Lectures on Polytopes: Updates, Corrections, and More

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February 18, 2007

The purpose of the following is to keep my book

> Günter M. Ziegler:
> **Lectures on Polytopes,**
> *Graduate Texts in Mathematics* **152**,
> Springer-Verlag New York Berlin Heidelberg, Revised edition, 1998.

up-to-date electronically. Thus, this is an electronic preprint, the newest, latest and hottest version of which you should always be able to get via our WWW-server, at

> http://www.math.tu-berlin.de/~ziegler

Note: these updates refer to the 1998 revised edition — see the separate notes on my homepage for updates to the 1995 original edition that went into the revision.

The following is a wild mix of small and large corrections, updates, references I overlooked, new work, additional comments, and more. However, just as in the book, there is no claim or even attempt to be complete or encyclopedic in the coverage of new and old results. This is my own personal selection and bias.

Note also that the second edition of Grünbaum's classic "Convex Polytopes" [234] has just appeared! In addition to an unaltered reprint of the original text, it contains contains over 100 pages of new material (written by Volker Kaibel, Victor Klee and myself): updates on topics covered in Grünbaum's book, new references, etc.

I am always eager to hear about your corrections, updates and comments!

Page 23:

The latest TSP record is the May 2004 optimal solution of a 24,978-city instance, by David Applegate, Bob Bixby, Vašek Chvátal and Bill Cook. See the TSP web site [549] for this, and for further references. (added 8/99)

Page 26:

Asymptotically the best known upper bound for the maximal number of facets of a d-dimensional 0/1-polytope is

$$f(d) \ \leq \ 30\,(d-2)!$$

for all large enough d, due to Fleiner, Kaibel & Rote [555] (8/99)

Page 48:

Several of the convex hull codes (ccd, lrs [542], and PORTA) are integrated in the `polymake` system [559, 560], which is *highly* recommended as a tool for the computation and the combinatorial analysis of example polytopes. You should get hands-on experience with all the examples appearing in the polytopes book, by generating, viewing, and analyzing them in the `polymake` framework! (8/99)

An effective ("polynomial in input plus output size") convex hull code for the special (and especially interesting, see [584]) case of 0/1-polytopes is provided by Bussieck & Lübbecke [548]. An implementation, `zerOne`, is available [574]. (10/00)

Page 84: The Hirsch Conjecture is sharp:

The lower bound result of Holt & Klee was extended by Fritzsche & Holt [556]: for all $n > d \geq 8$, there is a simple d-polytope with n facets whose diameter is at least $n - d$. (8/99)

Page 95, Reconstructing Polytopes:

A practical study about reconstructing simple polytopes from their graphs is Achatz & Kleinschmidt [540]. An extension of Kalai's theorem and proof to non-simple polytopes was given by Joswig [568]. (8/99)

Page 95, Balinski's Theorem:

A stronger, "directed" version of Balinski's theorem is hidden in Holt & Klee [567]: if P is a d-polytope and $\boldsymbol{cx}$ is a linear function in general position, then there are d vertex-disjoint $\boldsymbol{cx}$-monotone paths from the minimal vertex to the maximal vertex of P. (8/99)

Furthermore, Mihalisin & Klee [575] have proved a quite surprising converse to the 3-dimensional Holt-Klee theorem: indeed, acyclic orientation of planar 3-connected graph which has exactly one sink on every face, and which has 3 independent paths from the global source to the global sink, is realizable as the directed graph of a 3-polytope given by a linear function. (6/00)

Page 98, Problem 3.4 (iv)$^{(*)}$:

There are cubical 4-polytopes with the graph of the n-cube, for any $n \geq 4$. More generally, "neighborly cubical polytopes exist!" — see Joswig & Ziegler [569].
In particular, the graph of the n-cube is dimensionally ambiguous for all $n \geq 5$. This also follows from the existence of d-polytopes with the graph of the $(d+1)$-cube, for $d \geq 4$, which can be identified within Blind & Blind's [544] classification of all cubical d-polytopes with 2^{d+1} vertices. (11/98)

Page 100, Problem 3.11* – "Monotone Upper Bound Problem":

For $n > d \geq 2$ let $M_{ubt}(d,n) = f_0(C_d(n)^\Delta)$ be the maximal number of vertices of a (simple) d-polytope with n facets. It is still not clear whether equality holds in

$$M(d,n) \quad \leq \quad M_{ubt}(d,n)$$

for all $n > d \geq 2$. However, there is definite progress:

- Clearly equality holds for $d \leq 3$, and for $n = d+1$.
- Pfeifle [577, Chap. 5] has shown by an explicit inductive construction (and a tour-de-force in projective geometry) that equality holds for $d = 4$ and for *all* $n \geq 5$.
  This is non-obvious even for $n = 7$, where $P \cong C_4(7)^\Delta$ has seven combinatorial types of Hamiltonian paths that give an "abstract objective function" (AOF) that satisfies the Holt-Klee [567] conditions (cf. the comments on Balinski's theorem, p. 95). Of these seven types, exactly four are realizable geometrically, according to Pfeifle [577, Chap. 4].
  For $n = 8$, enumeration yields that $C_4(8)^\Delta$ has no Hamiltonian path that satisfies the AOF and HK conditions. This yields the negative answer to problem Problem 8.41$^{(*)}$ (page 290). However, there are two other combinatorial types of polar-to-neighborly simple 4-polytopes with 8 facets, and both yield realizable paths.
- The answer is also yes for $n = d+2$. This was shown by Gärtner, Solymosi, Tschirschnitz, Valtr & Welzl [558]. The construction by Gärtner et al. is in terms of the Gale$^\Delta$ diagrams of Welzl, which are also used by Pfeifle. (5/03)

Page 100, Problem 3.13$^{(*)}$ – "Perles' Conjecture":

This is false – counterexamples were constructed by Haase & Ziegler [562] (March 2000): The proof works by polarizing the problem, and then constructing counterexamples from 2-complexes that have no free edge, but no 2-dimensional homology either; specifically we use variants of the Borsuk's "dunce hat" or of "Bing's house." (6/00)

Pages 123, Problem 4.16*:

The currently best upper bounds are $f_3^s(n) \le 28.45^n$ (valid for *any* 3-polytope with at least one triangle facet, and $f_3(n) \le 533^{n^2}$ for all 3-polytopes with a large number n of vertices, due to Ribó Mor and Rote [580, Chap. 6]. This is based on bounds on the maximal number of spanning trees in a planar graph on n vertices. It improves the previously best bounds obtained by Stein [582] on the basis of Richter-Gebert's approach/treatment in [424].
(4/05)

Pages 123-124, Problem 4.19*:

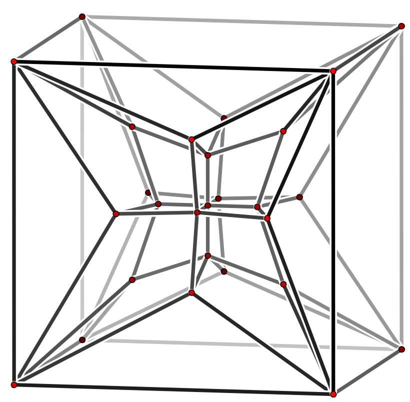
A suitable realization for the truncated tetrahedron, as well as for truncated cube and truncated octahedron, can indeed be constructed; explicit models were built by Andreas Paffenholz, July 2004. Data and pictures are now available at `http://www.math.tu-berlin.de/~paffenho/truncated3polys/`
However, there still doesn't seem to be a proof or disproof for general 3-polytopes. (4/05)

Pages 131-132, Piles of cubes:

More interesting structure connected with the "piles of cubes" polytopes constructed and studied here has recently been uncovered by Athanasiadis [541]. (11/98)

Page 137, A Pile of cubes polytope:

Here is a correct picture of the Schlegel diagram of the pile of cubes polytope $\tilde{P}_3(2,2,2)$, produced by Marc Pfetsch using `polymake` [559, 560] plus `javaview` [579]: (7/00)



Page 146, Problem 5.3 (iii)-(iv)$^{(*)}$:

It turns out that even $\Delta_3 \times \Delta_3$ has a non-regular triangulation – see De Loera [339].
(7/00)

Chapter 6:

See Eisenbud & Popescu [553] for an entirely different (algebraic geometry) view of Gale diagrams. (7/98)

Page 172, Counting d-polytopes with $d + 3$ vertices

Fusy [557] solves the enumeration problem for d-polytopes with $d+3$ vertices – extending the work by Perles reported in [561, Sect. 6.3], and correcting the earlier attempt by Lloyd [573].

Exercise 6.20*, page 189:

This exercise is excessively optimistic, and should have had an asterisk: it is quite easy to see that the classification problem for centrally symmetric polytopes would also give a classification of the ways to cut the d-cube by an affine hyperplane – and this is quite out of reach! Of course this can be done for small d, and should be done; the case $d = 4$ is contained in [234, Section 6.4]. (8/01)

Page 232, item (ii):

The boundary complex $\mathcal{C}(\partial P)$ of a polytope P is a *polytopal* complex. It is simplicial if and only if P is a simplicial polytope. Thanks to Ichiro Fukushima for this and the following corrections. (3/99)

Page 259, line 7*:

"$k = 1$" and "$k = 2$" should be "$k = 0$" and "$k = 1$." (Sperner's Lemma for $k = 0$ is trivial, for $k = 1$ it says $\frac{e}{v} \leq \frac{v-1}{2}$. Note that our statement does not include the case $k = 0$.) (3/99)

Page 263:

Part (iii) of the Kruskal-Katona thorem should be $f_{k-1} \geq \partial_{k+1}(f_k)$ rather than $f_{k-1} \geq \partial_k(f_k)$! (3/99)

Page 271, The upper bound theorem:

The maximal g-vector for given g_1 is misstated; is should have been

$$g_k = \left(\!\!\binom{g_1}{k}\!\!\right) = \binom{g_1 + k - 1}{k} = \binom{n - d + k - 2}{k}. \qquad (7/00)$$

Page 274, lines 1-8:

h_ℓ should read g_ℓ (3/99)

Pages 274-275, Nonunimodal f-vectors

Finally, in 2006 some of Eckhoff's unpublished work on non-unimodal f-vectors was published, also with current updates: See [551]!

Pages 275-277, Nonshellable balls:

Masahiro Hachimori [563, 564] has studied constructibility of simplicial balls (a concept that is weaker than shellability). He found, for example, that while my 10 vertex example of a non-shellable ball [539] is constructible, Furch/Bing's knotted hole balls are not constructible — this is stronger than just saying that they are not shellable. (For such balls based on Bing's house, see also the much earlier paper by Lickorish [572].)

(Hachimori also found that the facet list of Grünbaum's nonshellable ball, as given in [164], is incorrect: the facet "1 7 8 9" should be replaced by "1 7 8 10". A set of coordinates for Grünbaum's ball is also given in [564].) (8/98)

Newman's ball on page 276 has (30 vertices and) 72 facets, as correctly stated in [583].

Now Frank Lutz has constructed a non-shellable 3-ball with the minimal (!) number of 9 vertices and 18 facets. He has also obtained a non-shellable 3-sphere with 13 vertices and 56 facets, based on the trefoil knot. (Non-shellability follows from Hachimori & Ziegler [565].) (5/03)

Page 278, Many triangulated spheres:

There is a number of updates in connection with Kalai's "many triangulated spheres" paper [274]: First, Pfeifle [576] showed that all of the Kalai's "squeezed" 3-spheres are polytopal. This explains why Kalai's techniques were not sufficient to prove that there are asymptotically much more triangulated 3-spheres than there are combinatorial types of simplicial 4-polytopes. (Pfeifle also reproves Lee's result [330] that Kalai's spheres are shellable.)

However, in recent work Pfeifle & Ziegler [578] managed to prove that indeed the number of simplicial 3-spheres on n vertices is asymptotically much larger than the number of 4-polytopes with the same number of vertices. Thus for $d = 3$ "most" triangulated d-spheres are non-polytopal. (The same statement for $d > 3$ was Kalai's main result from [274].) (5/03)

Page 280, The "toric h-vector"

There is important, but difficult, recent progress connected to the "toric h-vector" of Stanley. It was known already that, due to the so-called hard Lefschetz theorem for toric varieties, that even for non-simplicial polytopes, the toric h-vectors of *rational* convex polytopes — satisfies the generalized lower bound inequalities

$$1 = h_0^{\mathrm{tor}} \leq h_1^{\mathrm{tor}} \leq \ldots \leq h_{\lfloor d/2 \rfloor}^{\mathrm{tor}}$$

which may also be written as

$$g_1^{\mathrm{tor}}, g_2^{\mathrm{tor}}, \ldots, g_{\lfloor d/2 \rfloor}^{\mathrm{tor}} \geq 0.$$

Now, based on the "combinatorial intersection homology" theory of Bartels, Brasselet, Fieseler & Kaup [543] and (independently) Bressler & Lunts [547], Karu has [571] extended these (non-trivial) equations to arbitrary, possibly non-rational polytopes.

The generalized lower bound inequality $g_2^{\rm tor}(P) \geq 0$ was previously established by Kalai [570] for general polytopes, using *much* more elementary rigidity type arguments, for the inequality "$g_2^{\rm tor}(P) \geq 0$," which for 4-polytopes, for example, translates into $f_{03} \geq 3f_0 + 3f_3 - 10$. See the comments below for p. 287. (10/03)

Page 281, Problem 8.1(iv)$^{(*)}$: Shelling crosspolytopes?

Tracy Hall [566] has proved that the crosspolytopes are *not* extendably shellable — via an unexpected connection to oriented matroid theory, and the existence of oriented matroids with an element (pseudosphere) that is not adjacent to a mutation (simplicial region), as first extablished by Richter-Gebert [581] and Bokowski & Rohlfs [546]. Explicit data are availble at `http://math.berkeley.edu/~hthall/stuck.html`. (10/03)

Page 282, Problem 8.4*: Stars and links of shellable complexes

This problem asked whether stars and links of shellable polytopal complexes are necessarily shellable as well. Matias Courdurier [550] has shown that the answer for "stars" is yes, and under extra assumptions (e. g. if the facets of the complex are simple) the answer is yes for links as well.

Page 279, The lower bound theorem:

Blind & Blind [545] gave an elementary proof of the lower bound theorem, including the characterization of equality (only stacked polytopes, for $d \geq 4$), using shellings. (4/98)

Page 287, Problem 8.29*, The f-vectors of d-polytopes

For the situation in dimension $d = 4$, see my survey [585] for the ICM 2002 in Beijing. It turns out that the key problem for $d = 4$ is whether the "fatness" parameter

$$\mathrm{F}(P) := \frac{f_1 + f_2 - 20}{f_0 + f_3 - 10}$$

can be arbitrarily large. This parameter is smaller than 3 for simple or simplicial polytopes, but it is larger than 5 for some 2-simple, 2-simplicial 4-polytopes, as constructed by Eppstein, Kuperberg & Ziegler [554], and approaches 9 for the "projected products of polygons" presented in [586].

For recent progress in higher dimensions, and a summary of the known linear inequalities in dimensions 5 to 8, see Ehrenborg [552]. (5/03)

Page 290, Problem 8.41$^{(*)}$, "Monotone Upper Bound Problem":

The answer is *no*: $C_4(8)^\Delta$ *cannot* be realized with an increasing monotone path, according to Pfeifle [577, Chap. 4]. See also the comments on Problem 3.11* (page 100). (5/03)

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