0.$$

The variance is t:

$$E(W_t^2) - E^2(W_t) = t.$$

The covariance and correlation:

$$\operatorname{cov}(W_s, W_t) = \min(s, t),$$

$$X_{t+\tau} = X_t + \tau f(X_t) + \boxed{\sigma\sqrt{\tau}\epsilon_t}$$

Wiener Process

Wiener Processes: Properties II

- For every ε>0, the function *w* takes both (strictly) positive and (strictly) negative values on (0,ε).
- The function *w* is continuous everywhere but differentiable nowhere (like the Weierstrass function).
- Points of local maximum of the function *w* are a dense countable set; the maximum values are pairwise different; each local maximum is sharp in the following sense: if *w* has a local maximum at *t* then $|w(s) - w(t)|/|s-t| \to \infty$ as *s* tends to *t*. The same holds for local minima.
- The function *w* has no points of local increase, that is, no *t*>0 satisfies the following for some ε in (0,*t*): first, *w*(*s*) ≤ *w*(*t*) for all *s* in (*t*-ε,*t*), and second, *w*(*s*) ≥ *w*(*t*) for all *s* in (*t*,*t*+ε). (Local increase is a weaker condition than that *w* is increasing on (*t*-ε,*t*+ε).) The same holds for local decrease.
- The function *w* is of unbounded variation on every interval.

Filtered Probability Space

Filtration:

sequence of σ-algebras on a measurable space. That is, given a measurable space $(\Omega, \mathcal{F})$, a filtration is a sequence of σ-algebras $\{\mathcal{F}_t\}_{t\geq 0}$ with $\mathcal{F}_t \subseteq \mathcal{F}$ for each t and

$$t_1 \leq t_2 \implies \mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}.$$

The exact range of the "times" t will usually depend on context: the set of values for t might be discrete or continuous, bounded or unbounded. For example,

$$t \in \{0, 1, \dots, N\}, \mathbb{N}_0, [0, T] \text{ or } [0, +\infty).$$

Similarly, a **filtered probability space** (also known as a **stochastic basis**) is a probability space with a filtration of its σ-algebra.

Filtered Probability Space:

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$$

The sigma algebra F_t represents the information available up until time t, and a process X is adapted if X_t is F_t-measurable. A Brownian motion B is understood to be an F_t-Brownian motion, which is just a standard Brownian motion with the property that B_{t+s} - B_t is independent of F_t for all $s,t \geq 0$.

Martingale

In full generality, a stochastic process $Y : T \times \Omega \to S$ is a **martingale with respect to a filtration $\Sigma_{\square}$ and probability measure P** if

- $\Sigma_{\square}$ is a filtration of the underlying probability space $(\Omega, \Sigma, \mathbf{P})$;
- Y is adapted to the filtration $\Sigma_{\square}$, i.e., for each t in the index set T, the random variable Y_t is a Σ_t-measurable function;
- for each t, Y_t lies in the L^p space $L^1(\Omega, \Sigma_t, \mathbf{P}; S)$, i.e.
  $$\mathbf{E}_{\mathbf{P}}(|Y_t|) < +\infty;$$
- for all s and t with $s < t$ and all $F \in \Sigma_s$,
  $$\mathbf{E}_{\mathbf{P}}\left([Y_t - Y_s]\chi_F\right) = 0,$$

$$X_{t+\tau} = X_t + \tau f(X_t) + \boxed{\sigma\sqrt{\tau}\epsilon_t}$$

Martingale

Stochastic Integrals

$$\int_0^t H\,dX \equiv \int_0^t H_s\,dX_s$$

What kind of Integral???

H and *X* are functions of unbounded variation, non-smooth almost everyvwhere

Riemann-Stiltjes Integral does not exist!

Convergence in Probability: $\lim_{n\to\infty} \Pr\left(|X_n - X| \geq \varepsilon\right) = 0$

Idea: define via partial sum

Martingale! $$I := \int_a^b X_t dY_t := \lim_{n\to\infty} \sum_{i=1}^{n} X_{(i-1)h}\left(Y_{ih} - Y_{(i-1)h}\right), h = \frac{b-a}{n}.$$

Not Martingale! $$S := \int_a^b X_t \circ dY_t := \lim_{n\to\infty} \sum_{i=1}^{n} X_{(i-0.5)h}\left(Y_{ih} - Y_{(i-1)h}\right).$$

Ito Integral: Properties

Ito Isometry: $$E\left((H\cdot B_t)^2\right) = E\left(\int_0^t H_s^2\,ds\right)$$

Linearity (obvious)…

Integration by parts: $$X_tY_t = X_0Y_0 + \int_0^t X_{s-}\,dY_s + \int_0^t Y_{s-}\,dX_s + [X,Y]_t$$

Expectation value of Ito integral is zero

Limit of $X_{t+\tau} = X_t + \tau f(X_t) + \sigma\sqrt{\tau}\epsilon_t$ for $\tau \to 0$ can be defined as

$$X_t = X_0 + \int_0^t \sigma_s\,dB_s + \int_0^t \mu_s\,ds$$

Two equivalent representations

Stochastic Differential Equation (SDE):

$$X_t = X_0 + \int_0^t \sigma_s \, dB_s + \int_0^t \mu_s \, ds$$

and

Partial Differential Equation (PDE):

$$\partial_t \pi(X,t) = \mathbf{G}\pi(X,t)$$