Numerics of Stochastic Processes (07.11.2008)

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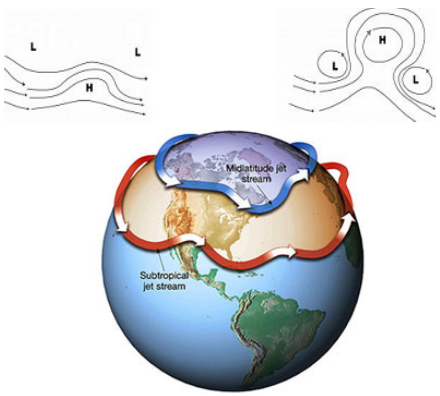
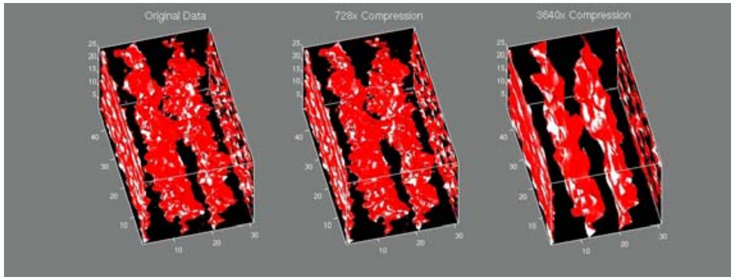
Complex Systems



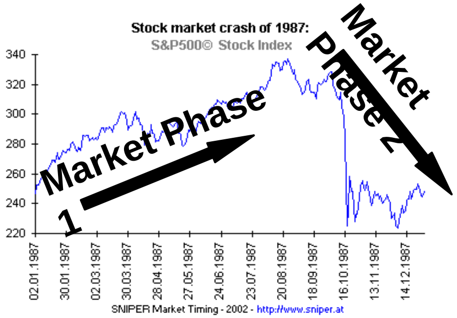
Weather Regimes

Meteorology



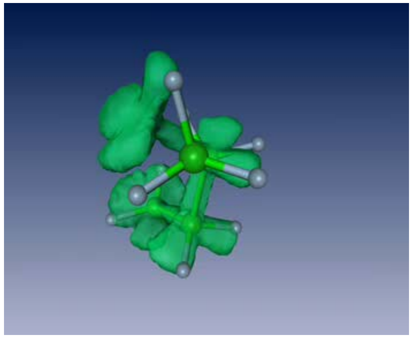
Flow Regimes

Fluid Mechanics



Market Phases

Finance



Molecular Conformations

Drug Design

Deterministic Description is Unavailable or Unfeasible!

Properties:

1) *non-stationarity*

2) a lot of *d.o.f.s* are involved (*multidimensionality*)

3) *stochasticity*

Today we look at:

1) Stochastic Processes and their deterministic interpretation

2) Memory

3) Concept of *Attractors*

Memo I: Probability

A probability space is a measure space such that the measure of the whole space is equal to 1.

In other words: a probability space is a triple $(\Omega, \mathcal{F}, P)$ consisting of a set Ω (called the sample space), a σ-algebra (also called σ-field) $\mathcal{F}$ of subsets of Ω (these subsets are called events), and a measure P on $(\Omega, \mathcal{F})$ such that $P(\Omega) = 1$ (called the probability measure).

Event	Probability
A	$P(A) \in [0,1]$
not A	$P(A') = 1 - P(A)$
A or B	$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(A) + P(B) \qquad \text{if A and B are mutually exclusive} \end{aligned}$
A and B	$\begin{aligned} P(A \cap B) &= P(A\|B)P(B) \\ &= P(A)P(B) \qquad \text{if A and B are independent} \end{aligned}$
A given B	$P(A\|B)$

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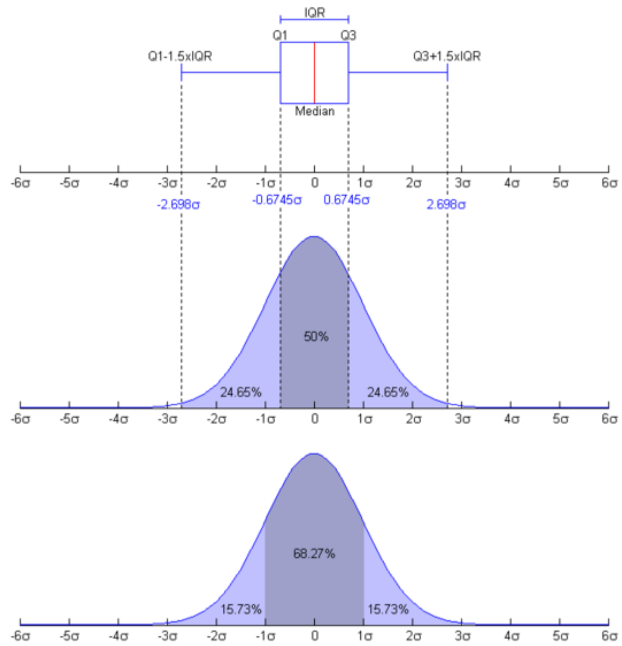
Memo II: Stochastic Process

Given a probability space $(\Omega, \mathcal{F}, P)$, a **stochastic process** (or **random process**) with state space X is a collection of X-valued random variables indexed by a set T ("time"). That is, a stochastic process F is a collection

$$\{F_t : t \in T\}$$

where each F_t is an X-valued random variable.

White Noise: $\epsilon_t \sim \mathcal{N}(0,1)$



Probability Density Function:

$$\Pr(4.3 \le X \le 7.8) = \int_{4.3}^{7.8} p(x)\,dx$$

Expectation Value:

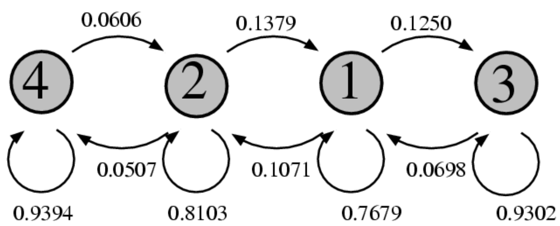
$$\mathrm{E}(X) = \int_\Omega X\,\mathrm{d}P \qquad \mu = \int x\,p(x)\,dx$$

Variance:

$$\mathrm{Var}(X) = \int (x-\mu)^2\,p(x)\,dx$$

Classification of Stochastic Process

Discrete State Space,
Discrete Time:
Markov Chain



Discrete State Space,
Continuous Time:
Markov Process

Stochastic Processes

Continuous State Space,
Discrete Time:
Autoregressive Process

$$X_{t+\tau} = \sum_{k=0}^{p} \alpha_k X_{t-k\tau} + \sigma \epsilon_t$$

Continuous State Space,
Continuous Time:
Stochastic Differential Equation

$$dX_t = f\left(X_t, t\right) dt + \sigma dW_t$$

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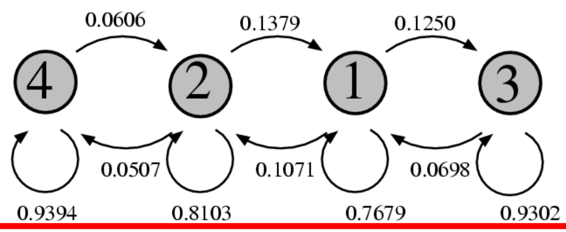

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TODAY

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Markov Chains

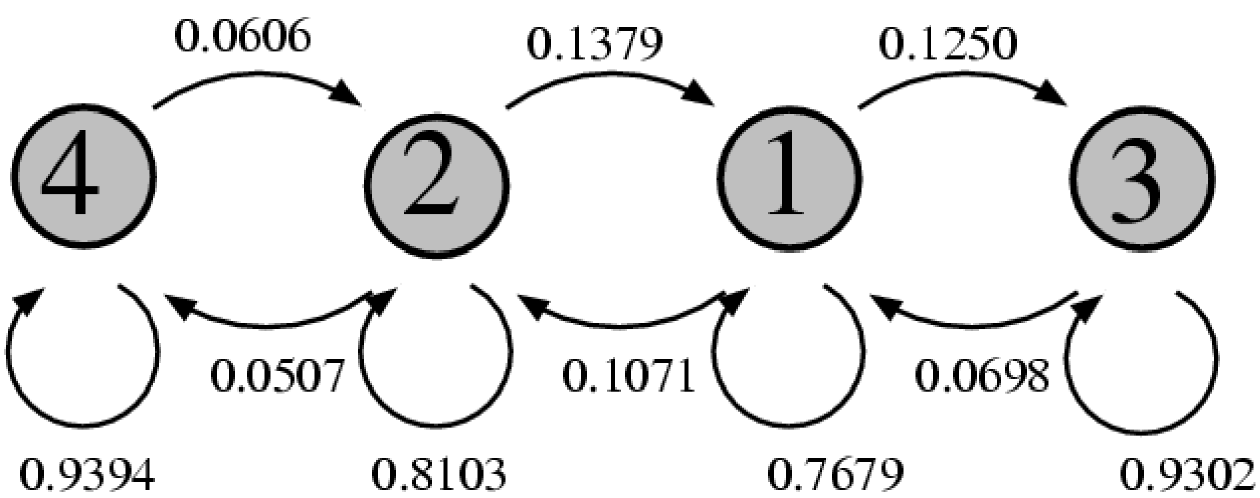
Realizations of the process: $X_t \in s_1, \ldots, s_m$

$$\{X_0, X_\tau, X_{2\tau}, \ldots, \mathbf{X}_{t-\tau}\}$$

Markov-Property:

$$\mathbb{P}\left[X_t = s_j | X_0, X_\tau, X_{2\tau}, \ldots, \mathbf{X}_{t-\tau} = s_j\right] = \mathbb{P}\left[X_t = s_j | \mathbf{X}_{t-\tau} = s_j\right] = P_{ij}\left(t, \tau\right)$$

Example:



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$$\pi(t) = (\pi_1(t), \pi_2(t), \dots, \pi_m(t))$$

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Dynamics:

$$\boxed{\pi(t+\tau) = \pi(t)P(t)}$$

This Equation is Deterministic!

Markov Processes

Infenitisimal Generator:

$$\mathcal{G}(t) = \lim_{\tau \to 0} \frac{P(t,\tau) - \mathcal{I}}{\tau}$$

Markov Process Dynamics:

$$\boxed{\dot{\pi} = \pi \mathcal{G}(t)}$$

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2) Monte-Carlo-Sampling just at the end!

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Concepts from the Theory of Dynamical Systems are Applicable

Thank you for attention!