BMS Basic Course

Topology

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— Lecture Notes, without guarantee —
— I am happy about any feedback, corrections, suggestions for improvement, etc. —
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Topology "for the working mathematician"

Topology is an important, classical mathematical discipline, which treats interesting objects (such as the Klein bottle, Bing's house, manifolds, lens spaces, knots, ...) and which has produced spectacular successes in 20th century mathematics. A full study of topology is hard (it is a huge field that encompasses many subtle tools and theories); our modest goal here is an introduction and overview "for the working mathematician".

Hence this is a Basic Course – primarily for mathematicians who do not head towards writing a thesis in topology, but who want to understand topological concepts, methods, and results that might be needed or useful tools at some point.

Thus in this course (a 4 hour course, with exercises) we will treat some fundamentals of (point set) topology as well as many important parts of algebraic topology: This is supposed to be precise and concrete enough to enable you to perform topological arguments, and to apply topological results and techniques. We will also include proof ideas and sketches, which explain why all of this "works" - but we will not do the more complicated or longer proofs in detail, which would be required study for anyone striving to be a serious research topologist.

Preface

Topology is an important, classical part of mathematics. It deals with interesting objects (the Klein bottle, Bing's house, manifolds, lens spaces, knots, . . .). To study it in detail is a considerable enterprise (a huge subject with many subtle sub-disciplines and methods); here, however, we are only setting out to get an overview of core parts, and a "potential user's introduction."

For a very theoretical part of mathematics such as topology it may sound strange to talk about *applications*. However, topology is not only one fo the most theoretical and most highly developed areas of so-called "pure mathematics" with remarkable successes and results *in* this subject. In the course of the twentieth century it has also

1. provided notions and concepts that are of core importance for all of mathematics, such as the notion of "compactness",
2. contributed a great variety of important *methods and tools* for the solution of mathematical problems in other areas — for example, there is a large variety of "fixed point theorems" that may be used for example for existence proofs for periodic solutions for systems of partial differential equations (which is, indeed, very close to Poincaré's[1] original motivation for starting off the subject . . .), and
3. the insight is growing that topological methods can also be *useful* directly for applications outside mathematics — see, for example, the recent volume "Topology for Computing" [51].

This is a Basic Course – primarily designed for mathematicians – meant for students who are not really planning to write a thesis in topology, but who want to understand topological notions, results, methods, and concepts, and hope to later feel at ease with using them as tools.

Thus in this course we will discuss the basics of (point set) topology as well as central parts of algebraic topology. We will try to be precise and concrete, so you can learn how to phrase topological facts reliably, and to apply them confidently. I also want to teach methods of proof and proof ideas, from which you can learn *why* all of this works, but we will not do all the gory details for more complicated or involved proofs. (Of course, anybody who wants to become a research topologist must work his/her way through these details at some point.)

Here is an overview (of topology, and of this course):

– *Point set topology* provides important definitions, concepts and foundations. It has long been an important area of research, but by now the "foundations have been clarified". We will spend only little time on this, but we will review resp. get to know key concepts such as continuity, compactness, separation axioms, etc.
– *Low dimensional topology* is concerned with the topology of surfaces (a.k.a. 2-dimensional manifolds) and with their analogs of dimension 3 and 4, and with related questions (which includes e.g. knot theory). This is a very hot topic, among other reasons due to the recent solution of the so-called Poincaré conjecture and Thurston's geometrization conjecture by G. Perelmann (St. Petersburg), which in effect yields a complete classification of 3-dimensional manifolds. We will not treat any of this in detail (the core of the matter is concerned with difficult estimates for PDEs, which arise from curvature-driven flows, particularly the Ricci flow), but we want to understand the basic definitions (e.g. manifolds), and survey the results.
– *Algebraic topology* provides algebraic tools and criteria that help to distinguish spaces, establish the (non-)existence of maps, etc. These tools are of course also important for low-dimensional topology,

[1] Jules Henri Poincaré (1854–1912), one of the greatest mathematicians of all times, founder of algebraic topology, `http://en.wikipedia.org/wiki/Henri_Poincare`

but also far beyond what is usually considered as topology. Main areas within algebraic topology include *homotopy theory* (the fundamental group!), *homology theory* (which yields the so-called homology groups), and *differential topology* (which treats in particular the case of smooth manifolds).

These lecture notes for the course are intentionally kept very brief. They are intended to give a reliable basis, which might save you from taking notes in the course — but they are not a substitute for attending the classes. For more detailed motivation, explanations, illustrations, and pictures I refer primarily to the class and its exercise sessions, but also to the references I give below. Please do spend some time with books such as those by Jänich [23] and Ossa [38] (in German) or by Stillwell [49], Munkres [37, 36] and Hatcher [20]!

And finally: Ask me, talk to me! Tell me, for example, if things (in class, or in these notes) are unclear, or not precise enough, or just don't look right/plausible. I am also interested in hearing about typos and about real mistakes, and I will update and correct these lecture notes correspondingly while we proceed.

1 Topological Spaces

In this section we collect the basic definitions, terms, and concepts as well as key results of the area known as *point set topology*. My main sources for this are Munkres [37] and Jänich [23]. The concepts are general and flexible enough to deal with "general" topological spaces. Being precise with such basics is important also since in the end we might be treating not only "nice, concrete, visualizable" topological spaces (such as triangulable, finite-dimensional spaces or differentiable manifolds), but unavoidably also "infinite-dimensional" objects such as function spaces (for example, loop spaces). Even to make the distinctions clear between "nice" and "not so nice" we have to have our concepts straight, which include "hausdorff" and "dimension", and we have to be fend off the pathologies of general topological spaces (see e.g. Seebach & Steen [47]).

Definition 1.1 (topological space, open sets)**.** A *topological space* is a pair $(X, \mathcal{O})$ that consists of a set X, the *ground set*, and a family $\mathcal{O} \subseteq 2^X$ of subsets, called the *open sets* of the topological space, and whose complements are called the *closed sets* of the space, such that

(TS1) $\emptyset, X \in \mathcal{O}$: the empty set and the ground set are open,
(TS2) any union of open sets is open, and
(TS3) any intersection of finitely-many open sets is open.

This implies: Finite unions, and arbitrary intersections, of closed sets are closed. An intersection of arbitrarily many open sets, or a union of arbitrarily many closed sets, are not open resp. closed in general.

Convention: $\mathcal{O}$ is usually not included in the notation, the topological space is denoted by X. $\mathcal{O}$ is also called "the topology" on X.

Definition 1.2 (neighborhood, basis)**.** An open subset $U \subseteq X$ that contains x is an *(open) neighborhood* of x. The open neighborhoods determine the topology (that is, the family $\mathcal{O}$ of open sets): a set is open if and only if it contains an open neighborhood for each of its points.

A *neighborhood basis* $\mathcal{U}_x$ for $x \in X$ is a set of open neighborhoods such that every open neighborhood of x contains a neighborhood from $\mathcal{U}_x$. A set of open sets $\mathcal{B}$ is a *basis* of the topology if it contains a neighborhood basis for each of its points.

Exercise. A collection of open sets $\mathcal{B} \subseteq \mathcal{O}$ is the basis of a topology if and only if every open set $U \in \mathcal{O}$ is the union of the open sets $U' \in \mathcal{B}$ that are contained in U.

In particular, any basis $\mathcal{B}$ determines the topology uniquely: $\mathcal{O}$ is the set of all unions of sets in $\mathcal{B}$. (Here you have to interpret $\emptyset$ a the union of an "empty set of open subsets".)

Examples.

1. $\mathbb{R}^n$ is a topological space, with
$$\mathcal{O} := \{U \subseteq \mathbb{R}^n : \text{ for every } x \in U \text{ there is an } \varepsilon\text{-neighborhood } B_\varepsilon(x) \text{ of } x \text{ contained in } U\}.$$
2. If X is a set, then $(X, 2^X)$ is a topological space. 2^X is the *discrete topology* on X.
3. If X is a set, then $(X, \{\emptyset, X\})$ is a topological space. $\{\emptyset, X\}$ is the *trivial topology* on X.
4. If (X, d) is a metric space, then
$$\mathcal{O}_d := \big\{U \subseteq X : \text{ for every } x \in U \text{ there is an } \varepsilon > 0 \text{ with } \{x \in X : d(x, y) < \varepsilon\} \subseteq U\big\}$$
is a topology; the *ε-neighborhoods* $U_\varepsilon(x) := \{x \in X : d(x, y) < \varepsilon\}$, for $x \in X$ and $\varepsilon > 0$, form a basis. If the metrix is $d(x, y) = 1 - \delta_{x,y}$, then the topology is discrete.
5. An interesting topology on $\mathbb{Z}$ is
$$\mathcal{P} := \{A \subseteq \mathbb{Z} : \text{ for every } a \in A \text{ there is an arithmetic sequence } a + \mathbb{Z}b, b \neq 0, \text{ contained in } A\}.$$

In this topology, every non-empty open set is infinite. Every sequence $a + \mathbb{Z}b$ is open, but also closed. Does this imply that $\mathbb{Z}\backslash\{-1, 1\} = \bigcup_{p\in\mathbb{P}}(0 + p\mathbb{Z})$ is closed?

Example (p-adic numbers). For a prime p set $|a|_p = p^{-\nu}$ for $a = \frac{b}{c}p^\nu$ with $(p, bc) = 1$, and $|0|_p = 0$. This is a norm on the set of rational numbers, the *p-adic norm*. This defines a metric and thus a topology on $\mathbb{Q}$, for which numbers are close together if they differ "only by high powers of p". Siehe Ebbinghaus et al. [11, Chap. 6].

Exercise. A collection $\mathcal{B} \subseteq 2^X$ of subsets of X is the basis of a topology if

(1) every $x \in X$ lies in some set $B \in \mathcal{B}$, and
(2) if x lies in the intersection of two sets $B', B'' \in \mathcal{B}$, then there is some $B \in \mathcal{B}$ with $x \in B \subseteq B' \cap B''$.

Exercise. The usual euclidean metric, the ℓ_1-metric, the taxicab metric ℓ_∞, and the more general ℓ_p-metrics all determine the same, the "usual" topology on $\mathbb{R}^n$.

Definition 1.3 (box topology/product topology)**.** On a product $X := \prod_{i\in I} X_i$ of topological spaces

- the products $\prod_{i\in I} U_i$ of open subsets $U_i \subseteq X_i$ form the basis of the *box topology* on X, while
- the products $\prod_{i\in I} U_i$ of open subsets $U_i \subseteq X_i$, where $U_i \subset X_i$ may hold only for finitely many factors, form the basis of the *product topology* on X.

If I is finite, then box topology and product topology coincide. In particular, the usual topology on $\mathbb{R}^n$ is also the product topology on $\prod_{i\in\{1,\dots,n\}} \mathbb{R} = \mathbb{R}^n$.

Definition 1.4 (subspace)**.** If $Y \subseteq X$ is a subset, for a topological space $(X, \mathcal{O})$, then Y is a *subspace* with the *induced* topology, whose open sets are of the form $U \cap Y$ for $U \in \mathcal{O}$.

Examples. The "usual" topology on $\mathbb{R}^n$ induces a topology on every subset as well. In particular, this defines a topology on
the *n-dimensional ball* $B^n := \{x \in \mathbb{R}^n : \|x\| \le 1\}$,
the *unit n-cube* I^n for $I := [0, 1]$,
the *unit sphere* $S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$, etc.

Definition 1.5 (continuous map, homeomorphism, embedding)**.** A map $f : X \to Y$ between topological spaces is *continuous* if the preimage $f^{-1}(U)$ of every open set $U \subseteq Y$ is open (in X).

A bijection $f : X \to Y$ is a *homeomorphism* if both f and f^{-1} are continuous. X and Y are then *homeomorphic*; we denote this by $X \cong Y$.

An *embedding* is an injective continuous map $f : X \to Y$ that induces a homeomorphism between X and the subspace $f(X) \subseteq Y$.

Exercise. Show that the subset $\{x \in \mathbb{R}^n : x_1 = 0\} \subset \mathbb{R}^n$ (with the induced topology) is homeomorphic to $\mathbb{R}^{n-1}$ (with the product topology).

Show that the open unit ball $\{x \in \mathbb{R}^n : \|x\| < 1\} \subset \mathbb{R}^n$ (with the induced topology) is homeomorphic to $\mathbb{R}^n$ (with the induced topology).

Exercise. Show: The product topology is the "coarsest" topology on the product set $\prod_i X_i$ (that is, the topology with the minimal collection of open sets), for which the projections $\pi_j : \prod_i X_i \to X_j$ are continuous.

If we talk about *maps* or *mappings* in the following we always mean *continuous* maps, unless explicitly stated otherwise. Similarly, all *spaces* in the following are *topological spaces*, unless they are not.

It is not a priori clear how to define the "dimension" of a (well-behaved) topological space. One would then want to show that $\mathbb{R}^n$ has dimension n, and that homeomorphic spaces have the same dimension — which implies that $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic, for $m > n$; one would also want to conclude that there is no embedding $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$ for $m > n$. This is not so easy; for this a very elaborate "dimension theory" has been developed with this aim (see e.g. Menger [32] oder Hurewicz & Wallmann [22]), a large part of this may now be considered obsolete: The "invariance of dimension" (first established in 1911 by Luitzen Brouwer[2] [8]) is best, most easily and most systematically proved using tools of *homology theory*, in particular via so-called local homology groups.

Example ([37, §44]). The *Peano curve* is a continuous, surjective map $P : [0,1] \to [0,1]^2$.

Definition 1.6 (connected/path connected)**.** A topological space is *connected* if it is not a disjoint union $X = A' \cup A''$ of two non-empty closed subsets.

It is *path connected* if for any $x', x'' \in X$ there is a continuous map $f : [0,1] \to X$ with $f(0) = x'$ and $f(1) = x''$, which we refer to as a *path from x' to x''*.

A *connected component* of a space X is a non-empty connected subset that is both closed and open.[3]
A *path component* of X consists of all points x'' that can be reached by a path from some fixed point x'.

Lemma 1.7. *Every path connected space is connected.*

Examples. The euclidean spaces $\mathbb{R}^n$, the n-balls $B^n = \{x \in \mathbb{R}^n : \|x\| \le 1\}$, and the n-cubes I^n for $I = [0,1]$ are path connected for $n \ge 0$.

The unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ is path connected for $n > 1$. However, S^0 is not connected (two points); $S^{-1} = \emptyset$ is connected.

Examples.

1. The *topologist's sine curve* $S := \{(x, \sin(\frac{1}{x})) : x > 0\} \cup \{(0,y) : -1 \le y \le 1\} \subset \mathbb{R}^2$ is connected, but not path connected.
2. The *Cantor set* $C : \{x \in [0,1) : x$ has a ternary expansion using only digits 0 and 2$\}$ is not discrete (the one-point subsets are not open), but it has no non-trivial paths (all path components consist of one point).

From a zoo of separation axioms, here are the four most important ones:

Definition 1.8 (separation axioms [37, Chap. 4])**.** A topological space X is a

- (T_1) T_1 *space* if every point is closed, that is, if for any $x', x'' \in X$, $x' \neq x''$ there is an open set U'' with $x' \notin U''$ and $x'' \in U''$;
- (T_2) T_2 *space*, or *hausdorff*[4], if for any $x', x'' \in X$, $x' \neq x''$ there are disjoint open sets U', U'' with $x' \in U'$ and $x'' \in U''$;
- (T_3) T_3 *space*, or *regular* if every point is closed (T_1) and for any closed $A'' \subseteq X$ and $x' \notin A''$ there are disjoint open sets U', U'' with $x' \in U'$ and $A'' \subseteq U''$;
- (T_4) T_4 *space*, or *normal* if every point is closed (T_1) and for any disjoint closed sets $A', A'' \subseteq X$ there are disjoint open sets U', U'' with $A' \subseteq U'$ and $A'' \subseteq U''$.

Clearly

$$\text{“normal } (T_4) \Longrightarrow \text{regular } (T_3) \Longrightarrow \text{hausdorff } (T_2) \Longrightarrow T_1\text{”.}$$

Example. The "real line with doubled origin" satisfies (T_1), but it is not hausdorff.

[2]Luitzen Egbertus Jan Brouwer, 1881–1966, topologist, "intuitionist", `http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Brouwer.html`

[3] This is the definition given in class. Better definition: define $x \sim y$ if x and y are contained in a connected subspace of X. The equivalence classes of this relation are called the *connected components* fx. This definition has the advantage that the space X is a disjoint union of its connected components. Compare the two approaches on the Cantor set (below).

[4]Felix Hausdorff, 1868–1942, topologist and poet (as Paul Mongré: "Der Arzt seiner Ehre"), driven into suicide by the Nazis, `http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Hausdorff.html`

Theorem 1.9 (Urysohn's lemma[5] [37, Thm. 33.1]). *If X is normal (i.e., T_4) and $A, B \subset X$ are disjoint closed subsets, then there is continuous interpolation between A and B, that is, there is a continuous map $f : X \to [0, 1]$ with $f(a) = 0$ and $f(b) = 1$ for all $a \in A, b \in B$.*

Theorem 1.10 (Urysohn's metrization theorem [37, Thm. 34.1]). *Every regular (T_3) topological space with a countable basis is metrizable, that is, there is a metric d on X that generates the given topology.*

Definition 1.11 (compactness). A topological space X is *compact* if every covering of X by open subsets has a finite subcollection that is also a covering.

A subset $C \subseteq X$ is *compact* if every covering of C by open subsets of X is also covered by a finite subcollection; equivalently: the topological space C (with the induced topology, considered as a subspace) is compact.

Exercise. Is it true that the compact subsets of $\mathbb{R}$ (with the usual topology) are exactly the finite unions of closed bounded intervals?

Proposition 1.12 (About compactness).

1. *Any closed subset of a compact set (i.e. every closed subspace of a compact space) is compact.*
2. *Every compact subset of a hausdorff space is closed.*
3. *Every image of a compact set by a continuous map is compact.*
4. *Any continuous function $f : X \to \mathbb{R}$ has a minimum and a maximum on every non-empty compact subset $C \subseteq X$.*
5. *(Heine Borel theorem) A subset $A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Theorem 1.13 ([37, Thm. 26.6]). *If X is compact and Y is hausdorff, then every bijective continuous map $f : X \to Y$ is a homeomorphism.*

Theorem 1.14 (Tychonoff's theorem[6] [37, Thm. 37.3]). *Every product of compact spaces (with product topology) is compact.*

Example. $[0, 1]^{\mathbb{N}}$ is a compact space.

Exercise. The unit ball in $\ell^2(\mathbb{N})$ is *not* compact.

Exercise. The product $[0, 1] \times [0, \frac{1}{2}] \times [0, \frac{1}{3}] \times \cdots$, known as the "Hilbert cube" with *product topology* is compact, according to Tychonoff's theorem. This set is also a subspace of the space $\ell^2(\mathbb{N}) = \{(x_1, x_2, \ldots) : \sum_{i\geq 1} x_i^2 < \infty\}$ of square-summable sequences, whose topology is defined by the ℓ^2-metric — and this subspace is also compact. Is it the same topological space?
(Compare [37, p. 128].)

Definition 1.15 (Compact-open topology, for function spaces). Let X and Y be topological spaces. then the set $\mathcal{C}(X, Y)$ of all continuous maps $X \to Y$ with the *compact-open topology* is a topological space: Its open sets are the unions of finite intersections of sets of the form

$$S(C, U) \;:=\; \{f \in \mathcal{C}(X, Y) : f(C) \subseteq U\}$$

for compact $C \subseteq X$ and open $U \subseteq Y$.

The sets $\mathcal{C}(X, Y)$ form a "subbasis" of the topology (that is, its finite intersections form a basis).

Example. If $X = \{x\}$ is a point, then $\mathcal{C}(X, Y)$ is homeomorphic to Y.

Exercise. Let X be a topological space, I a set. Then the product topology on $\prod_{i\in I} X$ (where all factors are equal to X) is exactly the compact-open topology on the space X^I of all continuous maps $f : I \to X$, if I gets the discrete topology.

[5] Pavel Samuilovich Urysohn, 1898–1924, Russian topologist, died at age 26 swimming in France, `http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Urysohn.html`

[6] Andrei Nikolaevich Tikhonov, 1906–1993, Russian topologist; he was 20 when he proved this, `http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Tikhonov.html`

2 Simplicial Complexes

Some topological spaces (and many interesting ones) can be "triangulated" – which yields a simple combinatorial model for the space in question. Viewed in the other direction: Building simplicial complexes is an effective combinatorial method for constructing interesting examples of topological spaces, some of which are very relevant for combinatorial or geometric problems or situations.

My sketch of this theory is based on books by Munkres [36] and Matoušek [31], where I tend to follow the notation and terminology of [31] .

Definition 2.1 (simplex, faces)**.** If $F = \{v_0, v_1, \dots, v_k\} \subset \mathbb{R}^n$ is a set of $k+1$ affinely independent points, then

$$\sigma = \operatorname{conv}\{v_0, \dots, v_k\} = \{\lambda_0 v_0 + \cdots + \lambda_k v_k \in \mathbb{R}^n : \lambda_i \geq 0, \sum\nolimits_{i=0}^{k} \lambda_i = 1\}$$

is a *k-dimensional simplex*, or *k-simplex*. The simplexes $\tau = \operatorname{conv}(G)$ for $G \subseteq F$ are referred to as the *faces* of σ. (This includes σ and $\emptyset$ as faces of σ. The other faces of σ are called *non-trivial*.)

Any two k-dimensional simplices are affinely isomorphic, so in particular they are homeomorphic (always referring to the subspace topology induced from the ambient space $\mathbb{R}^n$). A k-dimensional simplex is a point ("vertex") for $k = 0$, an interval ("edge") for $k = 1$, a triangle ("2-face") for $k = 2$, a tetrahedron for $k = 3$, etc. We write $V(\sigma)$ for the set of vertices of σ. Attention: every k-dimensional simplex has $k + 1$ vertices. Often the empty set is interpreted as a simplex of dimension -1 (with 0 vertices).

Definition 2.2 (geometric simplicial complex)**.** A *(geometric) simplicial complex* Δ is a set of simplices in $\mathbb{R}^N$ (for some $N \geq 0$) such that

(K1) $\emptyset \in \Delta$

(K2) For any face $\sigma \in \Delta$ also all faces $\tau \subseteq \sigma$ are in Δ

(K3) For any two faces $\sigma, \sigma' \in \Delta$ the intersection $\sigma \cap \sigma'$ is a face of σ and of σ'.

The *vertex set* $V(\Delta)$ of Δ is the set of all $v \in \mathbb{R}^N$ such that $\{v\}$ is a vertex of Δ.

The *dimension* of Δ, denoted $\dim \Delta$, is the largest dimension of a simplex in Δ.

A *subcomplex* is a non-empty subset of Δ which is again a complex, that is, which satisfies (K2). The *k-skeleton* of Δ is the subcomplex $\Delta^{(k)}$ that consists of all simplices of dimension at most k.

Examples. If σ is a simplex, then the set of all faces of σ is a complex (which is usually also called σ).

The set of all proper faces $\operatorname{conv}(G)$, $G \subset F$, of $\sigma = \operatorname{conv}(F)$ is called the *boundary (complex)* $\partial\sigma$ of σ. The boundary of a k-simplex is empty for $k = 0$, consists of two points $k = 1$, is the boundary of a triangle for $k = 2$, etc.

Definition 2.3. If Δ is a simplicial complex in $\mathbb{R}^n$, then the *polyhedron* of Δ is the topological space $\|\Delta\|$, which on the ground set $\bigcup \Delta$ (the *support* of Δ) is given by the following topology: A subset $A \subseteq \bigcup \Delta$ is closed resp. open if and only if $A \cap \sigma$ is closed resp. open for every simplex $\sigma \in \Delta$.

If Δ is finite, then the topology on $\|\Delta\|$ is the subspace topology on $\bigcup \Delta$ induced from $\mathbb{R}^n$.

Example. If σ is a k-simplex, then $\|\sigma\|$ is homeomorphic to B^k, while $\|\partial\sigma\|$ is homeomorphic to S^{k-1}.

Example. $\Delta := \{\emptyset, \{0\}\} \cup \{\{\frac{1}{n}\} : n \in \mathbb{N}\}$ is a 0-dimensional simplicial complex; the topology on $\|\Delta\|$ is thus discrete. In contrast, in the subspace topology on $\bigcup \Delta \subset \mathbb{R}$ the subset $(\bigcup \Delta)\backslash\{0\}$ is not closed.

Remark 2.4. The topology on $\|\Delta\|$ may be interpreted as the *quotient topology* with respect to the surjective map $\sum_{\sigma\in\Delta}\sigma \to \bigcup_{\sigma\in\Delta}\sigma = \|\Delta\|$ from the *sum* (disjoint union, with the obvious topology) to the union: It is the "finest" topology on $\|\Delta\|$ such that the map π is continuous. (See [36, §20].)

If in the following we refer to topological properties of a simplicial complex (like hausdorff, compact, connected, etc.) then this always refers to the topology of the polyhedron.

Lemma 2.5. *Every simplicial complex is hausdorff* (T_2).

A simplicial complex is compact if and only if it is finite (i.e. consists of finitely many simplices).

A simplicial complex is connected if and only if it is path-connected.

Definition 2.6 (triangulable)**.** A topological space X is *triangulable* if it is homeomorphic to (the polyhedron of) a simplicial complex Δ, that is, if $X \cong \|\Delta\|$.

Examples. The balls B^n and the spheres S^{n-1} are triangulable.

Example. The *standard triangulation* of $\mathbb{R}^n$ has vertex set $\mathbb{Z}^n$. The vertex sets of its faces are all sets $\{v^0,\dots,v^k\} \subset \mathbb{Z}^n$ for which all components of $v^j - v^i$ (for $j > i$) are either 0 or 1.

Examples. The "real line with doubled origin" is not triangulable (since it is not hausdorff).

The set $\mathbb{Q}$ of rational numbers (with the subspace topology induced from $\mathbb{R}$) is not triangulable. (Since it is countable, the corresponding complex would have to be 0-dimensional, but then this has discrete topology.)

Examples (Schönflies theorem/Alexander's horned sphere). If $f : S^1 \to \mathbb{R}^2$ is an embedding, then there is a triangulation of $\mathbb{R}^2$ such that $f(S^1)$ corresponds to the polyhedron of a subcomplex.

If $f : S^2 \to \mathbb{R}^3$ is an embedding, then there need not be a triangulation of $\mathbb{R}^3$ such that $f(S^2)$ corresponds to the polyhedron of a subcomplex.

Definition 2.7 (simplicial maps)**.** A *simplicial map* $f : \Delta \to \Delta'$ is a function $f : V(\Delta) \to V(\Delta')$ with the property that for every simplex $\sigma \in \Delta$ the image of the vertex set $V(\sigma)$ is the vertex set of a simplex in Δ', which is then denoted $f(\sigma)$.

For $\sigma \in \Delta$ and $f : \Delta \to \Delta'$ we automatically get $\dim f(\sigma) \le \dim \sigma$, but not necessarily equality.

Proposition 2.8. *Every simplicial map $f : \Delta \to \Delta'$ induces a continuous map $\|f\| : \|\Delta\| \to \|\Delta'\|$ of the corresponding polyhedra, by "linear extension to the simplices":*

$$\|f\| : \lambda_0 v^0 + \cdots + \lambda_k v^k \;\longmapsto\; \lambda_0 f(v^0) + \cdots + \lambda_k f(v^k).$$

*

Definition 2.9 (abstract simplicial complex)**.** An *abstract simplicial complex* K is a non-empty system $K \subseteq 2^V$ of finite subsets of a set V that is closed unter taking subsets, that is, such that for every set $S \in K$ all subsets of S are elements of K.

The union $V(K) := \bigcup K$ is referred to as the *vertex set* of K. The sets $S \in K$ are called the *faces* of K. The *dimension* of a face S is $\dim(S) := |S| - 1$. The dimension of K is the maximal dimension of a face of K.

Definition 2.10 (simplicial maps; isomorphic)**.** A *simplicial map* $f : K \to K'$ between abstract simplicial complexes K and K' is a function $f : V(K) \to V(K')$ that maps faces of K to faces of K', that is, such that $f(S) \in K'$ for all $S \in K$.

A simplicial map $f : K \to K'$ is an *isomorphism* if $f : V(K) \to V(K')$ is bijective and induces a bijection between the faces of K and those of K', i.e., if $K' = \{f(S) : S \in K\}$.

Definition 2.11 (vertex scheme)**.** For every geometric simplicial complex Δ the set

$$K_\Delta := \{V(\sigma) : \sigma \in \Delta\}$$

of vertex sets of simplices in Δ is an abstract simplicial complex, the *vertex scheme* of Δ.
If an abstract simplicial complex K is isomorphic to the vertex scheme K_Δ of a geometric complex, then Δ is a *realization* of K.

Lemma 2.12. *If Δ, Δ' are realizations for two isomorphic abstract complexes K, K', then $\|\Delta\|$ and $\|\Delta'\|$ are homeomorphic.*

Exercise. Every finite set system ("hypergraph") defines a simplicial complex, if you extend it by all subsets. So, for example,

$$\Delta \ := \ \big\{\{1,2,3\},\{1,4\},\{2,4\},\{4,5\}, \text{ and all their subsets } \big\}$$

is a simplicial complex. Draw a realization!

Examples. In combinatorial optimization one studies abstract simplicial complexes associated to graphs, such as the *independence complex* $I(G) \subseteq 2^V$ and the *matching complex* $M(G) \subseteq 2^E$ of a finite graph $G(V,E)$.

The *chessboard complex* $\Delta_{m,n}$ may be viewed as the matching complex of a complete bipartite graph, $\Delta_{m,n} := M(K_{m,n})$.

Examples. One can describe/construct topological spaces by combinatorially specifying a triangulation. For example, we may describe triangulations of a closed strip (cylinder), Möbius band, torus or of the so-called Klein bottle by "identifications on the bundary" on triangulated rectangles.

Proposition 2.13. *For any finite abstract simplicial complex K there is a canonical way to construct a realization, as follows: Let $V = V(K)$ be a vertex sets of K, and let $\mathcal{F}_c(V,\mathbb{R})$ be the $\mathbb{R}$ vector space of all functions $f : V \to \mathbb{R}$. Let $\mathcal{F}_c[K] \subset \mathcal{F}$ be the subset of all those functions $f \in \mathcal{F}(V,\mathbb{R})$, for which*

1. *the support $\{v \in V : f(v) \neq 0\}$ is a simplex in K,*
2. *all function values $f(v)$ are non-negative, and*
3. *the sum of all function values is* 1.

Then $\mathcal{F}[K]$ is the polyhedron of a simplicial complex that realizes K.

Exercise. Formulate and prove a version of this proposition that also works for infinite simplicial complexes. For this, you have to consider geometric simplicial complexes in infinite-dimensional real vector spaces.

Proposition. *Let $f : \Delta \to \Delta'$ be a simplicial map of geometric simplicial complexes. Then $\|f\|$ is a homeomorphism if and only if the corresponding simplicial map of abstract simplicial complexes $K_\Delta \to K_{\Delta'}$ is an isomorphism.*

Every simplicial complex on $n < \infty$ vertices is a subcomplex of an $(n-1)$-dimensional simplex, so it can be realized in $\mathbb{R}^{n-1}$.

Lemma 2.14. *Every finite (countable) simplicial complex of dimension $k < \infty$ can be geometrically realized in $\mathbb{R}^{2k+1}$.*

Theorem 2.15 (Steinitz' theorem 1922 [53, Lect. 3])**.** *Every triangulation of S^2 can be geometrically realized in $\mathbb{R}^3$ (even as the boundary complex of a simplicial convex polytope).*

The smallest triangulation of the torus $T = S^1 \times S^1$ has seven vertices.

The "Toblerone triangulation" of the torus $T = S^1 \times S^1$ on 9 vertices, as well as the minimal triangulation on 7 vertices (the Császár-Torus — see [29]) can be realized in $\mathbb{R}^3$. It has been an open problem for a long time, whether every triangulated torus has a geometric realization in $\mathbb{R}^3$: For this old problem, which goes back to Grünbaum [19, p. 253], a positive answer has just been published by Archdeacon, Bonnington & Ellis-Monaghan [3].

Realizations of surfaces are very much a topic of current research: See the Research Group *Polyhedral Surfaces* at TU Berlin,

`http://www.math.tu-berlin.de/geometrie/ps/`.

Definition 2.16 (manifold)**.** An *n-dimensional manifold* is a non-empty, hausdorff, second countable space M for which every point $x \in M$ has a neighborhood that is homeomorphic to $\mathbb{R}^n$.
(A space is *second countable* if its topology has a countable basis.)

A compact manifold (without boundary, as defined here) is also called *closed*.

Examples. We mention here: 1-dimensional manifolds; the 2-sphere with $g \geq 0$ handles; the spheres S^{n-1}, the projective spaces $\mathbb{R}\mathrm{P}^{n-1}$; the n-dimensional torus $T^n := (S^1)^n$, the groups $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$, as well as the non-compact examples $\mathbb{R}^n$, $\mathrm{SL}(n)$, $\mathrm{GL}(n)$, …

Theorem 2.17 (Rado 1925 [40]/Moise 1952 [35])**.** *All compact manifolds of dimension $n \leq 3$ are triangulable.*

By combining results by Casson and Freedman (see [2, p. xvi]) one obtains examples of 4-manifolds that are not triangulable. It is, however, not clear whether there are manifolds of dimension $n > 4$ that are not triangulable.

Minimal triangulations of manifolds are another topic of current reserach; see for example Björner & Lutz [4]. For example, it is not known how many vertices are needed to triangulate $S^m \times S^n$.

Examples. The *real projective plane* $\mathbb{R}\mathrm{P}^2$ has a six vertex triangulation (obtainable from the icosahedron). More generally from any centrally-symmetric polytope, for which no antipodal vertices have a common neighbor, one obtains a triangulation of $\mathbb{R}\mathrm{P}^{n-1}$ together with a simplicial map $S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}$.

Definition 2.18. *Semi-algebraic sets* are the subsets of $\mathbb{R}^n$ that may be obtained as the solution sets of finitely many polynomial equations and (strict or non-strict) inequalities, and their unions.

Theorem 2.19 (Lefschetz & Whitehead [27], Łojasiewicz [28], etc.)**.** *Every semialgebraic set is triangulable.*

3 Homotopy Theory

3.1 Homotopy equivalence, contractability

The homotopy groups of a topological space are *algebraic* invariants, which in principle can be used to distinguish spaces that at not only not homeomorphc, but stronger not even homotopy equivalent. Although in general they are notoriously difficult to compute (even for finite simplicial complexes), they are nevertheless fundamental ... We start with a discussion of "homotopies".

Definition 3.1 (homotopic, homotopies)**.** Let X, Y be topological spaces. Two continuous maps $f, g : X \to Y$ are *homotopic*, denoted $f \sim g$, if one can be deformed into the other, that is, if there is a continuous map $H : X \times I \to Y$ (a *homotopy*), which interpolates between f and g, with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

Lemma 3.2. *Homotopy defines an equivalence relation on the set of all (continuous) maps from X to Y. The set of all equivalence classes (the* homotopy classes*) is denoted $[X, Y]$.*

Example. The homotopy classes in $[\{x\}, Y]$ correspond to the path-connectivity components of Y.

Definition 3.3 (homotopy equivalence)**.** Two topological spaces X, Y are *homotopy equivalent*, denoted $X \simeq Y$, if there are continuous maps $f : X \to Y$ and $\bar{f} : Y \to X$ such that $\bar{f} \circ f$ is homotopic to the identity map $\mathrm{id}_X : X \to X$ on X, and $f \circ \bar{f}$ is homotopic to id_Y, that is, such that $f \circ \bar{f} \sim \mathrm{id}_X$ and $\bar{f} \circ f \sim \mathrm{id}_Y$.

Examples. $\mathbb{R}^n \simeq B^n \simeq \{0\}$; $\mathbb{R}^n \backslash \{0\} \simeq S^{n-1}$; $S^3 \backslash S^1 \simeq S^1$ (but beware of knot theory!). Similarly $S^n \backslash S^m \simeq S^{n-m-1}$

In this definition f and $\bar{f}$ need to be neither injective nor surjective.

Lemma 3.4. *Homotopy equivalence is an equivalence relation (as suggested by the name). The equivalence classes are known a* homotopy types.

Definition and Lemma 3.5 (contractible)**.** *A topological space X is* contractible *if it has the homotopy type of a point, that is, if for some (equivalently: every) point $y_0 \in X$ there is a homotopy between the constant map $c_{y_0} : X \to X$, $x \to y_0$, and the identity map $\mathrm{id}_X : X \to X$.*

Related terms: Retraction, deformation retraction, strong deformation rectraction. There are whole books in this context from the perspective of set-theoretic topology, see Borsuk [6] and Hu [21].

Definition 3.6 (collapsible)**.** A finite simplicial complex K is *collapsible* if it can be reduced to a single vertex by a sequence of *elementary collapses*: In such a step one removes one non-maximal face that is contained in exactly one maximal face, together with all faces that contain it.

Lemma 3.7. *Elementary collapses do not change the homotopy type. Every collapsible complex is contractible.*

There are serious attempts to replace homotopy equivalence as a basic notion by equivalence with respect to elementary collapsing and anti-collapsing steps. This leads to the notion of "simple homotopy type"; see Cohen [10]).

Example. A finite 1-dimensional simplicial complex (a finite "simple" graph) is collapsible if and only if it is a *tree*, that is, if it is connected and contains no cycle. Every elementary collapse removes a *leaf* [graph theory terminology].

Example (Borsuk's bottle; Bing's house; dunce hat). There are 2-dimensional finite simplicial complexes that are contractible, but not collapsible. They include "Bing's house with two rooms" (see e.g. [20, p. 4]), Borsuk's bottle (unpublished?, equivalent to Bing's house), and the "dunce hat".

3.2 *k*-Connectivity

Definition 3.8 (k-connected)**.** For $k \geq -1$, a topological space X is *k-connected*, if it is non-empty, and if every continuous map $f : S^\ell \to X$ with $0 \leq \ell \leq k$ is homotopic to a constant map. Equivalently: X is k-connected if for $\ell \in \{-1, 0, \ldots, k\}$ every $f : S^\ell \to X$ can be extended to a map $F : B^{\ell+1} \to X$.

For $\ell = -1$ the condition is that X is non-empty. For $\ell = 0$ it asks for path-connectivity. A 1-connected space is also called *simply connected*.

Lemma 3.9. *k-connectivity is an invariant of the homotopy type.*

Theorem 3.10. *S^n is $(n-1)$-connected, but not n-connected, for $n \geq -1$.*

Sketch of proof. Every continuous map $f : S^k \to S^n$ may be homotoped to a "well-behaved" (for example: piecewise-linear) map. (Compactness helps in proving this!)

Such a map is not surjective for $k < n$, and then easily shown to be homotopic to a constant map.

For the second part, one uses algebraic tools, such as the *degree* of a map: It counts (with signs) how often a generic point in the image is covered by the map. This quantity is equal for all "generic" points in the image, and it does not change under deformations. For the identity the degree is 1, for a constant map it is 0. □

The identity $\mathrm{id} : S^n \to S^n$ is *essential*, that is, not homotopic to a constant map (*null-homotopic*). This is a non-trivial result, which we note as follows.

Corollary 3.11. *S^n is not contractible.*

Theorem 3.12 (Brouwer's[7] fixed point theorem)**.** *Every continuous map $B^{n+1} \to B^{n+1}$ has a fixed point.*

Proof. Let $f : B^{n+1} \to B^{n+1}$ be fixed point free. Denote by $h(x)$ the intersection of S^n with the ray that starts at $f(x)$ and passes through x. Then $h : B^{n+1} \to B^{n+1}$ is continuous, it is the identity on the boundary, and thus yields a null-homotopy for id. □

Conversely, Brouwer's fixed point theorem implies that S^n is not contractible: Assume that $\mathrm{id} : S^n \to S^n$ is null-homotopic, then so is $-\mathrm{id}$. The null-homotopy yields a continuous map $\mathrm{Id} : B^{n+1} \to S^n \subset B^{n+1}$, which is fixed point free — contradiction!

A simple "combinatorial proof" for Brouwer's fixed point theorem was found by Sperner in 1928: See siehe [1, Chap. 25].

Theorem 3.13 (see [46, p. 405])**.** *A contractible space X is k-connected for all $k \geq -1$.*
Conversely, if X is (homotopy-equivalent to) a simplicial complex and k-connected for all $k \geq -1$, then it is contractible.

[7] `http://www.gap-system.org/~history/Biographies/Brouwer.html`

Proof. The first statement follows from Lemma 3.9.
For the converse claim, we have to construct a continuous extension $F : X \times I \to X$ of the map $f : X \times \{0,1\} \to X$ given by $f(x,0) := x$ and $f(x,1) = x_0$. This can be constructed cell-wise, by solving the extension problem by induction on the dimension of the skeleton. □

With simple connectivity (1-connectivity) we have discussed all the concepts that are needed to *formulate* a key result — one of the most important problems in mathematics (a Clay millennium problem!), which has recently been solved by Perelman (2003/2006), based on an Ansatz of Hamilton.

Theorem 3.14 (The Poincaré conjecture/Hamilton–Perelman theorem [39])**.** *Every simply-connected closed* 3*-manifold is homeomorphic to* S^3.

Exercise. The set (group) $\mathrm{SU}(2) \subset \mathbb{C}^{2\times 2}$ is a simply-connected closed 3-manifold. Show that it is homeomorphic to S^3.

3.3 The fundamental group

Pairs of topological spaces may be even more basic/fundamental for Algebraic Topology than just spaces.

Definition 3.15 (Pairs of Spaces)**.** A *pair of spaces* is a pair (X, A), where X is a topological space, and $A \subseteq X$ is a subspace. (X may be identified with $(X, \emptyset)$.) A pair $(X, \{x_0\})$ is a *pointed space*; the point $x_0 \in X$ is then referred to as the *base point*. *Continuous maps* $f : (X, A) \to (Y, B)$ between pairs of spaces are continuous maps $X \to Y$ that additionally satisfy $f(A) \subseteq B$. Maps of pairs $f, g : (X, A) \to (Y, B)$ are *homotopic* if there is a map $H : (X \times I, A \times I) \to (Y, B)$ with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. Homeomorphism and homotopy equivalence for pairs of spaces are also defined in analogy to the definitions for topological spaces.

Definition 3.16 (Fundamental group)**.** Let again $I = [0, 1]$ denote the unit interval and $\partial I = \{0, 1\}$ its endpoints.
For a topological space X with base point $x_0 \in X$ a continuous map $\gamma : (I, \partial I) \to (X, \{x_0\})$ is a *closed path* (a *loop*) in X.
The set $\pi_1(X; x_0) := [(I, \partial I), (X, \{x_0\})]$ of homotopy classes of closed paths is the *fundamental group* (or *first homotopy group*) of X (with respect to the base point x_0). A composition on this set is defined by $[\gamma] \circ [\gamma'] := [\gamma * \gamma']$, with

$$\gamma * \gamma'(t) := \begin{cases} \gamma(2t) & \text{for } t \in [0, \frac{1}{2}] \\ \gamma'(2t-1) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

Exercise. Prove that the composition is well-defined, and that with this composition $\pi_1(X; x_0)$ indeed is a group.

Remark 3.17. The fundamental group $\pi_1(X; x_0)$ "sees" only the path-component of X that contains the base point.

Remark 3.18. If X is path-connected, then the structure of the group $\pi_1(X; x_0)$ is independent of the base point. However, there is no canonical isomorphism: Any path γ from x_0 to x_1 induces an isomorphism $i_\gamma : \pi_1(X; x_0) \to \pi_1(X; x_1)$, but this isomorphism does depend on the homotopy class of γ. Nevertheless, one often writes $\pi_1(X)$ for the fundamental group of a (path-connected) space X.

Lemma 3.19. *A space X with $x_0 \in X$ is simply connected if and only if it is path-connected, and the fundamental group is trivial (that is, $\pi_1(X) = \{c_0\}$).*

Remark 3.20. This is not quite trivial, since "simple connectivity" refers to free homotopy, that is, its homotopies do not preserve a base point.

One is very interested in *loop spaces*: $\Omega(X, x_0) := \mathcal{C}^0((S^1, \{1\}), (X, \{x_0\}))$ gets the topology of a function space, with compact-open topology. Its path components are the elements of the fundamental group of X.

Examples. $\pi_1(\mathbb{R}^n) \cong \{c_0\}$; similarly for every contractible space.
$\pi_1(S^n) \cong \{c_0\}$ für $n > 1$.

Lemma 3.21. *For polyhedra $X = \|\Delta\|$ the fundamental group depends only on the 2-skeleton.*

Proof. Regularize, as sketched in the proof for Theorem 3.10: Every loop can be homotoped into the 1-skeleton, every homotopy into the 2-skeleton. □

Theorem 3.22 (Stillwell [49, Sect. 4.1])**.** *The fundamental group of a (w.l.o.g. connected) simplicial complex may be expressed in generators and relations as follows: Let $T \subseteq \Delta^{(1)}$ be a spanning tree in the 1-skeleton. For every vertex x_i let w_i be the unique simple path in T from the base vertex x_0 to x_i. Then every edge $e_{ij} = \{x_i, x_j\} \in \Delta^{(1)} \backslash T$ determines a loop $w_i e_{ij} w_j^{-1}$, and every triangle $\sigma = x_i x_j x_k \in \Delta^{(2)}$ yields a relation R_σ. With this,*

$$\left\langle\, g_{ij} : e_{ij} \in \Delta^{(1)} \backslash T \;\middle|\; R_\sigma : \sigma \in \Delta^{(2)} \,\right\rangle$$

is a presentation of $\pi_1(\Delta)$ by generators and relations.

Proof. Every loop based at x_0 may be deformed into a closed edge path through a sequence of vertices $x_0, x_1, \ldots, x_{N-1}, x_N = x_0$. This path is homotopic to a chain of the form

$$(w_0 e_{01} w_1^{-1})(w_1 e_{12} w_2^{-1}) \ldots (w_{N-1} e_{N-1,N} w_N^{-1}),$$

which is a concatenation of trivial loops (for $e_{i,i+1} \in T$) and of generators of the prescribed type (with $g_{ij} = g_{ji}^{-1}$).
Similarly, every homotopy between paths can be decomposed into single steps, which pass over triangles. Every cycle around a triangle $x_i x_j x_k x_i$ can be deformed into a concatenation of loops

$$(w_i e_{ij} w_j^{-1})(w_j e_{jk} w_k^{-1})(w_k e_{ki} w_i^{-1}),$$

and, depending on whether one, two or three edges do not lie in T, this yields a relation of the type g_{ij}, $g_{ij} g_{jk}$, or $g_{ij} g_{jk} g_{ki}$. □

Example. The fundamental groups of graphs (1-dimensional complexes) are free groups.
In particular, $\pi_1(S^1) \cong \mathbb{Z}$.

Proposition 3.23 (Every finitely-presented group is a fundamental group)**.** *For every finite presentation of a group $G = \langle g_1, \ldots, g_s \mid R_1, \ldots, R_t \rangle$ there is a 2-dimensional finite simplicial complex Δ_G with fundamental group $G \cong \pi_1(\Delta_G)$.*

Proof. Start with a 1-dimensional complex that consists of s triangle boundaries, identified in one point. Then glue in "2-cells" as prescribed by the relations, and triangulate these. □

Theorem 3.24 (The Seifert–van Kampen theorem [49, p. 125])**.** *If a space X can be written as a union $X = X_1 \cup X_2$ of two open sets with a path-connected intersection, if a common base point x_0 is chosen to lie in the intersection $X_1 \cap X_2$, and if the fundamental groups are given by*

$$\pi_1(X_1) = \langle a_1, \dots, a_m \mid R_1, \dots, R_n \rangle, \quad \pi_1(X_2) = \langle b_1, \dots, b_p \mid S_1, \dots, S_q \rangle$$

and

$$\pi_1(X_1 \cap X_2) = \langle c_1, \dots, c_x \mid T_1, \dots, T_y \rangle,$$

then one obtains the fundamental group of $X = X_1 \cup X_2$ presented as

$$\pi_1(X_1 \cup X_2) \; = \; \big\langle \, a_1, \dots, a_m, \, b_1, \dots, b_p \; \big| \; R_1, \dots, R_n, \, S_1, \dots, S_q, \, U_1 V_1^{-1}, \dots, U_x V_x^{-1} \, \big\rangle,$$

where U_i resp. V_i is a presentation of c_i by the generators a_j of $\pi_1(X_1)$ resp. by the generators b_k of $\pi_1(X_2)$.

Observe that in this description of $\pi_1(X_1 \cup X_2)$, the relations $T_1, \dots, T_y$ of $\pi_1(X_1 \cap X_2)$ no not play an (explicit) role.

Corollary 3.25. *$\pi_1(S^n)$ is trivial for $n > 1$.*

Proof. Cover S^n by two contractible open subsets, for example by open ε-neighborhoods of the upper resp. lower hemisphere. The intersection is then homotopy equivalent to S^{n-1}, hence path connected for $n > 1$. ☐

Example. A further, important application: The analysis of knot groups by Dehn (1914) and Schreier (1924); see Stillwell [49, Chap. 7].

In particular, the machinery described here allows one to classify the *torus knots* $T_{m,n}$ $(m, n \geq 2$ coprime), whose *knot groups* (fundamental groups of the complements) are given by

$$\pi_1(\mathbb{R}^3 \backslash T_{m,n}) \; \cong \; \langle a, b \mid a^m b^{-n} \rangle :$$

Except for reflection of the space, and $T_{m,n} \cong T_{n,m}$, the knot groups are not isomorphic, and thus the knots are not equivalent. See Stillwell [49, Sects. 4.2.1, 7.1].

Remark 3.26 (See [48, 49]). The *word problem* for groups (that is, given a finite group presentation and a word, to decide whether the word represents the unit element in the group) is not decidable by Novikov (1955). This fundamental algebraic/combinatorial result implies further undedicability results in Topology.

Indeed, it implies that the problem "Given a finite simplicial complex, decide whether it is simply-connected!" is not algorithmically decidable!

The homeomorphism problem for 2-dimensional complexes is effectively solvable (that is, we have an algorithm for the problem), but the homotopy type problem is undecidable: There is no finite algorithm that would decide for a finite 2-dimensional simplicial complex whether it is contractible.

The homeomorphism problem for the 3-dimensional sphere is effectively solvable (Rubinstein–Thompson [43]; compare King [24]), but the homeomorphism problem for the 5-dimensional sphere is not solvable. Similarly the homeomorphism problem for 4-dimensional manifolds is not solvable (Markov 1958). The classification problem for 3-dimensional manifolds is solved in a very strong way (via Perelman's proof for Thurston's "geometrization conjecture"), although the algorithmic consequences for this have certainly not been fully explored yet.

Already very early in the history of topology it has been observed (by Reidemeister[8]), that working with fundamental groups one hits the danger to merely translate difficult topological problems by difficult algebraic problems [49, p. 47].

3.4 Higher homotopy groups

The higher homotopy groups are often defined by

$$\pi_n(X, \{x_0\}) := [(S^n, \{e_1\}), (X, \{\{x_0\}\})],$$

but then it is not so easy to see that/why elements can be "added". One can interpret the homotopy groups also as equivalence classes of maps $(I^n, \partial I^n) \to (X, \{\{x_0\}\})$. The following lemma tells us that this is equivalent.

Lemma 3.27. *For every space with base point $(X, \{x_0\})$ there is a canonical bijection between the sets of homotopy classes of maps*

$$[(S^n, \{e_1\}), (X, \{x_0\})] \quad \longleftrightarrow \quad [(I^n, \partial I^n), (X, \{x_0\})].$$

Proof. Any $g : (I^n, \partial I^n) \to (X, \{x_0\})$ maps ∂I^n to $\{x_0\}$, so it is automatically constant on ∂I^n.
Let us now consider the *quotient space* $(I^n/\partial I^n, *)$, for which the complete boundary of the n-cube is identified into a single point $*$. By definition of the quotient topology every map $\bar{g} : (I^n/\partial I^n, *) \to (X, A)$ also induces a map $g : (I^n, \partial I^n) \to (X, A)$ that is constant on ∂I^n. Conversely, every map $g : (I^n, \partial I^n) \to (X, A)$ *that is constant on* ∂I^n induces a continuous map $\bar{g} : (I^n/\partial I^n, *) \to (X, A)$, via $g(x) := \bar{g}(*)$ for $x \in \partial I^n$, and $g(x) := \bar{g}(x)$ otherwise.
Thus we get a bijection $[(I^n, \partial I^n), (X, \{x_0\})] \quad \longleftrightarrow \quad (I^n/\partial I^n, *), (X, \{x_0\})]$.
Finally, $(I^n/\partial I^n, *)$ is homeomorphic to $(S^n, \{e_1\})$. ☐

(It is *not* true that the pairs of spaces $(S^n, \{e_{n+1}\})$ and $(I^n, \partial I^n)$ are homotopy equivalent!)

Definition 3.28 (Higher homotopy groups)**.** Let $(X, \{x_0\})$ be a space with base point. The *higher homotopy groups* of X are defined by

$$\pi_n(X; x_0) \;:=\; [(I^n, \partial I^n), (X, \{x_0\})].$$

For $n \geq 1$ a composition on this set is derived from glueing adjacent n-cubes: $[\gamma] \circ [\gamma'] := [\gamma * \gamma']$, with

$$\gamma * \gamma' \,(t_1, t_2, \ldots, t_n) := \begin{cases} \gamma(2t_1, t_2, \ldots, t_n) & \text{für } t_1 \in [0, \frac{1}{2}] \\ \gamma'(2t_1 - 1, t_2, \ldots, t_n) & \text{für } t_1 \in [\frac{1}{2}, 1]. \end{cases}$$

Lemma 3.29. *With this definition, $\pi_n(X; x_0)$ is a group for $n \geq 1$.*
The higher homotopy groups $\pi_n(X; x_0)$, $n \geq 2$, are even commutative (abelian).

For $n = 0$, there is no composition: $\pi_0(X, x_0)$ may be viewed as the pointed set of path components. Note that the new definition of the first homotopy group $\pi_1(X; x_0)$ is consistent with that of the fundamental group, as given in Definition 3.16.

[8] Kurt Reidemeister, 1893-1971, pioneer of group and knot theory, poet;
`http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Reidemeister.html`

Theorem 3.30 (Functor!)**.**

(1) *Any continuous map between topological spaces with base point* $f : (X, \{x_0\}) \to (Y, \{y_0\})$ *induces group homomorphism between the respective homotopy groups*

$$f_\# : \pi_k(X; x_0) \to \pi_k(Y; y_0).$$

(2) *Homotopic maps induce the same group homomorphism.*
(3) *The identity map* $\mathrm{id} : X \to X$ *induces the identity homomorphism on* $\pi_k(X; x_0)$.
(4) *For maps* $f : X \to Y$ *and* $g : Y \to Z$ *we get that* $(g \circ f)_\# = g_\# \circ f_\#$.
(5) *Hence homotopy equivalences induce group isomorphisms.*
(6) *Thus the homotopy groups (of spaces of base point) are invariants of the homotopy type: homotopy equivalent (pointed/path connected) spaces have isomorphic homotopy groups.*

Remark 3.31. If X is path-connected, then up to isomorphism the group $\pi_n(X; x_0)$ does not depend on the base point, and we write $\pi_n(X)$ for it.
The isomorphism is canonical if X is simply connected — otherwise one has to deal with *monodromy*: Moving the base point along a closed path induces a non-trivial (inner) automorphism of the fundamental group, and a non-trivial action of the fundamental group on the higher homotopy group, known as the *monodromy action*.

If $X = \|\Delta\|$ is a polyhedron, then the k-th homotopy group $\pi_k(X; x_0)$ depends only on the $(k+1)$-skeleton.

Homotopy groups behave well with respect to some basic constructions on topological spaces: This includes the formation of producs, but also so-called "suspensions".

Proposition 3.32 (Homotopy groups of products)**.** *Homotopy groups of products:*

$$\pi_n(X \times Y; (x_0, y_0)) \;\cong\; \pi_n(X; x_0) \times \pi_n(Y; y_0).$$

Definition 3.33 (Cone/suspension)**.** The *cone* $\mathrm{cone}(X)$ over a space X is the image of the product $X \times [0,1]$ (the *cylinder* over X) with respect to the equivalence relation $\sim$, which identifies all the points $(x, 1)$, $x \in X$.

The *suspension* $\mathrm{susp}(X)$ of a space X is the image of $X \times [-1, 1]$ with respect to the equivalence relation $\sim$, which identifies all the points $(x, 1)$, $x \in X$, and also all points $(y, -1)$, $y \in X$.

In both cases we choose the finest topology that makes the identification maps $p : X \times [0,1] \to (X \times [0,1])/\!\sim\; = \mathrm{cone}(X)$ bzw. $p : X \times [-1,1] \to (X \times [-1,1])/\!\sim\; = \mathrm{susp}(X)$ continuous, that is, the quotient topology.

For the case $X = \emptyset$ we define $\mathrm{cone}\,\emptyset := B^0$ to be a point, and $\mathrm{susp}\,\emptyset := S^0$ to consist of two points.

Lemma 3.34. *All cones* $\mathrm{cone}(X)$ *are contractible. Indeed, contraction can be written down explicitly as* $H([(x, s)], t) := (x, 1 - (1-s)(1-t))$.

A map $f : X \to Y$ *is null-homotopic (that is, homotopic to a continuous map), if and only if it can be extended to a map* $F : \mathrm{cone}(X) \to Y$.

If X *is* k*-connected, then its suspension* $\mathrm{susp}\,X$ *is* $(k+1)$*-connected.*

Obviously any suspension of a sphere is again (homeomorphic to) a sphere: $\mathrm{susp}(S^n) \cong S^{n+1}$, for $n \geq -1$. Similarly, $\mathrm{susp}\,B^n \cong B^{n+1}$.

If $f : X \to Y$ is any map, then we get a canonical suspension map $\mathrm{susp}(f) : \mathrm{susp}\,X \to \mathrm{susp}\,Y$. From this it is an elementary exercise to see that for every k there is a homomorphism

$$\sigma_k : \pi_k(X; x_0) \to \pi_{k+1}(\mathrm{susp}(X); x_0).$$

(The "obvious" homomorphism induced by inclusion $i_\# : \pi_k(X; x_0) \to \pi_k(\mathrm{susp}(X); x_0)$ is however trivial.)

Theorem 3.35 (Freudenthal's suspension lemma [20, Cor. 4.23])**.** *Let X be an $(n-1)$-connected, triangulable space with base point x_0 (for example, $X = S^n$).*
Then $\sigma_k : \pi_k(X; x_0) \to \pi_{k+1}(\mathrm{susp}(X); x_0)$ is an isomorphism for $k < 2n-1$, and surjective for $k = 2n-1$.

Freudenthal's suspension lemma is the basis for considering the so-called "stable homotopy groups", that is, instead of $\pi_n(X; x_0)$ we would look at $\pi_{n+m}(\mathrm{susp}^m X; x_0)$ for large m, that is, at the correspondingly higher homotopy group after many suspensions.

Theorem 3.36 (Homotopy groups of spheres I)**.**
$\pi_k(S^n) = \{e\}$ for $0 \le k < n$;
$\pi_n(S^n) \cong \mathbb{Z}$, where id_{S^n} is a generator;
$\pi_n(S^1) = \{e\}$ for $n > 1$;
$\pi_3(S^2) \cong \mathbb{Z}$, where the Hopf map *$S^3 \to S^2$, $(z_1, z_2) \mapsto (2z_1\bar{z}_2, z_1\bar{z}_1 - z_2\bar{z}_2)$ is a generator.*
Here the Hopf map is given as a map $\mathbb{C}^2 \supset S^3 \to S^2 \subset \mathbb{C} \times \mathbb{R}$.
One can also, for example, interpret it as a map $(z_1, z_2) \mapsto z_1/z_2$, $S^3 \to \mathbb{C} \cup \{\infty\} \cong \mathbb{C}\mathrm{P}^1$.

Theorem 3.37 (Homotopy groups of spheres II: Serre's Theorem [46, pp. 515/516])**.**
For any even $n \ge 2$, the homotopy group $\pi_{2n-1}(S^n)$ is a product of $(\mathbb{Z}, +)$ with a finite group. All other homotopy groups of spheres $\pi_k(S^n)$, $k > n$, are finite.

Proof. Uses the "spectral sequence" of a "fibration", known as the "Serre spectral sequence". □

Computing the homotopy groups of spheres is a key problem of topology, in particular of homotopy theory. It is, however, also very difficult, there is no simple "closed form" answer, and the complete picture is not available. See the last exercise on the last page of Spanier's classical monograph from the year 1966:

Exercise (Spanier [46, p. 520]). Prove that
(a): $\pi_5(S^2) \cong \mathbb{Z}_2$
(b): $\pi_6(S^3)$ is a group with 12 elements
(c): $\pi_7(S^4) \cong \pi_6(S^3) \oplus \mathbb{Z}$
(d): $\pi_{n+3}(S^n)$ for $n \ge 5$ is a group of order 24.

For a table of homotopy groups of spheres, see Hatcher [20, p. 339].

4 Homology

The homology groups of a topological space have similar "functorial" properties as the homotopy groups, but they are effectively computable (see the `topaz` module of `polymake` [17] [16]!) and in various respects much easier to handle, even if some "geometric" topologists don't want to believe that[9]. My main source for homology theory is Munkres [36, §5ff].

Here we start with a very concrete construction of the "simplicial" homology for (finite) simplicial complexes. This is a construction that is completely elementary and explicit. In contrast to many other things that after "Given a simplicial complex" can be "defined" or "constructed", the remarkable fact here is that the result is a *topological invariant*: Different triangulations of the same space usually have different numbers of vertices, edges, etc. — but, as we will see, they have the same homology!

To describe a target meaning: "$H_k(X) \cong \mathbb{Z}^r$" should mean that X has "r k-dimensional holes".

Definition 4.1 (Orientation)**.** Let $\{v_0, \dots, v_k\}$ be the vertex set of a k-simplex σ. Two linear orderings of the vertex set are called equivalent, if they differ by an even permutation. The equivalence classes are called the *orientations* of σ. (Thus for $k > 0$ every k-simplex has exactly two orientations.) We write $[v_0, \dots, v_k]$ for the orientation that is given by $v_0 < \cdots < v_k$, and $-[v_0, \dots, v_k]$ for the other orientation.

If the vertex set of Δ is ordered linearly, then this automatically yields an orientation for each of its simplices.

For the following we fix an abelian group G as our *coefficient group*: We will construct the "homology of a simplicial complex with coefficients in G." Here we are primarily thinking of the additive group $G = \mathbb{Z}$; however, other important cases are $G = \mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{Z}_2$, always interpreted as a group with respect to addition. In the case that G is the additive group of a field, the chain groups to be defined now will turn out to be vector spaces, and the group homomorphisms will be linear (vector space) maps, which helps for calculations. If nothing different is specified, we will always assume that $G = \mathbb{Z}$ is chosen. In particular, in all cases where the notation does not include information about the coefficient group, we are referring to integer coefficients.

Definition 4.2 (Chains)**.** Let Δ be a simplicial complex. A *k-chain* in Δ with coefficients in G is a formal linear combination

$$\sum_{\sigma \in \Delta^{(k)}} c_\sigma \sigma$$

of oriented k-simplexes in Δ, with coefficients $c_\sigma \in G$, where only finitely-many coeffients are allowed to be non-zero, and every simplex appears only in one of the two possible orientations.

The set of all k-chains in Δ with coefficients in G is the *k-th chain group* $C_k(\Delta; G)$ of Δ. Two k-chains are added by adding the coefficients in front of the same oriented simplicies (with the same orientations), and set $c_\sigma \sigma = (-c_\sigma)\sigma'$ whenever σ' is the other orientation of σ.

For $k < 0$ and for $k > \dim \Delta$ we set $C_k(\Delta; G) := 0$, where here and in the following "0" is used as shorthand for the "trivial group" $(\{0\}, +)$ with exactly one element.

$C_k(\Delta; \mathbb{Z})$ is a free abelian group of rank f_k, where $f_k = f_k(\Delta)$ denotes the number of k-faces of Δ: Choosing one orientation for each k-simplex also determines a basis.

[9] See e.g. Stillwell [49, p. 171]): "as history shows, homology theory is loaded with subtleties, and an inordinate amount of preparation is required for correct definitions and the desired theorems."

Definition 4.3 (Boundary map)**.** The k-th *boundary map* is the group homomormphism $\partial_k : C_k(\Delta; G) \to C_{k-1}(\Delta; G)$ that is defined as follows by prescribing its values on a basis:

$$\partial_k : \ [v_0, \dots, v_k] \ \mapsto \ \sum_{i=0}^{k} (-1)^i [v_0, \dots, \widehat{v_i}, \dots, v_k]$$

Thus the ("algebraic") boundary of a vertex is zero, $\partial_0[v_0] = 0$. The boundary of an edge is "end vertex minus beginning vertex", that is, "head minus tail" for an oriented/directed edge, $\partial_1[v_0, v_1] = [v_1] - [v_0]$. The boundary of a triangle is the sum of its three directed boundary edges, $\partial_2[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$, etc.

Exercise. Check that the boundary map is well-defined (that is, that after an even permutation of the vertex order the definition assigns the same boundary), and that $\partial_k \sigma' = -\partial_k \sigma$, if σ, σ' denote the two orientations of a k-simplex Δ.

Definition 4.4 (Cycles)**.** The *k-th cycle group* of Δ (with coefficients in G) is the group

$$Z_k(\Delta; G) \ := \ \ker(\partial_k) \ = \ \{c \in C_k(\Delta; G) : \partial_k c = 0\}.$$

The cycle group $Z_k(\Delta; \mathbb{Z})$ is thus a subgroup of a free abelian group, so it is free itself [36, Lemma 11.2]. In the finitely-generated case the rank of the cycle group is at most the rank of the chain group. (Attention: this refers to the case $G = \mathbb{Z}$ of integer coefficients. One may deal with it analogously in the case when G is the additive group of a field – then we are dealing with vector spaces, and the cycle group is indeed a vector subspace. For more general G, say $G = \mathbb{Z}_4$, one may still refer to $C(\Delta; G) \cong G^{f_k}$ as "free", and use a basis (of cardinality f_k), but then a subgroup of G^{f_k} need not be of the form G^r anymore.

Examples. How do cycles "look like"?
Intuition: Think of the image of a sphere into the space in question, and try to capture its essence abstractly . . . (compare Kreck [26])

Definition 4.5 (Boundaries)**.** The *k-th boundary group* of Δ (with coefficients in G) is the group

$$B_k(\Delta; G) \ := \ \operatorname{im}(\partial_{k+1}) \ = \ \{\partial_{k+1} d : d \in C_{k+1}(\Delta; G)\}.$$

The group of boundaries $B_k(\Delta; Z)$ is a subgroup of $C_k(\Delta; \mathbb{Z})$, so again it its a free abelian group.

Lemma 4.6 ($\boldsymbol{\partial^2 = 0}$)**.** *The following relation holds:*
Boundaries of boundaries are zero;
that is, all boundaries are cycles;
that is, $B_k(\Delta; G) \subseteq Z_k(\Delta; G)$*;*
that is, $\partial_k \circ \partial_{k+1} = 0 \qquad$ *for all* k*.*

Proof. It suffices to calculate this on a set of basis elements of $C_k(\Delta; G)$:

$$\begin{aligned}
\partial_{k-1} \circ \partial_k [v_0, \dots, v_k] \ &= \ \partial_{k-1} \sum_{i=0}^{k} (-1)^i [v_0, \dots, \widehat{v_i}, \dots, v_k] \ = \ \sum_{i=0}^{k} (-1)^i \partial_{k-1} [v_0, \dots, \widehat{v_i}, \dots, v_k] \\
&= \ \sum_{i=0}^{k} (-1)^i \sum_{j=0}^{i-1} (-1)^j [v_0, \dots, \widehat{v_j}, \dots, \widehat{v_i}, \dots, v_k] \ + \ \sum_{i=0}^{k} (-1)^i \sum_{j=i+1}^{k} (-1)^{j-1} [v_0, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots, v_k] \\
&= \ \sum_{0 \le j < i \le k} (-1)^{i+j} [v_0, \dots, \widehat{v_j}, \dots, \widehat{v_i}, \dots, v_k] \ + \ \sum_{0 \le i < j \le k} (-1)^{i+j-1} [v_0, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots, v_k] \ = \ 0.
\end{aligned}$$

□

Definition 4.7 (Homology)**.** Let Δ be a simplicial complex. The *k-th homology group of Δ with coefficients in G* is

$$H_k(\Delta; G) \; := \; Z_k(\Delta; G) \, / \, B_k(\Delta; G).$$

The "integer" homology groups $H_k(\Delta; \mathbb{Z})$ are quotient groups of the cycle groups — they are not free abelian in general: We get a result of the form

$$H \; = \; \mathbb{Z}^{\beta} \; \oplus \; \mathbb{Z}_{t_1} \oplus \mathbb{Z}_{t_2} \oplus \cdots \oplus \mathbb{Z}_{t_r}$$

with $t_i \geq 2$ and $t_1 \mid t_2 \mid \cdots \mid t_r$. Here $\operatorname{rank} H := \beta$ is the *rank* of the group H: it is the maximal number of elements for which all linear combinations with $\mathbb{Z}$-coefficients are distinct. The group $T(H) := \mathbb{Z}_{t_1} \oplus \mathbb{Z}_{t_2} \oplus \cdots \oplus \mathbb{Z}_{t_r}$ is the *torsion subgroup* of all elements of finite order in H. We have $H/T(H) \cong \mathbb{Z}^{\beta}$ — this is the structure theorem for finitely-generated abelian groups [36, Thm. 4.3]).

Thus for the k-th homology group we get a decomposition

$$H_k(\Delta; \mathbb{Z}) \; = \; \mathbb{Z}^{\beta_k} \; \oplus \; \mathbb{Z}_{t_1} \oplus \mathbb{Z}_{t_2} \oplus \cdots \oplus \mathbb{Z}_{t_r}.$$

Here $\beta_k = \operatorname{rank} H_k(\Delta; \mathbb{Z})$ is called the k-th *Betti number* of Δ. The numbers t_i, with $t_1 \mid t_2 \mid \cdots \mid t_r$ (as above) are called the *torsion coefficients* of the k-th homology group of Δ.

If G is the additive group of a coefficient *field*, then we are working with vector spaces, and will get a quotient vector space.

Lemma 4.8. *$H_k(\Delta; G) = \{0\}$ for $k > \dim \Delta$ and for $k < 0$.*
$H_k(\Delta; \mathbb{Z})$ is a free abelian group for $k = \dim \Delta$.

Proposition 4.9. *The 0-th homology goup is free, $H_0(\Delta; G) \cong G^{\beta_0}$.*
The 0-th betti number is the number of connected components of Δ.

Proof. $Z_0(\Delta; G) = C_0(\Delta; G)$ is free, with basis $\{[v] : v \in \Delta^{(0)}\}$.

$B_0(\Delta; G)$ may be identified as the subgroup of all chains that have coefficient sum 0 on all connected components.

Thus if you choose a vertex v_0 in each component of Δ, then the corresponding equivalence classes $[v_0]$ form a basis for $Z_0(\Delta; G)/B_0(\Delta; G) = H_0(\Delta; G)$.

Equivalently, we might map each cycle to the vector

$$(\text{sum of coefficients on the } i\text{-th component} : 1 \leq i \leq \beta_0\},$$

which is surjective with kernel $B_0(\Delta; G)$. □

Remark 4.10. Homology of finite simplicial complexes is efficiently computable: using elementary (integrally-invertable) row and column operations every integer matrix can be can be brought into *Smith normal form (SNF)*. This can be done fast both in theory (i.e., in polynomial time) and also in practice (as implemented e.g. in `topaz`).

In case of field coefficients it suffices to compute ranks:

$$\begin{aligned} \dim H_k(\Delta_k; F) &= \dim Z_k(\Delta_k; F)/B_k(\Delta_k; F) \\ &= \dim Z_k(\Delta_k; F) - \dim B_k(\Delta_k; F) \\ &= \dim \ker \partial_k - \dim \operatorname{im} \partial_{k+1} \\ &= f_k - \operatorname{rank} \partial_k - \operatorname{rank} \partial_{k+1}. \end{aligned}$$

Corollary 4.11. *For $G = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$ the k-dimensional homology has the same rank β_k:*

$$\operatorname{rank} H_k(\Delta_k;\mathbb{Z}) \;=\; \dim_{\mathbb{Q}} H_k(\Delta_k;\mathbb{Q}) \;=\; \dim_{\mathbb{R}} H_k(\Delta_k;\mathbb{R}) \;=\; f_k - \operatorname{rank}\partial_k - \operatorname{rank}\partial_{k+1}.$$

More generally: The "universal coefficient theorems" of homology theory provide prescriptions for how to compute $H_k(\Delta_k; G)$ when the groups $H_k(\Delta;\mathbb{Z})$ and $H_{k-1}(\Delta;\mathbb{Z})$ are given [36, Thm. 55.1].

Definition 4.12 (Reduced homology)**.** To construct the *reduced homology groups* we define the boundary of a vertex not as zero, but to be $\partial[v_0] := []$, corresponding to the empty set.
This yields $\widetilde{C}_{-1}(\Delta; G) \cong G$. The *augmentation homomorphism* $\varepsilon = \partial_0 : \widetilde{C}_0(\Delta; G) \to \widetilde{C}_{-1}(\Delta; G)$ is surjektive.
Thus we always have $H_0(\Delta; G) \cong \widetilde{H}_0(\Delta; G) \oplus G$, and $\widetilde{H}_k(\Delta; G) = H_k(\Delta; G)$ for $k \neq 0$.

Lemma 4.13 (Homology of a cone)**.** *Every cone has zero reduced homology:*

$$\widetilde{H}_k(\operatorname{cone}(\Delta);\mathbb{Z}) \;=\; 0 \qquad \textit{für alle } k.$$

Lemma 4.14 (Homology of a simplex boundary)**.** *The boundary of an n-simplex has the following homology:*

$$\widetilde{H}_k(\partial\sigma_n; G) \;=\; \begin{cases} G & \textit{for } k = n-1, \\ 0 & \textit{otherwise.} \end{cases}$$

Thus we have also computed the homology of contractible spaces, and of the $(n-1)$-sphere, modulo a proof of topological invariance (see below).
It is common to write $H_*(\Delta; G)$ for the sequence of homology groups of Δk, that is

$$H_*(\Delta; G) \;:=\; (H_0(\Delta; G), H_1(\Delta; G), \ldots, H_d(\Delta; G))$$

for $d = \dim \Delta$.

Examples (Homology of some surfaces; cf. [36, §6]).
2-sphere: $H_*(S^2;\mathbb{Z}) = (\mathbb{Z}, 0, \mathbb{Z})$.
torus: $H_*(T;\mathbb{Z}) = (\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z})$.
sphere with g handles: $H_*(M_g;\mathbb{Z}) = (\mathbb{Z}, \mathbb{Z}^{2g}, \mathbb{Z})$.
projective plane: $H_*(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) = (\mathbb{Z}, \mathbb{Z}_2, 0)$.
connected sum of two projective planes: $H_*(\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2;\mathbb{Z}) = (\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2, 0)$.
Klein bottle: $H_*(K;\mathbb{Z}) = (\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2, 0)$.
projective plane with a handle: $H_*(\mathbb{R}\mathrm{P}^2 \# T;\mathbb{Z}) = (\mathbb{Z}, \mathbb{Z}^2 \oplus \mathbb{Z}_2, 0)$.

Lemma 4.15 (Orientability of manifolds)**.** *Let M be a connected, closed, triangulated d-dimensional manifold. Then either $H_d(M;\mathbb{Z}) \cong \mathbb{Z}$, in which case we call M* orientable*; or we have $H_d(M;\mathbb{Z}) \cong 0$, in which case M is* not orientable*. In both cases $H_d(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$.*
(Orientability does not depend on the triangulation.)

Example. The chessboard complexes $\Delta_{m,n}$ have been studied intensively in the last few years. Their homology contains 3-torsion: Only recently Shareshian und Wachs [45] managed to prove that they contain *only* 3-torsion (and possibly 9-torsion). See also [50].

Theorem 4.16 (Topological invariance; functor!)**.** *The homology groups are invariants of the homotopy type: If Δ, Δ' are homotopy equivalent, then $H_k(\Delta; G) \cong H_k(\Delta'; G)$ for all k.*
Further every continuous map $h : \|\Delta\| \to \|\Gamma\|$ induces group homomorphisms $h_* : H_k(\Delta) \to H_k(\Gamma)$, which satisfy $(k \circ h)_* = k_* \circ h_*$. The identity map $\mathrm{id} : \|\Delta\| \to \|\Delta\|$ *induces the identity* $\mathrm{id}_* : H_k(\Delta) \to H_k(\Delta)$ *in homology, for all k. Homotopic maps induce the same map in homology.*

Proof overview: (1) Simplicial maps induce homomorphisms $f_\#$ of chain complexes, and thus maps f_* in homology.
(2) Chain-homotopic maps yield the same homomorphism in homology: $f_\# - g_\# = \partial D + D\partial$ implies $f_* = g_*$.
(3) Subdivisions, and simplicial approximation
(4) The subdivision operator
(5) Homotopic maps are chain-homotopic after a suitable subdivision. □

Definition 4.17 (Chain complex, chain map)**.** A *chain complex* C_* is a sequence $(C_k)_{k\in\mathbb{Z}}$ of abelian groups, with homomorphisms $\partial_k : C_k \to C_{k-1}$ that satisfy $\partial_{k-1}\partial_k = 0$.

The *homology of a chain complex* with coefficients in the abelian group G is given by

$$H_k(C_*; G) \ := \ (\ker \partial_k)/(\operatorname{im} \partial_{k+1}).$$

A *chain map* $f_\# : C_* \to C'_*$ is a family of group homomorphisms $f_{\#,k} : C_k \to C'_k$ that satisfy $\partial'_k f_{\#,k} = f_{\#,k-1}\partial_k : C_k \to C'_{k-1}$.

Our primary example is the chain complex $C_*(K; G) = (C_k(K; G))_{k\in\mathbb{Z}}$ of a simplicial complex K. We will check now that every simplicial map induces a chain map, and this in turn induces a homomorphism in homology.

Lemma 4.18 (Simplicial maps induce chain maps)**.**
Let $f : K \to L$ be a simplicial map, then

$$f_{\#,k} : [v_0, \ldots, v_k] \ \mapsto \ \begin{cases} [f(v_0), \ldots, f(v_k)] & \textit{if the } f(v_i) \textit{ are pairwise distinct,} \\ 0 & \textit{othewise} \end{cases}$$

induces a chain map $f_\# : C_*(K) \to C_*(L)$.

Proof. Case distinction: $\partial f_\# = f_\#\partial$ applied to $[v_0, \ldots, v_k]$ yields

$$\sum_{i=0}^{k} (-1)^i [f(v_0), \ldots, \widehat{f(v_i)}, \ldots, f(v_k)]$$

if all $f(v_i)$ are distinct, and it yields 0 otherwise. □

Lemma 4.19 (Chain maps induce homomorphisms in homology [36, Thm. 12.2])**.** *For any chain map $f_\# : C_* \to C'_*$ setting $f_*[c] := [f_\# c]$ yields homomorphisms $f_k : H_k(C_*) \to H_k(C'_*)$.*
Here the identity chain map $\mathrm{id} : C_* \to C_*$ *induces the identity* $\mathrm{id}_* : H_*(C_*) \to H_*(C_*)$ *in homology, and we have $(f \circ g)_* = f_* \circ g_*$ in general.*

Proof. The main part to check is that f_* is well-defined, so the result does not depend on the representative that we have picked from the homology (equivalence) class $[c]$. For this let $c + \partial c'$ be a different representative of $[c] = [c+\partial c']$; for this we get $f_*[c+\partial c'] = [f_\# c + f_\# \partial c'] = [f_\# c + \partial f_\# c'] = [f_\# c]$. □

Definition 4.20 (Chain homotopies)**.** A *chain homotopy* between chain maps $f_\#, g_\# : C_* \to C'_*$ is a family of index-raising homomorphisms $D_k : C_k \to C'_{k+1}$ that satisfy $f_{\#,k} - g_{\#,k} = \partial_{k+1} D_k + D_{k-1}\partial_k : C_k \to C'_k$, for which we write $f_\# - g_\# = \partial D + D\partial$ for short.

Lemma 4.21 ([36, Thm. 14.2])**.** *Chain homotopic maps $f_\#, g_\# : C_* \to C'_*$ induce the same map $f_* = g_* : H_k(C_*) \to H_k(C'_*)$ in homology.*

Proof. Computation: Let $[c]$ be a homology class (so c is a cycle, $\partial c = 0$), then

$$f_*[c] - g_*[c] = [f_\# c] - [g_\# c] = [(f_\# - g_\#)c] = [(D\partial + \partial D)c)] = [\partial(Dc)] + [D(\partial c)] = 0.$$

Here the first equality holds by Definition 4.19 of the homomorphism in homology, the second and fourth one since we are dealing with group homomorphisms, the third by chain-homotopy, and the last one since the homology class of a boundary is zero (first term) and since c is a cycle (second term). □

Definition 4.22 (Subdivision)**.** A *subdivision* of a (geometric) simplicial complex Δ is a complex Δ' for which every simplex $\sigma' \in \Delta'$ is contained in a simplex $\sigma \in \Delta$, and such that also every simplex $\sigma \in \Delta$ is a union of finitely many simplices $\sigma' \in \Delta'$.

If Δ' is a subdivision of Δ, then the two complexes have the same support $\bigcup \Delta = \bigcup \Delta'$, and the same polyhedron $\|\Delta\| = \|\Delta'\|$, so the topological spaces (polyhedra) defined by Δ and Δ' are the same.

Definition 4.23 (Link, open star, closed star)**.** Let σ be a nonempty face in a (geometric) simplicial complex Δ.

- The *(closed) star* of σ is the subcomplex $\mathrm{Star}_\Delta\,\sigma$ of all faces $\sigma' \in \Delta$ such that there is some $\tau \in \Delta$ with $\sigma \cup \sigma' \subseteq \tau$.
- The *deletion* of σ is the subcomplex $\mathrm{del}_\Delta\,\sigma$ of all faces $\sigma' \in \Delta$ with $\sigma' \not\supseteq \sigma$.
- The *link* of σ is the subcomplex $\mathrm{link}_\Delta\,\sigma$ of all faces $\sigma' \in \Delta$ that satisfy $\sigma' \cap \sigma = \emptyset$ and $\sigma \cup \sigma' \subseteq \tau$ for some $\tau \in \Delta$.

Thus we have

$$\mathrm{Star}_\Delta\,\sigma \cap \mathrm{del}_\Delta\,\sigma \;=\; \mathrm{link}_\Delta\,\sigma * \partial\sigma.$$

In particular, for vertices $\mathrm{link}_\Delta\,v \;= \mathrm{Star}_\Delta\,v \,\cap\, \mathrm{del}_\Delta\,v$.

In a geometric simplicial complex Δ the links $\mathrm{link}_\Delta\,\sigma$ and the stars $\mathrm{Star}_\Delta\,\sigma$ are subcomplexes, so they represent closed subsets. One also considers the *open star*

$$\mathrm{star}_\Delta\,\sigma \;:=\; \|\Delta\| \setminus \|\,\mathrm{del}_\Delta\,\sigma\| \;=\; \|\,\mathrm{Star}_\Delta\,\sigma\| \setminus \|\,\mathrm{link}_\Delta\,\sigma * \partial\sigma\|.$$

This is an open subset of $\|\Delta\|$. It can also be described as all the points that lie in the relative interior of a simplex $\tau \in \Delta$ that contains σ.

The open stars of the vertices form an open cover of the polyhedron $\|\Delta\|$.

Examples. The *stellar subdivision* of a complex Δ with respect to a non-empty face σ is obtained as follows. Delete σ and all faces that contain σ, and then add a new simplex $\mathrm{conv}(\sigma' \cup v_\sigma)$ for each face

$$\sigma' \;\in\; \mathrm{Star}_\Delta\,\sigma \cap \mathrm{del}_\Delta\,\sigma \;=\; \mathrm{link}_\Delta\,\sigma * \partial\sigma,$$

where v_σ is the barycenter of σ. Topological description: the open star of σ is removed, and instead we "glue in" a cone over $\mathrm{Star}_\Delta\,\sigma \cap \mathrm{del}_\Delta\,\sigma = \mathrm{link}_\Delta\,\sigma * \partial\sigma$.

The *barycentric subdivision* $\mathrm{sd}\,\Delta$ has as its vertex set the set of all barycenters of non-empty faces of Δ, while the simplices of $\mathrm{sd}\,\Delta$ correspond to the chains of faces of Δ (with respect to inclusion).

For finite simplicial complexes Δ the barycentric subdivision $\mathrm{sd}\,\Delta$ can also be constructed as a sequence of stellar subdivisions: Subdivide all nonempty faces in any order such that for faces $\sigma \subset \sigma'$ the face σ' is subdivided before the face σ.

Definition 4.24 (star condition)**.** A continuous map of polyhedra $h : \|\Delta\| \to \|\Gamma\|$ satisfies the *star condition* if every open star of a vertex of a vertex is mapped into the open star of a vertex, that is, if for every vertex $v \in \|\Delta\|$ there is a vertex $w \in \Gamma$ (not unique in general) such that

$$h(\mathrm{star}_\Delta\,v) \;\subseteq\; \mathrm{star}_\Gamma\,w.$$

Lemma 4.25. *Let $h : \|\Delta\| \to \|\Gamma\|$ satisfy the star condition. For every vertex $v \in \Delta^{(0)}$ choose an arbitrary vertex $f(v) \in \Gamma$ with $h(\mathrm{star}_\Delta v) \subseteq \mathrm{star}_\Gamma f(v)$. This defines a simplicial map $f : \Delta \to \Gamma$ such that f and h are homotopic.*

If f, g are two such maps, then the corresponding chain maps $f_\#, g_\#$ are chain-homotopic.

Definition 4.26 (Simplicial approximation)**.** A *simplicial approximation* of $h : \|\Delta\| \to \|\Gamma\|$ is a simplicial map $f : \Delta \to \Gamma$ that satisfies the star condition $h(\mathrm{star}_\Delta v) \subseteq \mathrm{star}_\Gamma f(v)$ for all vertices $v \in \Delta$.

Theorem 4.27 (Simplicial approximation [36, §16])**.** *If $h : \|\Delta\| \to \|\Gamma\|$ is continuous, then there is a simplicial approximation $f : \Delta' \to \Gamma$, for a subdivision Δ' of Δ.*

If Δ is finite, then there is even a simplicial approximation $f : \mathrm{sd}^N \Delta \to \Gamma$ based on an N-fold barycentric subdivision, for some $N \geq 0$.

Proof. In the second (finite, compact) case one repeats barycentric subdivisions until the star condition is satisfied. This can be achieved since in every barycentric subdivision the largest diameter of a simplex is reduced by a constant factor, and thus after finitely-many steps gets smaller than the Lebesgue number of the covering of $\|\Delta\|$ by the pre-images $h^{-1}(\mathrm{star}_\Gamma v)$ of the open stars in Γ: The *Lebesgue number* of an open covering is the largest number λ such that every λ-neighborhood of a point is contained in an open set of the covering. This number is positive for every open covering of a compact (!) metrizable space. □

Corollary 4.28. *Any continuous map $h : \|\Delta\| \to \|\Gamma\|$ with $\dim \Delta < \dim \Gamma$ is homotopic to a map that is not surjective.*

Corollary 4.29. *For $m < n$ all continuous maps $h : S^m \to S^n$ are nullhomotopic.*

Lemma 4.30 (Subdivision operator [36, Thm. 17.2])**.** *If Δ' is a subdivision of Δ, then there is a unique chain map $\lambda : C_*(\Delta) \to C_*(\Delta')$ with $|\lambda(\sigma)| \subset |\sigma|$ for all $\sigma \in \Delta$, that maps vertices to vertices, $\lambda : [v] \to [v]$.*

Furthermore, there is a simplicial approximation $g : \Delta' \to \Delta$ of the identity. The chain maps $g_\#$ and λ are inverses up to chain-homotopy. In particular, $g_* : H_*(\Delta') \to H_*(\Delta)$ and $\lambda_* : H_*(\Delta) \to H_*(\Delta')$ are inverse isomorphisms between the homology of Δ and of Δ'.

Note: The subdivision operator λ is not a simplicial map! The inverse simplicial map g is not unique and there is no canonical choice.

Definition 4.31 (Construction of h_*)**.** Let $h : \|\Delta\| \to \|\Gamma\|$ be continuous, let $f : \Delta' \to \Gamma$ be a simplicial approximation, and let $\lambda : C_*(\Delta) \to C_*(\Delta')$ be the corresponding subdivision operator.

Then $h_* := f_* \circ \lambda_* : H_k(\Delta) \to H_k(\Gamma)$ is the *homomorphism in simplicial homology induced by h*, for $k \geq 0$.

Lemma 4.32 ([36, Thm. 18.1])**.** *The maps $h_* : H_k(\Delta) \to H_k(\Gamma)$ are well-defined, that is, for any other choice of a subdivision Δ' and a simplicial approximation f one obtains* the same *group homomorphisms h_*.*

This finally implies the homotopy invariance of homology, which we had already claimed in Theorem 4.16:

Theorem 4.33 (Homotopy invariance of homology [36, Thms. 19.2, 19.5])**.** *Homotopic maps $h, \ell : \|\Delta\| \to \|\Gamma\|$ induce chain-homotopic maps chain maps $h_\#, \ell_\# : C_k(\Delta') \to C_k(\Gamma')$ and thus the same group homomorphisms $h_* = \ell_* : H_k(\Delta) \to H_k(\Gamma)$ in homology.*

Any homotopy equivalence $h : \|\Delta\| \to \|\Gamma\|$ induces isomorphisms $h_* : H_k(\Delta) \to H_k(\Gamma)$ in homology.

Corollary 4.34 (Invariance of dimension)**.** *The n-spheres are not homeomorphic for different n, and not homotopy equivalent either.*
The real vector spaces $\mathbb{R}^n$ are not homeomorphic for different n.

5 Euler- and Lefschetz Numbers

5.1 Mapping degree

As an application of homology theory we get a whole bunch of invariants: The most elementary ones are the Euler characteristic of a space, and the degree of a map of spheres, and the strongest perhaps the Lefschetz number.
This is based on strong connection between whatever happens on simplices (and thus on chains), the corresponding maps in homology, and the Hopf "trace formula".
My presentation of this is based on Munkres [36, §§21, 22].

Definition 5.1 (Mapping degree)**.** Every continuous map $h : S^n \to S^n$ induces a group homomorphism $h_* : \widetilde{H}_n(S^n;\mathbb{Z}) \to \widetilde{H}_n(S^n;\mathbb{Z})$, which maps every element of $H_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$ to its d times itself. The integer $\deg h := d \in \mathbb{Z}$ defined by this is the *degree* (or *mapping degree*) of h.

Note: the group $H_n(S^n;\mathbb{Z})$ is isomorphic to $\mathbb{Z}$, but the isomorphism is not unique: For this we have to choose a fundamental cycle $c_n \in Z_n(S^n;\mathbb{Z})$, which is a sum of all n-simplices in which the simplices get a consistent orientation — that is, an *orientation* of the sphere. If c_n is such a fundamental cycle, then $-c_n$ is the other one. The generator for $H_n(S^n;\mathbb{Z})$ is then $[c_n]$, and the isomorphism $H_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$ identifies this with $1 \in \mathbb{Z}$.

Lemma 5.2. *For continuous maps $g, h : S^n \to S^n$ we have:*
(1) *Homotopic maps $g \sim h$ have the same degree* $\deg g = \deg h$.
(2) *The identity has degree* 1*, that is,* $\deg \mathrm{id} = 1$.
(3) *Composition leads to multiplication of the mapping degrees,* $\deg(g \circ h) = \deg g \cdot \deg h$.
(4) *If h has an extension to $\widehat{h} : B^{n+1} \to S^n$, then* $\deg h = 0$.
(5) *The reflection r in a hyperplane, $(x_0, x_1, \dots, x_n) \mapsto (-x_0, x_1, \dots, x_n)$, has degree* $\deg = -1$.
(6) *The antipodal map $a : x \mapsto -x$ has degree* $\deg a = (-1)^{n+1}$.

We will not prove here that the converse of part (1) is also true: Two maps g, h have the same degree $\deg g = \deg h$ *if and only if* they are homotopic. This may be explained via the connection between the n-th homotopy group $\Pi_n(X; x_0)$ and the n-th homology group $H_n(X;\mathbb{Z})$ — the *Hurewicz homomorphism* is an isomorphism for the n-sphere.

Proof. (1), (2) and (3) follow from the general properties of the homomorphism in homology induced by continuous maps. For (4) consider $h = \widehat{h} \circ i : S^n \to B^{n+1} \to S^n$ and the corresponding homomorphism $h_* = \widehat{h}_* \circ i_* : H_n(S^n) \to H_n(B^{n+1}) = \{0\} \to H_n(S^n)$. For (5) we may assume that S^n carries the "octahedral" triangulation, which is induced by the $n + 1$ coordinate hyperplanes in $\mathbb{R}^{n+1}$. Then the fundamental cycle $c \in C_n(S^n;\mathbb{Z})$ is clearly mapped to $r_\# c = -c$. This also implies (6), since a may be obtained by composition of $n + 1$ reflections. □

Corollary 5.3. *There is no* retraction $B^{n+1} \to S^n$*, that is, no continuous map $B^{n+1} \to S^n$ that would fix all the points $x \in S^n$.*

This corollary is equivalent to the fact that S^n is not contractible (Corollary 3.11).

Proof. The retraction would yield an extension of the identity map $\mathrm{id} : S^n \to S^n$, which by part (4) of Lemma 5.2 then must have degree 0, while it should have degree 1 by part (2). □

This implies, as we have seen, the Brouwer fixed point theorem 3.12. However, we can derive further fixed point theorems using the mapping degree.

Corollary 5.4. *Every map $h : S^n \to S^n$ with $\deg h \neq (-1)^{n+1}$ has a fixed point.*

Proof. If h has not fixed point, then we can easily construct a homotopy to the antipodal map a. □

Corollary 5.5. *Every map $h : S^n \to S^n$ with $\deg h \neq 1$ maps some point $x \in S^n$ to its antipode $-x$.*

Proof. Consider $a \circ h$. □

Corollary 5.6 (The "hairy ball theorem"/"Der Satz vom Igel")**.** *The sphere S^n has a tangential vector field without a zero if and only if n is odd.*
Equivalently: A continuous map $v : S^n \to S^n$ with $\langle x, v(x)\rangle = 0$ for all $x \in S^n$ exists if and only if n is odd.

Proof. For even $n + 1$ we can construct such a vector field by

$$v(x_0, x_1, \dots, x_{n-1}, x_n) \ := \ (x_1, -x_0, \dots, x_n, -x_{n-1}).$$

Any vector field without a zero can be normalized to satisfy $\|v(x)\| = 1$. Thus it defines a map $v : S^n \to S^n$ that has neither fixed points nor antipodal points, so according to the last two corollaries it has the degrees $\deg v = 1$ and $\deg v = (-1)^{n+1}$. □

A much deeper question asks which n-spheres are *parallelizable*, that is, would admit n independent vector fields that are linearly independent (and thus non-zero). The answer (exactly for $n \in \{0, 1, 3, 7\}$!) is closely tied to the existence of division algebras; the vector fields may be obtained from the complex numbers for $n = 1$ ($v(z) := iz$ for $z \in S^1 \subset \mathbb{R}^2 \cong \mathbb{C}$), for $n = 3$ from the multiplication with the units i, j, k of Hamilton's non-commutative algebra of quaternions $\mathbb{H}$, for $n = 7$ from Cayley's non-commutative and non-associative 8-dimensional algebra of octonians $\mathbb{O}$, with $S^7 \subset \mathbb{R}^8 \cong \mathbb{O}$. The non-existence of real division algebras for all dimensions different from $1, 2, 4, 8$ is indeed proved with methods of algebraic topology — no other proof is known!

An even deeper question asks for the maximal number of linearly independent vector fields on the n-sphere. It was answered in 1962 by Adams using the (now) so-called "Adams spectral sequence" and lots of "K-Theory" (a generalized homotopy theory).

For an excellent and exciting discussion of these questions I can refer you to Hirzebruch's chapter in the "Numb3rs" volume by Ebbinghaus et al. [11, Chap. 11].

5.2 Euler characteristics

Let K be a finite simplicial complex. From the definition $H_k(K; G) := \ker \partial_k / \operatorname{im} \partial_{k+1}$ we had already derived that

$$\operatorname{rank} H_k(K; G) \ = \ \big(\operatorname{rank} C_k(K; G) \ - \ \operatorname{rank} \partial_k\big) \ - \ \operatorname{rank} \partial_{k+1}, \qquad (1)$$

and indeed this holds both in the case $G = \mathbb{Z}$, where "rank" refers to the rank of the abelian group, and in the case when G is the additive group of a field: In this case "rank" is the dimension of a finite-dimensional vector space.

The equation (1) can tell us a lot of interesting things:

- $\operatorname{rank} C_k(K;G) = f_k$ is the number of k-dimensional faces of K. This number does not depend on the coefficient group G.
  The parameters f_k of an at most n-dimensional complex K may be collected in the so-called *f-vector*

$$f_*(K) \;:=\; (f_0, f_1, f_2, \dots, f_n).$$

- $\operatorname{rank} H_k(K;G) = \beta_k$ is the k-th Betti number of K. This number in general does indeed depend on the group of coefficients; thus it would be advisable to write $\beta_k(K;G)$ instead.
  The *Betti vector* of the complex K is

$$\beta_*(K;G) \;:=\; (\beta_0, \beta_1, \beta_2, \dots, \beta_n).$$

- It seems natural to pack the formula (1) into alternating sums, so that the ranks of the boundary operators cancel. This leads us directly to the Euler–Poincaré equation, which we state next.

Example. If $\mathbb{RP}^2_6$ denotes the minimal triangulation of the projective plane, on 6 vertices (which is obtained by identification of opposite vertices of the icosahedron.), then $f(\mathbb{RP}^2_6) = (6, 15, 10)$, $\beta(\mathbb{RP}^2_6, \mathbb{Z}) = (1, 0, 0)$, and $\beta(\mathbb{RP}^2_6, \mathbb{Z}_2) = (1, 1, 1)$, while with integral coefficients we have $\beta(\mathbb{RP}^2_6, \mathbb{Z}) = (1, 0, 0)$.

Theorem 5.7 (Euler–Poincaré formula)**.** *For every finite simplicial complex of dimension at most n,*

$$f_0 - f_1 + f_2 - f_3 \pm \cdots + (-1)^n f_n \;=\; \beta_0 - \beta_1 + \beta_2 - \beta_3 \pm \cdots + (-1)^n \beta_n. \tag{2}$$

Of course, we also get for reduced homology the equation

$$-1 + f_0 - f_1 + f_2 - f_3 \pm \cdots + (-1)^n f_n \;=\; \widetilde{\beta}_0 - \beta_1 + \beta_2 - \beta_3 \pm \cdots + (-1)^n \beta_n.$$

Observe: The summands on the left-hand side are independent of G; this is not true for the right-hand side. The summands on the right do not depend on the triangulation chosen for $\|K\|$, they are *topological invariants*, which in turn is not true for the components on the left side of the equation.

Definition 5.8 (Euler characteristic of a space)**.** The *Euler characteristic* of a triangulable topological space with homology groups of finite rank is the alternating sum

$$\chi(X) \;:=\; \beta_0 - \beta_1 + \beta_2 - \beta_3 \pm \ldots \;=\; \sum_{i \ge 0} (-1)^i \operatorname{rank} H_i(X;\mathbb{Z}).$$

The *reduced Euler characteristic* $\widetilde{\chi}(X)$ is similarly given by the reduced homology groups. Thus $\widetilde{\chi}(X) = -1 + \chi(X)$.

Examples. Every contractible space has the same Euler characteristic as $\mathbb{R}^n$,

$$\chi(X) \;=\; \chi(\mathbb{R}^n) \;=\; 1,$$

and thus the reduced Euler characteristic $\widetilde{\chi}(X) = \widetilde{\chi}(\mathbb{R}^n) = 0$.

The spheres have the Euler characteristics

$$\chi(S^n) \;=\; 1 + (-1)^n$$

and thus the reduced Euler characteristics $\widetilde{\chi}(S^n) = (-1)^n$.

The real projective spaces $\mathbb{RP}^n$ have the Euler characteristics

$$\chi(\mathbb{RP}^n) \;=\; \frac{1 + (-1)^n}{2} \;=\; \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even,} \end{cases}$$

since in the 2-fold *covering map* $S^n \to \mathbb{RP}^n$ (cf. Section 6.2) the Euler characteristic is halved: Any triangulation of $\mathbb{RP}^n$ yields a centrally symmetric triangulation of S^n, in which every k-simplex in $\mathbb{RP}^n$ corresponds to exactly two k-simplices of S^n, for $k \ge 0$.

Corollary 5.9. *For every triangulated n-sphere (such as the boundary complex of an $(n+1)$-dimensional simplicial polytope with f_i simplices of dimension i we have*

$$f_0 - f_1 + f_2 \mp \dots (-1)^n f_n \;=\; 1 + (-1)^n.$$

Example. Consider $\Delta_n^{(k)}$, the k-dimensional skeleton of the n-dimensional simplex ($1 \le k \le n$). Then

$$f_i \;=\; \begin{cases} \binom{n+1}{i+1} & \text{for } 0 \le i \le k \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\chi(\Delta_n^{(k)}) = \sum_{i=0}^{k}(-1)^i\binom{n+1}{i+1}$. Comparison with the full simplex yields that $\beta_0 = 1$ and $\beta_i = 0$ otherwise, except (possibly) for β_k. Thus from the Euler–Poincaré formula we get

$$\beta_k \;=\; \sum_{i=-1}^{k}(-1)^{k-i}\binom{n+1}{i+1}.$$

Indeed, by comparison with the star of a vertex (which is contractible), we get more explicitly

$$\beta_k \;=\; \binom{n}{k+1}.$$

The k-th homology group of a k-dimensional simplicial complex is always free, thus

$$H_i(\Delta_n^{(k)};\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ \mathbb{Z}^{\beta_k} & \text{for } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise (Homology of Swiss cheese (“Emmentaler”)). Discuss the homology of a piece of Emmentaler: Here the complex is 3-dimensional, but it retracts (homotopy equivalence!) to a 2-dimensional one. The piece is connected, so $H_0(K;\mathbb{Z}) \cong \mathbb{Z}$, and the only other possibly nonzero homology groups are $H_1(K;\mathbb{Z})$ and $H_2(K;\mathbb{Z})$. Can they both be non-zero? Are they free, or could there be torsion? What can you say about the Euler characteristic?

5.3 The Hopf trace formula

Generalization of (1): the ranks that appear here can also interpreted as the traces of identity maps — and thus generalize to the traces of chain maps of simplicial self-maps. This leads directly to the Hopf trace formula.

The trace of a quadratic matrix is given by $\operatorname{trace} A = \sum_i a_{ii}$. If we interpret A as a transition matrix, which records the probabilities or numbers a_{ij} of transitions from i to j, then the trace is a measure for how often we “stay where we are”. In such an interpretation (or algebraically) it is easy to see that $\operatorname{trace}(AB) = \operatorname{trace}(BA)$: this is $\sum_{i,j} a_{ij}b_{ji}$. (Similarly for non-square matrices of compatible formats, $A \in \mathbb{Q}^{m\times n}$ and $B \in \mathbb{Q}^{n\times m}$.) If B is invertible, then this implies $\operatorname{trace}(B^{-1}AB) = \operatorname{trace} A$; so the trace of an endomorphism is well-defined (independent of a choice of basis). This is also true if we work with integer matrices $A \in \mathbb{Z}^{n\times n}$ that represent homomorphisms $f_A : \mathbb{Z}^n \to \mathbb{Z}^n$.

Theorem 5.10 (Hopf trace formula)**.** *Let K be a finite simplicial complex, and let $f : K \to K$ be a simplicial map, then this induces endomorphisms $f_\# : C_k(K;\mathbb{Z}) \to C_k(K;\mathbb{Z})$. For these*

$$\sum_{k\ge 0}(-1)^k \operatorname{trace}(f_\#, C_k(K;\mathbb{Z})) \;=\; \sum_{k\ge 0}(-1)^k \operatorname{trace}(f_*, H_k(K;\mathbb{Z})/T(H_k(K;\mathbb{Z})))$$

in case of integer coefficients, where $T(H) := \{x \in H : kx = 0 \textit{ for some } k > 0\}$ *denotes the* torsion subgroup *of* H*, and*

$$\sum_{k\geq 0}(-1)^k \operatorname{trace}(f_\#, C_k(K;G)) \;=\; \sum_{k\geq 0}(-1)^k \operatorname{trace}(f_*, H_k(K;G))$$

in case of a coefficient field G*.*

Proof. $f_* : H_k(X;\mathbb{Z}) \to H_k(X;\mathbb{Z})$ induces a map

$$\bar{f}_* : H_k(X;\mathbb{Z})/T(H_k(X;\mathbb{Z})) \;\longrightarrow\; H_k(X;\mathbb{Z})/T(H_k(X;\mathbb{Z}))$$

since every group homomorphism, such as f_*, maps torsion elements to torsion elements.

As a second ingredient we need an analogue of the rank formula, as discussed at the beginning of Section 5.2, but with "trace instead of rank". In the following, we sketch the proof only for the case of fields, essentially following Ossa [38, Satz 5.9.3]. The integral case can be found in Munkres' book [36, Thm. 22.1].

By definition of the groups of chains, cycles and boundaries and the induced chain map we have a commutative diagram, where the rows are exact by definition:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_n & \longrightarrow & C_n & \xrightarrow{f_n} & B_{n-1} & \longrightarrow & 0 \\
 & & \downarrow f_n & & \downarrow f_n & & \downarrow f_{n-1} & & \\
0 & \longrightarrow & Z_n & \longrightarrow & C_n & \xrightarrow{f_n} & B_{n-1} & \longrightarrow & 0
\end{array}$$

and another one of the same type, which reflects the definition of the homology groups and the induced map in homology:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B_n & \longrightarrow & Z_n & \longrightarrow & H_n & \longrightarrow & 0 \\
 & & \downarrow f_n & & \downarrow f_n & & \downarrow H_n(f) & & \\
0 & \longrightarrow & B_n & \longrightarrow & Z_n & \longrightarrow & H_n & \longrightarrow & 0
\end{array}$$

The proof is now finished by applying to both diagrams the following Lemma 5.11, which yields

$$\operatorname{trace}(f_n, C_n) \;=\; \operatorname{trace}(f_n, Z_n) + \operatorname{trace}(f_{n-1}, B_{n-1})$$

and

$$\operatorname{trace}(f_n, Z_n) \;=\; \operatorname{trace}(f_n, B_n) + \operatorname{trace}(f_{*,n}, H_n),$$

and thus

$$\operatorname{trace}(f_n, C_n) \;=\; \operatorname{trace}(f_{n-1}, B_{n-1}) + \operatorname{trace}(f_n, B_n) + \operatorname{trace}(f_{*,n}, H_n),$$

and then taking alternating sums. □

Lemma 5.11. *If in a diagram of finite-dimensional vector spaces and linear maps of the form*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \longrightarrow & V & \xrightarrow{\pi} & W & \longrightarrow & 0 \\
 & & \downarrow f' & & \downarrow f & & \downarrow f'' & & \\
0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0
\end{array}$$

the rows are exact and the squares commute, then $\operatorname{trace} f = \operatorname{trace} f' + \operatorname{trace} f''$*.*

Proof. Let $\mathcal{B}' := \{u_i\}$ be a basis of U, let $\mathcal{B}'' := \{w_j\}$ be a basis of W, and choose $\{\tilde{w}_j\}$ with $\tilde{w}_j \in \pi^{-1}(w_j)$, then $\mathcal{B} := \{u_i\} \cup \{\tilde{w}_j\}$ is a basis of V (with the usual proof for the dimension formula, in linear algebra).

Now we represent f with respect to the basis $\mathcal{B}$, and find

$$M_{\mathcal{B}}(f) \;=\; \begin{pmatrix} M_{\mathcal{B}'}(f') & 0 \\ 0 & M_{\mathcal{B}''}(f'') \end{pmatrix}$$

and we are done. □

The special case $f = \mathrm{id}$ is the Euler–Poincaré formula 5.7.

5.4 Lefschetz number and fixed point theorem

In analogy to the right-hand side of the Euler–Poincaré formula, which is the Euler characteristic, the right-hand side of the Hopf trace formula yields an important numerical invariant of continuous self-maps:

Definition 5.12 (Lefschetz number of a self-map)**.** Let K be a finite complex, and let $h : \|K\| \to \|K\|$ be a continuous self-map. Then

$$\Lambda(h) \;:=\; \sum_{k \geq 0} (-1)^k \,\mathrm{trace}(h_*, H_k(K;\mathbb{Z})/T(H_k(K;\mathbb{Z})))$$

is the *Lefschetz number* of h.

The Lefschetz number is an invariant of the homotopy class: Homotopic maps induce the same homomorphisms in homology, and thus they have the same Lefschetz number.

Intuition: The Lefschetz number is a measure for the Euler-characteristic of the fixed point set.

Theorem 5.13 (Lefschetz fixed point theorem [36, Thm. 22.3])**.** *Let K be a finite complex, and let $h : \|K\| \to \|K\|$ be a continuous map. If $\Lambda(h) \neq 0$, then h has a fixed point.*

Sketch of proof. Assume that h has no fixed point. The Lefschetz number is independent of the triangulation, in particular it does not change if we subdivide K. Thus we may subdivide K until for all vertices v the condition $h(\mathrm{Star}_K v) \cap \mathrm{Star}_K v = \emptyset$ is satisfied. (This may be obtained by repeated barycentric subdivisions, using that K is compact.)

In the second step we subdivide K further, such that h has a simplicial approximation $f : K' \to K$. It is homotopic to h, so it has the same Lefschetz number $\Lambda(f) = \Lambda(h)$.

Now let $\lambda : C_*(K) \to C_*(K')$ be the subdivision operation. Then $f_\# \circ \lambda : C_*(K) \to C_*(K)$ is a chain map that induces h_*. And for this chain map all traces are 0: Every simplex $\sigma \in K$ is mapped by λ to a sum of simplices $\sigma' \in K'$, whose images under $f_\#$ are disjoint to σ due to the star condition that we have imposed above. The trace formula now yields $\Lambda(h) = 0$. □

Lemma 5.14. $\mathrm{trace}(f_*, H_0(K;\mathbb{Z}))$ *is the number of components of K that are mapped to themselves. If K is connected, then $f_* = \mathrm{id}_* : H_0(K;\mathbb{Z}) \to H_0(K;\mathbb{Z})$ and thus* $\mathrm{trace}(f_*, H_0(K;\mathbb{Z})) = 1$.

Examples. $\Lambda(\mathrm{id}_K) = \chi(K)$. $\Lambda(\mathrm{const} : K \to K) = 1$. $\Lambda(f : S^n \to S^n) = 1 + (-1)^n \deg(f)$.

Example. Let $n > 0$. For $h : S^n \to S^n$ we have $\Lambda(h) = 1 + (-1)^n \deg h$. Thus the Lefschetz fixed point theorem implies that $\deg a = (-1)^{n+1}$, since the antipodal map a has no fixed points.

A topological space is *acyclic* if all its reduced homology groups are zero, that is, if $H_0(X;\mathbb{Z}) \cong \mathbb{Z}$, and $H_k(X;\mathbb{Z}) \cong \{0\}$ for all $k \neq 0$. Every contractible space is acyclic, but there are also acyclic spaces that are not contractible. (See Example 6.3 below.) Similarly, we define spaces to be $\mathbb{Q}$-acyclic, or F-acyclic for some field F, if all reduced homology groups with coefficients in $\mathbb{Q}$ resp. F are trivial.

Corollary 5.15. *For finite acyclic complexes K, every continuous map $h : \|K\| \to \|K\|$ has a fixed point.*

This corollary is a strong generalization of the Brouwer fixed point theorem 3.12! Note that it also implies that any map of a finite tree to itself, but also for any map $\mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2$ has a fixed point. ($\mathbb{R}\mathrm{P}^2$ is $\mathbb{Q}$-acyclic.)

5.5 The Borsuk–Ulam theorem

Here is an extremely useful result with many applications (see [31]!!) that we get from the Lefschetz fixed point theory nearly "for free".

Theorem 5.16 (Borsuk's antipodal theorem [5]; see [31])**.** *If a continuous map $f : S^n \to S^m$ is* antipodal *(that is, $f(-x) = -f(x)$ for all $x \in S^n$), then $n \leq m$.*

Proof. Let $f : S^m \to S^n$ be continuous and antipodal, with $n > m$. The usual embedding $g : S^m \hookrightarrow S^n$ is also antipodal. Thus we also get an antipodal composition $g \circ f : S^n \to S^n$.

Further we can subdivide S^n finely enough, and centrally symmetrically, such that $g \circ f$ has a simplicial approximation. Because of the central symmetry the simplices counted by $\Lambda(g \circ f)$ appear in pairs, which implies that $\Lambda(g \circ f) \equiv 0 \bmod 2$: The Lefschetz number is even.

However, $g : S^m \to S^n$ is null-homotopic for $m < n$, as one can see from an explicit homotopy that uses the upper hemisphere of S^{m+1}. Thus also $g \circ f$ is nullhomotopic, that is homotopic to a constant map, so it has Lefschetz number 1: Contradiction! □

Theorem 5.17 (Equivalent versions of the Borsuk–Ulam theorem)**.**

- (BU1) (Borsuk's "Satz I") *An antipodal map $f : S^n \to S^n$ cannot be null-homotopic.*
- (BU2) (Borsuk's "Satz II") *Every continuous map $f : S^n \to \mathbb{R}^n$ identifies some antipodes, i.e., there is some $x \in S^n$ with $f(x) = f(-x)$.*
- (BU3) *Every symmetric map $f : S^n \to \mathbb{R}^n$, $f(-x) = -f(x)$, has a zero.*
- (BU4) (Borsuk's "Satz III", Lyusternik–Schnirelman (1930); version of Greene (2002), see Aigner & Ziegler [1]) *In every covering $F_0, \dots, F_n$ of S^n, for which every set F_i with $i > 0$ is either open or closed, one of the sets A_i contains a pair of antipodal points.*

Proof. We have already proved (BU1): We have shown that if f is antipodal, then $\Lambda(f)$ is even, while any nullhomotopic f has Lefschetz number $\Lambda(f) = 1$.

The next few results are easy to derive directly from the original version (BU) of Theorem 5.16, e.g. in the order (BU)$\Longrightarrow$(BU3)$\Longrightarrow$(BU2).

For the implication (BU2)$\Longrightarrow$(BU4) let a covering $S^n = F_0 \cup F_1 \cup \cdots \cup F_n$ be given as specified, and assume that there are no antipodal points in any of the sets F_i. We define a map $f : S^n \to \mathbb{R}^d$ by

$$f(x) \;:=\; \big(d(x, F_1), d(x, F_2), \dots, d(x, F_n)\big).$$

Here $d(x, F_i)$ denotes the distance of x from F_i. Since this is a continuous function in x, the map f is continuous. Thus the Borsuk–Ulam theorem (BU2) tells us that there are antipodal points $x^*, -x^*$ with $f(x^*) = f(-x^*)$. Since F_0 does not contain antipodes, we get that at least one of x^* and $-x^*$ must be contained in one of the sets F_k with $k \geq 1$. After exchanging x^* with $-x^*$ if necessary, we may assume that $x^* \in F_k$, $k \geq 1$. In particular this yields $d(x^*, F_k) = 0$, and from $f(x^*) = f(-x^*)$ we get that $d(-x^*, F_k) = 0$ as well.

If F_k is closed, then $d(-x^*, F_k) = 0$ implies that $-x^* \in F_k$, and we arrive at the contradiction that F_k contains a pair of antipodal points.

If F_k is open, then $d(-x^*, F_k) = 0$ implies that $-x^*$ lies in $\overline{F_k}$, the closure of F_k. The set $\overline{F_k}$, in turn, is contained in $S^n \backslash (-F_k)$, since this is a closed subset of S^n that contains F_k. But this means that $-x^*$ lies in $S^n \backslash (-F_k)$, so it cannot lie in $-F_k$, and x^* cannot lie in F_k, a contradiction. □

Here is the first, spectacular, application of Topology — specifically, of the Borsuk–Ulam theorem — to combinatorics: Lovász 1978 solution of the Kneser conjecture. The full story is told in [31], here is only a sketch.

Definition 5.18 (Kneser graphs)**.** For $n \geq 2k \geq 4$, the Kneser graph $\mathrm{KG}\binom{[n]}{k}$ has as it vertex set all the k-subsets of the n-set $[n] := \{1, \ldots, n\}$, where two vertices are connected by an edge if the corresponding edges are connected.

Thus $\mathrm{KG}\binom{[n]}{k}$ has $\binom{n}{k}$ vertices and $\frac{1}{2}\binom{n}{k}\binom{n-k}{k}$ edges. $\mathrm{KG}\binom{[2k]}{k}$ is a matching on $\binom{2k}{k}$ vertices, while $\mathrm{KG}\binom{[5]}{2}$ is the "Petersen graph".

Theorem 5.19 (The Kneser conjecture (1955); Lovász (1978))**.** *The Kneser graph* $\mathrm{KG}\binom{[n]}{k}$ *cannot be colored with less than* $n - 2k + 2$ *colors:*

$$\chi\big(\mathrm{KG}\tbinom{[n]}{k}\big) \;=\; n - 2k + 2.$$

Proof (Greene [18]). First, we note that

$$\tbinom{[n]}{k} \longrightarrow [n - 2k + 2], \qquad S \longmapsto \min\{\min S, [n - 2k + 2]\}$$

is a correct coloring with $n - 2k + 2$ colors, which assigns different colors to any two disjoint k-sets.

Now we assume that the Kneser graph $\mathrm{KG}\binom{[n]}{k}$ can be colored with $d := n - 2k + 1$ colors, and let

$$c : \tbinom{[n]}{k} \longrightarrow [d] = [n - 2k + 1]$$

be such a coloring.

Let $X \subset S^d$ be a set of $|X| = n$ points *in general position* on the d-sphere: Here "general position" requires that no $d + 1$ of the points in X lie on a great $(d - 1)$-dimensional subsphere.

For $i = 1, \ldots, d$ define

$$F_i \;:=\; \{x \in S^d : H(x) \text{ contains an } i\text{-colored } k\text{-set of } X\}.$$

Here $H(x) := \{y \in S^d : \langle x, y\rangle > 0\}$ is the hemisphere centered at x.

These sets F_i are open by construction, and they are antipode-free since the coloring is correct: Otherwise there is $x, -x \in S^d$ such that both $H(x)$ and $H(-x)$ contain an i-colored k-subset of X, and these two k-sets are disjoint.

Then define

$$F_0 \;:=\; S^d \setminus (F_1 \cup \cdots \cup F_d).$$

This F_0 is closed by construction, which is irrelevant for (BU4). More importantly, is antipode-free: Otherwise there is $x, -x \in S^d$ such that both $H(x)$ and $H(-x)$ contain at most $k - 1$ points from X, so the corresponding "equator" contains at least $n - 2(k - 1) = n - 2k + 2 = d + 1$ points of X, in contradiction to "general position".

Thus we have obtained a covering $F_0, F_1, \ldots, F_d$ that contradicts (BU4). □

6 Manifolds

The study and (as far as possible ...) the classification of manifolds (cf. Definition 2.16) are one of the main themes and an (unsolvable, compare Remark 3.26) main task of topology.

6.1 Classification of 2-dimensional manifolds

Recall that an n-dimensional manifold M is a second countable Hausdorff topological space such that each point $x \in M$ is contained in a neighborhood U_x that is homeomorphic to either $\mathbb{R}^n$ or $\mathbb{R}^n_+$. The 2-dimensional manifolds are also known as *surfaces*. In the following, we primarily concern ourselves with connected, closed (compact, without boundary) surfaces. These are all triangulable (Theorem 2.17).

Examples that we know include: the sphere S^2, the torus $T^2 = S^1 \times S^1$, the real projective plane $\mathbb{R}\mathrm{P}^2$, the Klein bottle K^2, the sphere with g handles M_g (where $M_0 = S^2$, $M_1 = T^2$), the projective plane with g handles, etc.

There are two basic operations we can perform to produce a new surface from a given surface. The first, which we call "attaching a handle", involves first removing a pair of open disks and gluing in a cylinder $S^1 \times I$ along its boundary. Adding a handle to the sphere S^2 produces the torus T^2. The second operation is called "adding a cross-cap"; here we cut out a single disk and identify opposite points on the boundary of the hole we have just created. Equivalently, we identify that boundary with the boundary of a Möbius strip (that is, of a real projective plane from which a disk has been removed). Adding a cross-cap to S^2 yields $\mathbb{R}\mathrm{P}^2$.

Exercise. The operation "attaching a handle" reduces the Euler characteristic by 2. It preserves orientability.
"Adding a cross-cap" reduces the Euler characteristic by 1. The resulting surface is not orientable.

Definition 6.1 (Orientable and non-orientable surfaces of genus g)**.**
The surfaces M_g obtained by adding $g \geq 0$ handles to a 2-sphere is known as the *orientable surface of genus* g.

The surface M'_g obtained by adding $g \geq 1$ cross-caps to a 2-sphere is called the *non-orientable surface of genus* g.

Recall that orientablity of a connected closed surface can be recognized by $H_2(M;\mathbb{Z}) \simeq \mathbb{Z}$. Since $\chi(M_g) = 2 - 2g$ resp. $\chi(M'_g) = 2 - g$, this implies that each of the surfaces in Definition 6.1 are actually distinct, and can also be distinguished by their homology. One natural question to ask whether there exist other homeomorphism types of surfaces, but this turns out not to be the case according to the theorem that we next prove.

One consequence will be that connected surfaces that agree with respect to *orientability* and *Euler characteristics* are homeomorphic. Note that this also implies that there are relations of the form "three cross-caps = one cross-cap and a handle".

We will sketch the proof of the so-called *ZIP proof* of Conway, see Francis & Weeks [14] for more discussion and especially the nice drawings. Conway (known for his clever terminology) uses ZIP to refer to its standing as the *Zero Irrelevancy Proof* as well as to the 'zips' involved in the constructions.

For the proof, we (temporarily) allow possibly disconnected surfaces with boundary. The main objects of study will be surfaces with the additional information given by four types of *zip-pairs*. A zip-pair is pair of connected pieces of the boundary of a surface, together with instructions (orientations) on how to zip them up (identify). Note that if the boundary of a surface is nonempty, then each component of the boundary is homeomorphic to a circle.

Definition 6.2 (Types of zip-pairs)**.** When each part of a zip-pair occupies a single component of the boundary (so that each part lies on its own circle), we have two types of zip-pairs depending on how the orientations match. In one case, zipping up produces a *handle* and in the other we obtain a *cross-handle*. In the latter we must intersect the surface with itself to accomplish the zipping in 3-space; this is merely an artifact of the drawing.

When both parts of the zip-pair completely occupy a single component of the boundary (each occupies say one half of a circle), we again obtain two types of zippings. In one case the orientations are as in a 'usual' zipper and we simply close the hole to obtain a *cap*. In the case that the orientations are in opposite directions, zipping up produces a *crosscap*.

Definition 6.3. A *perforation* is obtained by removing an open disk from a surface.

Definition 6.4. A surface is called *ordinary* if it is homeomorphic to a finite collection of spheres, each with a finite number of handles, crosshandles, crosscaps, and perforations.

Theorem 6.5 (Preliminary version)**.** *Every surface is ordinary.*

Proof. Suppose that X is an arbitrary triangulated compact surface (possibly disconnected with a finite number of components and possibly with boundary). Each edge in the triangulation is contained in either 1 or 2 triangles, since X is a manifold with boundary. In the case that an edge is contained in 2 triangles we associate a zip pair along each edge with the orientations determined by the way the edges meet in X. Unzip all the zip pairs, so that X decomposes into a finite (since X is compact) number of triangles with zip-pairs. Each triangle is ordinary (homeomorphic to a sphere with a single perforation). Lemma 6.6 (below) tells us that zipping up a single zip-pair in an ordinary surface produces an ordinary surface, and hence by induction X is ordinary. □

Lemma 6.6.
Let X be a surface with a zip-pair. If X is ordinary, then the surface obtained by zipping up the zip-pair is also ordinary.

Proof. We first suppose that each part of the zip pair completely occupies a connected component (circle) of the boundary. If the pair is contained in a single path-component of X then we can deform X so that the circles are close and then by zipping obtain either a handle or cross-handle, depending on the orientation. If the each part is in different components of X then zipping simply connects these components. Either way we are still ordinary.

If the two parts of the zip-pair occupy a single circle, then zipping them produces a cap or a cross cap.

Finally, if the zips do not completely occupy their boundary circle(s), we see that zipping them together produces a surface with one of the above modifications, together with a certain number of perforations. In this case the resulting surface is again ordinary. □

We next determine some relations between the zipping operations.

Lemma 6.7.
A crosshandle is equivalent to two crosscaps,

Proof. (idea) Consider a surface with a Klein bottle perforation, i.e. a square perforation with two zip-pairs installed, each part of a pair parallel to the other and occupying opposite sides, with one matching and one opposing orientation. One can check that zipping together this configuration one pair at a time produces a crosshandle in one case and those for two crosscaps in the other. □

Lemma 6.8.
Handles and crosshandles are equivalent in the presence of a crosscap.

Proof. (idea) Consider a pair of circles in with two zip-pairs installed, each part of a pair occupying a different circle, with one pair of orientations matching and the other in different directions. Zipping together this configuration one pair at a time yields a handle and crosscap in the one case, and a crosshandle and a crosscap in the other. □

Theorem 6.9 (Classification of 2-manifolds; compare e.g. Ossa [38, Abschnitt 3.8])**.**
Every connected closed surface is homeomorphic to exactly one of M_g, a sphere with g handles ($g \geq 0$), or M'_g, a sphere with g crosscaps ($g \geq 1$).

Proof. By the preliminary Theorem 6.5, a connected closed surface is homeomorphic to a single sphere with a finite number of handles, crosshandles, and crosscaps (no perforations since we're assuming no boundary). Suppose at least one crosscap or crosshandle is present (otherwise we are done). By Lemma 6.7, each crosshandle is homeomorphic to two crosscaps, and hence only crosscaps and handles are present. But handles are equivalent to crosshandles in this situation by Lemma 6.8 and hence only crosscaps and crosshandles are present. But again crosshandles are the same as two crosscaps so in the end we have a sphere with a finite number of crosscaps. If no crosscaps or crosshandles are present then we're done. □

Corollary 6.10. *Connected closed 2-manifolds that have the same (i.e. isomorphic) homology with $\mathbb{R}$-coefficients are homeomorphic.*

Proof. We have $H_*(M_g;\mathbb{R}) \cong (\mathbb{R}, \mathbb{R}^{2g}, \mathbb{R})$, and $H_*(M'_g;\mathbb{R}) \cong (\mathbb{R}, \mathbb{R}^{g-1}, 0)$. □

Note that the same corollary also holds with rational coefficients instead of $\mathbb{R}$ coefficients.
However, the homology with coefficients taken in the two-element group/field $\mathbb{F}_2 = \mathbb{Z}_2$ is *not* sufficient to identify a closed connected manifold:
$H_*(M_g;\mathbb{F}_2) \cong (\mathbb{F}_2, {\mathbb{F}_2}^{2g}, \mathbb{F}_2)$, and $H_*(M'_g;\mathbb{F}_2) \cong (\mathbb{F}_2, {\mathbb{F}_2}^{g}, \mathbb{F}_2)$.

Proposition 6.11. *The fundamental group of the orientable surface M_g of genus g ($g \geq 0$) has a presentation of the form*

$$\pi_1(M_g) \;\cong\; \big\langle\, a_1, b_1, a_2, b_2, \ldots, a_g, b_g : a_1 b^1 a_1^{-1} b_1^{-1}\, a_2 b_2 a_2^{-1} b_2^{-1} \,\cdots\, a_g b_g a_g^{-1} b_g^{-1} = 1 \,\big\rangle.$$

The fundamental group of the non-orientable surface M'_g ($g \geq 1$) of genus g is

$$\pi_1(M'_g) \;\cong\; \big\langle\, a_1, a_2, \ldots, a_g : a_1 a_1\, a_2 a_2 \,\cdots\, a_g a_g = 1 \,\big\rangle$$

Proof. This is derived from a normal form of the surface: a $4g$-gon with identifications between the boundary edges for M_g, and similarly for M'_g. □

Theorem 6.12 (Abelianization of the fundamental group [20, Sect. 2.A])**.** *Let X be triangulable and connected. The the first homology group $H_1(X;\mathbb{Z})$ of X is canonically isomorphic to the* abelianization $\pi(X)^{\mathrm{ab}} := \pi_1(X)/[\pi_1(X), \pi_1(X)]$ *of the fundamental group $\pi_1(X) = \pi_1(X;x_0)$.*

Proof. Every closed loop in X represents a homology class, and homotopic loops determine the same homology class. (This is proved either by simplicial approximation, or via the presentation of the fundamental group in terms of the not-tree edges and triangles with respect to a spanning tree.) Composition of loops corresponds to addition of homology classes. Thus there is a canonical homomorphism $p : \pi_1(X) \to H_1(X;\mathbb{Z})$.

The commutator subgroup $[\pi_1(X), \pi_1(X)]$ is contained in the kernel of the homomorphism, as $H_1(X; \mathbb{Z})$ is abelian. Thus we have a canonical homomorphism $\bar{p} : \pi_1(X)/[\pi_1(X), \pi_1(X)] \to H_1(X; \mathbb{Z})$.
The homomorphism $\bar{p}$ is surjective, since for every 1-cycle one can construct a loop that induces it (e.g. again using a spanning tree).
The homomorphism is injective: see Hatcher [20, pp. 167,168], in particular the geometric explanation given at the end of his proof. □

The first homology group therefore vanishes if X is path-connected and $\pi_1(X)$ is a *perfect group*, which by definition means $\pi_1(X) = [\pi_1(X), \pi_1(X)]$; see the construction of the Poincaré homology sphere discussed below.

Using this result we can then determine the homology groups (with $\mathbb{Z}$ coefficients) of surfaces from the fundamental groups: $H_*(M_g; \mathbb{Z}) \cong (\mathbb{Z}, \mathbb{Z}^{2g}, \mathbb{Z})$, and $H_*(M'_g; \mathbb{Z}) \cong (\mathbb{Z}, \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2, 0)$. We see that one can distinguish connected closed surfaces by their fundamental groups, since in particular the abelianizations are all non-isomorphic.

The connection between fundamental group and homology has a higher-dimensional analogue — except then everything gets *simpler* since the higher homotopy groups are abelian. For any space X and positive integer k there exists a group homomorphism (called the Hurewicz map) $h_k : \pi_k(X) \to H_k(X; \mathbb{Z})$.

Theorem 6.13 (Hurewicz's theorem)**.** *Suppose X is an $(n-1)$-connected space (this means that $\pi_i(X) = 0$ for all $i < n$). Then the Hurewicz map is an isomorphism if $k \le n$ and an epimorphism if $k = n + 1$.*

For a proof and further discussion, see Spanier [46, p. 398]. In particular, in the first dimension where homology and homotopy group are non-trivial, they are isomorphic — if the space is simply connected. Hurewicz's theorem has further extensions, for example that any map between simply-connected (say triangulable) spaces which induces isomorphisms in homology (with integer coefficients) in all dimensions is necessarily a homotopy equivalence (Whitehead's theorem). We may note that this result is *important*, *deep*, but not really *difficult* to prove (the homotopy equivalence is constructed by induction on dimension on the skeleton; see Hatcher [20, Thm. 4.5]).

6.2 Coverings

Definition 6.14 (Coverings)**.** A surjective map $p : \widetilde{M} \to M$ is called a *covering* if for every $x \in M$ there is a neighborhood where the preimage $p^{-1}(U_x)$ is a disjoint union of open sets in $\widetilde{M}$ that are all mapped by p to U_x homeomorphically. M is called the *base* of the covering, and $\widetilde{M}$ the *covering space*.

We note that if M is a manifold, then so is $\widetilde{M}$.

Two coverings $p : \widetilde{M} \to M$ and $p' : \widetilde{M}' \to M$ are said to be *isomorphic* if there exists a homeomorphism $f : \widetilde{M} \to \widetilde{M}'$ such that $p = p' \circ f$.

If $p : \widetilde{M} \to M$ is a covering of a connected space M, then the preimage $p^{-1}(x)$ has the same cardinality for each $x \in M$. In the finite case $|p^{-1}(x)| = k < \infty$ we talk about a *k-fold covering*.

Example. Consider S^1 as a subset of $\mathbb{C}$. Then $p : \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$ defines a covering such that each fiber $p^{-1}(x)$ is a countably infinite number of points.

Example. The map $p : S^1 \to S^1$, $z \mapsto z^k$ defines a k-fold covering (for $k > 0$).

We will see below that these are essentially the only (connected) coverings of the circle.

Example. Consider $\mathbb{RP}^2$ as the set of lines through the origin in $\mathbb{R}^3$. Then the map $p : S^2 \to \mathbb{RP}^2$, $x \mapsto \langle x \rangle$ defines a 2-fold covering.

Although coverings are defined in fairly straightforward geometric terms, from an algebraic topological perspective much of their importance stems from the fact they enjoy certain "lifting properties". These lead to important connections between types of coverings and the (algebraic) topology of M.

Proposition 6.15 (Homotopy lifting property)**.** *Let $p : \widetilde{M} \to M$ be a covering and suppose we have maps $f : Y \to \widetilde{M}$ and $H : Y \times I \to M$ such that $p \circ f = H|_{Y\times\{0\}}$. Then there exists a unique map (called a "lift") $\widetilde{H} : Y \times I \to \widetilde{M}$ such that $H = p \circ \widetilde{H}$ and $f = \widetilde{H}|_{Y\times\{0\}}$.*

One can think of H as a homotopy between a pair of maps $f_0, f_1 : Y \to M$ such that there exists a lift of $f_0 := f$ to $\widetilde{M}$. If $p : \widetilde{M} \to M$ is a covering, then once can lift the entire homotopy to $\widetilde{M}$ (hence the name of the property). Taking Y to be a point gives the following.

Corollary 6.16 (Unique path lifting property)**.** *Let $p : \widetilde{M} \to M$ be a covering and suppose $\gamma : I \to M$ is a path in M with $x_0 = \gamma(0)$. Then given any point $\widetilde{x_0} \in p^{-1}(x_0)$ there exists a unique lift $\widetilde{\gamma} : I \to \widetilde{M}$ such that $f = p \circ \widetilde{p}$.*

Proof. Although this of course follows from the homotopy lifting property, one can prove this directly as follows (to also get an idea of the general case). Cover M with open sets U_i such that $p^{-1}(U_i)$ is a disjoint collection of open sets, each mapped homeomorphically onto U_i. Pull this back via γ to get a cover of I and take a finite subcover. Pick points $t_1 < t_2 < \cdots < t_k$ in I such that each t_i sits in the intersection of overlapping sets in this cover. Now define $\widetilde{\gamma}$ inductively by taking $\widetilde{\gamma}(0) = \widetilde{x}$ and $\widetilde{\gamma}([t_i, t_{i+1}])$ according to $p^{-1}(\gamma([t_i, t_{i+1}])$. □

Corollary 6.17. *If $p : \widetilde{M} \to M$ is a covering then the induced map $p_* : \pi_1(\widetilde{M}, \widetilde{x_0}) \to \pi_1(M, x_0)$ is injective.*

Proof. A pointed map $f : S^1 \to \widetilde{M}$ is in the kernel of p_* precisely when there exists a homotopy $H : S^1 \times I \to M$ between $p \circ f$ and the constant map at x. Apply the homotopy lifting property to lift H and conclude that the lift at $S^1 \times \{1\}$ must be the constant map at $\tilde{x}$. □

This in turn can be used to provide a useful criterion for the existence of lifts in general in terms of and algebraic condition on fundamental groups.

Proposition 6.18 (Lifting criterion)**.** *Suppose $p : \widetilde{M} \to M$ is a covering and $f : (Y, y) \to (M, x)$ is a pointed map with Y triangulable. Then a lift $\widetilde{f} : (Y, y) \to (\widetilde{M}, \widetilde{x})$ of f exists if and only if $f_*\big(\pi_1(Y, y)\big) \subseteq p_*\big(\pi_1(\widetilde{M}, \widetilde{x})\big)$.*

Theorem 6.19. *Suppose M is triangulable. Then for every subgroup $H \subseteq \pi_1(M, x)$ there exists a covering space $p : M_H \to M$ such that $p_*\big(\pi_1(M_H, \widetilde{x})\big) = H$ for some $\widetilde{x} \in H$.*

In particular, since p_* is always injective, this ensures the existence of a simply-connected covering space $p : M_0 \to M$ which is called the *universal cover* of M. It is unique up to isomorphism.

We see a nice interplay between coverings and the fundamental group of the base space. In fact, the above proposition is one half of a tight correspondence provided by the following theorem, sometimes called the Galois correspondence for its striking resemblance to the correspondence between field extensions and subgroups of the Galois group.

Theorem 6.20 (Galois correspondence for coverings)**.** *Suppose M is triangulable. Then there is a bijection between the set of base point preserving isomorphism classes of path-connected covering spaces and the set of subgroups of $\pi_1(M, x)$, induced by associating the subgroup $p_*\big(\pi_1(\widetilde{M}, \tilde{x})\big)$ to the covering space $(\widetilde{M}, \widetilde{x})$.*

If base points are ignored, the bijection is with conjugacy classes of subgroups of $\pi_1(M, x_0)$. See Fulton [15, Part VI] or Hatcher [20, Chapter 1.3] or Munkres [37, Chap. 13] for proofs of these results.

As an application, we see that all isomorphism types of coverings of the circle S^1 are obtained by the coverings described in the examples.

Lemma 6.21 (Euler characteristics of coverings)**.** *If M is compact and triangulable, and $p : \widetilde{M} \to M$ is a k-fold covering, then $\chi(\widetilde{M}) = k\chi(M)$.*

If there is a k-fold covering $p : M \to M$ with $1 < k < \infty$, as for example in the cases of S^1 and $T = S^1 \times S^1$, then the Euler characteristic equation implies that $\chi(M) = 0$.

Example. Surfaces can be constructed in terms of many different representations, for example as covering spaces. For example, M'_h may be obtained by adding $\frac{h-1}{2}$ handles to $\mathbb{R}\mathrm{P}^2 = M'_1$ (if h is odd) and by adding $\frac{h-2}{2}$ handles to K^2 (if h is even). From this it is easy to see that M'_h has a two-fold covering, which is orientable, and thus homeomorphic to some M_g (by the classification theorem). The Euler characteristic computation according to Lemma 6.21 yields for this $2 - 2g = \chi(M_g) = 2\chi(M'_h) = 2(2 - h)$, hence $g = h - 1$.

Example. We have a 2-fold covering $p : S^n \to \mathbb{R}\mathrm{P}^n$, and thus

$$\chi(\mathbb{R}\mathrm{P}^n) \;=\; \tfrac{1}{2}(1 + (-1)^n) \;=\; \begin{cases} 0 & n \text{ odd,} \\ 1 & n \text{ even.} \end{cases}$$

Proposition 6.22. *Every connected non-orientable manifold M has a canonical connected orientable 2-fold covering $p : \widetilde{M} \to M$.*

Observe, however: $\mathbb{R}\mathrm{P}^n$ is orientable for odd $n \geq 1$. In this case the antipodal map $a : S^n \to S^n$ is orientation-preserving. However, for every $n > 1$ the map $S^n \to \mathbb{R}\mathrm{P}^n$ is the universal (simply-connected) covering of $\mathbb{R}\mathrm{P}^n$. For $n = 1$ this is given by $\mathbb{R} \to S^1 \cong \mathbb{R}\mathrm{P}^1$, $t \mapsto e^{it}$.

Proposition 6.23. *Let X be a topological space, on which a finite group (of "symmetries") acts freely, that is, such that*

- *for every $g \in G$ there is an associated homeomorphism $\varphi(g) : X \to X$,*
- *$\varphi(1) = \mathrm{id}_X : X \to X$ and $\varphi(gh) = \varphi(g) \circ \varphi(h) : X \to X$, and*
- *$\varphi(g)(x) \neq x$ for all $g \neq 1$.*

Then the projection map $p : X \to X/G$, $x \mapsto [x] = \{\varphi(g)(x) : g \in G\}$ is a covering.

The same is true also in the case of infinite groups G, if we require that the group action is "proper discontinuous", that is, such that every $x \in X$ has a neighborhood U_x such that all the sets $g(U_x)$ are disjoint.

Examples. $G = \mathbb{Z}_2$ acts freely on S^n, with $\varphi(0) = \mathrm{id}$ and $\varphi(1) = a$ (this is the "antipodal $\mathbb{Z}_2$-action").

$G = \mathbb{Z}_p$ acts freely on $S^{2n-1} \subset \mathbb{C}^n$ via multiplication with p-th roots of unity ($p \geq 2$).

$G = \mathbb{Z}_p$ has no free action on S^{2n}, for $p > 2$: this may be deduced from Lemma 6.21, since this would require $\chi(S^{2n}/\mathbb{Z}_p) = \frac{2}{p}$.

Lemma 6.24 ([20, p. 71])**.** *If X is path-connected, simply-connected and "locally path-connected" (this condition is satisfied if X is triangulable), and if the group G acts properly discontinuously and freely on X, then $\pi_1(X/G) \cong G$.*

Proof. This can/should be derived in connection with Theorem 6.20. □

Example. The group $G = \mathbb{Z}$ acts freely on $X = \mathbb{R}$ via $\varphi(z)(x) := x + z$. We obtain $X/G = \mathbb{R}/\mathbb{Z} \cong S^1$ with $\pi_1(S^1) \cong \mathbb{Z}$.

The group $G = \mathbb{Z}^2$ acts freely on $X = \mathbb{R}^2$ with $\varphi(z_1, z_2)(x_1, x_2) := (x_1 + z_1, x_2 + z_2)$. We obtain $X/G = \mathbb{R}^2/\mathbb{Z}^2 \cong T^2$ with $\pi_1(T^2) \cong \mathbb{Z}^2$.

6.3 Some 3-manifolds

While the fundamental questions about 2-manifolds are solved by Theorem 6.9, the 3-manifolds clearly pose much more difficult problems (cf. the Poincaré conjecture 3.14). In this section we want to mainly present some examples of interesting (classes of and construction methods for) 3-manifolds.

Example (The dodecahedron space, a Poincaré homology sphere; see [4]). The *dodecahedron space* Σ^3 may be obtained from a solid regular 3-dimensional dodecahedron by identification ("glueing") each pentagonal face with the opposite face after a rotation by $\pi/5 = 36°$. Please verify that the resulting space is a 3-manifold that has the homology of a the 3-sphere: $H_*(\Sigma^3; \mathbb{Z}) \cong (\mathbb{Z}, 0, 0, \mathbb{Z})$. The fundamental group is, however, non-trivial, it has 120 elements (compare Seifert–Threlfall [44, §62]). It may be identified as the "binary icosahedral group", as one can see from a presentation of the dodecahedron space as a quotion of the 3-sphere — see below. With elementary means (spanning tree presentation!) one can obtain the presentation

$$\pi_1(\Sigma^3; x_0) \;\cong\; \langle a, b : a^5 = (ab)^2 = b^3 \rangle,$$

from which again one can see that the abelianization is trivial (as it should be, by Proposition 6.12). Let me refer you to the discussion and computation in Seifert & Threlfall [44, §62], the original source from 1934. The whole Chapter 9 of this book is interesting for us and very accessible!

The 2-dimensional space that may be obtained from the *boundary* of the dodecahedron by the identifications obtained above is a simple example of a space that is acyclic (all reduced homology groups vanish), but which has a non-trivial fundamental group (namely $\widetilde{I}$).

The following class of manifolds was introduced by Tietze (1908).

Examples (Lens spaces). Let $1 \leq q < p$, with p and q relatively prime.

On the solid ball B^3 identify the boundary points $x \in \partial B^3 = S^2$ and $\varphi(x)$, where $\varphi(x)$ is obtained from x by "rotation around the polar axis by the angle $2\pi q/p$, and then reflection in the equator plane".

The quotient space is a compact orientable 3-manifold, the *lens space* $L(p, q)$. For example, we have $L(2, 1) \cong \mathbb{R}\mathrm{P}^3$.

Here is an alternative construction: we may obtain $L(p, q)$ also as a quotient $S^3/\mathbb{Z}_p$, where the $\mathbb{Z}_p$-action on S^3 is given by $(z_1, z_2) \mapsto ((\xi_p)^q z_1, \xi_p z_2) = (e^{2\pi i q/p} z_1, e^{2\pi i/p} z_2)$. (Cf. Hatcher [20, Example 2.43].)

Since thus $L(p, q)$ can be constructed from a *free* action of $\mathbb{Z}_p$ on S^3, we get $\pi(S^2/\mathbb{Z}_p) \cong \mathbb{Z}_p$. The interior of the 3-ball won't affect the fundamental group, thus we have $\pi_1(L(p, q)) \cong \mathbb{Z}_p$, and thus also $H_1(L(p, q); \mathbb{Z}) \cong \mathbb{Z}_p$. The homology of the lens spaces is $H_*(L(p, q); \mathbb{Z}) = (\mathbb{Z}, \mathbb{Z}_p, 0, \mathbb{Z})$. Thus homology and fundamental group are independent of q!

Theorem 6.25 (Classification of the lens spaces)**.**
Homeomorphism classification: $L(p, q) \cong L(p, q')$ *if and only if*

$$qq' \equiv \pm 1 \bmod p \qquad \textit{or} \qquad q' \equiv \pm q \bmod p.$$

Homotopy equivalence classification: $L(p, q) \simeq L(p, q')$ *if and only if*

$$q' \equiv \pm a^2 q \bmod p \quad \textit{for some } a \in \mathbb{Z}.$$

The “then” parts of this theorem are elementary geometry, but the “only if” parts are difficult (where the main parts were achieved by Reidemeister resp. Whitehead); cf. Munkres [36, §§40,69].

Corollary 6.26. *The lens spaces $L(7,1)$ and $L(7,2)$ are homotopy equivalent, but not homeomorphic!*

Here is another interesting aspect: $L(3,1)$ has no orientation-reversing homeomorphism! (Kneser 1929, see Seifert–Threlfall [44, Footnote 48].)

6.4 More examples

Here are a few more construction methods for 3-manifolds (all of them important!) in very brief sketches. See Seifert & Threlfall [44, Kap. 9] (classical source) *and* Stillwell [49, Chap. 8] (modern).

Proposition 6.27 (Heegaard decomposition (Heegaard 1898))**.**
Every orientable compact 3*-manifold without boundary may be obtained by glueing two solid handlebodies of genus g (whose boundaries are homeomorphic to M_g). Any decompositon of M into two handlebodies whose boundaries are homeomorphic to M_g is a* Heegaard decomposition of genus g.

Proof. We had already said that M is triangulable. For such a triangulation now consider a “tubular neighborhood” of the 1-skeleton. Its boundary is a connected orientable 2-manifold, thus is is of type M_g for some $g \geq 0$.
Now verify that both the tubular neighborhood as well as the closure of its complement in M are homeomorphic to a handle body. □

The 3-manifolds of Heegard genus 1, which may be obtained by glueing two solid tori, are exactly S^3, $S^2 \times S^1$, and the lens spaces.

Proposition 6.28 (Construction of 3-manifolds by surgery . . .)**.**
Every connected orientable 3-manifold may be obtained from the 3*-sphere by surgery along a knot or a link (that is, several disjoint knots): for this one removes one or several solid tori from S^3 (by “drilling a hole along a knot or link”), and glues them back in differently (“with a twist”).*

Proposition 6.29 (. . . and as branched coverings (Hilden 1974/Montesinos 1976))**.**
Every connected orientable 3-manifold can be obtained from the 3*-sphere as a covering with 3-fold branching along a knot.*

I’d recommend the knot theory book by Rohlfsen [41] for such things. A nice, up-to-date and very visual-concrete overview of (further) construction methods for 3-manifolds, such as the *Seifert manifolds* is given by Lutz [30].

6.5 Some Lie groups

Further interesting and important examples are produced by the theory of *Lie groups*: These are groups that are differentiable manifolds, and for which the group operations are continuous and differentiable. You may think of matrix groups for our purposes.

Examples (Some Lie groups).
The matrix groups $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{R})$, and $\mathrm{SL}(n,\mathbb{C})$ (for $n \geq 1$) are manifolds of dimensions n^2, $2n^2$, n^2-1 resp. $2n^2-2$.

Here $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{C})$ are examples of *complex manifolds*, a very important structure with import structure theory, which will however not be treated in this course.

The matrix groups $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{U}(n)$, and $\mathrm{SU}(n)$ (for $n \geq 1$) are even compact manifolds, of dimensions $\frac{n^2-n}{2}$, $\frac{n^2-n}{2}$, n^2 resp. $n^2 - 1$. Most of these are connected (Counterexample: $\mathrm{O}(1) \cong S^0$; $\mathrm{O}(n)$ has two components.)

Further examples: symplectic groups, spin groups, $O(n, k)$, in particular $\mathrm{O}(3, 1)$.

Note $\mathrm{SO}(2) \cong \mathrm{U}(1) \cong S^1$.

Another important homeomorphism is $\mathrm{SO}(3) \cong \mathbb{R}\mathrm{P}^3$: For $\mathbb{R}\mathrm{P}^3 \cong B^3/\sim \mapsto \mathrm{SO}(3)$ one associates to each vector $v \in B^3$ a rotation with the axis $\mathbb{R}v$ and the angle $\pi|v|$.

The group $\mathrm{SO}(3)$ has a double cover $p : \mathrm{SU}(2) \to \mathrm{SO}(3)$, which is a group homomorphism. In particular $\mathrm{SU}(2) \cong S^3$. This is the group structure on S^3 that is also given by multiplication of unit quaternions. See Knörrer [25, Satz 6.5].

In $\mathrm{SO}(3)$ we may identify as a subgroup the rotations (orientation-preserving symmetries) of the regular dodecahedron/icosahedron, the *icosahedral group* I, with $|I| = 60$. Via the two-fold covering we obtain the *binary icosahedral group* $\widetilde{I} \subset \mathrm{SU}(2)$, which has 120 elements, and whose abelianization is trivial. The quotient $\mathrm{SU}(2)/\widetilde{I}$ yields the dodecahedral space as the quotient of a group action — a covering.

7 Exact Sequences

Definition 7.1 (Exact sequences)**.** A (finite or infinite) sequence of abelian groups and homomorphisms

$$\ldots \to A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} A_3 \xrightarrow{i_3} A_4 \to \ldots$$

is called a *complex* if the composition $i_k \circ i_{k-1}$ of two adjacent maps always yields zero, that is, if $\operatorname{im} i_{k-1} \subseteq \ker i_k$ holds for all k.
The complex is an *exact sequence* if additionally we have $\operatorname{im} i_{k-1} = \ker i_k$, that is, if the complex "has no homology".

Example. A *short exact sequence* is an exact sequence of the form

$$0 \to A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} A_3 \to 0.$$

Exactness here means that i_2 is surjective, with $\ker i_2 \cong A_1$, or equivalently that i_1 is injective, with $\operatorname{coker} i_1 \cong A_3$.

In many situations two terms of an exact sequence are given and the third term has to be reconstructed. For this, however, information about the maps is needed: If, for example, we have $A_1 = A_2 = \mathbb{Z}$, then $A_3 \cong \mathbb{Z}_n$ if i_1 is multiplication with $\pm n$; this yields $A_3 \cong \{0\}$ in the case of $n = 1$.
In particular the middle group A_2 is not in general determined by A_1 and A_3: If

$$0 \to \mathbb{Z} \to A \to \mathbb{Z}_2 \to 0,$$

is exact, then we could have $A \cong \mathbb{Z}$ or $A \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

Exercise. In a short exact sequence the rank of the middle group is the sum of the ranks of the two other groups.

Analogously one develops definitions and basic properties for short exact sequences of vector spaces (and linear maps), and for exact sequences of chain complexes (and chain maps).

Example (Homology of a pair). Fix a coefficient group G (say $G = \mathbb{Z}$), which is used throughout in the following, without being explicitly given by the notation.
Let X be a simplicial complex, $A \subseteq X$ a subcomplex. Then $C_k(A)$ is a subgroup of $C_k(X)$, and the quotient group $C_k(X, A) := C_k(X)/C_k(A)$ is again a free group: The simplices of X that do not lie in A induce a basis. Here

$$0 \to C_k(A) \to C_k(X) \to C_k(X, A) \to 0$$

is a short exact sequence.
The boundary operator of $C_*(X)$ also induces a boundary operator for $C_*(X, A)$: For $c \in C_k(X)$ define $\partial[c] := [\partial c]$; if $a \in C_k(A)$ then ∂a lies in $C_{k-1}(A)$ (since A is a subcomplex!), and we get $\partial[c + a] = [\partial c + \partial a] = [\partial c]$, such that the boundary operator ∂ on $C_k(X, A)$ is well-defined.

Definition 7.2 (Relative homology)**.** The *relative homology of X modulo A* is the homology of the chain complex $C_*(X, A)$.

Examples (reduced/relative homology).
$H_k(X; G) \cong H(X, \emptyset; G)$: ordinary homology is (isomorphic to) homology relative to the empty set.
$\widetilde{H}_k(X; G) \cong H_k(X, \{x_0\}; G)$: Reduced homology is (isomorphic to) homology relative to a a (base)point.

Interpretations:

(1) Relative Homology uses cycles whose boundary doesn't have to be zero, it just should be supported only on A ("lie in A").
(2) Relative homology of (X, A) is reduced homology of a quotient space X/A, in which A has been contracted into a "base point", that is, to relative homology of the pair $(X/A, [A])$:

$$H_k(X, A; G) \;\cong\; \widetilde{H}_k(X/A; G).$$

(Compare Munkres [36, p. 230, Exercise 2].)

Example. We have

$$H_k(B^n, S^{n-1}; \mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z} & \text{for } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

There is a short exact sequence of chain complexes:

$$0 \;\to\; C_*(A) \;\overset{i}{\to}\; C_*(X) \;\overset{p}{\to}\; C_*(X, A) \;\to\; 0.$$

Due to this sequence there is a close connection between the homology of X, of A, and of X modulo A.

Proposition 7.3 (Zig-zag Lemma, [36, Lemma 24.1]: "Where the long exact sequences come from")**.** *Every short exact sequence of chain complexes induces a long exact sequence in homology.*

Theorem 7.4 (The long exact sequence of a pair [36, Thm. 23.3])**.** *If X is a simplicial complex and $A \subseteq X$ is a subcomplex, then there is a long exact sequence*

$$\ldots \;\to\; H_{k+1}(X, A) \;\overset{\partial_*}{\to}\; H_k(A) \;\overset{i_*}{\to}\; H_k(X) \;\overset{p_*}{\to}\; H_k(X, A) \;\overset{\partial_*}{\to}\; H_{k-1}(A) \;\overset{i_*}{\to}\; \cdots$$

where the homomorphisms i_* and p_* are induced by the inclusion $i : A \to X$ resp. by the quotient map $p : (X, \emptyset) \to (X, A)$. The connecting homomorphism *$\partial_* : H_k(X, A) \to H_{k-1}(A)$ is induced by the boundary operator on $C_*(X)$: Every homology class in $H_k(X, A)$ is represented by a chain $c \in C_k(X)$ whose boundary lies in $C_{k-1}(A)$; the boundary operator with this is obtained by $\partial_*[c] := [\partial c]$.*

Analogously we also get this long exact sequence for reduced homology.

Proposition 7.5 (Excision (version for complexes) [36, Thm. 9.1])**.** *If X' and A are subcomplexes of X with $X \setminus X' \subseteq A$ (that is, $A \cup X' = X$), then*

$$H_k(X', X' \cap A) \;\cong\; H_k(X, A).$$

Interpretation: the relative homology does not "see" what happens in A: We can thus add a cone over A, or cut out parts of A, without any changes in the relative homology.

This holds in particular for $X' := \overline{X \backslash A}$, the complex generated by $X \backslash A$:

$$H_k(\overline{X \backslash A}, \overline{X \backslash A} \cap A) \;\cong\; H_k(X, A).$$

Theorem 7.6 (Mayer[10]–Vietoris[11]sequence [36, Thm. 25.1])**.** *If $X = A \cup B$ for subcomplexes $A, B \subset X$, then there is a long exact sequence*

$$\to\; H_k(A \cap B) \;\to\; H_k(A) \oplus H_k(B) \;\to\; H_k(X) \;\overset{\partial}{\to}\; H_{k-1}(A \cap B; \mathbb{Z}) \;\to$$

and analogously for reduced homology (in the case when $A \cap B \neq \emptyset$.

[10]Walther Mayer, 1887–1948, later Einsteins assistent: `http://en.wikipedia.org/wiki/Walther_Mayer`

[11]Leopold Vietoris, 1891–2002! See `http://www.ams.org/notices/200210/fea-vietoris.pdf`

Proof. It is rather natural to come up with the following short exact sequence of chain complexes, where we only have to insert a minus sign in order to make it exact:

$$0 \;\to\; C_*(A\cap B) \;\stackrel{(i'\oplus i'')}{\longrightarrow}\; C_*(A)\oplus C_*(B) \;\stackrel{(i''',-i'''')}{\longrightarrow}\; C_*(A\cup B) \;\to\; 0.$$

This short exact sequence of chain complexes induces the Mayer–Vietoris sequence. □

Examples. The *disjoint union* of two spaces can be treated via the Mayer–Vietoris sequence, where $A\cap B=\emptyset$, and $H_i(\emptyset;G)=0$ for all i:

$$H_k(A\uplus B;G) \;\cong\; H_k(A;G) \,\oplus H_k(B;G).$$

The *wedge (one point union)* of two spaces is most easily treated via the Mayer–Vietoris sequence for reduced homology, with $A\cap B=\{x_0\}$, and $\widetilde{H}_i(\{x_0\};G)=0$ for all i:

$$\widetilde{H}_k(A\uplus B;G) \;\cong\; \widetilde{H}_k(A;G) \,\oplus \widetilde{H}_k(B;G).$$

Thus, for example, for a bouquet (wedge) of n k-spheres, we get

$$\widetilde{H}_i(\bigvee_n S^k;\mathbb{Z}) \;\cong\; \begin{cases}\mathbb{Z}^n & \text{for } i=n\\ 0 & otherwise.\end{cases}$$

The *suspension* $\Sigma X=\operatorname{susp} X = X*S^0$ of a space X can be written as a union of two cones: $\operatorname{susp} X = (X*x_0)\cup(X*x_1)$. The Mayer–Vietoris sequence for this yields

$$\widetilde{H}_k(\operatorname{susp} X)\cong \widetilde{H}_{k-1}(X).$$

More generally, one can treat the join with a discrete set of $p\geq 1$ points, which we identify with $\mathbb{Z}_p=\{x_0,x_1,\ldots,x_{p-1}\}$; here we either use the Mayer–Vietoris sequence and induction, or we use relative homology, where

$$H_k(X*\mathbb{Z}_p, X*x_0) \;\cong\; H_k(X*\{x_0,x_1\},X*x_0)^{p-1}.$$

Thus we get

$$\widetilde{H}_k(X*\mathbb{Z}_p)\cong \widetilde{H}_{k-1}(X)^{p-1}.$$

By induction we can thus treat the spaces

$$E_n\mathbb{Z}_p \;:=\; \underbrace{\mathbb{Z}_p*\cdots*\mathbb{Z}_p}_{n+1 \text{ factors,}}$$

which have the important property to be n-dimensional, $(n-1)$-connected complexes with a free $\mathbb{Z}_p$-action. We get

$$\widetilde{H}_k(E_n\mathbb{Z}_p;\mathbb{Z}) \;\cong\; \begin{cases}\mathbb{Z}^{(p-1)^{n+1}} & k=n\\ 0 & \text{otherwise.}\end{cases}$$

As special cases this contains the homology of the n-dimensional sphere $S^n\cong E_n\mathbb{Z}_2$, and of the complete bipartite graph $K_{3,3}\cong E_1\mathbb{Z}_3$.

Similarly, one can try to compute more generally the homology of a join, by writing $X*Y$ as a union of two parts, where one is homotopy equivalent to X and the other one to Y (indeed, admits a retraction to X resp. Y), and the intersection is homeomorphic to $X\times Y$. We can easily write down a Mayer–Vietoris sequence for this. The problem we face then is to determine the homology of the product $X\times Y$, for which we now state the so-called Künneth theorem.

Theorem 7.7 (Künneth theorem [36, Thm. 59.3]). *For arbitrary topological spaces X, Y there is an exact sequence*

$$0 \to \bigoplus_{p+q=k} H_p(X) \otimes H_q(Y) \to H_k(X \times Y) \to \bigoplus_{p+q=k} H_{p-1}(X) * H_q(Y) \to 0$$

where the torsion products "$*$" vanish if either we use field coefficients, or if at least one of the factors is free. (For all other cases we refer to Munkres [36, p. 331].)

The algebraic way of describing homology via chain complexes is important also since it allows us to use *the same formal apparatus* to define *different homology theories* — which then turn out to have some advantages and differences, but largely the same/analogous properties.

Our main example here is the *singular homology* theory, for which we give a very brief description/ "definition":

Definition 7.8 (Singular homology [36, §29]).
Let X be an arbitrary topological space.
A *singular k-simplex in X* is a continuous map $\Delta_k \to X$ from the (standard) k-simplex Δ_k to X.
A *singular k-chain with coefficients in G* is a finite formal linear combination of singular k-simplices in X. The singular k-simplices in X form the *singular chain group $S_k(X; G)$*.
Now we define the boundary of a simplex, and this yields a boundary operator $\partial_k : S_k(X; G) \to S_{k-1}(X; G)$.
Then we define cycles and boundaries just as for the simplicial homology theory, and this yields the *singular homology groups* of X with coefficients in G.

The singular homology groups are usually also denoted by $H_k(X; G)$, which makes sense only since (a non-trivial theorem!) for complexes singular and simplicial homology groups are canonically isomorphic:

Theorem 7.9 ([36, §34, Theorems 34.3 and 35.5]). *For triangulable spaces, simplicial and singular homology are canonically isomorphic: There is a welldefined isomorphism $\eta_* : H_*^{\text{simp}}(K; G) \cong H_*^{\text{sing}}(K; G)$ that commutes with the boundary map ∂_* and with the homomorphisms induced by continuous maps.*

Exercise. Describe a canonical map from the simplicial chain group $C_k^{\text{simp}}(K; G)$ to the singular chain group $C_k^{\text{sing}}(K; G)$ of a simplicial complex K. Show that it is a chain map, and induces a map in homology.

Note: singular homology groups are definied for *arbitrary* topological spaces, and it is rather easy to show that they are invariants of homotopy type. On the other hand, it is much more difficult to "compute" singular homology groups with bare hands (without use of theorems, exact sequences, etc.) since already the chain groups do not have a finite or even countable rank, and thus we have no bases at hand.

One verifies that singular homology shares most of the fundamental properties of simplicial homology:

- singular homology is a functor,
- homotopic maps induce the same map in homology
- homotopy equivalent spaces have isomorphic singular homology
- excision for relative singular homology
- long exact sequence of a pair
- Mayer–Vietoris sequence in the case of a covering $X = U_1 \cup U_2$ by two open sets,
- etc.

Since we now can work with open subsets of a space, we also have access to further constructions such as local homology.

Definition 7.10 (Local homology [36, §35])**.**
The *local homology* of X at the point $x_0 \in X$ is the relative homology $H_k(X, X\backslash\{x_0\})$.

This version of the definition needs explanation/interpretation in the setting of simplicial complexes/ simplicial homology, of course: "It works" if you interpret $X\backslash\{x_0\}$ as the (geometric) simplicial complex obtained from X by removing all simplices that contain the point x_0.

Looking at local homology in particular yields the "invariance of dimension".

Corollary 7.11 (Invariance of dimension)**.**
Manifolds of different dimension cannot be homemorphic: for interior (non-boundary) points of an n-dimensional manifold X we get $H_n(X, X\backslash\{x_0\}) = \mathbb{Z}$, but $H_k(X, X\backslash\{x_0\}) = 0$ for $k \neq n$.

Similarly, the boundary is a topological invariant of a manifold: The local homology vanishes at boundary points: $H_k(X, X\backslash\{y_0\}) = 0$ for all *k if y_0 is a boundary point.*

Let's close the chapter on exact sequences with an application to homotopy groups: A more algebraic wording of the Seifert–van Kampen theorem, via a short exact sequence (of non-abelian groups!):

Theorem 7.12 (Seifert–van Kampen [37, p. 431])**.** *If a space X is a union $X = X_1 \cup X_2$ of two open subsets with a path-connected intersection and a common base point $x_0 \in X_1 \cap X_2$, with inclusions $i_i : X_i \hookrightarrow X$, then*

$$\pi_1(X_1) * \pi_1(X_2) \longrightarrow \pi_1(X_1 \cup X_2) \longrightarrow 1$$

is exact: $\pi(X)$ is the image of the surjective map induced by (i_1, i_2). Its kernel is the nomalizer of the image of $\pi_1(X_1 \cap X_2) \to \pi_1(X_1) * \pi_1(X_2)$.

8 Cell complexes

We have seen that working with simplicial complexes may be unwieldy and laborious due to the large number of simplices in practically any triangulation of an interesting topological space. Here we will describe two alternative concepts, "CW complexes" and "regular CW complexes". CW complexes were introduced by J. H. C. Whitehead in the fifties, and they have very soon been accepted and used as a very basic structure – see [20, Chap. 0]. (Note that the abbreviation CW is *not* derived from the initials of the inventor, but denotes "closure-finite" and "weak topology".)

A *cell* is a topological space that is homeomorphic to B^k. Here k is the *dimension* of the cell, which is also referred to as a k*-cell*. Instead of B_k we could also use $[0,1]^k$ a k-simplex as our "model" — the combinatorial structure of cells is not used (this is a major difference to the case of simplicial complexes).

An *open cell* is a topological space that is homeomorphic to $\operatorname{int} B^k$ (that is, to $\mathbb{R}^k$).

Definition 8.1 (CW complexes; regular CW complexes [36, §38])**.** A *CW complex* is a Hausdorff space X which is a disjoint union $X = \biguplus e_\alpha$ of subspaces that are open cells, such that

- For every cell e_α there is a *characteristic map*: a continuous map $f_\alpha : B^k \to X$ that maps $\operatorname{int} B^k$ homeomorphically to e_α, and maps the boundary $\partial B^k = S^{k-1}$ continuously into a finite (!) union of cells e_β that all have smaller dimension than k. The image $f_\alpha(B_k)$ is denoted $\bar{e}_\alpha$.
- A subset $A \subset X$ is closed if and only if every intersection $A \cap \bar{e}_\alpha$ is closed.

A CW complex is *regular* if all the characteristic maps are homeomorphisms, and the image is a subcomplex (that is, a finite union of cells). In that case (but not for general CW complexes), every closure of an open cell $\bar{e}_\alpha$ is a subcomplex, and it is a ball.

Exercise. In a CW complex, the closure of an open cell e_α is the image $f_\alpha(B_k) = \bar{e}_\alpha$ of the corresponding attaching map.

Exercise. Finite CW complexes are compact. A subset of a CW complex is compact if and only if it is closed and hits only finitely many open cells.

Examples.
S^n: CW complex consisting of one 0- and one n-cell.
M_g: one vertex, $2g$ edges, one 2-cell.
$\mathbb{R}\mathrm{P}^n$: CW structure with exactly one cell in each dimension k, for $0 \le k \le n$. The k-skeleton is a $\mathbb{R}\mathrm{P}^k$.
$\mathbb{C}\mathrm{P}^n$: one $2k$-cell for $0 \le k \le n$, no odd-dimensional cells. The $2k$-skeleton is a $\mathbb{C}\mathrm{P}^k$.

Every non-empty CW-complex has at least one vertex. It is connected if and only if it is path-connected. (The 1-skeleton is an "(undirected) graph" in the terminology of graph theory, where multiple edges and loops are allowed.)

An alternative (equivalent) description of CW complexes is as follows [20, p. 7]: Every CW complex X can be built up "by induction over the dimension of the skeleton". The 0-skeleton X^0 consists of a set of vertices (with discrete topology). The k-skeleton is obtained by attaching k-cells to the $(k-1)$-skeleton X^{k-1}, where B^k is attached via a map $f_\alpha : \partial B^k = S^{k-1} \to X^{k-1}$ that may hit only finitely many cells in its image.

Definition 8.2 (CW pair)**.** A *CW pair* is a pair (X, A) that consists of a CW complex X and a subcomplex A.

In every CW complex X the k-skeleton X^k is a subcomplex, so (X, X^k) is a CW pair. Similarly, (X^{k+1}, X^k) is a CW pair.

CW complexes are more flexible and easier to handle than simplicial complexes in many situations:

Proposition 8.3 (Constructions [20, pp. 8,9])**.**
If X and Y are CW complexes, and one of them is locally-finite, then the product $X \times Y$ is again a CW complex (with product topology).

If X is a CW complex then its suspension, $\operatorname{susp} X$*, is again a CW complex, with one $(k+1)$-cell for each non-empty k-cell of X, plus two additional vertices.*

If (X, A) is a CW pair with $A \neq \emptyset$, then X/A is again a CW complex. It has one vertex as a "base point" that corresponds to A, and one k-cell for each k-cell of X that does not lie in A.

We write $f_k = f_k(X)$ for the number of k-dimensional cells of X (if this is finite).

Lemma 8.4. *Every regular CW complex X is triangulable: The barycentric subdivision* $\operatorname{sd} X$ *is a simplicial complex that is homeomorphic to X.*

Thus regular CW complexes are determined up to homeomorphism by the discrete/combinatorial data of the "face poset", the partial order determined by the closed cells and their inclusion relations. Thus they can be described combinatorially — where typically they have much fewer cells than are needed for a triangulation of the same space.

Not every CW complex has a triangulation (example/proof: see [36, p. 218]).

Examples. S^n has the (minimal) structure of a regular CW complex with two vertices, two edges, etc.: this is generated by repeated suspension from the empty set ((-1)-sphere). Its barycentric subdivision yields the "octahedral" triangulation of S^n with $2(n+1)$ vertices and 2^n facets (n-simplices).

Every convex n-dimensional polytope has the face structure of a regular cell complex (homeomorphic to B^n).

Lemma 8.5. *If X is a CW complex, then $H_k(X^k, X^{k-1}; \mathbb{Z}) \cong \mathbb{Z}^{f_k}$, and $H_i(X^k, X^{k-1}; \mathbb{Z}) = 0$ otherwise.*

Definition 8.6 (Cellular homology)**.** Let X be a X CW complex. We define the *cellular chain groups* by

$$D_k(X; \mathbb{Z}) \ := \ H_k(X^k, X^{k-1}; \mathbb{Z})$$

and the *cellular boundary operator* $\partial_k : D_k(X; \mathbb{Z}) \to D_{k-1}(X; \mathbb{Z})$ as the composition

$$\partial_k : H_k(X^k, X^{k-1}; \mathbb{Z}) \ \stackrel{\partial_*}{\to} \ H_{k-1}(X^{k-1}; \mathbb{Z}) \ \stackrel{i_*}{\to} \ H_{k-1}(X^{k-1}, X^{k-2}; \mathbb{Z})$$

where ∂_* is the boundary operator of the long exact sequence of the pair (X^k, X^{k-1}), while i_* is induced by the inclusion $i : (X_{k-1}, \emptyset) \subset (X^{k-1}, X^{k-2})$, so it is a map from the long exact sequence of the pair (X^{k-1}, X^{k-2}).

The homology of the chain complex $(D_k(X; \mathbb{Z}), \partial_k)$ is called the *cellular homology* of X.

Lemma 8.7. *The cellular boundary operator as defined in Def. 8.6 satisfies $\partial_{k-1} \circ \partial_k = 0$, so it defines a chain complex.*

Proof. For this we look at the composition $\partial_k \circ \partial_{k+1}$:

$$H_k(X^{k+1}, X^k; \mathbb{Z}) \stackrel{\partial_*}{\to} H_k(X^k; \mathbb{Z}) \ \stackrel{i_*}{\to} H_k(X^k, X^{k-1}; \mathbb{Z}) \stackrel{\partial_*}{\to} H_{k-1}(X^{k-1}; \mathbb{Z}) \stackrel{i_*}{\to} H_{k-1}(X^{k-1}, X^{k-2}; \mathbb{Z})$$

and notice that the two maps in the center follow each other in the long exact sequence of the pair (X^k, X^{k-1}), so their composition is zero. □

Theorem 8.8. *For every CW complex X, the cellular homology of X is canonically isomorphic to simplicial homology of X (if X is triangulable) and to singular homology of X.*

Along the patterns we know by now, one also defines cellular homology with coefficients, in particular with $\mathbb{Q}$- and with $\mathbb{Z}_2$-coefficients. (The latter often is much easier to compute since then we don't have to worry about orientations of cells/signs.)

Examples. The homology of $\mathbb{C}\mathrm{P}^n$ follows from the cell decomposition claimed above, as

$$H_{2k}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z} & \text{for } 0 \le k \le n, \\ 0 & \text{otherwise,} \end{cases}$$

since the cellular chain complex has the form

$$0 \;\to\; \mathbb{Z} \;\to\; 0 \;\to\; \mathbb{Z} \;\to\; \ldots \;\to\; \mathbb{Z} \;\to\; 0 \;\to\; \mathbb{Z} \;\to\; 0$$

and thus all boundary homomophisms are zero maps.
For $\mathbb{R}\mathrm{P}^n$ one has to work a bit harder, and gets

$$H_i(\mathbb{R}\mathrm{P}^n;\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ \mathbb{Z}_2 & \text{for odd } i,\ 0 < i < n, \\ \mathbb{Z} & \text{for odd } i = n, \\ 0 & \text{otherwise,} \end{cases}$$

but also

$$H_i(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \;\cong\; \begin{cases} \mathbb{Z}_2 & \text{for } 0 \le i \le n, \\ 0 & \text{othewise,} \end{cases}$$

which is more useful and more important for many applications.
(...in particular, the whole theory of "characteristic classes" is based on this.)

Note: If X has a cell decomposition with f_k k-cells, then the cellular chain group $D_k(X;\mathbb{Z}) \cong \mathbb{Z}^{f_k}$ has rank f_k. The (cellular) homology group $H_k(X;\mathbb{Z})$ is a quotient group of a subgroup of this, so it also has rank at most f_k. Thus we have

$$\beta_k \;\le\; f_k.$$

This is the trivial form of the so-called Morse inequalities, which can be sharpened considerably. So, for example, we have not only $\beta_0 \le f_0$ and $\beta_1 \le f_1$, but even $\beta_1 - \beta_0 \le f_1 - f_0$. Etc.

A perspective: What is "Morse Theory"?
A generic "height" or distance function on a smooth manifold is a "Morse function", all of whose critical points are isolated and non-degenerate. From Morse functions one can derive CW decompositions (with one k-cell for each critical point of index k); they also yield the Morse complex which computes the corresponding cellular (co)homology, or at least bounds on homology via the Morse inequalities.
The classical reference for all of this is the book by Milnor [33].

9 Cohomology

Let's start with an algebraic description.

Definition 9.1 (Cochain complex)**.** A *cochain complex*

$$\mathcal{C}^* = (C^k, d^k)_{k\in\mathbb{Z}}$$

is a sequence of abelian groups C^k (note the notation: the upper index denotes dimension!) and homomorphisms $d^k : C^k \to C^{k+1}$ (note: dimension goes up!), with the condition $d^{k+1} \circ d^k = 0$ for all k.

The groups

$$H^k(\mathcal{C}^*) := \ker(d^k : C^k \to C^{k+1}) / \operatorname{im}(d^{k-1} : C^{k-1} \to C^k)$$

are called the *cohomology groups* of the complex.

The "co" in the terminology denotes a dualization — which also explains "why the maps go into the other direction".

Lemma 9.2. *If $\mathcal{C}_* = (C_k, \partial_k)$ is a chain complex and G is an abelian group, then setting $C^k := \operatorname{Hom}(C_k, G)$ and $d^k(f) := f \circ \partial_{k+1}$ yields a cochain complex $\mathcal{C}^* := (C^k, d^k)$.*

If C and G are abelian groups, then $\operatorname{Hom}(C, G)$ is an abelian group as well. If $f : C \to C'$ is a group homomorphims, then $f^* : \operatorname{Hom}(C', G) \to \operatorname{Hom}(C, G)$, $h \mapsto f \circ h$ is again a group homomorphism. Moreover, a direct computation shows that dualization is compatible with composition: If $f : C \to C'$ and $g : C' \to C''$, then $(g \circ f)^= f^* \circ g^*$.

For computations, one sorts out that $\operatorname{Hom}(\mathbb{Z}^f, \mathbb{Z}) \cong \mathbb{Z}^f$, and more generally $\operatorname{Hom}(\mathbb{Z}^f, G) \cong G^f$. If furthermore $f : \mathbb{Z}^f \to \mathbb{Z}^{f'}$ is a homomorphism that is represented by the matrix $A \in \mathbb{Z}^{f'\times f}$, then f^* is represented by the transposed matrix $A^T \in \mathbb{Z}^{f\times f'}$.

Definition 9.3 (Simplicial/cellular/singular cohomology)**.** Let X be a topological space, possibly given by a simplicial or CW complex, and let G be a group of coefficients. Let $\mathcal{C}_*(X;\mathbb{Z})$ be the corresponding chain complex for (simplicial, cellular, resp. singular) homology of X with integer coefficients.

The cohomology of the cochain complex obtained from this via Lemma 9.2

$$\mathcal{C}^*(X;G) := \big(C^k(X;G) := \operatorname{Hom}(C_k(X;\mathbb{Z}), G),\ d^k : f \mapsto f \circ \partial_{k+1}\big)$$

is then the *simplicial, cellular, resp. singular cohomology of X with coefficients in G.*

Example. For the real projective plane we have a CW cell decomposition into one vertex, one edge and one 2-cell, and thus the cellular chain complex

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0,$$

and this yields the homology groups

$$H_0(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) = \mathbb{Z}, \qquad H_1(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) = \mathbb{Z}_2, \qquad H_2(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) = 0.$$

Dualization yields the (cellular) cochain complex

$$0 \leftarrow \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0,$$

and thus the cohomology of the projective plane:

$$H^0(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) = \mathbb{Z}, \qquad H^1(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) = 0, \qquad H^2(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) = \mathbb{Z}_2.$$

In analogy to the geometric description of chains, cycles, boundaries, etc. in simplicial theory one can also describe cochains, cocycles, coboundaries etc. geometrically.

A k-dimensional *cochain* is then a function that assigns a value $f(\sigma_k) \in G$ to every k-dimensional simplex σ_k. The *coboundary* is then obtained as follows:

$$df(\tau_{k+1}) = f(\partial\tau_{k+1})$$

is the sum of all f-values of k-simplices in the boundary of τ_{k+1}, with the right signs (as given by the boundary operator ∂ of the chain complex). See [36, §42].

The "k-dimensional cohomology with coefficients in G" is a *contravariant functor* from the category of topological spaces (and continuous maps) to the category of (sequences of) abelian groups (and group homomorphisms):

Theorem 9.4 (Cohomology is a contravariant functor)**.** *The construction $X \to H^k(X;G)$ assigns to each topological space an abelian group, and to each continuous map a group homomorphism. However, the homomorphism goes "in the other direction": $f : X \to Y$ induces $H^k(f) : H^k(Y;G) \to H^k(X;G)$. This is in contrast to "k-dimensional homology with coefficients in G", which is a* (covariant) functor*, since $f : X \to Y$ induces $H_k(f) : H_k(X;G) \to H_k(Y;G)$.*

Moreover, just like homology, cohomology depends only on the homotopy type, so indeed we get a functor from *homotopy types* to groups: Homotopic maps yield the same homomorphism in cohomology, homotopy equivalences induce isomorphisms in cohomology, and thus homotopy equivalent sapces have isomorphic cohomology.

Moreover, cohomology can be computed directly from homology. Thus spaces with isomorphic (finitely-generated) homology also have isomorphic cohomology. More precisely, $H^k(X;\mathbb{Z})$ has the same rank as $H_k(X;\mathbb{Z})$, but the same torsion subgroups as $H_{k-1}(X;\mathbb{Z})$. More canonical versions of this use a short exact sequence. See [36, Cor. 45.6/Thm. 53.1].

Thus homology and cohomology are, at least, rather "similar". This is reflected for example in the fact that homology and cohomology are (non-canonically) isomorphic, if field coefficients are used and homology is finite-dimensional. Why should we be interested in cohomology at all, if it does not contain any additional information? One good reason is that cohomology has an additional multiplicative structure, it forms a ring, the *cohomology ring*.

(A second reason will come up in the next chapter: Poincaré duality is a duality between the homology and the cohomology of a manifold — so we need them both.)

Definition 9.5 (Cup product)**.** Let R be a ring of coefficients (e.g. $\mathbb{Z}$ or a field). The *cup product* is the homomorphism

$$\cup : H^k(X;R) \otimes H^\ell(X;R) \to H^{k+\ell}(X;R),$$

which is induced by the diagonal map

$$\Delta : X \to X \times X, \qquad x \mapsto (x,x)$$

followed by the product map

$$\times : H^k(X;R) \otimes H^\ell(X;R) \to H^{k+\ell}(X \times X;R)$$

of the cohomological version of the Künneth theorem [36, Thm. 60.5].

In simplicial theory, the cup product can be represented by explicit combinatorial formulas. Essentially,

$$\langle c^k \cup c^\ell, \sigma_{p+q} \rangle \ =$$

$$\langle c^k,\ \text{face of } \sigma_{p+q} \text{ spanned by the first } p+1 \text{ vertices} \rangle \cdot \langle c^\ell,\ \text{face spanned by the last } q+1 \text{ vertices} \rangle.$$

However, the cellular chain complex is not sufficient to derive this! See Munkres [36, p. 292].

A basic computation yields that

$$d(c^k \cup \bar{c}^\ell) \ = \ (dc^k \cup \bar{c}^\ell) + (-1)^k (c^k \cup d\bar{c}^\ell)$$

for a k-chain c^k and an ℓ-chain c^ℓ, which implies that the product of two cocycles yields a cocycle, and also that the product of a cocycle with a coboundary is a coboundary, and thus that the cup-product of cohomology classes is well-defined.

Theorem 9.6. *Let X be a topological space. The cohomology $\bigoplus_{k \geq 0} H^k(X; R)$ with multiplication given by cup product has the structure of an associative ring with a unit. It is not commutative, but satisfies*

$$\alpha^k \cup \beta^\ell \ = \ (-1)^{k\ell} \beta^\ell \cup \alpha^k.$$

Every continuous map $f : X \to Y$ induces a ring homomorphism. Homotopic maps induce the same ring homomorphism. Thus the cohomology ring *is a homotopy invariant.*

Examples. $H^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[u]/(u^{n+1})$ [36, Thm. 68.3] and $H^*(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) \cong \mathbb{Z}[t^2]/(t^{2n+2})$ are "truncated polynomial algebras". From this one can derive that for $m < n$ every map $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^m$ will map the fundamental group trivially; this yields, for example, a new proof of the Borsuk–Ulam theorem 5.16; see [36, p. 405].

Concerning the information that is "hidden" in the product structure of cohomology see the rather recent works [52] and [13].

Remark 9.7 (de Rham cohomology). On smooth manifolds, the differential forms form a cochain complex, with the exterior derivative as the coboundary operator. The cohomology groups derived from this are known as the *de Rham cohomology*. They are defined with $\mathbb{R}$-coefficients (there is no natural construction with $\mathbb{Z}$-coefficients). However, differential forms can be multiplied: The wedge (exterior) product of differential forms induces the cub product for de Rham cohomology.

10 Manifolds II: Poincaré duality

First let us note that for practically any sequence of abelian groups one $H_0, H_1, H_2, \ldots$ one can construct a CW complex X that has exactly these homology groups (see Munkres [36, p. 231, Problem 4]). The only restrictions are of the type

- H_0 is always free,
- if X has dimension n, then $H_n(X;\mathbb{Z})$ is free, and $H_i(X;\mathbb{Z}) = 0$ for $i > n$,

There is no such result for the homology of manifolds — the homology of manifolds has extra conditions/restrictions. For the following let us consider n-dimensional manifolds that are connected, closed (compact, no boundary), and triangulable. One can extend the discussion to *homology manifolds*, for which the links of vertices are only required to have the homology of a sphere (which is a much weaker condition than to be a sphere).

Lemma 10.1. *Let M be a connected, closed, triangulable n-manifold.*
Then either $H_n(M;\mathbb{Z}) = \mathbb{Z}$ (if M is orientable), or $H_n(M;\mathbb{Z}) = 0$ (if M is not orientable).
In the first case, $H^n(M;\mathbb{Z}) = \mathbb{Z}$, in the second case $H^n(M,\mathbb{Z}) \cong \mathbb{Z}_2$.
In the non-orientable case $H_{n-1}(M;\mathbb{Z})$ has a $\mathbb{Z}_2$-torsion summand (and is free otherwise).

So clearly there are substantial restrictions to the possible homology groups. The following result has much greater scope.

Theorem 10.2 (Poincaré duality [36, Thm. 65.1])**.** *Let M be a connected closed triangulable n-dimensional (homology) manifold. Then*

$$H_k(M;G) \;\cong\; H^{n-k}(M;G) \qquad \textit{for all } k$$

for an arbitrary group of coefficients G if M is orientable, and for $G = \mathbb{Z}_2$ also in the non-orientable case.

For the following let $\beta_k := \operatorname{rank} H_k(M;\mathbb{Z})$ denote the k-th Betti number of M.
Exercise. For every finite CW complex (which need not be a manifold), we have

$$\beta_k \;=\; \operatorname{rank} H_k(M;\mathbb{Z}) \;=\; \dim H_k(M;\mathbb{Q}) \;=\; \operatorname{rank} H^k(M;\mathbb{Z}) \;=\; \dim H^k(M;\mathbb{Q}).$$

(Hint: elementary linear algebra!)

You might check that this, as well as the claims for Poincaré duality, are valid in the case of homology and cohomology of the surfaces M_g resp. M'_g, which we have computed directly resp. may be derived easily in cellular theory.

Corollary 10.3. *For orientable n-manifolds*

$$\beta_k \;=\; \beta_{n-k},$$

so the sequence of Betti numbers is symmetric.
For non-orientable manifolds this is also true if one uses $\mathbb{Z}_2$-coeffients, that is, $\beta_k := \dim_{\mathbb{Z}_2} H_k(M;\mathbb{Z}_2)$.
In particular, for odd-dimensional manifolds (orientable or not), the Euler-characteristic vanishes: $\chi(M) = 0$ of $\dim(M) = n$ is odd. For even dimensional manifolds, we have $\chi(M) \equiv \beta_{n/2} \pmod 2$, where the $\mathbb{Z}_2$ Betti number $\beta_{n/2} = \dim H_{n/2}(M;\mathbb{Z}_2)$ has to be used if M is not orientable.

Proof (Poincaré duality). Let us consider a regular CW decomposition X of M, and the dual decomposition X^*. Every k-cell σ_k of X corresponds to an $(n-k)$-cell σ_k^* in X^*. If M is oriented, then every orientation of a k-cell of X canonically corresponds to an orientation of the associated dual cell. (Example, for $n = 2$, $k = 1$: every directed edge on a surface corresponds to a dual edge, whose orientantion may be described by "rotation in counter-clockwise orientation", which refers to the global orientation of the surface).

Thus for computation of the cellular homology of X, and of the cellular cohomology of X^*, we get complexes

$$\begin{array}{ll} \mathcal{D}_* & 0 \to D_n \to D_{n-1} \to \ \dots \ \to D_1 \to D_0 \to 0 \\ \mathcal{D}^* & 0 \to D^0 \to D^1 \to \ \dots \ \to D^{n-1} \to D^n \to 0 \end{array}$$

with isomorphic groups $D_k \cong D^{n-k}$ and with *the same* maps/matrices. Thus the complexes also yield isomorphic homology, $H_k(X; G) \cong H^{n-k}(X^*; G)$.

In the non-orientable case the orientation part of this argument does not work, but if one uses $\mathbb{Z}_2$-coefficients then it is not needed either. $\square$

This (sketch of) a proof is problematic in so far as the construction of the "dual cell complex" works only under some cautionary assumptions, as is demonstrated by the following famous result. In general the "cells" of the dual complex are not topological balls, since the boundaries into which we have to glue the cells are not spheres but only homology spheres.

Possible/sufficient "cautionary assumptions" are:

- for dimension $n \leq 3$ there are no problems,
- one can work in a category of so-called *PL manifolds* (see Rourke & Sanderson [42]), or
- one can work with homology manifolds and the "dual block complex", for which cellular homology and cohomology can still be made to work (see Munkres [36, §64]).

Theorem 10.4 (Double Suspension Theorem (Edwards [12], Cannon [9]))**.**
For an arbitrary homology 3*-sphere* Σ^3 *(that is, a* 3*-manifold with the homology of a* 3*-sphere), the double suspension* $S^1 * \Sigma = \operatorname{susp}\operatorname{susp}\Sigma^3$ *is homeomorphic to* S^5*.*

We can easily construct a triangulation of the Poincaré-sphere. Its double suspension will be obtained as a simplicial complex with four additional vertices. This yields a triangulation of S^5 for which some edges have Σ^3 as their link; the dual block complex then has a "block" that is homeomorphic to $\operatorname{cone}\Sigma^3$, which is not a cell.

Poincaré duality is connected with further important properties of manifolds and their cohomology. Here we state an important manifestation in the cohomology ring.

Theorem 10.5 (Dual pairing)**.** *Let* M *be a connected triangulable closed* n*-manifold. If* M *is orientable, then the cup product induces a map*

$$\cup: \ H^k(M;\mathbb{Z})/\text{torsion} \ \otimes \ H^{n-k}(M;\mathbb{Z})/\text{torsion} \ \ \to \ \ \mathbb{Z}$$

that is a dual pairing. *Furthermore,*

$$\cup: \ H^k(M;K) \ \otimes \ H^{n-k}(M;K) \ \ \to \ \ K$$

is a non-degenerate bilinear form for an arbitrary field of coefficients K *if the manifold is orientable, and for* $K = \mathbb{Z}_2$ *in the non-orientable case.*

This implies for example our claims about the cohomology rings of real and complex projective spaces in Chapter 9!

The pairing of Theorem 10.5 is particularly interesting in the case $k = n - k$, that is, for $k = n/2$ (where n is even). The $\cup$ defines a bilinear form on $H^{n/2}(M;\mathbb{Z})/\mathrm{torsion}$, which is symmetric for $n \cong 0 \bmod 4$ and antisymmetric for $n \cong 2 \bmod 4$.

This again can be worked out elementarily for surfaces. For 4-manifolds it is connected with a spectacular result of modern algebraic topology. In its formulation we use that for simply-connected M the group $H^2(M;\mathbb{Z})$ does not have torsion (since otherwise $H_1(M;\mathbb{Z})$ would have torsion as well).

Theorem 10.6 (Classification of simply-connected 4-manifolds; Milnor (1958) und Freedman (1986))**.** *The correspondence*

$$\delta : \left\{\begin{array}{l}\textit{homotopy types of}\\ \textit{simply connected}\\ \textit{orientable triangulable}\\ 4\textit{-manifolds}\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\textit{equivalence classes of}\\ \textit{non-degenerate}\\ \textit{symmetric bilinear forms}\\ (\mathbb{Z}^N; \langle\cdot,\cdot\rangle)\end{array}\right\},$$

which to each 4*-manifold associates the group* $H^2(M;\mathbb{Z})$ *and on it the bilinear form given by cup product, is injective and surjective.*

Thus the homotopy types of such manifolds are completely classified by purely algebraic objects. We refer to Milnor & Husemoller [34] for more information about this.

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Software

Examples — computations can be done using "`topaz`" by Michael Joswig and Ewgenij Gawrilow, a module for the `polymake` software system, `http://www.math.tu-berlin.de/polymake/`