Adelic methods in algebraic geometry

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1 Motivation

In order to motivate the concepts we will encounter in these notes, and simultaneously test their applicability, we will take a closer look at the interplay of Tate vector spaces and the theory of algebraic curves. We will see how the formalism of Tate objects and adèles can be used to shed light on various classical results and constructions. The Riemann-Roch theorem, and a zoo of classical reciprocity laws (such as the sum of residues theorem) will follow from entirely elementary considerations about infinite-dimensional vector spaces.

This section closely follows Dustin Clausen's treatment [Cla], which was also the authors' port of entry to this fascinating subject. We strongly recommend Clausen's text, particularly to the reader wishing to see applications of this philosophy to the geometric Langlands programme.

1.1 Central extensions of loop groups

Consider the compact Lie group $G = U(n)$. The associated loop group LG is defined to be infinite-dimensional Lie group of smooth maps $S^1 \longrightarrow G$. Its Lie algebra $L\mathfrak{g}$ is the vector space of smooth maps $S^1 \longrightarrow \mathfrak{g}$. A central construction in the theory of loop groups, as presented in [PS86], is its canonical central extension

$$1 \longrightarrow S^1 \longrightarrow \widehat{LG} \longrightarrow LG \longrightarrow 1.$$

Central extensions $1 \longrightarrow A \longrightarrow \widehat{\Gamma} \longrightarrow \Gamma \longrightarrow 1$ can be obtained as follows: at first one constructs an action of Γ on X, which carries a transitive and faithful action of BA (in other words is a BA-torsor). We can either think of BA as a homotopy type, that is, really the classifying space of A, or, if A is a discrete group, realise it as the groupoid with a single object which has A has group of automorphisms.

If the action of Γ on X intertwines with the BA-action giving the torsor structures, we have actually constructed a homomorphism $\Gamma \longrightarrow BA$, which classifies a central extension of Γ by A.

In the case of the loop group LG, one notices that there is a canonical central extension of the Lie algebra

$$0 \longrightarrow \mathbb{R} \longrightarrow \widehat{L\mathfrak{g}} \xrightarrow{p} L\mathfrak{g} \longrightarrow 0,$$

which corresponds to an alternating pairing $b(-,-)\colon L\mathfrak{g} \times L\mathfrak{g} \longrightarrow \mathbf{R}$, such that we have $[x,y]_{\widehat{L\mathfrak{g}}} = [p(x), p(y)] + b(x,y) \cdot \mathbf{1}$. In the case of a loop group one then chooses b to be (up to a constant) equal to

$$\int_{S^1} \langle f, dg \rangle = \int_{S^1} \langle f, g' \rangle d\theta.$$

The definition already indicates a close connection to geometric concepts such as monodromy and the residue pairing. The latter is one of the reasons why it seems feasible to study analogous constructions purely within the realm of algebraic geometry.

Lemma 1.1. *Let M be a simply-connected, connected manifold. Given a closed 2-form ω, such that $\frac{\omega}{2\pi}$ represents an integral cohomology class, there exists a unitary line bundle with a connection (L, ∇), such that $F(\nabla) = \omega$. Moreover for two such line bundles (L_1, ∇_1) and (L_2, ∇_2), there exists an isomorphism, unique up to multiplication an element of $U(1)$.*

One applies this lemma to X being the identity component of the homogeneous space LG/T. It is clear that the identity component of LG acts on X, and therefore also on the space of unitary line bundles with connections equal to ω. Hence we are in the situation discussed above, and we see at least how a canonical central extension of the identity component is constructed.

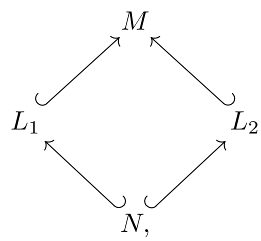
1.2 Tate vector spaces

For the rest of this section we fix a field k, which we assume to be algebraically closed to simplify the exposition. We give a preliminary definition of Tate vector spaces, which we will revisit later in a more general context.

Definition 1.2. *A* Tate vector space V *is a pair* (V, Gr_V), *consisting of a* k*-vector space* V, *and a set of subsets* $\mathsf{Gr}_V = \{L \subset V\}$, *satisfying the conditions (A1-4) below. Subsets belonging to* Gr_V *will be called* lattices.

(A1) We have $\bigcap_{L\in\mathsf{Gr}_V} L = \{0\}$, *and* $\bigcup_{L\in\mathsf{Gr}_V} L = \{V\}$.

(A2) For $L_1, L_2 \in \mathsf{Gr}_V$ *there exist lattices* $M, N \in \mathsf{Gr}_V$, *such that*



with the respective subquotients being finite-dimensional

(A3) If $L_1 \subset M \subset L_2$, *such that* $L_1, L_2 \in \mathsf{Gr}_V$, *then* $M \in \mathsf{Gr}_V$.

(A4) The natural map

$$V \longrightarrow \varprojlim_{L\in\mathsf{Gr}_V} V/L$$

is an isomorphism.

A familar example of such a structure is given by the vector space $F = k((t))$ of formal Laurent series. Consider the collection Gr'_F of subspaces $t^n k[[t]]$ for $n \in \mathbb{Z}$. It is clear that the axioms (A1,2,4) are satisfied, so we simply define Gr_V as the saturation or sandwich-closure of Gr'_F.

The fourth axiom indicates that Tate vector spaces shouldn't be viewed as plain vector spaces, but rather as topological vector spaces. Endowing each of the quotients V/L with the discrete topology, and V with the resulting inverse limit topology, we obtain Lefschetz's *linearly compact vector spaces.* We call this functor from the category of Tate spaces to topological vector spaces the *topological realisation*, and denote it by V^{top}.

Definition 1.3. *A topological vector space is called linearly locally compact, if it can be obtained as the topological realisation of a Tate space.*

This construction is as good as it gets. Topological realisation is fully faithful, that is we can reconstruct the pair (V, Gr_V) from the associated linearly locally compact vector space. Speaking from an aesthetic perspective it is relieving that the existence of a collection of lattices Gr_V is a mere property of a topological vector space, rather than an extra structure imposed on V. Let's see why this is true, and how topology allows one to recover the image of lattices L in the topological realisation V^{top}.

Definition 1.4. *A topological vector space is called* linearly locally compact, *if it is equivalent to the inverse limit of finite-dimensional (discrete) vector spaces.*

Linearly compact spaces W are precisely the topolgical realisations of Tate spaces V, where $V \in \mathsf{Gr}_V$ is itself a lattice. Indeed, we just let V be the underlying vector space of W, and stipulate $L \subset V$ to belong to Gr_V, if it arises as the kernel of a continuous surjection $V \twoheadrightarrow V/L$ with V/L a finite-dimensional discrete vector space.

If $W = V^{\mathsf{top}}$ is a linearly locally compact vector space, we can reconstruct (V, Gr_V) by forgetting the topology on W, and stipulating that a subspace $L \subset V$ is a lattice, if and only if the associated subspace of V is linearly compact, and agrees with the kernel of a surjection onto a discrete vector space V/L.

Definition 1.5. *A Tate vector space V is called* discrete*, if $\{0\} \in \mathsf{Gr}_V$ is a lattice. This happens precisely when V^{top} is discrete in the topological sense.*

Let's imagine for a second that our base field k is finite. In this case, linearly compact vector spaces are actually compact as topological spaces, because they are obtained as inverse limit of finite sets, and similar Tate spaces have locally compact topological realisations. A subspace of a linearly locally compact is a lattice if and only if it is compact and co-discrete (that is, V/L is discrete).

The next lemma claims that every continuous homomorphism from a linearly compact space to a discrete one, factors through a finite-dimensional subspace. In the case of a finite field this is clear, because continuous images of compact spaces are compact, and compact subsets of discrete spaces are finite.

Lemma 1.6. *Every morphism $L \longrightarrow D$ from a linearly compact Tate space L to a discrete Tate space D factors through a finite-dimensional subspace of D.*

As an immediate consequence we obtain that lattices are commensurable in the sense that they only differ by finite-dimensional corrections (which is also a consequence of (A2) of our definition of Tate vector spaces).

Remark 1.7. *Let $M \subset N$ be an ordered pair of lattices in a Tate space, then the quotient L_2/L_1 is finite-dimensional.*

We define the relative dimension of two lattices $L_1, L_2 \in \mathsf{Gr}_V$ to be the difference

$$\dim(L_1 - L_2) = \dim(L_1/N) - \dim(L_2/N) = \dim(M/L_2) - \dim(M/L_1),$$

where M, N are common upper, respectively lower bounds, which are guaranteed to exist by axiom (A2).

1.3 Torsors and central extensions

Every Tate vector space V gives rise to a whole zoo of torsors, acted on by $\mathsf{Aut}(V)$, the group of continuous automorphisms of V. As a simple warm-up exercise we define the $\mathbb{Z}$-torsor of dimension theories.

Definition 1.8. *A dimension theory is a map $d\colon \mathsf{Gr}_V \longrightarrow \mathbb{Z}$, assigning an integer to every lattice, such that for every pair of lattices we have $d(L_1) - d(L_2) = d(L_1 - L_2)$. The set of dimension theories for V will be denoted by $\mathsf{Dim}(V)$.*

Since any two lattices in V are commensurable, we see that for every lattice L, and integer $n \in \mathbb{Z}$ there exists precisely one dimension theory d, satisfying $d(L) = n$. This implies that $\mathsf{Dim}(V)$ is a non-empty set, and that the natural $\mathbb{Z}$-action $d + k\colon L \mapsto d(L) + k$ endows $\mathsf{Dim}(V)$ with the structure of a $\mathbb{Z}$-torsor. Since $\mathsf{Aut}(V)$ acts on the set Gr_V of lattices, we obtain an induced action on $\mathsf{Dim}(V)$.

A dimension theory should be viewed as a *renormalisation* of the infinite-dimensional object V. In general it is not possible to speak of the dimension of a lattice $L \subset V$ in the classical sense. Instead of worrying about definition, we choose the dimension of a lattice arbitrarily, and only ask for internal consistency of this assignment. Such renormalisations play a role in Kontsevich's theory of motivic integration, and in the work of Kapranov–Vasserot [KV07] about categories of D-modules on infinite-dimensional varieties.

As we observed in the first subsection, this gives rise to a map $\nu\colon \mathsf{Aut}(V) \longrightarrow \mathbb{Z}$. We can compute the value of ν on an automorphism f as the relative dimension $\dim(gL - L)$, for an arbitrary lattice $L \in \mathsf{Gr}_V$. The group of units of the field $F = k((t))$ acts on the Tate space F by automorphisms, and therefore we obtain an induced map $\nu\colon F^\times \longrightarrow \mathbb{Z}$. This is the valuation map, which is characterised by the property that $t^{-\nu(f)}f \in k[[t]]^\times$.

The choice of a dimension theory for a Tate space is often compared to the choice of a Haar measure for a local field F. Just like for dimension theories, there is no canonical choice of a Haar measure on F. And if we pullback a measure μ via the multiplication map $x \mapsto ax$, we obtain $a^*\mu = q^{\nu(a)}\mu$, where q denotes the cardinality of the residue field of F.

Let's take this to the next level. This time we put the emphasis on top exterior powers of vector spaces, rather than their dimension. This is sensible because determinants behave formally similar to dimensions. For every short exact sequence $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ of finite-dimensional vector spaces we have a natural isomorphism

$$\det(V) \simeq \det(U) \otimes \det(W), \tag{1}$$

in particular we have $\det(U \oplus W) \simeq \det(U) \otimes \det(W)$.

Definition 1.9. *A* determinant theory *assigns to every lattice $L \in \mathsf{Gr}_V$ a line (that is, a 1-dimensional vector space) $\mathcal{L}(L)$, such that for every ordered pair of lattices $M \subset N$, we have $\mathcal{L}(N) \simeq \mathcal{L}(M) \otimes \det(N/M)$, which satisfies for every ordered triple $L \subset M \subset N$ a compatibility condition*

$$\begin{array}{ccc} \mathcal{L}(N) & \longrightarrow & \mathcal{L}(M) \otimes \det(N/M) \\ \downarrow & & \downarrow \\ \mathcal{L}(L) \otimes \det(N/L) & \longrightarrow & \mathcal{L}(L) \otimes \det(N/M) \otimes \det(M/L). \end{array}$$

We denote the category (or rather groupoid) of determinant theories by $Det(V)$. The same argument as in the case of dimension theories shows that a determinant theory is uniquely (up to a unique isomorphism) characterised by the line associated to a single lattice $L \in \mathsf{Gr}_V$. In particular, we see that $Det(V)$ is a torsor over the groupoid of lines $Bk^\times$. The natural action of $\mathsf{Aut}(V)$ on Gr_V induces therefore as before an action on $Det(V)$, which gives rise to a homomorphism $\mathsf{Aut}(V) \longrightarrow k^\times$. The corresponding central extension of $\mathsf{Aut}(V)$,

$$1 \longrightarrow k^\times \longrightarrow \widehat{\mathsf{Aut}(V)} \longrightarrow \mathsf{Aut}(V) \longrightarrow 1$$

is the promised algebraic analogue of the canonical central extension of loop groups. Indeed, we can embed $L\,\mathrm{GL}_n = \mathrm{GL}_n((t))$ in the automorphism group of the Tate vector space F^n, and thus obtain a central extension of $L\,\mathrm{GL}_n$ by pullback.

Just like the dimension torsor $\mathsf{Dim}(V)$ gives rise to a numerical invariant $\nu(f)$ of an automorphism $f \in \mathsf{Aut}(V)$, the central extension induced by $Det(V)$ can be used to attach an element of $k^\times$ to a pair of commuting automorphisms of V.

Lemma 1.10. *Let $1 \longrightarrow A \longrightarrow \widehat{\Gamma} \xrightarrow{p} \Gamma \longrightarrow 1$ be a central extension, and $f, g \in \Gamma$ to commuting elements. Then, $[p^{-1}(f), p^{-1}(g)] \in A$ is a well-defined element of A. If Γ is commutative, we denote the pairing $\Gamma \times \Gamma \longrightarrow A$ by $(-,-)_{\widehat{\Gamma}}$.*

In the case of the central extension of $L\,\mathbb{G}_{\mathrm{m}}$, the resulting pairing is a well-known construction in number theory.

Lemma 1.11. *The pairing $(-1)^{\nu(f)\nu(g)}(f,g)^{-1}_{\widehat{L\,\mathbb{G}_{\mathrm{m}}}}$ agrees with the tame symbol*

$$(-1)^{\nu(f)\nu(g)} \frac{f^{\nu(g)}}{g^{\nu(t)}}|_{t=0} \in k^\times.$$

The modification of the commutator pairing by the sign $(-1)^{\nu(f)\nu(g)}$ can be explained in a more natural way, by replacing determinants by graded determinants.

Definition 1.12. *A graded line is a pair (L, n), where L is a 1-dimensional space, and $n \in \mathbb{Z}$ is an integer. The graded determinant of a finite-dimensional vector space V is the pair $(\det V, \operatorname{rk} V)$.*

The tensor product of two graded lines (M, m), (N, n) is simply defined by the tensor product of the two lines, and addition of integers. However, we modify the natural isomorphism $M \otimes N \xrightarrow{\simeq} N \otimes M$ by multiplication with the sign $(-1)^{mn}$.

By consistently adding the word *graded* to Definition 1.9 one obtains the definition of a *graded determinant theory.*

1.4 Adèles

The time has come to see the theory of Tate objects in action in a global situation. Let X be a smooth curve over $k = \overline{k}$ (sometimes, we will assume X to be projective). For every line E/X we define the vector space of adèles to be

$$\mathbb{A}_X(E) = \prod_{x \in X_{cl}}{}' (E_x \otimes \widehat{F}_x) = \{(f_x)_{x \in X_{cl}} | f_x \in E_x \otimes \widehat{\mathcal{O}}_x \text{ for almost all } x\}.$$

For $E = \mathcal{O}_X$ the trivial line bundle, we obtain a ring, denoted by $\mathbb{A}$. In general, $\mathbb{A}(E)$ is an $\mathbb{A}$-module.

However, $\mathbb{A}_X(E)$ also carries the structure of a Tate vector space. The collection of lattices is given by the saturation of $\mathsf{Gr}'_{\mathbb{A}(E)}$, consisting of the subspaces

$$\mathbb{O}_D(E) = \{(f_x)_{x \in X_{cl}} | f_x \in t^{-\nu_x(D)}(E_x \otimes \widehat{\mathcal{O}}_x) \text{ for almost all } x\},$$

indexed by divisors on X. We denote $\mathbb{O}_0(E)$ also by $\mathbb{O}(E)$, and refer to it as integral adèles.

Theorem 1.13 (Weil)**.** *There is a canonical equivalence* $\mathsf{Pic}(X) \cong [\mathbb{G}_{\mathrm{m}}(F) \setminus \mathbb{G}_{\mathrm{m}}(\mathbb{A}) / \mathbb{G}_{\mathrm{m}}(\mathbb{O})] = [F^\times \setminus \mathbb{A}^\times / \mathbb{O}^\times]$.

Weil's theorem holds in fact for a wide range of groups. For example we can describe $\mathsf{Bun}_n(X)$, the stack of rank n bundles on X, as the double quotient $[\mathrm{GL}_n(F) \setminus \mathrm{GL}_n(\mathbb{A}) / \mathrm{GL}_n(\mathbb{O})]$. The case of line bundles however, is easy to prove, since $\mathbb{A}^\times / \mathbb{O}^\times$ is equivalent to the free abelian group generated by the closed points of X, that is, the group of divisors Div_X. The corresponding action of $F^\times$ on Div_X takes a divisor D and modifies it by the principal divisor associated to a non-zero invertible function f. Therefore, we see that Weil's theorem for line bundles is just another way to state the classical identity

$$\mathsf{Pic}(X) \cong \mathsf{Div}_X / \mathsf{Div}_X^{\mathsf{princ}},$$

that line bundles on X can be presented by divisors modulo principal divisors.

Lemma 1.14. *Let $f \in \mathbb{A}^\times$ be an invertible adèle (also known as an* idèle*). The order $\nu(f)$ computed with respect to* $\mathsf{Dim}(\mathbb{A})$ *agrees with the degree* $\deg(f) = \deg(D)$ *of the divisor* $D \in \mathsf{Div}_X = \mathbb{A}^\times / \mathbb{O}^\times$ *associated to* f.

Indeed, we just have to choose a dimension theory d, and compute the difference $d \circ f - d$. If we choose d to be the unique dimension theory which assigns to the lattice $\mathbb{O} \subset \mathbb{A}$ the integer 0, we see directly that $\nu(f)$ is equal to the relative dimension $\dim(f\,\mathbb{O} - \mathbb{O}) = \deg(f)$.

We could have chosen a completely different dimension theory, and still have obtained the same integer $\nu(f)$. Another interesting choice for a projective curve X is

$$\chi\colon\ \mathbb{O}_{D'} \mapsto \chi(\mathcal{O}(D')) = \dim H^0(X, \mathcal{O}(D')) - \dim H^1(X, \mathcal{O}(D')).$$

The long exact sequence in cohomology implies readily that χ is a dimension theory, and we see that $\nu(f) = \chi(\mathcal{O}(D)) - \chi(\mathcal{O})$, which can be summarised as the assertion of the Riemann-Roch theorem

$$\chi(\mathcal{O}(D)) - \chi(\mathcal{O}) = \deg D.$$

Replacing the dimension torsor by $Det(V)$, or rather its graded analogue, one obtains the following multiplicative analogue of the Riemann-Roch theorem:

Theorem 1.15 (Weil reciprocity)**.** *Let f, g be two non-zero rational functions on a projective curve X, then we have that the product below is well-defined, and* $\prod_{x \in X_{\mathrm{cl}}} (-1)^{\nu(f)\nu(g)} \frac{f^{\nu(g)}}{g^{f(x)}}(x) = 1$.

Let us explain how this works. We denote the field of rational functions on X by K. The simple observation that two rationally equivalent divisors have the same cohomology groups, implies that the central extension of $K^\times$, obtained from the action on $\mathbb{A}^\times$ is trivial. Hence, the commutator pairing is equal to 1, for elements of $K^\times$.

1.5 An adelic viewpoint on cohomology

When discussing the proofs of Riemann-Roch and Weil reciprocity, using adèles, we were making us of cohomology theory, which on a priori seems to be external to the theory of adèles. However, one can use adèles to compute cohomology groups.

Proposition 1.16. *Let E be a coherent sheaf on X, and $\mathbb{A}(E)$, $\mathbb{O}(E)$, and $K(E) = K \otimes_{\mathcal{O}} E$ be the abelian groups of adèlic, integral adèlic, and rational sections. Then, we have that the cohomology groups of the complex*

$$\mathbb{O}(E) \oplus F(E) \xrightarrow{\mathsf{incl}-\mathsf{incl}} \mathbb{A}(E)$$

are canonically equivalent to $H^i(X, E)$. In fact, this complex is quasi-isomorphic to the complex of derived global sections $R\Gamma(X, E)$.

This proposition is easy to prove. At first one observes that replacing X by a Zariski open subset, one obtains a complex of flasque sheaves, which happens to be a resolution of E. The latter property is a consequence of the tautology that a rational section of E, without poles, is a regular section.

The complex above is quasi-isomorphic to

$$[\mathbb{O}(E) \longrightarrow \mathbb{A}(E)/F(E)], \tag{2}$$

and using this presentation one can give an adelic proof of Serre duality.

Theorem 1.17 (Adelic Serre duality)**.** *For a smooth projective curve, the residue pairing* $\mathrm{res}\colon\ \mathbb{A} \times \mathbb{A}(\Omega^1_X) \longrightarrow \mathbb{A}$ *is perfect, that is induces an equivalence between $\mathbb{A}(\omega_X)$ and the (topological) dual of $\mathbb{A}_X$. In particular we see that $\mathbb{A}(E)^\vee \cong \mathbb{A}(E^\vee \otimes \Omega^1_X)$. These isomorphisms descend to isomorphisms $(\mathbb{A}(E)/\mathbb{O}(E))^\vee \cong \mathbb{O}(E)^\vee$, and $F(E)^\vee \cong \mathbb{A}(E^\vee \otimes \Omega^1_X)/F(E^\vee \otimes \Omega^1_X)$.*

The first two assertions can be checked by a direct computation, and actually don't rely on projectivity. The third assertion does use projectivity. Even just to define a pairing, we use the residue formula for rational 1-forms.

Using the adelic duality theorem above, we obtain a proof of Serre duality, because the (topological) dual of the complex (2) is now equivalent to

$$[F(E^\vee \otimes \Omega^1_X) \longrightarrow \mathbb{A}(E^\vee \otimes \Omega^1_X)/\mathbb{O}(E^\vee \otimes \Omega^1_X)],$$

which is just another way to write down the complex of Proposition 1.16 for $E^\vee \otimes \Omega^1_X$.

2 From adèles for curves to multidimensional adèles

In this section we will discuss the transition from the 1-dimensional case to higher dimensions. This is inspired by higher class field theory (established by Kato and Saito).

2.1 Classical story: function field case

We fix X/k a classical curve, and E a line bundle on X. The integral adèles are defined to be

$$\mathbb{O}(E) = \prod_{x \in X_0} \widehat{\mathcal{O}}_x \otimes E,$$

and the adèles are defined to be

$$\mathbb{A}_E = \mathsf{colim}_D\, \mathbb{O}_{\mathcal{O}_D \otimes (E)} = \prod_{x \in X_{cl}}{}' (E_x \otimes \widehat{F}_x) = \{(f_x)_{x \in X_{cl}} | f_x \in E_x \otimes \widehat{\mathcal{O}}_x \text{ for almost all } x\}.$$

Over a finite fields, these are compact abelian topological groups, respectively locally compact abelian topological groups, simply because $k[[t]]$ and $k((t))$ has this property. The modern viewpoint is to work functorially, that is replace every appearance of factors such as $k((t))$ by the functor, sending a commutative ring R to $R((t))$. An equivalent perspective is to work with Tate vector spaces, respectively bundles.

Recall the archetypical example of a Tate vector space: the space of formal Laurent series $F = k((t))$ over a field k. We defined Gr_F as the saturation $\{t^n k[[t]] | n \in \mathbb{Z}\}'$. Similarly one defines $\mathsf{Gr}_{\mathbb{A}_E} = \{\mathbb{O}_{\mathcal{O}_D} \otimes E\}'$, which endows the adèles with the structure of a Tate space.

Recall that we discussed the notion of a dimension theory, which was a map $\mathsf{Gr}_V \longrightarrow \mathbb{Z}$, consistent with relative dimension of lattices.

In the case of F, we obtained a dimension torsor $\mathsf{Dim}(F)$, such that the resulting map $k((t))^\times \xrightarrow{\nu} \mathbb{Z}$ agrees with the t-valuation of F.

Let's discuss the analogous situation for the adèles: we saw that $\mathsf{Pic}(X) \cong [k(X)^\times \setminus \mathbb{A}^\times / \mathbb{O}^\times]$, and the dimension torsor of the Tate object $\mathbb{A}$ induced a map $\deg\colon \mathsf{Pic}(X) \longrightarrow \mathbb{Z}$, which agrees with the degree of line bundles. We then used this observation to reprove the Riemann-Roch formula for curves. The key step to make this connection was to use the dimension theory given by the Euler characteristic of the cohomology of line bundles. Now that we've seen the applicability of adèles we'll try to generalise this theory to higher-dimensions.

2.2 Adèles for schemes (after Parshin, Beilinson)

In the case of curves, adèles were realised as a sort of restricted product, ranging over closed points of X, or the places of $k(X)$. In the higher-dimensional case, their place will be taken by flags of irreducible subsets. The local factors for a surface, are expected to be of the shape $\widehat{\mathcal{O}_{\eta_2 \eta_1}}$, where η_1 is the generic point of a curve, containing a closed point η_2 (the subscript encodes the codimension of the closure of a point).

The local factors have the structure of an n-local field.

$$
\begin{array}{ccccc}
k((s))((t)) & & & & \\
\uparrow & & & & \\
k((s))[[t]] & \twoheadrightarrow & k((s)) & & \\
 & & \uparrow & & \\
 & & k[[t]] & \twoheadrightarrow & k
\end{array}
$$

The next step is to take some case of restricted product. However, there's a recursive definition, which helps us avoiding a lot of guesswork what the right definition should be.

Let's what kind of structure we can put on higher-dimensional adéles. We'll work with the local factors, in order to get a feeling for this problem. Consider a 2-local field, such as $K = k((s))((t))$. Is there a well-behaved topology on it?

Example 2.1. *Let $\mathcal{T}$ be any topology on K, such that $k((s))[[t]] \twoheadrightarrow k((s))$ is a quotient topology map, then $\mathcal{O}_K$ (the ring of integers), is not a topological ring.*

This is disappointing. Yekutieli has therefore studied adèles using the theory of semi-topological rings. Fesenko proposed to work with sequential topological rings, however these approaches are not directly compatible. Kato suggested that topology is simply not the right structure to study higher local fields, and proposed to work with categories of iterated Ind-Pro objects. This works, because $k((s))$ can be defined as

$$\mathsf{colim}_j \, \mathsf{lim}_i s^{-j} k[s]/(s^i).$$

Remark 2.2. *Pro-systems carry more information than their topological realisations. There are pro-systems of sets, such that the resulting space is empty, but the pro-system carries a lot of information which would be lost if we didn't retain it in a formal manner.*

2.3 Dimension theories for 2-local fields

We fix an equicharacteristic 2-local field $K = k((s))((t))$, and choose a splitting $k((s))[[t]] \oplus t^{-1}k((s))[t^{-1}]$. The problem is that a general automorphism won't respect such a splitting.

Subquotients of lattices (with respect to the variable t), will look like $t^2/k((s))[[t]]/t^{10}k((s))[[t]] \cong k((s))^{\oplus 8}$, which is infinite-dimensional over the base field k. In order to rectify this, we also have to consider s-lattices.

This shows that for 2-local fields the natural construction is to work with pairs of automorphisms. In the definition below we denote by $\mathcal{O}_1$ the ring $k((s))[[t]]$. The definition below is intentionally incomplete, and will be corrected below.

Definition 2.3 (Beilinson, Yekutieli)**.** *Suppose V is a finite-dimensional K-vector space, a lattice is a finitely generated $\mathcal{O}_1$-submodule of V, such that $K \cdot L = V$. We define $E^{Yek}(V_1, V_2)$ (the analogue of continuous homomorphisms) to be the vector space of k-linear maps $\phi\colon V_1 \longrightarrow V_2$, such that for every pair of lattices $L_1 \subset V_1$, $L_2 \subset V_2$, there exist lattices $L_1' \subset V_1$ and $L_2' \subset V_2$, such that $L_1' \subset L_1$, $L_2 \subset L_2'$, and $\phi(L_1') \subset L_2$, and $\phi(L_1) \subset L_2'$, and with the resulting maps $\overline{\phi}\colon L_1/L_1' \longrightarrow L_2'/L_2$ belonging to $E^{Yek}(L_1/L_1', L_2'/L_2)$.*

The problem with this definition is that in order for the last condition to make sense, we have to be able to consider the subquotients L_1/L_1', and L_2'/L_2 as modules over $k((s))$. This is possible once we've chosen splittings.

3 Beilinson-Parshin adèles

3.1 The definition

Adèles were first introduced by Parshin for surfaces in 1976, and for Noetherian schemes by Beilinson in 1980. Let X be a Noetherian scheme, and $\eta_0, \eta_1 \in X$. We define $\eta_0 \geq \eta_1$ to be equivalent to $\eta_1 \in \overline{\{\eta_0\}}$.

Suppose $K \subset S(X)_\eta$, define ${}_\eta K = \{(\eta_1 \geq \cdots \geq \eta_n) | \eta \geq \eta_1 \geq \cdots \geq \in K\}$.

If $\mathcal{F}$ is a coherent sheaf on X, define for

- $n = 0$: $\mathsf{A}(K, \mathcal{F}) = \prod_{\eta \in K} \varprojlim (\mathcal{F} \otimes \mathcal{O}_\eta / \mathfrak{m}_\eta^i)$.
- $n \geq 1$: $\mathsf{A}(K, \mathcal{F}) = \prod_{\eta \in X} \varprojlim_i \mathsf{A}({}_\eta K, \mathcal{F} \otimes \mathcal{O}_{X,\eta} / \mathfrak{m}_\eta^i)$,

and if $\mathcal{F}$ is quasi-coherent, we stipulate $\mathsf{A}(K, \mathcal{F}) = \mathsf{colim}_i\, \mathsf{A}(K, \mathcal{F}_i)$ with $\mathsf{colim}\, \mathcal{F}_i \cong \mathcal{F}$ a colimit of coherent sheaves. By induction this defines a unique family of functors $\mathsf{A}(K, -)$ from quasi-coherent sheaves on X to $\mathcal{O}_X$-modules. We are mostly interested in the values of these functors for coherent sheaves, however note that we cannot unravel the induction without including the case of quasi-coherent sheaves in the definition above.

Example 3.1. *We fix a curve $X = \mathsf{Spec}\, k[s,t]/(s^3 + s^2 - t^2)$, which we allow to be singular. Let Δ be the flag $(0) \geq (s,t)$. Since we have $s^3 + s^2 - t^2 = (s\sqrt{1+s} + t)(s\sqrt{1+s} - t)$, one sees that $\mathsf{A}(\Delta, \mathcal{O}_X) = k((t_1)) \oplus k((t_2))$.*

This is a special case of the following theorem.

Theorem 3.2 (Beilinson, first published proof by Yekutieli)**.** *Let X be a Noetherian reduced excellent scheme of dimension n, and $\Delta = (\eta_0 \geq \cdots \geq \eta_n)$ a flag with* codim $\overline{\{\eta_i\}} = i$, *then*

$$\mathsf{A}({}_{\eta_0}\Delta, \mathcal{O}_X) \subset \prod \mathcal{O}_1 \subset \prod_{\textit{finite}} K_i = \mathsf{A}(\Delta, \mathcal{O}_X),$$

where K_i are n-local fields.

We want an efficient calculus of lattices. We follow the suggestion of Kato and Beilinson from 1980 that Tate vector spaces can be emulated using categories of ind-pro objects. We fix an exact category C, e.g. an abelian category C.

Definition 3.3. *We define $\mathsf{Ind}^a(\mathsf{C})$ to be the full subcategory of $\mathsf{Ind}\,\mathsf{C}$ of admissible ind objects, that is, objects which allow a presentation $C_1 \hookrightarrow C_2 \hookrightarrow \cdots$ with transition maps being admissible monomorphisms.*

Inverting directions of morphisms, and replacing admissible monomorphisms by admissible epimorphisms, one obtains the definition of $\mathsf{Pro}^a(\mathsf{C})$.

Definition 3.4. *An object $X \in \mathsf{Ind}^a\, \mathsf{Pro}^a(\mathsf{C})$ is called an* elementary Tate object*, if it has a lattice, that is there exists $L \in \mathsf{Pro}^a(\mathsf{C})$, and an admissible monomorphism $L \hookrightarrow X$, such that $X/L \in \mathsf{Ind}^a(\mathsf{C})$. We denote the resulting exact category by $\mathsf{Tate}^{\mathrm{el}}(\mathsf{C})$. The idempotent completion is denoted by $\mathsf{Tate}(\mathsf{C})$, and its objects are called Tate objects.*

We define categories of n-Tate objects by iteratively applying the Tate construction. Beilinson's adèles can be promoted to take values in exact categories of these iterated Tate objects.

The endomorphism algebra à la Yekutieli of a higher local field, as we defined it in the last section, is equivalent to the endomorphism algebra of n-Tate objects.

3.2 Generalised dimension and determinant theories

Define the Sato complex $\mathsf{Gr}^{\leq}_n(\mathsf{C}) = \{(L_0 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V)\}$, where the L_i denote lattices in an elementary Tate object V.

Definition 3.5.

$$\mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^{\times} \longleftarrow \mathsf{Gr}^{\leq}_{\bullet}(\mathsf{C}) \longrightarrow S_{\bullet}\mathsf{C}^{\times},$$

where the map to the left forgets lattices, and the map to the right sends the collection of lattices to $(0 \hookrightarrow L_1/L_0 \hookrightarrow \cdots \hookrightarrow L_n/L_0)$. *Taking geometric realisation, we obtain the map* $\mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^{\times} \longrightarrow BK_{\mathsf{C}}$, *respectively* $\Omega K_{\mathsf{Tate}^{\mathrm{el}}(\mathsf{C})} \longrightarrow K_{\mathsf{C}}$, *which are called the* index map.

The index map generalises the construction of the dimension torsor, as it produces for every elementary Tate object a torsor over the K-theory space K_{C}, together with an action of $\mathsf{Aut}(V)$. For instance, if $\mathsf{C} = \mathsf{Proj}\ R$, then we have the rank map from K_{C} to $\mathbb{Z}$, and by changing the structures group of torsors, we obtain the dimension torsor $\mathsf{Dim}(V)$.

3.3 Adèles and cohomology

As we have seen above adèles $\mathsf{A}(K, \mathcal{F})$ depend functorially on a quasi-coherent sheaf $\mathcal{F}$ and a subset $K \subset |X|_n$, where $|X|_n$ denotes the set of flags $\eta_0 \geq \cdots \geq \eta_n$. By replacing a point $\eta \in X$ by its closure, we obtain a flag of irreducible closed subsets $Y_0 \supset \cdots \supset Y_n$.

Flags have a simplicial structure. We can omit or repeat certain elements. Formally, a simplicial set is defined to be a functor $\Delta^{\mathsf{op}} \longrightarrow \mathsf{Set}$, where Δ denotes the category of finite non-empty totally ordered sets. Every object of in Δ is equivalent to $[n] = \{0, 1, \ldots, n\}_{\leq}$ with the ordering induced from $\mathbb{N}$. A poset $S_{\leq}$, such as $|X|_{\leq}$ induces a simplicial set $S_{\bullet}$. One defines S_n to be the set of flags $s_0 \leq \cdots \leq s_n$, and the functor $\Delta^{\mathsf{op}} \longrightarrow \mathsf{Set}$ associates to an object $[n] \in \Delta$ the set of order-preserving maps $[n] \longrightarrow S_{\leq}$.

Dual to this simplicial structure on the set of flags of irreducible closed subsets of $|X|$ is a co-simplicial structure on the adèles. A co-simplicial sheaf is a functor $\Delta \longrightarrow \mathsf{Sh}(X)$. For any quasi-coherent sheaf $\mathcal{F}$ we have a co-simplicial sheaf of $\mathcal{O}_X$-modules $\mathsf{A}^{\bullet}(\mathcal{F})$. The functor assigns to $[n] \in \Delta$ the sheaf $\mathsf{A}(|X|_n, \mathcal{F})$. Morphisms $[k] \longrightarrow [n]$ induce maps of sheaves (which are compositions of boundary, respectively face maps). We refer the reader to Huber's account in [Hub91] for a detailed treatment of this finer structure. If $\mathcal{F} = \mathcal{O}_X$ we obtain a co-simplicial sheaf of rings on X. Taking global sections, we get a co-simplicial ring $\mathbb{A}^{\bullet}_X$ associated to a scheme X.

Example 3.6. *How does this structure look like in the case of an irreducible curve X? Ignoring degenerate examples, length 2 flags are of the shape $x \leq \eta$, where x is a closed point, and η is the generic point of X. Omitting either the first or the last point yields the flags of length 1, given by x respectively η. On the level of adèles these two types of truncating a length 2 flag to a length 1 flag corresponds to the inclusions*

$$F \hookrightarrow \mathbb{A} \hookleftarrow \mathbb{O},$$

where F denotes the function field of X.

One of the central insights in [Bei80] is that adèles can be used to compute cohomology. At first we observe that we have a canonical augmentation $\mathcal{F} \longrightarrow \mathsf{A}^{\bullet}_X(\mathcal{F})$, which allows to turn the co-simplicial sheaf into a complex

$$F \longrightarrow \mathsf{A}^0(\mathcal{F}) \longrightarrow \mathsf{A}^1(\mathcal{F}) \longrightarrow \cdots, \tag{3}$$

either by the usual trick of taking alternating sums of boundary maps, or using the Dold-Kan construction.

Theorem 3.7 (Beilinson)**.** *For a Noetherian scheme X and a quasi-coherent sheaf $\mathcal{F}$ on X, the complex* (3) *defines a flasque resolution of $\mathcal{F}$. In particular, the complex of global sections $\mathbb{A}^0(\mathcal{F}) \longrightarrow \mathbb{A}^1(\mathcal{F}) \longrightarrow \cdots$ is quasi-isomorphic to $R\Gamma(X, \mathcal{F})$.*

The special case of curves is easy yet instructive to prove. Let us assume without loss of generality that $\mathcal{F}$ is coherent. The sheaf $\mathsf{A}(\mathcal{F})$ of adelic sections of $\mathcal{F}$, assigns to an open subset $U \subset X$ the abelian group $\prod'_{x \in U_{cl}}(E_x \otimes \widehat{F}_x)$, while $\mathsf{O}(\mathcal{F})(U) = \prod_{x \in U}(E_x \otimes \widehat{\mathcal{O}}_x)$, and $F(\mathcal{F})(U) = \mathcal{F}_\eta$, for $U \neq \emptyset$. The co-simplicial structure is covered by the diagram

$$F(\mathcal{F}) \times \mathsf{O}(\mathcal{F}) \rightrightarrows \mathsf{A}(\mathcal{F}) \times F(\mathcal{F}) \times \mathsf{O}(\mathcal{F}) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots,$$

where the resulting maps are assembled from the obvious inclusions. The complex associated to this by the Dold-Kan correspondence is the length 2 complex

$$[F(\mathcal{F}) \oplus \mathsf{O}(\mathcal{F}) \longrightarrow \mathsf{A}(\mathcal{F})].$$

This complex is a resolution of $\mathcal{F}$ boils down to the observation that a rational section of $\mathcal{F}$ without any poles is a (regular) section of $\mathcal{F}$. Moreover one needs that locally any adelic section of $\mathcal{F}$ can be written as the difference of an integral adelic, and a rational section.

The proof of the theorem in full generality requires more sophisticated commutative algebra. We refer the reader to Huber's [Hub91], and will devote the rest of this section to explain a slightly different viewpoint.

3.4 Adelic descent theory

Recall Weil's theorem, which allows one to reconstruct vector bundles on a curve X from so-called adelic descent data.

Theorem 3.8 (Weil)**.** *There is a canonical equivalence $\mathrm{Vect}_n(X) \cong [\mathrm{GL}_n(F) \backslash \mathrm{GL}_n(\mathbb{A}) / \mathrm{GL}_n(\mathbb{O})]$, expressing the groupoid of rank n vector bundles on X as a double quotient.*

In this paragraph we will relate this classical observation to Beilinson's Theorem 3.7, computing the cohomology of quasi-coherent sheaves. Co-simplicial rings, and particularly the notion of cartesian modules provide a flexible way to encode descent data.

Definition 3.9. *Let $R^\bullet$ be a co-simplicial ring. A co-simplicial $R^\bullet$-module $M^\bullet$ is called* cartesian*, if for every map $[k] \longrightarrow [n]$ in Δ, the natural morphism $M^k \otimes_{R^k} R^n \longrightarrow M^n$ is an equivalence.*

Let $R \longrightarrow S$ be a ring homomorphism. There is a co-simplicial ring $S^\bullet$ which assigns to $[n]$ the tensor product $S^{\otimes_R (n+1)} = S \otimes_R \cdots \otimes_R S$.

Theorem 3.10 (Faithfully flat descent)**.** *If $R \longrightarrow S$ is faithfully flat, then tensoring along $R \longrightarrow S^\bullet$ induces an equivalence of categories $\mathsf{Mod}(R) \xrightarrow{\simeq} \mathsf{Mod}^{\mathsf{cart}}(S^\bullet)$.*

For a Noetherian scheme X we have an augmented co-simplicial sheaf of rings $\mathcal{O}_X \longrightarrow \mathsf{A}_X^\bullet$, and although it behaves quite different to the co-simplicial ring $S^\bullet$, it is still reasonable to ask if a similar result holds for this resolution.

Theorem 3.11. *(a) For any Noetherian scheme X, tensoring along the augmentation $\mathcal{O}_X \longrightarrow \mathsf{A}_X^\bullet$ induces an equivalence $\mathsf{Perf}(X) \xrightarrow{\simeq} \mathsf{Perf}^{\mathsf{cart}}(\mathsf{A}_X^\bullet)$.*

(b) The global sections functor induces an equivalence $\mathsf{Perf}^{\mathsf{cart}}(\mathsf{A}_X^\bullet) \xrightarrow{\simeq} \mathsf{Perf}^{\mathsf{cart}}(\mathbb{A}_X^\bullet)$.

The functor from part (a) is fully faithful by Theorem 3.7. In the light of descent theory, we now recognise Beilinson's theorem as a cohomological descent statement. The theorem above claims further that adelic descent data for perfect complexes are effective, which is a generalisation of Weil's observation to arbitrary dimension.

Before discussing applications of Beilinson's result we comment on the proof of Theorem 3.11, which also implicitly contains a proof of Theorem 3.7. In order to explain the argument we compare it with faithfully flat descent. The key observation needed there is the following: let $R \longrightarrow S$ be a faithfully flat ring homomorphism. The co-base change $S \longrightarrow S \otimes_R S$ has a canonical retraction, given by the multiplication map $S \otimes_R S \longrightarrow S$ (also known as the co-diagonal map). One then reduces the entire argument to the case of a ring homomorphism $R \xrightarrow{f} S$ with a retraction r. The latter allows to construct a deformation retraction of the co-simplicial ring $S^\bullet$ onto its canonical augmentation R. In particular one obtains that the co-simplicial ring $S^\bullet$ is homotopy equivalent to the constant co-simplicial ring R. This implies that the category $\mathsf{Mod}(R)$ is equivalent to $\mathsf{Mod}(S^\bullet)$.

Adelic descent theory relies on a similar observation. One shows that the co-simplicial sheaf of rings $\mathsf{A}^\bullet_X \otimes \mathcal{O}_\eta$, where η denotes a generic point of X, deformation retracts onto $\mathcal{O}_\eta$. In a next step one concludes that the assertion of the theorem *spreads out* to an open dense subset $U \subset X$ (see example below), and continues by Noetherian induction. That is, layer by layer one decreases the locus where adelic descent could fail, in each step revealing another generic point, to which the observation above can be applied.

The deformation retraction of $\mathsf{A}^\bullet_X \otimes \mathcal{O}_\eta$ onto $\mathcal{O}_\eta$ is dual to the observation that a simplicial set $S_\bullet$, corresponding to a poset $S_\leq$ with a maximal element m, deformation retracts onto the constant simplicial set $\{m\}$. The corresponding simplicial homotopy moves the vertices of a simplex $s_0 \leq \cdots \leq s_n$ one-by-one to the maximal element m. Just like a discrete time analogue of the contraction of a star-shaped domain onto its centre.

Let's see how spreading out from the generic point to an open subset works in a particular case.

Example 3.12. *We consider the scheme $\mathsf{Spec}\,\mathbb{Z}$, in which case $\mathbb{A}^0 = \mathbb{Q} \times \prod_{p\ prime} \mathbb{Z}_p$. Does the following adelic descent data $M^\bullet$ for a perfect complex on $\mathsf{Spec}\,\mathbb{Z}$ exist? We stipulate $M^0 \otimes_{\mathbb{A}^0} \prod_{p\ prime} \mathbb{Z}_p = \prod_{p\ prime} \mathbb{F}_p$, and $M^0 \otimes_{\mathbb{A}^0} \mathbb{Q} = 0$. It is clear that there cannot be a perfect complex M of $\mathbb{Z}$-modules, such that $M \otimes \widehat{\mathbb{Z}}_p = \mathbb{F}_p$, and $M \otimes \mathbb{Q} = 0$. So let's convince ourselves, without alluding to adelic descent, that there cannot be a cartesian $\mathbb{A}^\bullet$-module with M^0 as described above. Since $\mathbb{A}^1_{red} = (\mathbb{A}^0 \otimes \mathbb{Q})$, we obtain from the descent condition that $M^0 \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. However, it is clear that $\prod_{p\ prime} \mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{Q} \neq 0$, since the element $(1)_{p\ prime}$ of the product above, is not annihilated by any non-zero integer.*

3.5 Applications of Beilinson's resolution

In this very last paragraph we study the interaction of the Tate structure on adèles together with the fact that adèles compute cohomology. We will explain how certain reciprocity phenomena can be explained adelically. Although the same results could be proven purely in terms of algebraic K-theory, the adelic viewpoint feels however particularly illuminating, since it avoids using algebraic K-theory as some sort of blackbox.

We begin with the case of a curve, which relies on finite-dimensionality of cohomology. We denote the adèles of X by $\mathbb{A}$ and consider them to be a Tate vector space over k. It is endowed with a continuous action by $\mathbb{A}^\times$, and therefore we obtain an associative map $\mathbb{A}^\times \longrightarrow K_k$. We have also seen that this map encodes the dimension torsor, respectively $\sum_{x \in X_{cl}} \nu_x(f)$ the order function of adèles, and also captures more refined information related to the tame symbol.

Theorem 3.13. *(General Weil reciprocity) Let X be a smooth projective and connected curve over an algebraically closed field k. We denote by F the function field of X. The composition $F^\times \longrightarrow \mathbb{A}^\times \longrightarrow K_k$ is homotopic to the constant map.*

As a consequence one obtains directly that the map $\sum_{x \in X_{cl}} \nu_x \colon F^\times \longrightarrow \mathbb{Z}$ is constant, and equal to 0. Working with graded determinant theories instead, one sees that $\prod_{x \in X_{cl}} (f,g)_x = 1$ for $f, g \in F^\times$, where (f,g) denotes the tame symbol.

How is Theorem 3.13 established? The intuitive idea is the following: one defines an enriched dimension theory, by assigning to a lattice $L \subset \mathbb{A}$ the complex of vector spaces $[F \oplus L \longrightarrow \mathbb{A}]$. This complex is perfect, since every lattice can be sandwiched between lattices $\mathbb{O}_D$ corresponding to divisors, and therefore its cohomology groups are equal to the finite-dimensional spaces $H^i(X, \mathcal{O}(D))$. We can now prove the theorem by showing that for an arbitrary lattice L, the complexes $[F \oplus fL \longrightarrow \mathbb{A}]$ and $[F \oplus fL \longrightarrow \mathbb{A}]$ are quasi-isomorphic. However, this is clear, since the second is obtained from the first by multiplying with $f \in F^\times$.

Osipov–Zhu's article [OZ11] applied similar arguments to show a local reciprocity law for tame symbol on a surface X. For every irreducible curve $C \subset X$, and $x \in C$ there is a tame symbol $(f,g,h)_{x,C} \in k^\times$. We fix $x \in X$ and assume without loss of generality that $X = \mathsf{Spec}\, \mathcal{O}_x$. The reciprocity law then asserts that

$$\prod_{C \ni x} (f, g, h)_{x,C} = 1,$$

for $f, g, h \in \text{Frac } \mathcal{O}_x$. In order to establish the reciprocity law, one cannot rely on finite-dimensionality of cohomology. Instead, one completes X at 0, and shows that the cohomology groups of $X \setminus \{x\}$ give rise to Tate vector spaces. This structure can then be used to similar effect.

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