Today’s lecture

- Today we will discuss the simplest non-trivial example of a stack.
- A torsor for a group $G$ is a principal $G$-bundle.
- The moduli stack of torsors is denoted by $BG$.
- It looks like a point, with stabilizer group $G$. 
Today’s lecture

- Discussing torsors allows us to define quotient stacks $[X/G]$.
- This yields an abundance of examples of algebraic stacks.
- Many algebraic stacks are of this type.
- However, next week we will discuss the stack $\text{Bun}_G(X)$ of $G$-bundles on a curve $X$.
- $\text{Bun}_G(X)$ cannot be written as a quotient stack.
Let \( C \) be a category, the structure of a group object on \( X \in C \) is equivalent to a factorisation of functors 

\[
\begin{tikzcd}
C^{\text{op}} & \text{Grp} \\
\text{Hom}_C(\cdot, X) & \text{Set}
\end{tikzcd}
\]

where \( \text{Grp} \to \text{Set} \) is the forgetful functor from groups to sets.
Lemma

If $C$ has finite products, the structure of a group object on $X \in C$ is equivalent to the following collection of morphisms:

(a) $e: \bullet \to X$, where $\bullet \in C$ is a final object,
(b) $m: X \times X \to X$,
(c) $\iota: X \to X$,

satisfying the group axioms: (...)
(...) satisfying the group axioms:

(a) *(unit)*

\[
\begin{array}{c}
X \overset{\sim}{\longrightarrow} X \times \bullet \\
\downarrow \id_X \downarrow \quad \downarrow m \circ (\id_X \times e) \\
\quad X, \\
\end{array}
\]

(b) *(associativity)*

\[
\begin{array}{c}
X \times X \times X \overset{m \times \id_X}{\longrightarrow} X \times X \\
\downarrow \id_X \times m \quad \downarrow m \\
X \times X \overset{m}{\longrightarrow} X, \\
\end{array}
\]
(...) satisfying the group axioms:

(c) (inverses)

\[ X \times X \xrightarrow{\text{id}_X \times \iota} X \times X \]

\[ \Downarrow \quad \Downarrow m \]

\[ \bullet \xrightarrow{e} X. \]
Example

Fix a base ring $R$. A group $R$-scheme is a group object in the category $\text{Sch}_R$.

(a) Let $\mathbb{A}^1_R : \text{Alg}_R \to \text{Set}$ be the functor sending an $R$-algebra $R \to S$ to the set underlying $S$. We have $\mathbb{A}^1_R \cong \mathbb{A}^1_\mathbb{Z} \times_{\text{Spec} \mathbb{Z}} \text{Spec} R$, in particular we see that $\mathbb{A}^1_R$ is a scheme. It carries a natural structure of a group object, given by the group-valued functor, sending $R \to S$ to the abelian group $(S, +)$. 

Torsors and quotient stacks
Example

Fix a base ring $R$. A group $R$-scheme is a group object in the category $\text{Sch}_R$.

(b) Similarly, we may define $\mathbb{G}_{m,R}$ as the group-valued functor, sending an $R$-algebra $R \to S$ to the set of invertible elements $S^\times$. It is a commutative group $R$-scheme.
Group objects

Example

Fix a base ring $R$. A group $R$-scheme is a group object in the category $\text{Sch}_R$.

(c) We have a group scheme $\text{GL}_{n,R}$, sending an $R$-algebra $S$ to the group of invertible $(n \times n)$-matrices.
If $G$ is an affine group $R$-scheme, then the $R$-algebra $\Gamma(G)$ of regular functions on $G$ is endowed with the structure of a Hopf $R$-algebra.

**Definition**

A Hopf $R$-algebra consists of an $R$-algebra $S$, together with $R$-algebra maps

(a) $e^\#: S \to R$, the *co-unit*,
(b) $m^\#: S \to S \otimes_R S$, called *comultiplication*,
(c) $\iota^\#: S \to S$, the *co-inverse*,

such that $(S, e^\#, m^\#, \iota^\#)$ satisfies the axiom of a group object (see the lemma stated earlier) in $\text{Alg}^\text{op}_R \cong \text{Aff}_R$. 
Let $G$ be an algebraic space of finite type over a field $k$. If $G$ has the structure of a group object, then $G$ is automatically a scheme:

- every algebraic space of finite type over a field has an open dense subset, which is a scheme,
- using the group operation we can translate this open subset, hence creating a Zariski open covering by schemes.
Let \((C, \mathcal{T})\) be a site, s.t. for every \(U \in C\), the representable functor \(\text{Hom}_C(-, U)\) is a sheaf.

consider a group-valued sheaf \(G : C \to \text{Grp}\) (i.e., a group object in \(\text{Sh}(C)\)).
Let $Y \in \text{Sh}_T(C)$ be endowed with a $G$-action, and $\pi: Y \to X$ be a morphism $G$-invariant morphism in $\text{Sh}_T(C)$. If for every $U \to X$ there exists $\{U_i \to U\}_{i \in I}$, s.t. 

$Y \times_X U_i \cong U_i \times G$ as a $G$-space,

we call $\pi$ a $G$-torsor.
We say that $\pi: Y \to X$ is a trivial torsor, if we have an isomorphism $Y \cong X \times G$.

A $G$-torsor is free if and only if there exists a section $s: X \to Y$, $\pi \circ s = \text{id}_X$. 
Proof.

- A section $s$ induces a map of $G$-torsors $X \times G \overset{(s \times \text{id}_G)}{\longrightarrow} Y \times G \rightarrow Y$. We show later that every map of torsors is an isomorphism.
- Vice versa, given an isomorphism $X \times G \cong Y$, as torsors, it induces a section

$$X \overset{\cong}{\rightarrow} X \times \bullet \overset{\text{id}_X \times e}{\longrightarrow} X \times G \overset{a}{\rightarrow} Y.$$ 

This concludes the proof.
A map of $G$-torsors over $X$ is a commutative diagram

$$
\begin{array}{c}
Z \\
\downarrow \\
Y \\
\downarrow \\
X,
\end{array}
$$

where $Z \to X$ and $Y \to X$ are $G$-torsors.

Every map of $G$-torsors is an isomorphism.
Proof.

- By Yoneda’s lemma, it suffices to show that for every \( U \in \mathcal{C} \), and \( U \to X \), the map \( Z \times_X U \to Y \times_X U \), of \( G \)-torsors over \( U \) is an isomorphism.

- Since \( X \), \( Y \), \( Z \) are sheaves, and a map of sheaves is an isomorphism, if and only if locally is an isomorphism, it suffices to prove the assertion for trivial \( G \)-torsors.

- (...)
Proof.

(...)

Let $\phi: U \times G \to U \times G$ a $G$-equivariant map of trivial $G$-torsors.

For every $V \in C$, and every morphism $V \to U$, we obtain a $G$-equivariant map of $G(V)$-torsors $U(V) \times G(V) \to U(V) \times G(V)$.

Every map of torsors in sets is an equivalence, since the transitivity and freeness of the action guarantee bijectivity of the map.
Corollary

Every commutative diagram of sheaves, and $G$-equivariant maps

\[
\begin{array}{ccc}
Z & \longrightarrow & Y \\
\downarrow & & \downarrow \\
W & \longrightarrow & X,
\end{array}
\]

where $Z \to W$ and $Y \to X$ are $G$-torsors, is cartesian.
Corollary

*Every commutative diagram of sheaves, and G-equivariant maps*

\[
\begin{array}{ccc}
Z & \rightarrow & Y \\
\downarrow & & \downarrow \\
W & \rightarrow & X,
\end{array}
\]

*where* \(Z \rightarrow W\) *and* \(Y \rightarrow X\) *are G-torsors, is cartesian.*

Proof.

The commutative diagram induces a map of G-torsors \(Z \rightarrow W \times_X Y\). But every map of G-torsors is an isomorphism.
• $C = \text{Aff}_R$

• there are various topologies $\mathcal{T}$ to consider:

• \textbf{Zar}, \textbf{et}, \textbf{smth} (equivalent to \textbf{et}), \textbf{fppf}, etc.

• We will study how the notion of torsors behaves with respect to various topologies.

• First we introduce the stack of torsors $BG$. 

Torsors and quotient stacks
We denote by $BG : \text{Aff}^{\text{op}}_R \rightarrow \text{Gpd}$ the prestack, which sends $R$ to the groupoid of $G$-torsors on $\text{Spec } R$.

To really obtain a strict functor, one needs to strictify - like in the definition of $\widetilde{\text{Mod}}$.

Since torsors are defined by glueing trivial torsors, one sees easily that $BG$ is a stack.

(All of the above refers to torsors using the fppf topology.)
For every affine scheme $U \in \text{Aff}_R$, the groupoids $B \text{GL}_n(U)$ and rank $n$ vector bundles $\text{VB}_n(U)$ (i.e., finite projective modules, locally of rank $n$) are equivalent.

Every finite projective module is Zariski locally free.

Hence, we see that every $\text{GL}_n$-torsor, defined with respect to the fppf or étale topology, is also a $\text{GL}_n$-torsor with respect to the topology $\text{Zar}$. 
Proof.

- We fix an affine scheme $U$,
- in order to establish an equivalence of groupoids $B\ GL_n(U) \cong VB_n(U)$,
- we have to define mutually inverse functors $F: B\ GL_n(U) \to VB_n(U)$, and $G: VB_n(U) \to B\ GL_n(U)$.
- (...)
Proof.

- (...)  
- We will only construct the functor $G$, and verify locally that it is an equivalence.
- Since $\text{VB}_n$ and $BG$ are stacks, this is sufficient to conclude the proof.
- The functor $G : \text{VB}_n(U) \to B \text{GL}_n(U)$ sends a vector bundle $E$ to the torsor of frames.
- Its total space is the sheaf on $\text{Aff} / U$, sending $V \to U$ to the set of bases of $f^* E$. 
Proof.

- (...) 

Since the set of bases of a vector space is freely and transitively acted on by $\text{GL}_n(V)$, we obtain that $G$ is a fully faithful functor.

- It is also easily seen to be locally essentially surjective (hence globally essentially surjective, by the stack property), since every $\text{GL}_n$-torsor is locally trivial, thus the image of the trivial vector bundle.
We can also give a direct description of the functor $F : B \text{GL}_n(U) \to VB_n(U)$.

From a $\text{GL}_n$-torsor $Y \to X$, extract a cocycle, by choosing an fppf covering $\{U_i \to U\}_{i \in I}$, over which $Y$ becomes trivial (the cocycle is obtained by comparing trivialisations over the fibre products $U_i \times_U U_j$).

(...)

Torsors and quotient stacks
Since automorphisms of a vector bundle are $\text{GL}_n$-valued regular functions, we can also view this as a cocycle for the glueing of vector bundles.

Hence, obtain a well-defined vector bundle $E_Y$ associated to the torsor $Y$.

A more conceptual description of the total space of $E_Y$:

$$E_Y \cong \mathbb{A}^n \times^{\text{GL}_n} Y = (\mathbb{A}^n \times Y)/\text{GL}_n.$$
Corollary (Hilbert’s Theorem 90)

Let $K$ be a field, with separable closure $L$. Then, the Galois cohomology group $H^1_{\text{Gal}}(K, \text{GL}_n(L))$ vanishes.
Corollary (Hilbert’s Theorem 90)

Let $K$ be a field, with separable closure $L$. Then, the Galois cohomology group $H^1_{\text{Gal}}(K, \text{GL}_n(L))$ vanishes.

Proof.

- $H^1_{\text{Gal}}(K, \text{GL}_n(L)) \cong H^1_{\text{et}}(\text{Spec } K, \text{GL}_n)$, since a morphism of field $K \to L$ is étale, if and only if it is separable.

- The group $H^1_{\text{et}}(\text{Spec } K, \text{GL}_n)$ is equivalent to the set of isomorphism classes of $\text{GL}_n$-torsors over $\text{Spec } K$, i.e., rank $n$ vector spaces over $K$.

- There is precisely one vector space of dimension $n$. 
Lemma

Let $G$ be a smooth affine group $R$-scheme. Then every $G$-torsor with respect to the smooth topology is also a $G$-torsor with respect to the étale topology.
## Lemma

*Let $G$ be a smooth affine group $R$-scheme. Then every $G$-torsor with respect to the smooth topology is also a $G$-torsor with respect to the étale topology.*

- Since every smooth morphism has a section in the étale topology, it suffices to construct a covering of $\{U_i \to X\}$ of $X$ in the smooth topology, such that $Y \times_X U_i \cong U_i \times_{\text{Spec } R} G$ as a $G$-torsor.

- We have mentioned that every smooth morphism has a section in the étale topology.

- It is a worthwhile exercise to think through the details of the proof.
Let $G$ be a smooth affine group $R$-scheme. Then, a $G$-torsor $Y \rightarrow X$, with respect to the fppf is also a $G$-torsor in the étale (and hence also smooth) topology.

The proof of this result will take the next couple of slides.
We want descent for smooth and affine morphisms.
We will use the classification of smooth maps of affine schemes in terms of Jacobi matrix.
Hence we also have to check that being of finite presentation descends along faithfully flat maps.
We will also need that affineness descends.
Lemma

For ring homomorphisms \( R \to S \), and \( R \to R' \) faithfully flat, the co-base change \( R' \to S \otimes_R R' \) is of finite presentation if and only if \( R \to S \) is.
Proof.

- Being of finite presentation is certainly invariant under tensor products. Let's focus on the descent result.
- We had an abstract characterisation of finite presentation:
  - namely that for every directed system \((T_i)_{i \in I}\) of \(R\)-algebras, the natural map
    \[
    \varprojlim_{i \in I} \text{Hom}_R(S, T_i) \to \text{Hom}_R(S, \varprojlim_{i \in I} T_i)
    \]
    is an equivalence.
- (...)
Faithfully flat descent revisited

Proof.

Faithfully flat descent for ring homomorphisms implies that we have a commutative diagram,

\[
\begin{array}{c}
\lim_{\to} \text{Hom}_R(S, T_i) \quad \rightarrow \quad \text{Hom}_R(S, \lim_{\to} T_i) \\
\downarrow \quad \quad \quad \downarrow \\
\lim_{\to} \text{Hom}_{R'}(S', T'_i) \quad \sim \quad \text{Hom}_{R'}(S', \lim_{\to} T_i R') \\
\downarrow \quad \quad \quad \downarrow \\
\lim_{\to} \text{Hom}_{R' \otimes_R R'}(S'', T''_i) \quad \sim \quad \text{Hom}_{R' \otimes_R R'}(S'', \lim_{\to} T_i'').
\end{array}
\]

The middle and bottom horizontal arrows are isomorphisms,

the equalizer property implies that also the top horizontal map is an isomorphism.
Lemma

Let $V \to U$ be a morphism of affine schemes, and $U' \to U$ a faithfully flat morphism of affine schemes. Then, the base change $V' = V \times_U U' \to U'$ is smooth, respectively étale, if and only if the map $V \to U$ is smooth, respectively étale.
Proof.

1. Choose a presentation $U = \text{Spec } R$, $V = \text{Spec } R[t_1, \ldots, t_m]/(f_1, \ldots, f_n)$.
2. We have to show that the Jacobi matrix $J = \left( \frac{\partial f_i}{\partial t_j} \right)$ is surjective (respectively bijective),
3. if and only if the Jacobi matrix $J'$, for $V' \rightarrow U'$ is surjective (respectively bijective).
4. (...)
Proof.

- (...) Since we have a presentation $V' = \text{Spec } R'[t_1, \ldots, t_m]/(f_1, \ldots, f_n)$, we obtain for $J'$ the matrix associated to $J$ with respect to the natural map from $R$-matrices to $R'$-matrices.
- The Jacobi matrix $J$ can be seen as an $R$-linear map from the $R$-module $R^m$ to $R^n$.
- Since $R \to R'$ is faithfully flat, we obtain from the faithfully flatness that $J$ is surjective (respectively bijective) if and only if the Jacobi matrix for the map $V' \to U'$ is surjective (respectively bijective).
Lemma

Let $F \to U$ be a map of fppf sheaves, and $V \to U$ an fppf morphism of affine schemes, such that $F \times_U V$ is affine, then $F$ is an affine scheme.
Proof.

- We use the notation $U = \text{Spec } R$, and $V = \text{Spec } S$.
- We will show that the ring $T' = \Gamma(F \times_U V)$ is naturally endowed with a descent datum,
- hence defines an $R$-algebra $T$ by faithfully flat descent (for modules, also holds for algebras).
- We have a natural isomorphism $\text{Spec } T \times_U V \to F \times_U V$, which satisfies the descent condition, hence an isomorphism of stacks $F \cong \text{Spec } T$.
- (...)
Proof.

We have a commutative diagram:

\[
\begin{array}{ccc}
F \times_U (V \times_U V \times_U V) & \xrightarrow{\sim} & \text{Spec } T' \otimes_S (S \otimes_R S \otimes_R S) \\
\downarrow & & \downarrow \\
F \times_U (V \times_U V) & \xrightarrow{\sim} & \text{Spec } T' \otimes_S (S \otimes_R S) \\
\downarrow & & \downarrow \\
F \times_U V & \xrightarrow{\sim} & \text{Spec } T' \\
\downarrow & & \downarrow \\
F & \xrightarrow{\sim} & \text{Spec } T,
\end{array}
\]

We therefore obtain from faithfully flat descent, and the condition that \( F \) is a sheaf that \( F \) is equivalent to \( \text{Spec } T \).
Every torsor is trivial when pulled back to itself.

**Lemma**

- Let $\pi : Y \to X$ be a $G$-torsor over an affine $R$-scheme $X$,
- where $G$ is a smooth affine group $R$-scheme.
- Then, $\pi$ is a covering in the smooth topology,
- and we have a canonical equivalence of $G$-torsors $Y \times_X Y \cong Y \times G$. 
The tautological trivialisation

Lemma

- Let $\pi : Y \to X$ be a $G$-torsor over an affine $R$-scheme $X$, where $G$ is a smooth affine group $R$-scheme.
- Then, $\pi$ is a covering in the smooth topology,
- and we have a canonical equivalence of $G$-torsors $Y \times_X Y \cong Y \times G$.

- For torsors in sets $Y \to X$, one can identify $Y \times_X Y$ with the set of pairs $(y_1, y_2)$, such that $y_2 \in G y_1$.
- Since the action on the fibres is free and transitive, this set is in bijection with $Y \times G$. 
Lemma

Let \( \pi : Y \to X \) be a \( G \)-torsor over an affine \( R \)-scheme \( X \), where \( G \) is a smooth affine group \( R \)-scheme. Then, \( \pi \) is a covering in the smooth topology, and we have a canonical equivalence of \( G \)-torsors 
\[ Y \times_X Y \cong Y \times G. \]

Proof.

Since \( \pi \) is torsors, it is fppf locally of the shape 
\[ p_1 : X \times G \to X. \]
Descent implies that \( \pi \) is a smooth, affine morphism. 

(...)
The tautological trivialisation

Lemma

Let \( \pi : Y \to X \) be a \( G \)-torsor over an affine \( R \)-scheme \( X \), where \( G \) is a smooth affine group \( R \)-scheme. Then, \( \pi \) is a covering in the smooth topology, and we have a canonical equivalence of \( G \)-torsors

\[ Y \times_X Y \cong Y \times G. \]

Proof.

(...) It suffices to produce a map of \( G \)-torsors \( Y \times G \to Y \times_X Y \) over \( Y \).

The required map is given by \( \text{id}_Y \times a \), i.e. the identity in the first, and the \( G \)-action in the second component.
Algebraicity of quotient stacks

Theorem

- For a smooth affine group $R$-scheme $G$, the stack $BG$ is algebraic.
- An atlas is given by the map $p: \text{Spec } R \to BG$, corresponding to the trivial $G$-torsor on $\text{Spec } R$. 

Torsors and quotient stacks
Proof.

- We will show that $p$ is a surjective, smooth, affine morphism.
- By the definition of algebraic stacks, this implies the assertion.
- Surjectivity of $p$ amounts to the statement that for every $G$-torsor $Y$ on an affine scheme $U$, there exists an fppf covering $\{U_i \to U\}_{i \in I}$, such that $Y \times_U U_i$ is equivalent to the trivial $G$-torsor on $U_i$.
- However, this local triviality property is the defining property of torsors.
- (....)
Proof.

- (....)
- We now have to check that \( p \) is representable and smooth.
- Let \( U \to BG \) be an arbitrary map, classified by a \( G \)-torsor \( Y \) on \( U \), where \( U \) is an affine \( R \)-scheme.
- We claim that the fibre product \( \text{Spec} R \times_{B} GU \) is equivalent to \( Y \). This shows that \( U \to BG \) is affine and smooth.
- In order to see that \( \text{Spec} R \times_{B} GU \) is equivalent to \( Y \), we observe that for every affine \( R \)-scheme \( V \) we have that the fibre product of groupoids \( \bullet \times_{BG(V)} U(V) \) agrees with the set of isomorphisms from the trivial \( G(V) \)-torsor to the \( G(V) \)-torsor \( Y(V) \).
- This set is canonically equivalent to the set \( Y(V) \).
In order to get started, we consider a set $X$ with a group action, and describe maps into the quotient groupoid $[X/G]$ in terms of torsors.

**Example**

- We have a groupoid $[X/G]$, whose object are the elements of $X$, and morphisms from $x$ to $y$ correspond to $g \in G$ with $g \cdot x = y$.
- If $U$ is a set, then the groupoid of morphisms $U \to [X/G]$ is equivalent to the groupoid of pairs $(\pi: Y \to U, f: Y \to X)$, where $\pi$ is a $G$-torsor (i.e. a $G$-set with a free and transitive action),
- and $f$ is a $G$-equivariant map.
Definition

Let $G$ be a group-valued fppf sheaf acting on an fppf sheaf $X$. The quotient stack $[X/G]$ is defined to be the prestack, sending an affine $R$-scheme $U$ to the groupoid of pairs $(\pi : Y \to U, f : Y \to X)$, where $\pi$ is a $G$-torsor, and $f$ is a $G$-equivariant map.
Theorem

- If $G$ is a smooth affine group scheme, and $X$ an algebraic $R$-space,
- then $[X/G]$ is an algebraic stack.
- The canonical projection $p: X \to X/G$, corresponding to the trivial pair $(X \times G \to X, X \times G \to X)$ is a surjective, smooth, affine morphism,
- hence an atlas.

See the notes for a proof.
Objects on $[X/G]$ correspond to $G$-equivariant objects on $X$.

E.g.: A $G$-invariant map $X \rightarrow Z$ corresponds to a map $[X/G] \rightarrow Z$.

A $G$-equivariant $H$-torsor corresponds to an $H$-torsor on $[X/G]$.

$G$-equivariant cohomology of $X$ is equivalent to cohomology of $[X/G]$. 

Torsors and quotient stacks