Numerics of Stochastic Processes II (14.11.2008)

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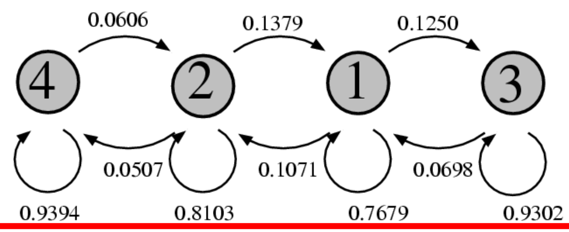
„Mathematics in key technologies"

Short Reminder

Discrete State Space,
Discrete Time:
Markov Chain



Discrete State Space,
Continuous Time:
Markov Process

Last Time

Stochastic Processes

Continuous State Space,
Discrete Time:
Markov Process

$$X_{t+\tau} = \sum_{k=0}^{p} \alpha_k X_{t-k\tau} + \sigma \epsilon_t$$

Continuous State Space,
Continuous Time:
Stochastic Differential Equation

$$dX_t = f\left(X_t, t\right) dt + \sigma dW_t$$

Short Reminder: Markov Processes

Infenitisimal Generator:

$$\mathcal{G}(t) = \lim_{\tau \to 0} \frac{P(t,\tau) - \mathcal{I}}{\tau}$$

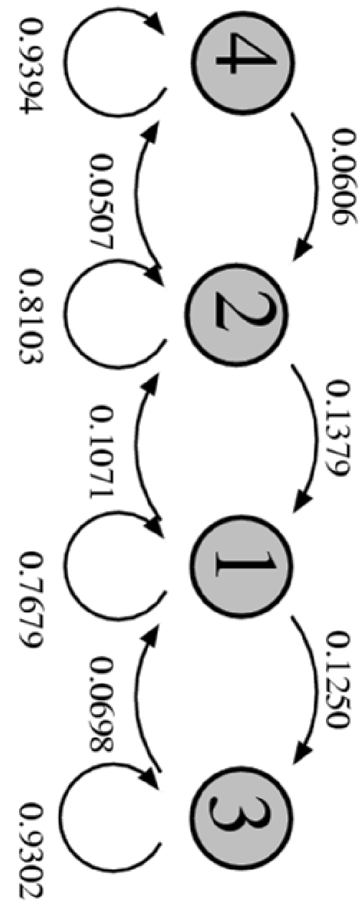
Markov Process Dynamics:

$$\dot{\pi} = \pi \mathcal{G}(t)$$

This Equation is Deterministic: ODE!

1) Numerical Methods from ODEs like Runge-Kutta-Method become available for stochatic processes

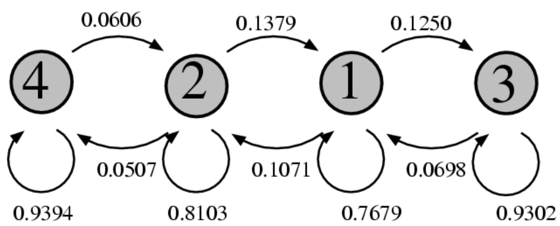
2) Monte-Carlo-Sampling just at the end!



Concepts from the Theory of Dynamical Systems are Applicable

Short Reminder

Discrete State Space,
Discrete Time:
Markov Chain



Discrete State Space,
Continuous Time:
Markov Process

Stochastic Processes

Continuous State Space,
Discrete Time:
Markov Process

Continuous State Space,
Continuous Time:
Stochastic Differential Equation

Today

$$X_{t+\tau} = X_t + \tau f(X_t) + \sigma \sqrt{\tau} \epsilon_t$$

$$dX_t = f(X_t, t)\, dt + \sigma dW_t$$

Markov Process in **R**

Realizations of the process: $X_t \in \mathbb{R}$

$$\{X_0, X_\tau, X_{2\tau}, \ldots, \mathbf{X}_{t-\tau}\}$$

Process:

$$X_{t+\tau} = X_t + \tau f(X_t) + \sigma\sqrt{\tau}\epsilon_t$$

Next-Step Probability:

$$P(X_{t+\tau}|X_t) = \mathbf{N}(X_t + \tau f(X_t), \tau\sigma^2)$$

Next-Step Distribution:

$$\pi(X, t+\tau) = \exp\{\frac{\tau\sigma^2}{2}\partial_X^2 + \tau f(X)\partial_X\}\pi(X, t)$$

This Equation is Deterministic!

Markov Process in **R**

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Next-Step Distribution:

$$\pi(X, t+\tau) = \exp\{\frac{\tau\sigma^2}{2}\partial_X^2 + \tau f(X)\partial_X\}\pi(X,t) \qquad \tau \rightarrow 0$$

This Equation is Deterministic!

Markov Process in **R**

Infenitisimal Generator:

$$\mathbf{G} = \lim_{\tau \to 0} \frac{\exp\{\frac{\tau\sigma^2}{2}\partial_X^2 + \tau f(X)\partial_X\} - \mathbf{I}}{\tau}$$

Markov Process Dynamics:

$$\partial_t \pi(X,t) = \mathbf{G}\pi(X,t)$$

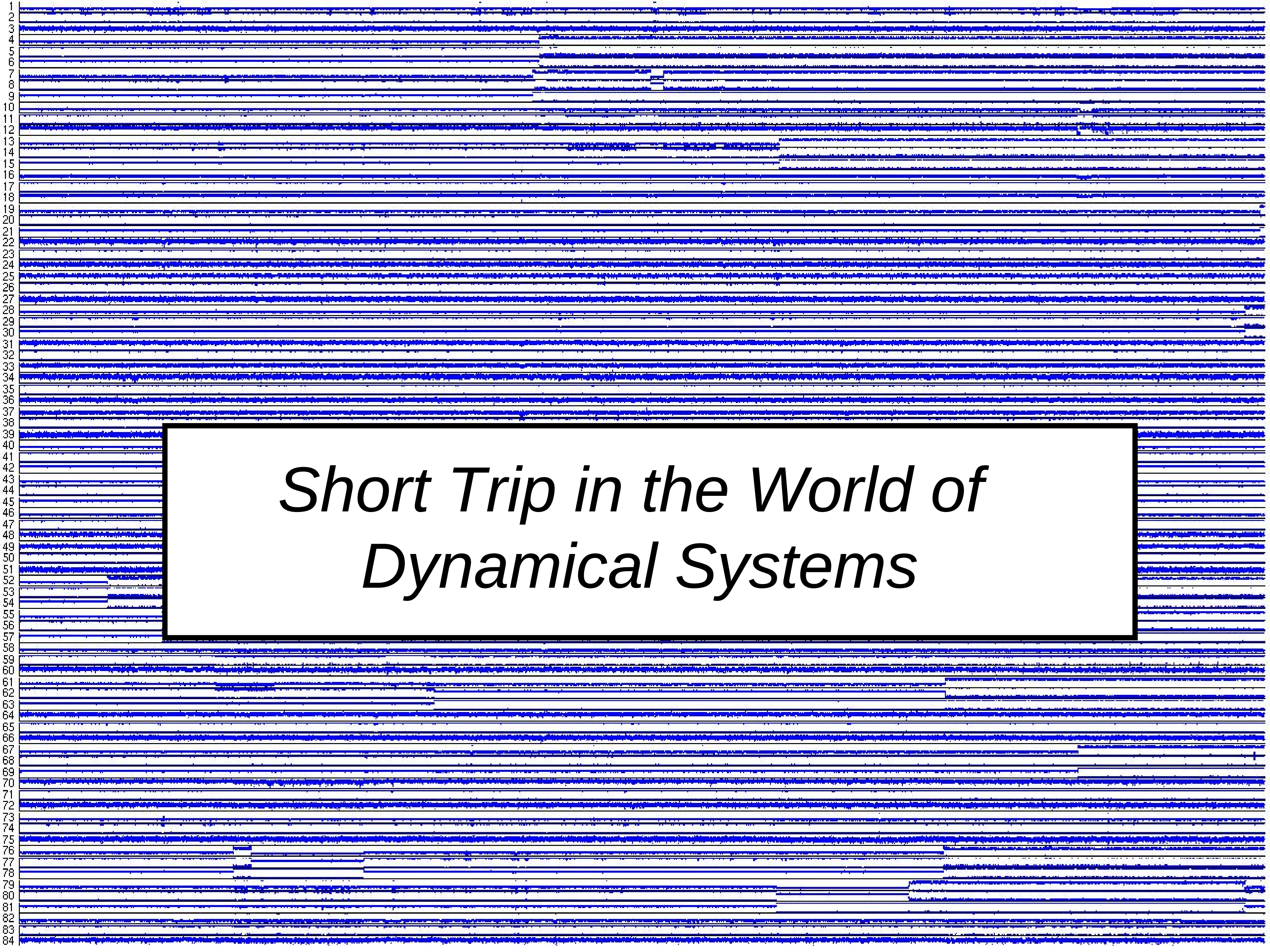
This Equation is Deterministic: PDE!

1) Numerical Methods from PDEs like Finite-Element-Methods (FEM) become available for stochatic processes

2) Monte-Carlo-Sampling just at the end!

Concepts from the Theory of Dynamical Systems are Applicable

Short Trip in the World of Dynamical Systems

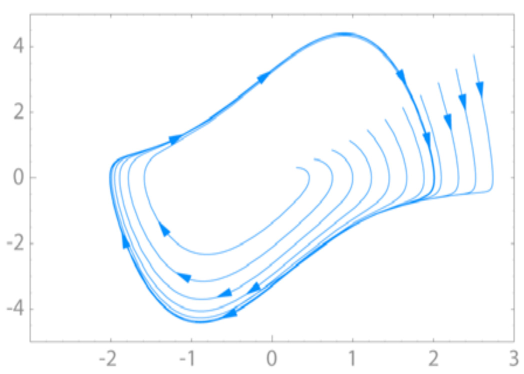


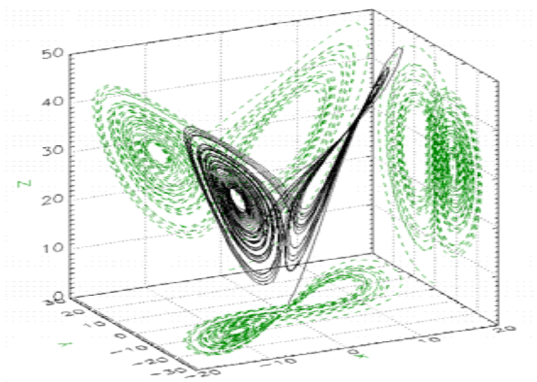
Dynamical systems viewpoint

Essential Dynamics of many dynamical systems is defined by *attractors*.

Attractor $\mathbf{A}$ stays invariant under the *flow operator* f_τ

Figures from http:\\www.wikipedia.org

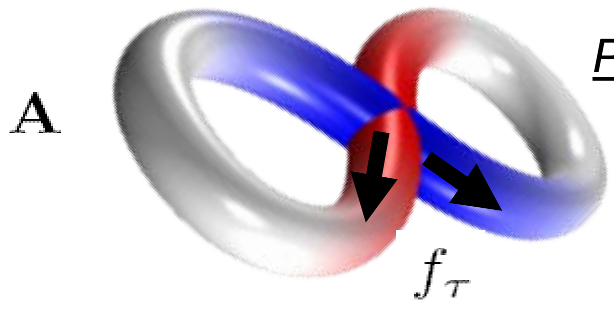




Separation of informative (attractor) and non-informative (rest) parts of the space leads to dimension reduction

How to localize the attractor?

Attractors can have very complex, even fractal geometry



Problem: Euclidean distance cannot be used as a measure for the relative neighbourghood of attractor elements.

Strategy: the data have to be "embedded" into Eucllidean space

Whitney embedding theorem (Whitney, 1936) : sufficiently smooth connected m-dimensional manifolds can be smoothly embedded in $(2m+1)$-dimensional Euclidean space.

How to construct the embedding?

Takens' Embedding (Takens, 1981)

$z_t \in \mathbf{M} \subset \mathbf{R}^d$ *and* $f_\tau : \mathbf{M} \rightarrow \mathbf{M}$ *is a smooth map.*

Assume that f_τ *has an attractor* $\mathbf{A}$ *with dimension*

$m << d$. *Let* $\alpha : \mathbf{A} \rightarrow \mathbf{R}^1$, $\alpha \in \mathcal{C}^2$ *be a "proper"*

measurement process. Then

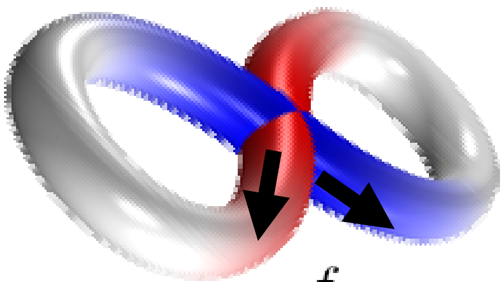
$$\phi_\alpha(z) = \left(\alpha(z), \alpha(f_\tau(z)), \ldots, \alpha(f_{2m\tau}(z))\right)$$

is a $(2m+1)$ *-dimensional embedding in Euclidean space*

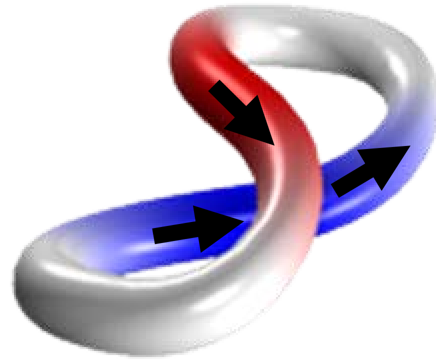
Illustration of Takens‘ Embedding

The image of $\phi_\alpha(z)$ is unfolded in $\mathbf{R}^{2m+1}$

$\mathbf{A}$



f_τ



Problems:

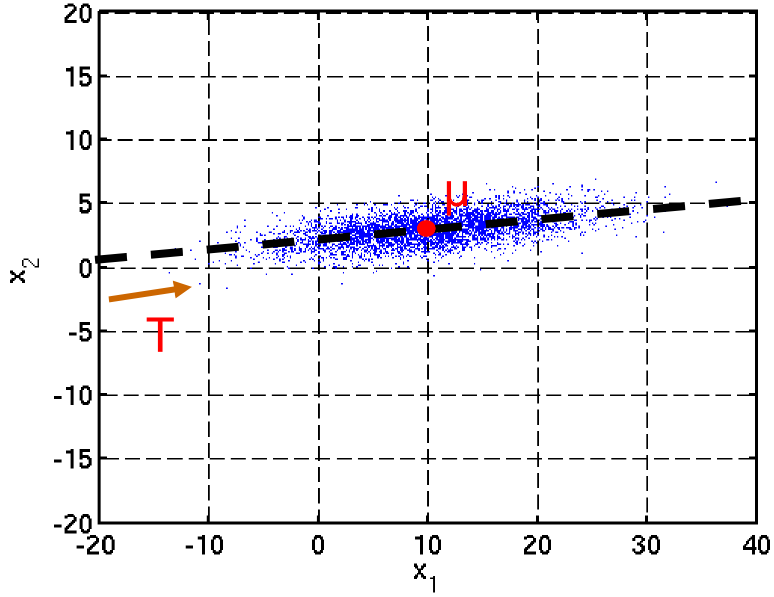
- how to “filter out“ the attractive subspace?
- how to connect the changes in attractive subspace with hidden phase?

Strategy:

Let $\{z_t\}_{t=1,\ldots,T}, z_t \in \mathbf{R}^d$. Define a new variable $\{x_t^q\} = \{z_t, z_{t-1}, \ldots, z_{t-q}\}$. Let the attractor $\mathbf{A}$ for each of the hidden phases be contained in a distinct *linear manifold* defined via $\mathbf{T} \in \mathbf{R}^{nxm}, m << d$

Topological dimension reduction

(*H. 07*): for a given time series x_t look for a minimum of the *reconstruction error*



Topological dimension reduction

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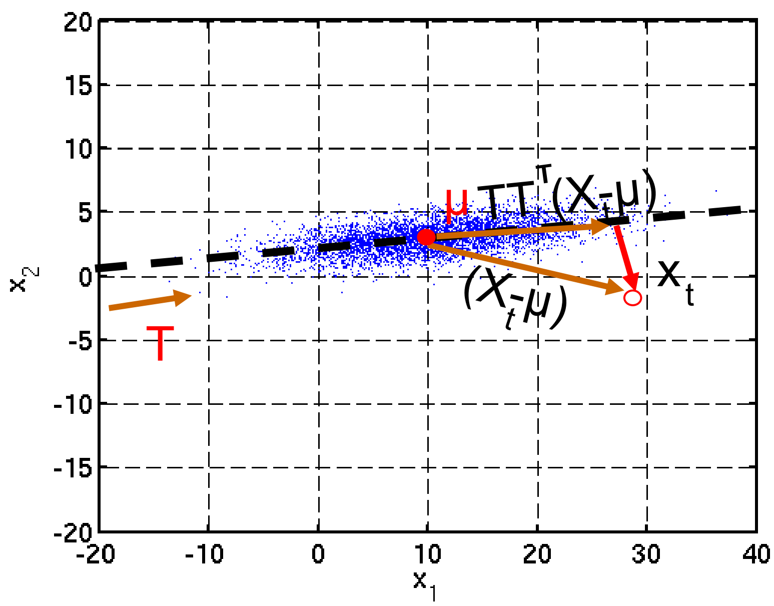
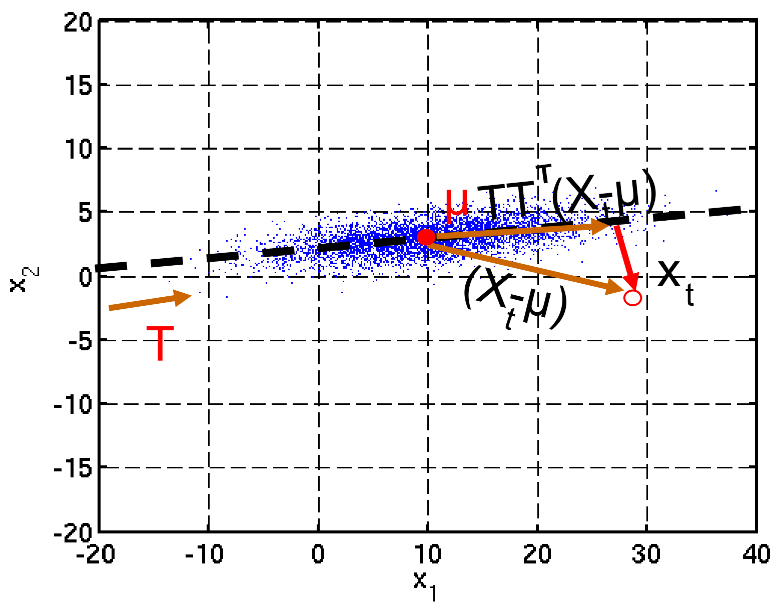


Topological dimension reduction

(*H. 07*): for a given time series x_t look for a minimum of the *reconstruction error*

$$\left\| (x_t - \mu) - \mathbf{T}\,\mathbf{T}^{\mathsf{T}} (x_t - \mu) \right\|_2^2$$



Analysis of embedded data

PCA + Takens' Embedding *(Broomhead&King, 1986)*

Let $\{z_t\}_{t=1,\ldots,T}, z_t \in \mathbf{R}^d$. Define a new variable $\{x_t^q\} = \{z_t, z_{t-1}, \ldots, z_{t-q}\}$. Let the attractor from Takens' Theorem be in a *linear manifold* defined via $\mathbf{T} \in \mathbf{R}^{nxm}, m << d$. Then

$$\sum_{t=1}^{T} \left\| x_t - \mathbf{T}\mathbf{T}^{\mathsf{T}} x_t \right\|_2^2 \rightarrow \min$$
$$\mathbf{T}^{\mathsf{T}}\mathbf{T} = Id,$$

Reconstruction from m *essential coordinates*: $\mathbf{T}\,\mathbf{T}^{\mathsf{T}} x_t$

PCA+Takens: data compression/reconstruction

1.

$$z = \left\{ z_1,\, z_2,\, z_3,\, \dots \right\}$$

$\xrightarrow{Embedding}$

2.

$$x = \begin{Bmatrix} z_1 & z_2 & z_3 & \dots \\ z_2 & z_3 & z_4 & \dots \\ z_3 & z_4 & z_5 & \dots \\ \dots & \dots & \dots & \dots \\ z_q & z_{q+1} & z_{q+2} & \dots \end{Bmatrix}$$

↓ Projection

3.

$$x_p(t) = \mathbf{T}^{\mathsf{T}} x(t)$$

← ReconstructionI

4.

$$x_r(t) = \mathbf{T} x_p(t)$$

$$x_r = \begin{Bmatrix} z_1^{(1)} & z_2^{(2)} & z_3^{(3)} & \dots \\ z_2^{(1)} & z_3^{(2)} & z_4^{(3)} & \dots \\ z_3^{(1)} & z_4^{(2)} & z_5^{(3)} & \dots \\ \dots & \dots & \dots & \dots \\ z_q^{(1)} & z_{q+1}^{(2)} & z_{q+2}^{(3)} & \dots \end{Bmatrix}$$

ReconstructionII →

5.

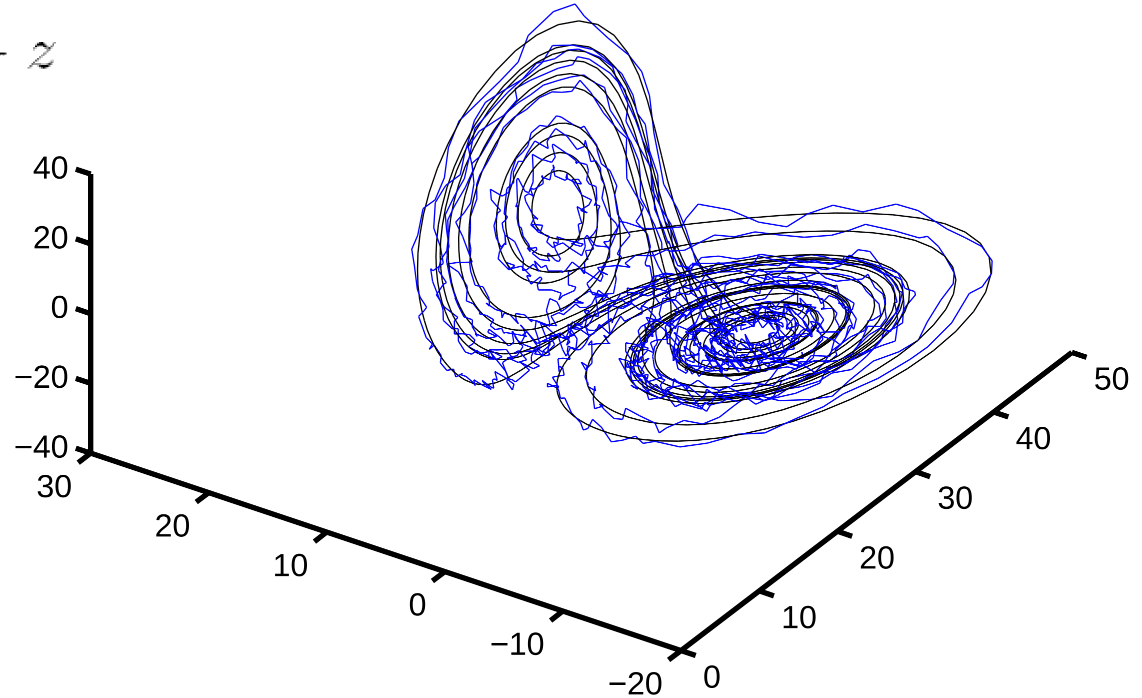
$$z_r = \left\{ z_1^{(1)},\, \frac{1}{2}\left(z_2^{(2)} + z_2^{(1)}\right),\, \dots \right\}$$

$$z_r^{(t)} = \frac{1}{\min\{q, t\}} \sum_{j=0}^{\min\{q-1, t-1\}} z_t^{(t-j)}$$

Example: Lorenz

Lorenz-Oszillator with measurement noise

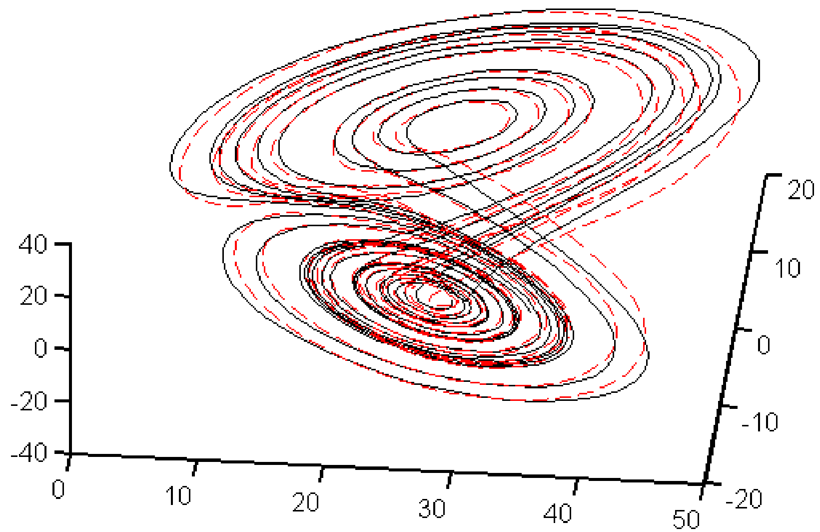
$$\dot{x} = \frac{8}{3}x + yz$$
$$\dot{y} = -10y + 10z$$
$$\dot{z} = -xy + 28y - z$$



Example: Lorenz

$$\mathbf{T}\,\mathbf{T}^{\top} x_t$$

Reconstructed trajectory
(*red*, for *m=2*)



Themen

1) **Markov-Prozesse (I): Dynamische Systeme und Identifikation der metastabilen Mengen (Maren)**

2) **Markov-Prozesse (II): Generatorschätzung (??)**

3) **Large Deviations Theory (Olga)**

4) **Dimensionsreduktion in Datenanalyse: graphentheoretische Zugänge I (Paul)**

5) **Dimensionsreduktion in Datenanalyse: graphentheoretische Zugänge II(Ilja)**

6) **Prozesse mit Errinerung (I): opt. Steuerung, Kalman-Filter und ARMAX-Model (??)**

7) **Prozesse mit Errinerung (II): Change-Point analysis (Peter)**

8) **Varianz von Parameterschätzungen (Bootstrap-Algorithmus) (Beatrice)**

9) **Modelvergleich in Finanzsektor (Lars)**