Proofs for THE BOOK

Günter M. Ziegler
ziegler@math.tu-berlin.de

Plan

1. On Proofs
2. The BOOK of Proofs
3. Two-Coloring Proofs
4. Odd Dissections of a Square
5. Enumerating the Rationals
6. Kneser's Conjecture
7. Places

Plan

Thesis 1: Mathematics is difficult!

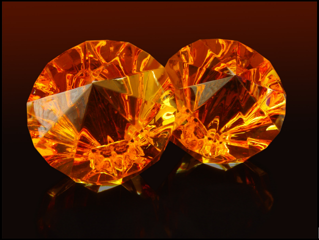
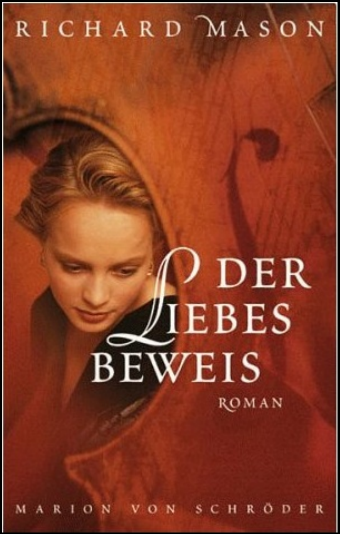


Figure: Pixelio.de

*"I hope that the popular press will continue to portray mathematics as being like a diamond:
extremely hard material, but valuable and highly prized both for its industrial applications and for its intrinsic beauty."*

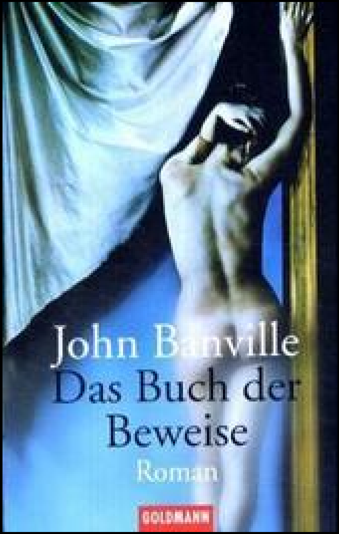
— Prof. Harold P. Boas, 2003

Thesis 2: There are many types of proof



amazon.de

Thesis 2: There are many types of proof



amazon.de

BOOK Proofs . . .

http://www.jwuniverse.co.uk/

Thesis 2: There are many types of proof

simple	—	complicated
surprisingly simple	—	surprisingly complicated
short	—	long
with an idea	—	routine
entertaining	—	boring
correct	—	nearly correct (= wrong)
elementary	—	technical
"trivial"	—	tricky
using a computer	—	visual
new	—	old
geometric	—	algebraic
	...	

Thesis 3: Proofs have history

The "Four Color Theorem"

— Guthrie 1852
— Kempe 1879 : Heawood 1890
— Heesch 1969 — Appel, Haken, Koch, 1976
— Robertson, Sanders, Seymour, Thomas, 1996
— Werner, Gonthier 2004

http://www.freewebs.com/desargues/4ct1.htm

Thesis 3: Proofs have history

The "Four Color Theorem"

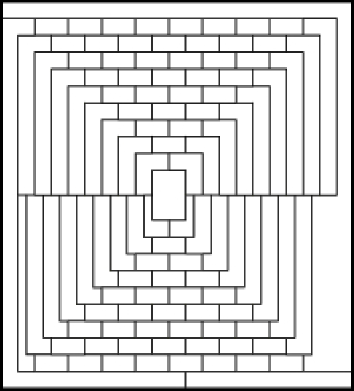
— Guthrie 1852

— Kempe 1879 : Heawood 1890

— Heesch 1969 — Appel, Haken, Koch, 1976

— Robertson, Sanders, Seymour, Thomas, 1996

— Werner, Gonthier 2004



http://mathworld.wolfram.com/Four-ColorTheorem.html

Thesis 3: Proofs have history

The "Four Color Theorem"

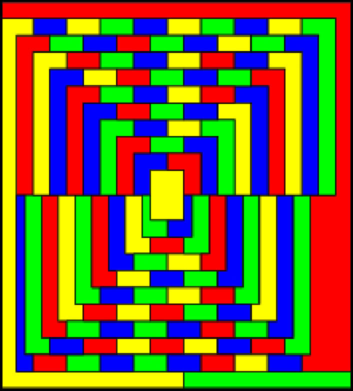
— Guthrie 1852

— Kempe 1879 : Heawood 1890

— Heesch 1969 — Appel, Haken, Koch, 1976

— Robertson, Sanders, Seymour, Thomas, 1996

— Werner, Gonthier 2004



http://mathworld.wolfram.com/Four-ColorTheorem.html

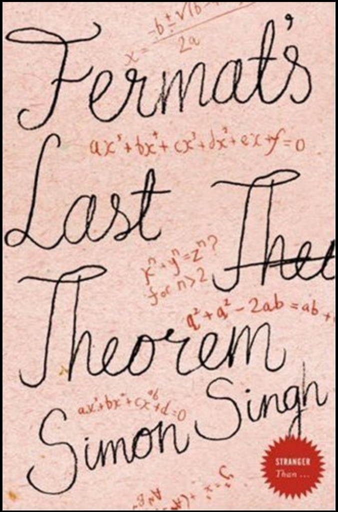
Thesis 3: Proofs have history

"Fermat's Last Theorem":
For $n \geq 3$

$$x^n + y^n = z^n$$

has no solutions
in positive integers
x, y and z.

— Fermat, 1637
— Wiles, Taylor 1993
(published 1995).



http://www.amazon.com

Thesis 4: We need proofs

The equation

$$z^3 = x^3 + y^3$$

has no solution. (Euler 1753)

Thesis 4: We need proofs

The equation

$$z^3 = x^3 + y^3$$

has no solution. (Euler 1753)

The equation

$$z^4 = x^4 + y^4 + u^4$$

has no solution either? (Euler 1772)

Thesis 4: We need proofs

The equation

$$z^3 = x^3 + y^3$$

has no solution. (Euler 1753)

The equation

$$z^4 = x^4 + y^4 + u^4$$

has no solution either? (Euler 1772)

. . . it does:

$$422481^4 = 95800^4 + 217519^4 + 414560^4 \ !$$

(Elkies 1986)

Thesis 4: We need proofs

Proofs, Proofs, Who Needs Proofs?

AUGUST 18, 2010

by rjlipton

Why prove statements we seem to be sure are true?

Michael Atiyah, actually Sir Michael Atiyah, is one of the great mathematicians in the world. He has received awards from the Fields Medal, to the Abel Prize, for his seminal work in many aspects of algebraic geometry, topology, and operator theory. Besides his deep and beautiful results he has some striking insights into the nature of proof. For example, one of his **quotes** is:

> *I think it is said that Gauss had ten different proofs for the law of quadratic reciprocity. Any good theorem should have several proofs, the more the better. For two reasons: usually, different proofs have different strengths and weaknesses, and they generalise in different directions—they are not just repetitions of each other.*

Today I want to talk about the P=NP question, and not even mention **the claimed** proof. Oh, I

http://rjlipton.wordpress.com/

Thesis 5:
Proofs should not be done in public!

“Proofs should be communicated only by consenting adults in private”
— Victor Klee (U. Washington)

Oberwolfach photo library, http://www.mfo.de

Thesis 5:
Proofs should not be done in public!

Sidney Harris book cover, http://www.amazon.com

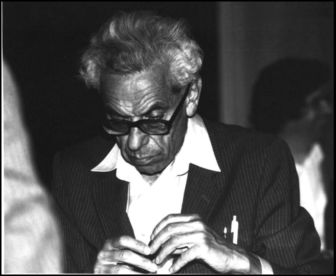
Thesis 5:
Proofs should not be done in public!

K. H. Hofmann for "Proofs from THE BOOK"

Plan

THE BOOK of Proofs . . .

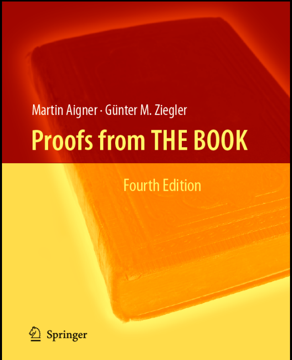
Paul Erdős (1913-1996)



Oberwolfach photo library, http://www.mfo.de

K. H. Hofmann for “Proofs from THE BOOK”

THE BOOK of Proofs . . .



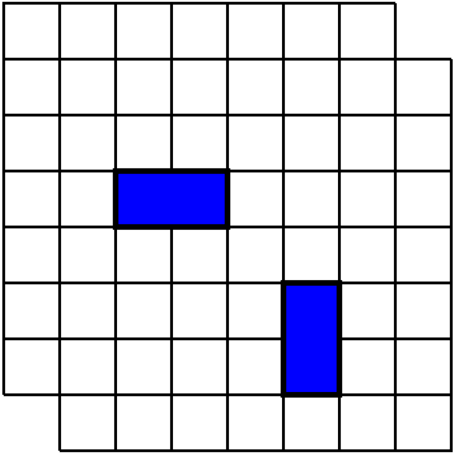
THE BOOK of Proofs . . .

Plan

Two-Coloring Proofs

Theorem.

The "chessboard without corners" cannot be covered by dominos.

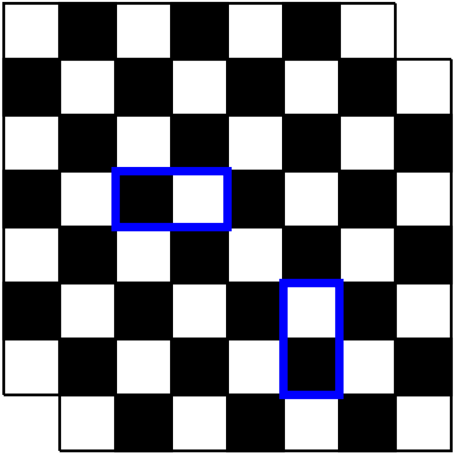


Two-Coloring Proofs

Theorem.

The "chessboard without corners" cannot be covered by dominos.

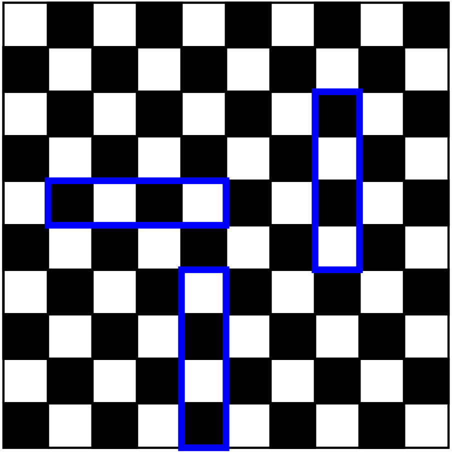
Proof:



Two-Coloring Proofs

Theorem.

The "10 × 10 chessboard" cannot be covered by quadrominos.

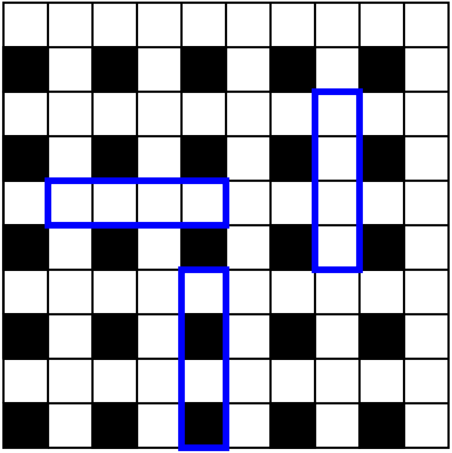


Two-Coloring Proofs

Theorem.

The "10 × 10 chessboard" cannot be covered by quadrominos.

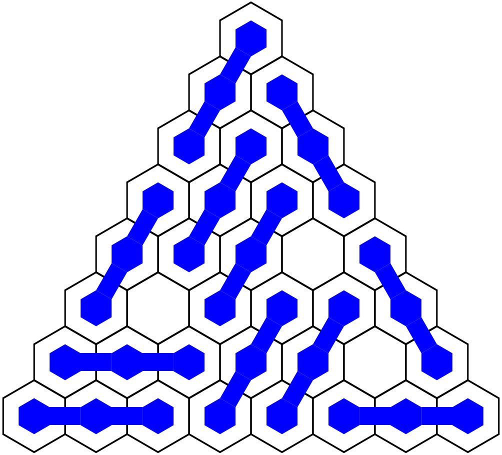
Proof:



Two-Coloring Proofs

Theorem. [Conway & Lagarias 1990]

The following board cannot be covered by triminos.

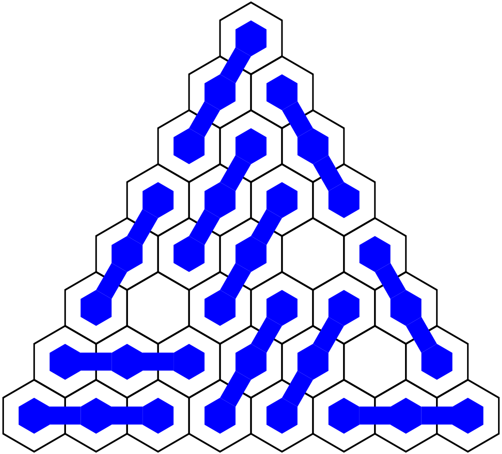


Two-Coloring Proofs

Theorem. [Conway & Lagarias 1990]

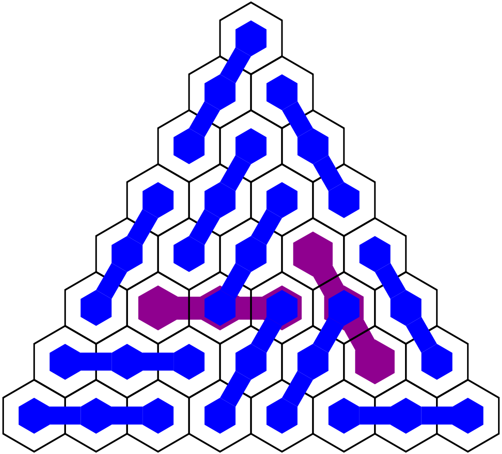
The following board cannot be covered by triminos.

. . . and there is no coloring proof for this!



Two-Coloring Proofs

Theorem. [Conway & Lagarias 1990]
The following board cannot be covered by triminos.
. . . and there is no coloring proof for this!

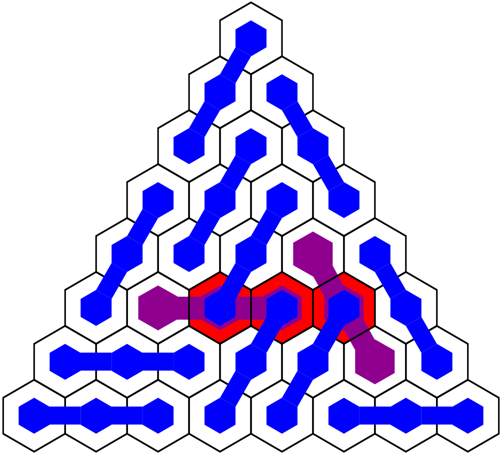


Two-Coloring Proofs

Theorem. [Conway & Lagarias 1990]

The following board cannot be covered by triminos.

. . . and there is no coloring proof for this!



Two-Coloring Proofs

Theorem. [Conway & Lagarias 1990]
The triangular board cannot be covered by triminos.

Proofs:

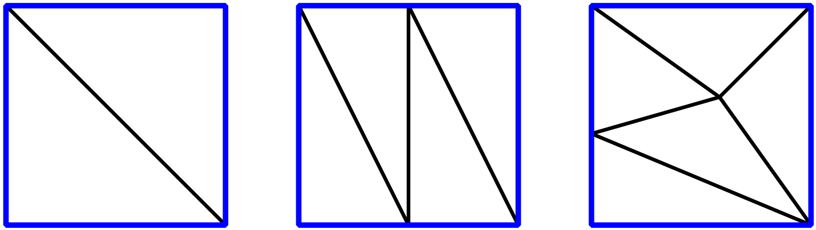
- group-theoretic proof
  (using “Conway’s tiling group”, 1990)
- elementary proof
  (by Doug West, 1991)

Plan

1. On Proofs
2. The BOOK of Proofs
3. Two-Coloring Proofs
4. Odd Dissections of a Square
5. Enumerating the Rationals
6. Kneser's Conjecture
7. Places

Odd Dissections of a Square

Theorem. [Thomas 1968/Monsky 1970]
A square cannot be cut into an odd number of equal-area triangles!



Odd Dissections of a Square

Proof — Part 1: We color the rational plane:

[Thomas 1968/Lenstra]

Color a point (x, y) according to which entry of

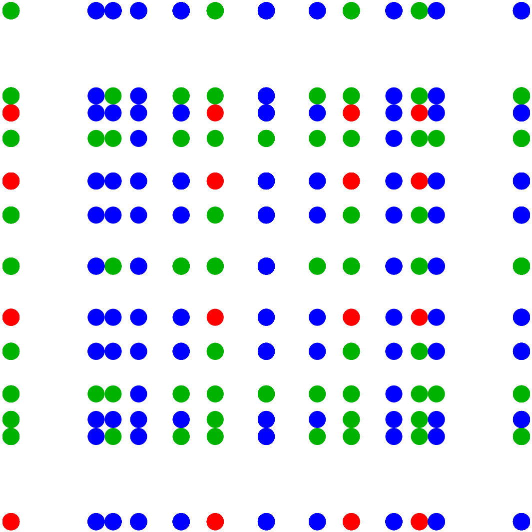
$$(x, y, 1) \quad = \quad (\frac{a}{b}, \frac{c}{d}, 1)$$

is the oddest
(that is, contains the largest power of 2 in the denominator):

- **blue:** if x is the oddest of the three numbers,
- **green:** if y is odder than x and at least as odd as 1,
- **red:** if both x and y have an even numerator.

Odd Dissections of a Square

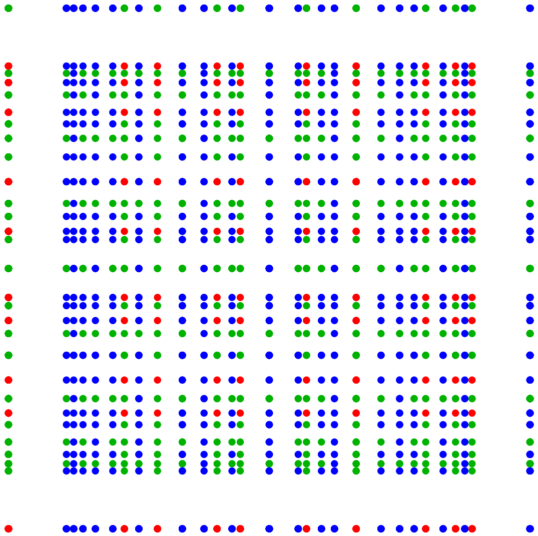
Color the plane:

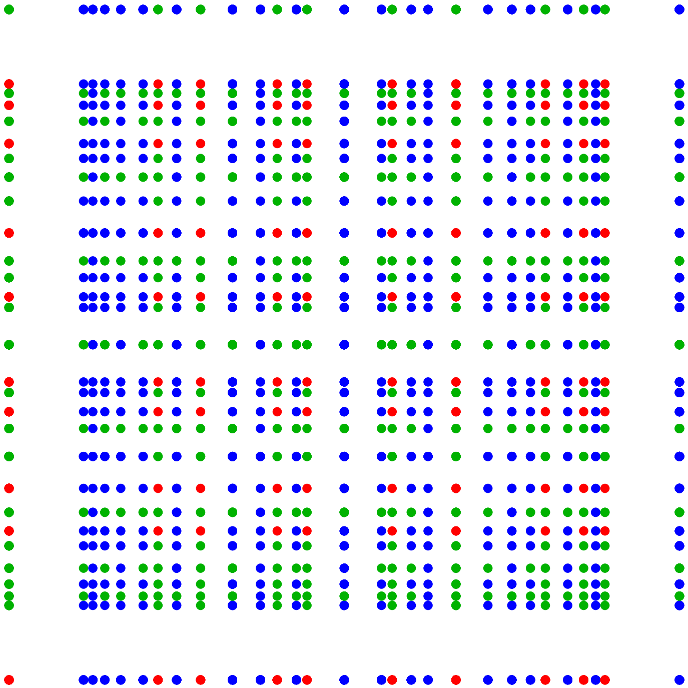


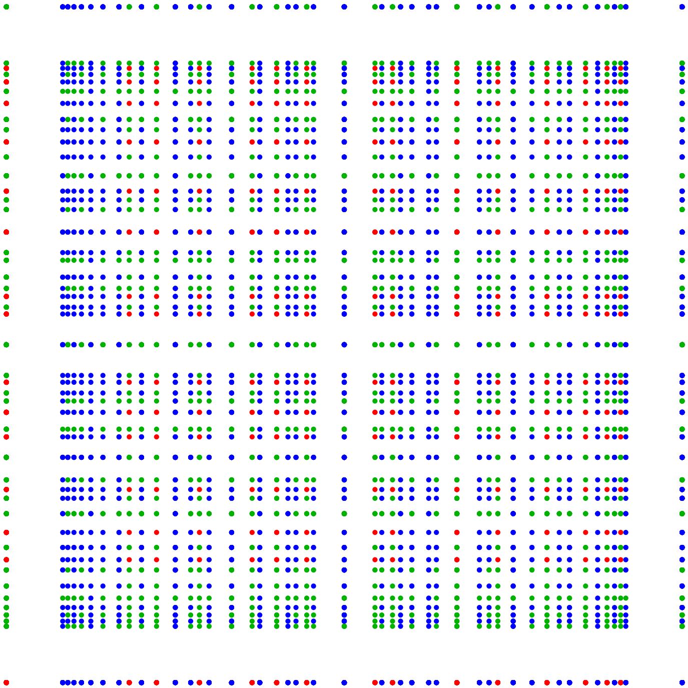
Postscript graphics by Ronald Wotzlaw

Odd Dissections of a Square

Color the plane:

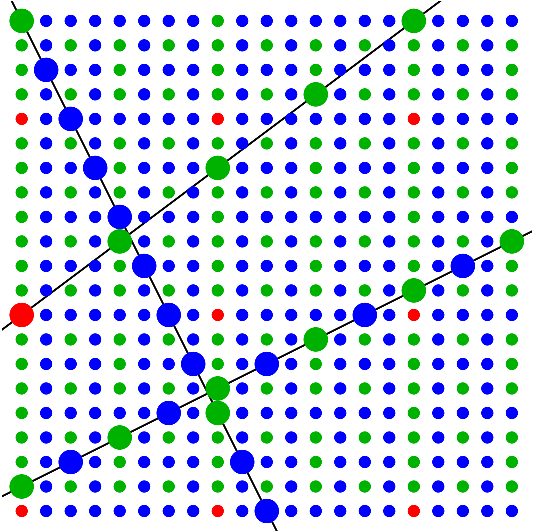






Odd Dissections of a Square

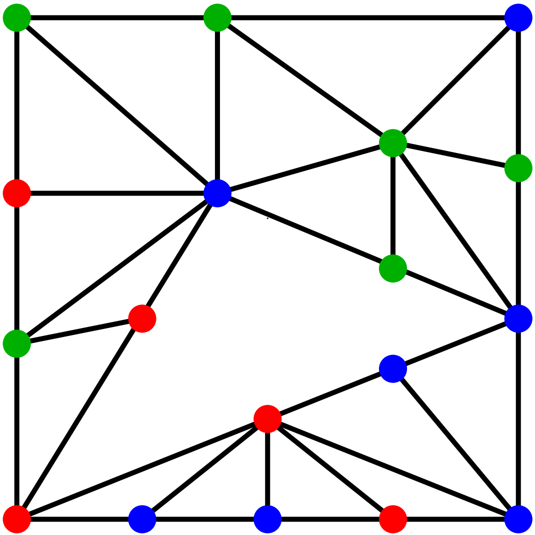
Lemma 1: On any line there are only two colors.



Postscript graphics by Ronald Wotzlaw

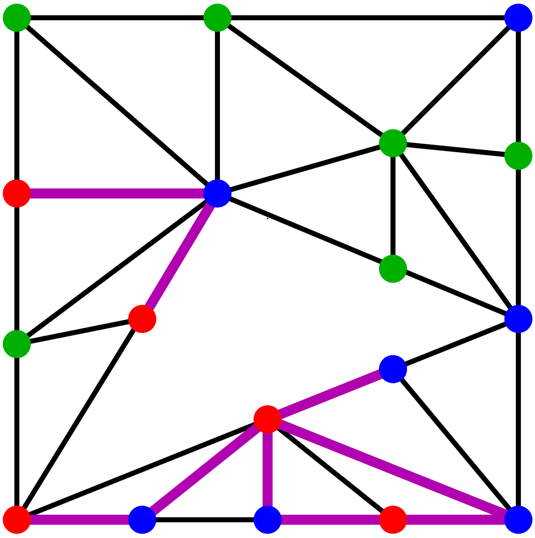
Odd Dissections of a Square

Lemma 2: Every dissection has a rainbow triangle!



Odd Dissections of a Square

Lemma 2: Every dissection has a rainbow triangle!



Odd Dissections of a Square

Lemma 3: The area $A = \dfrac{p}{q}$ of any rainbow triangle has even denominator.

Odd Dissections of a Square

Lemma 3: The area $A = \dfrac{p}{q}$ of any rainbow triangle has even denominator.

Using the *2-adic valuation of the reals*
to measure *how odd* a real number is
the whole plane can be colored so that

- the corners of the unit square have the right colors,
- every line has only two different colors, and
- no rainbow triangle has area $\frac{1}{2n+1}$.

□

Odd Dissections of a Square

Lemma 3: The area $A = \frac{p}{q}$ of any rainbow triangle has even denominator.

Using the *2-adic valuation of the reals*
to measure *how odd* a real number is
the whole plane can be colored so that

- the corners of the unit square have the right colors,
- every line has only two different colors, and
- no rainbow triangle has area $\frac{1}{2n+1}$.

Thus a square cannot be divided into triangles of area $\frac{1}{2n+1}$!

$\square$

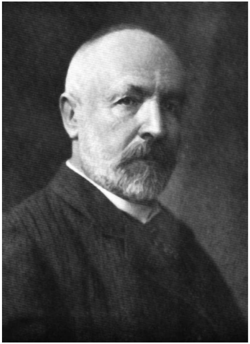
Plan

1. On Proofs
2. The BOOK of Proofs
3. Two-Coloring Proofs
4. Odd Dissections of a Square
5. Enumerating the Rationals
6. Kneser's Conjecture
7. Places

Enumerating the Rationals

Cantor: The fractions can be enumerated!

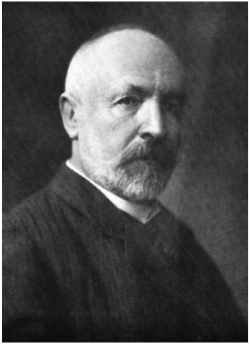
Georg Cantor (1845–1918)



Enumerating the Rationals

Cantor: The fractions can be enumerated!

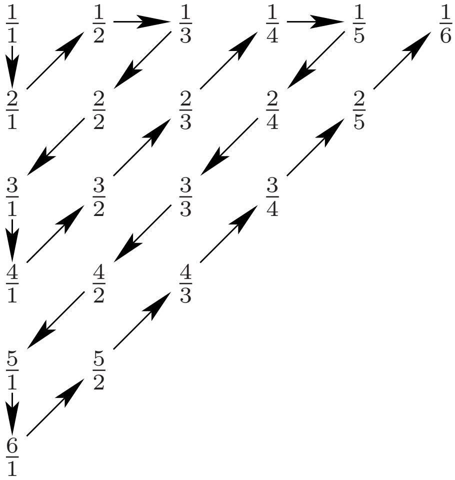
Cantor: The reals *cannot* be enumerated!



Georg Cantor (1845–1918)

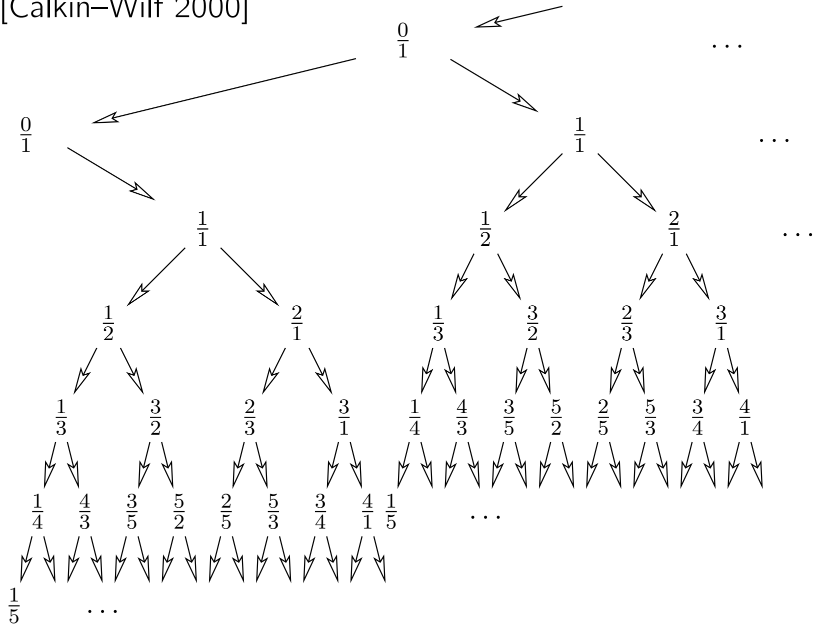
Enumerating the Rationals

Cantor: The fractions can be enumerated!



Enumerating the Rationals

[Calkin–Wilf 2000]



Enumerating the Rationals

Theorem. The rule

$$n\tfrac{a}{b} \quad \longmapsto \quad \frac{1}{n+1-\frac{a}{b}}$$

generates the sequence

$$\tfrac{1}{1} \mapsto \tfrac{1}{2} \mapsto \tfrac{2}{1} \mapsto \tfrac{1}{3} \mapsto \tfrac{3}{2} \mapsto \tfrac{2}{3} \mapsto \tfrac{3}{1} \mapsto \tfrac{1}{4} \mapsto \tfrac{4}{3} \mapsto \ldots$$

which contains every positive fraction exactly once.

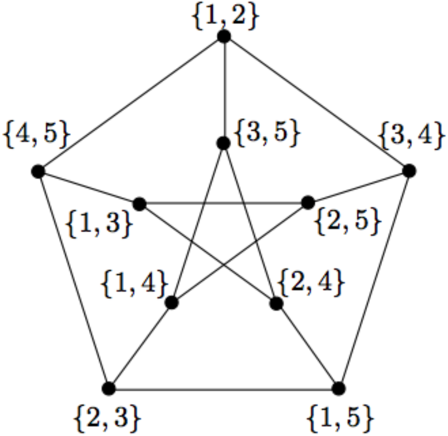
[Moshe Newman]

Plan

1. On Proofs
2. The BOOK of Proofs
3. Two-Coloring Proofs
4. Odd Dissections of a Square
5. Enumerating the Rationals
6. Kneser's Conjecture
7. Places

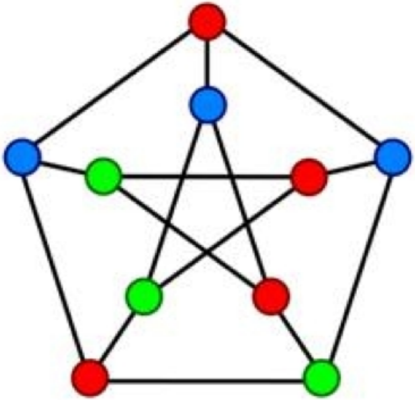
Kneser's Conjecture

Kneser graph $KG(5,2)$: the "Petersen graph"



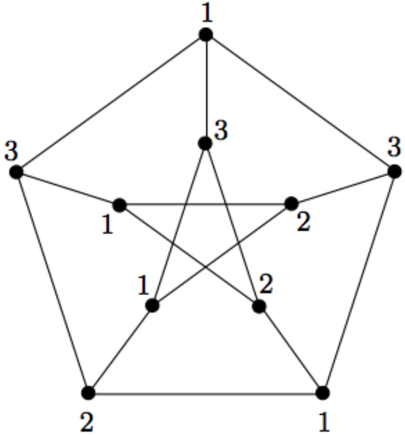
Kneser's Conjecture

Kneser graph $KG(5,2)$ colored:



Kneser's Conjecture

Kneser graph $KG(5,2)$ colored:



Kneser's Conjecture

The graphs $KG(n, k)$ can be colored by $n - 2k + 2$ colors.

Kneser's conjecture (1955): Less colors don't suffice

Aufgabe 360: k und n seien zwei natürliche Zahlen, $k \leqq n$; N sei eine Menge mit n Elementen, N_k die Menge derjenigen Teilmengen von N, die genau k Elemente enthalten; f sei eine Abbildung von N_k auf eine Menge M, mit der Eigenschaft, daß $f(K_1) \neq f(K_2)$ ist falls der Durchschnitt $K_1 \cap K_2$ leer ist; $m(k, n, f)$ sei die Anzahl der Elemente von M und $m(k, n) = \operatorname{Min}_f m(k, n, f)$. Man beweise: Bei festem k gibt es Zahlen $m_0 = m_0(k)$ und $n_0 = n_0(k)$ derart, daß $m(k, n) = n - m_0$ ist für $n \geqq n_0$; dabei ist $m_0(k) \geqq 2k - 2$ und $n_0(k) \geqq 2k - 1$; in beiden Ungleichungen ist vermutlich das Gleichheitszeichen richtig.

Heidelberg. MARTIN KNESER.

Oberwolfach photo library, http://www.mfo.de

Kneser's Conjecture

Proofs:

László Lovász (1978)

Oberwolfach photo library, http://www.mfo.de

Kneser's Conjecture

Proofs:

László Lovász (1978)

. . . using the Borsuk–Ulam Theorem

Oberwolfach photo library, http://www.mfo.de

Kneser's Conjecture

Proofs:

László Lovász (1978)

. . . using the Borsuk–Ulam Theorem

Imre Bárány (1978)

Oberwolfach photo library, http://www.mfo.de

Kneser's Conjecture

Proofs:

László Lovász (1978)

. . . using the Borsuk–Ulam Theorem

Imre Bárány (1978)

. . . using the Borsuk–Ulam Theorem

Oberwolfach photo library, http://www.mfo.de

Kneser's Conjecture

Proofs:

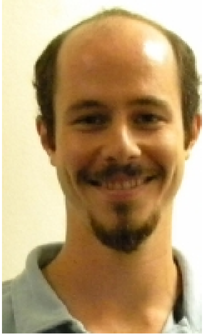
László Lovász (1978)

. . . using the Borsuk–Ulam Theorem

Imre Bárány (1978)

. . . using the Borsuk–Ulam Theorem

Josh Greene (2002)



Oberwolfach photo library, http://www.mfo.de

Kneser's Conjecture

Proofs:

László Lovász (1978)

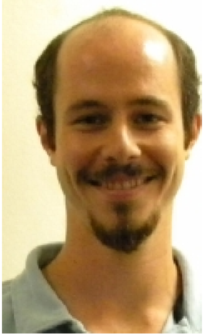
. . . using the Borsuk–Ulam Theorem

Imre Bárány (1978)

. . . using the Borsuk–Ulam Theorem

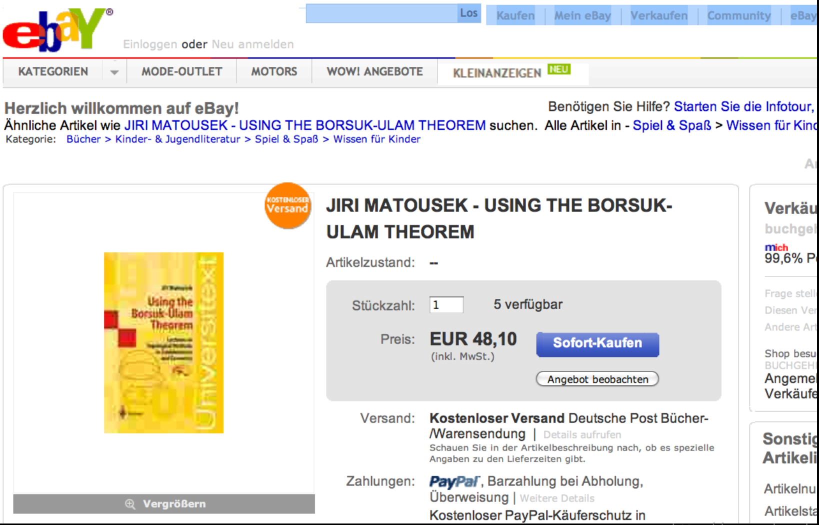
Josh Greene (2002)

. . . using the Borsuk–Ulam Theorem



Oberwolfach photo library, http://www.mfo.de

Kneser's Conjecture



http://www.ebay.de

Kneser's Conjecture

Proof: Let X be a set of n points in general position on an $(n-2k+1)$-dimensional sphere.
Assume there is a coloring

$$\binom{n}{k} \longrightarrow \{1, 2, \ldots, n-2k+1\}.$$

Define

$$A_i := \{x \in S^{n-2k+1} : \text{the open hemisphere } H_x \text{ with pole } x \text{ contains a } k\text{-subset of } X \text{ of color } i\}.$$

for $i = 1, 2, \ldots, n-2k+1$, and

$$A_0 := \{\text{all points on the sphere not covered by these}\}.$$

Kneser's Conjecture

[Lyusternik-Shnirel'man 1930]

The Borsuk–Ulam Theorem:
If you cover the d-sphere by $d+1$ sets,
all of them open, or all of them closed,
then one of the sets contains antipodal points.

Kneser's Conjecture

[Greene 2002]

The Borsuk–Ulam Theorem:

If you cover the d-sphere by $d+1$ sets,

all of them either open or closed,

then one of the sets contains antipodal points.

Kneser's Conjecture

[PFTB 2009]

The Borsuk–Ulam Theorem:

If you cover the d-sphere by $d + 1$ sets,
all but one of which are either open or closed,
then one of the sets contains antipodal points.

Plan

1. On Proofs
2. The BOOK of Proofs
3. Two-Coloring Proofs
4. Odd Dissections of a Square
5. Enumerating the Rationals
6. Kneser's Conjecture
7. Places

Places

At the desk:

“The mathematician is a mythological beast:
half chair, half human.”

— Simon Golin

Places

At the coffee machine:

"A mathematician is a machine
that converts coffee into theorems."

— Paul Erdős

Places

In the library:

Places

In the library:

In bed:

Places

In the library:

In bed:

In church:

Places

In the library:

In bed:

In church:

On the beach:

Places

In the library:

In bed:

In church:

On the beach:

etc.!

Places

Foto: Sven Paustian/Piper

Places

Berlin Mathematical School, http://www.math-berlin.de

K. H. Hofmann for "Proofs from THE BOOK"

Acknowledgement:
I am grateful to all friends and colleagues who contributed to this presentation. I have attempted to give picture credits for the materials used for these lecture slides as complete as possible. However, for some of the figures I do not know or remember or have access to the original source. In case you have right on any of the pictures used in the presentation, if they are copyrighted, or not given with the right credits, please let me know - I will correct this and/or remove them immediately.

Günter M. Ziegler (ziegler[at]math.tu-berlin.de)