A tight colored Tverberg theorem for maps to manifolds

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April 15, 2011

Abstract

We prove that any continuous map of an N-dimensional simplex Δ_N with colored vertices to a d-dimensional manifold M must map r points from disjoint rainbow faces of Δ_N to the same point in M: For this we have to assume that $N \geq (r-1)(d+1)$, no r vertices of Δ_N get the same color, and our proof needs that r is a prime. A face of Δ_N is a *rainbow face* if all vertices have different colors.

This result is an extension of our recent "new colored Tverberg theorem", the special case of $M = \mathbb{R}^d$. It is also a generalization of Volovikov's 1996 topological Tverberg theorem for maps to manifolds, which arises when all color classes have size 1 (i.e., without color constraints); for this special case Volovikov's proofs, as well as ours, work when r is a prime power.

1 Introduction

Recently, we formulated a new version of the 1992 "colored Tverberg conjecture" by Bárány and Larman [2], and proved this new version in the case of primes.

Theorem 1.1 (Tight colored Tverberg theorem [4])**.** *For $d \geq 1$ and a prime $r \geq 2$, set $N := (d+1)(r-1)$, and let the $N+1$ vertices of an N-dimensional simplex Δ_N be colored such that all color classes are of size at most $r-1$.*

Then for every continuous map $f : \Delta_N \to \mathbb{R}^d$, there are r disjoint faces $F_1, \dots, F_r$ of Δ_N such that the vertices of each face F_i have all different colors, and such that the images under f have a point in common: $f(F_1) \cap \dots \cap f(F_r) \neq \emptyset$.

Here a *coloring* of the vertices of the simplex Δ_N is a partition of the vertex set into color classes, $C_1 \uplus \dots \uplus C_m$. The condition $|C_i| \leq r-1$ implies that there are at least $d+2$ different color classes. In the following, a face all whose vertices have different colors, $|F_j \cap C_i| \leq 1$ for all i, will be called a *rainbow face.*

Theorem 1.1 is tight in the sense that it fails for maps of a simplex of smaller dimension, or if some r vertices of the simplex have the same color. It implies an optimal result for the Bárány–Larman conjecture in the case where $r+1$ is a prime, and an asymptotically-optimal bound in general; see [4, Corollary 2.4 and 2.5]. The special case where all vertices of Δ_N have different colors, $|C_i| = 1$, is the topological Tverberg theorem of Bárány, Shlosman & Szűcs [3].

In this paper we present an extension of Theorem 1.1 that treats continuous maps $\Delta_N \to M$ from the N-simplex to an arbitrary d-dimensional manifold M in place of $\mathbb{R}^d$.

*The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement no. 247029-SDModels. Also supported by the grant ON 174008 of the Serbian Ministry of Science and Environment.

**Supported by Deutsche Telekom Stiftung.

***The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement no. 247029-SDModels.

Theorem 1.2 (Tight colored Tverberg theorem for M)**.** *For $d \geq 1$ and a prime $r \geq 2$, set $N := (d+1)(r-1)$, and let the $N+1$ vertices of an N-dimensional simplex Δ_N be colored such that all color classes are of size at most $r-1$.*

Then for every continuous map $f : \Delta_N \to M$ to a d-dimensional manifold M, the simplex Δ_N has r disjoint rainbow faces whose images under f have a point in common.

Theorem 1.2 without color constraints (that is, when all color classes are of size 1, and thus all faces are rainbow faces) was previously obtained by Volovikov [14], using different methods. His proof (as well as ours in the case without color constraints) works for prime powers r; see Section 3.2.

The prime power case for the colored version, Theorem 1.2, seems however out of reach at this point, even in the case $M = \mathbb{R}^d$. Similarly, there currently does not seem to be a viable approach to the case without color constraints, even for $M = \mathbb{R}^d$, when r is not a prime power. This is the remaining open case of the topological Tverberg conjecture [3].

The conclusion of Theorem 1.2 remains valid if we only consider a continuous map $f : R \to M$, where $R = C_1 * \ldots * C_m$ denotes the subcomplex of rainbow faces in Δ_N. This is non-trivial in general. See the discussion in Section 3.3.

2 Proof

We prove Theorem 1.2 in two steps:

- First, a geometric reduction lemma implies that it suffices to consider only manifolds M that are of the form $M = \widetilde{M} \times I^g$, where $I = [0,1]$ and $\widetilde{M}$ is another manifold. More precisely we will need for the second step that
$$(r-1)\dim(M) > r \cdot \operatorname{cohdim}(M), \qquad (1)$$
where $\operatorname{cohdim}(M)$ is the cohomology dimension of M. This is done in Section 2.1.
- In the second step, we can assume (1) and prove Theorem 1.2 for maps $\Delta_N \to \widetilde{M}$ via the configuration space/test map scheme and Fadell–Husseini index theory, see Sections 2.2 and 2.4.

In the second step we rely on the computation of the Fadell–Husseini index of joins of chessboard complexes that we obtained in [5].

2.1 A geometric reduction lemma

In the proof of Theorem 1.2 may assume that M satisfy the above inequality (1) by using the following reduction lemma repeatedly.

Lemma 2.1. *Theorem 1.2 for parameters (d, r, M, f) can be derived from the case with parameters $(d', r', M', f') = (d+1, r, M \times I, f')$, where the continuous map f' is defined in the following.*

Proof. Suppose we have to prove the theorem for the parameters (d, r, M, f). Let $d' = d+1$, $r' = r$, and $M' = M \times I$. Then $N' := (d'+1)(r-1) = N + r - 1$. Let $v_0, \ldots, v_N, v_{N+1}, \ldots, v_{N'}$ denote the vertices of $\Delta_{N'}$. We regard Δ_N as the front face of $\Delta_{N'}$ with vertices $v_0, \ldots, v_N$. We give the new vertices $v_{N+1}, \ldots, v_{N'}$ a new color. Define a new map $f' : \Delta_{N'} \to M'$ by

$$\lambda_0 v_0 + \ldots + \lambda_{N'} v_{N'} \longmapsto (f(\lambda_0 v_0 + \ldots + \lambda_{N-1} v_{N-1} + (\lambda_N + \ldots + \lambda_{N'}) v_n), \lambda_{N+1} + \ldots + \lambda_{N'}) \,.$$

Suppose we can show Theorem 1.2 for the parameters (d', r', M', f'). That is, we found a Tverberg partition $F_1', \ldots, F_r'$ for these parameters. Put $F_i := F_i' \cap \Delta_N$. Since f' maps the front face Δ_N to $M \times \{0\}$ and since $\Delta_{N'}$ has only $r - 1 < r$ vertices more than Δ_N, already the F_i will intersect in $M \times \{0\}$. Hence the r faces $F_1, \ldots, F_r$ form a solution for the original parameters (d, r, M, f). This reduction is sketched in Figure 1. □

If the reduction lemma is applied $g = 1 + \left\lfloor \frac{d}{r-1} \right\rfloor$ times, the problem is reduced from the arbitrary parameters (d, r, M, f) to parameters (d'', r'', M'', f'') where $M'' = M \times I^g$. Thus M'' has vanishing cohomology in its g top dimensions. Therefore $(r-1)\dim(M'') > r \cdot \operatorname{cohdim}(M'')$.

Having this reduction in mind, in what follows we may simply assume that the manifold M already satisfies inequality (1).

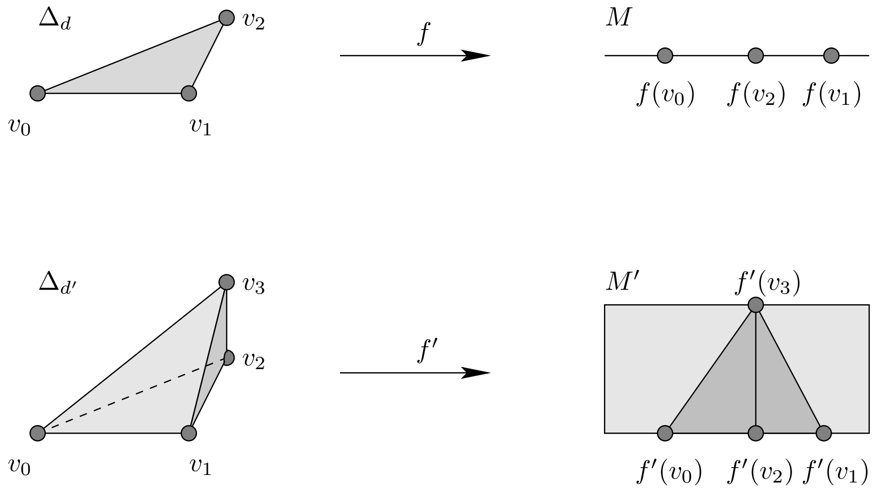

Figure 1: Exemplary reduction in the case $d = 1$, $r = 2$, $N = 2$.

2.2 The configuration space/test map scheme

Suppose we are given a continuous map

$$f : \Delta_N \longrightarrow M,$$

and a coloring of the vertex set $\mathrm{vert}(\Delta_N) = [N+1] = C_0 \uplus \cdots \uplus C_m$ such that the color classes C_i are of size $|C_i| \leq r-1$. We want to find a colored Tverberg partition, that is, pairwise disjoint rainbow faces $F_1, \ldots, F_r$ of Δ_N, $|F_j \cap C_i| \leq 1$, whose images under f intersect.

The test map F is constructed using f in the following way. Let $f^{*r} : (\Delta_N)^{*r} \longrightarrow_{\mathbb{Z}_r} M^{*r}$ be the r-fold join of f. Since we are interested in pairwise disjoint faces $F_1, \ldots, F_r$, we restrict the domain of f^{*r} to the simplicial r-fold 2-wise deleted join of Δ_N, $(\Delta_N)^{*r}_{\Delta(2)} = [r]^{*(N+1)}$. This is the subcomplex of $(\Delta_N)^{*r}$ consisting of all joins $F_1 * \ldots * F_r$ of pairwise disjoint faces. (See [11, Chapter 5.5] for an introduction to these notions.) Since we are interested in colored faces F_j, we restrict the domain further to the subcomplex

$$R^{*r}_{\Delta(2)} = (C_0 * \cdots * C_m)^{*r}_{\Delta(2)} = [r]^{*|C_0|}_{\Delta(2)} * \cdots * [r]^{*|C_m|}_{\Delta(2)}.$$

This is the subcomplex of $(\Delta_N)^{*r}$ consisting of all joins $F_1 * \ldots * F_r$ of pairwise disjoint rainbow faces. The space $[r]^{*k}_{\Delta(2)}$ is known as the *chessboard complex* $\Delta_{r,k}$, [11, p. 163]. We write

$$K := (\Delta_{r,|C_0|}) * \cdots * (\Delta_{r,|C_m|}). \tag{2}$$

Hence we get a *test map*

$$F' : K \longrightarrow_{\mathbb{Z}_r} M^{*r}.$$

Let $T_{M^{*r}} := \{\sum_{i=1}^{r} \frac{1}{r} \cdot x \ : \ x \in M\}$ be the thin diagonal of M^{*r}. Its complement $M^{*r} \backslash T_{M^{*r}}$ is called the topological r-fold r-wise deleted join of M and it is denoted by $M^{*r}_{\Delta(r)}$.

The preimages $(F')^{-1}(T_{M^{*r}})$ of the thin diagonal correspond exactly to the colored Tverberg partitions. Hence the image of F' intersects the diagonal if and only if f admits a colored Tverberg partition.

Suppose that f admits no colored Tverberg partition, then the test map F' induces a $\mathbb{Z}_r$-equivariant map

$$F : K \longrightarrow_{\mathbb{Z}_r} M^{*r}_{\Delta(r)}. \tag{3}$$

We will derive a contradiction to the existence of such an equivariant map using the Fadell–Husseini index theory.

2.3 The Fadell–Husseini index

In this section we review equivariant cohomology of G-spaces via the Borel construction. We refer the reader to [1, Chap. V] and [7, Chap. III] for more details.

Let in the following H^* denote singular or Čech cohomology with $\mathbb{F}_r$-coefficients, where r is prime, and G a finite group. Let EG be a contractible free G-CW complex, for example the infinite join $G * G * \ldots$, suitably topologized. The quotient $BG := EG/G$ is called the *classifying space of* G. To every G-space X we can associate the *Borel construction* $EG \times_G X := (EG \times X)/G$, which is the total space of the fibration $X \hookrightarrow EG \times_G X \xrightarrow{pr_1} BG$.

The *equivariant cohomology* of a G-space X is defined as the ordinary cohomology of the Borel construction,

$$H^*_G(X) \;:=\; H^*(EG \times_G X).$$

If X is a G-space, we define the *cohomological index* of X, also called the *Fadell–Husseini index* [8, 9], to be the kernel of the map in cohomology induced by the projection from X to a point,

$$\mathrm{Ind}_G(X) \;:=\; \ker\left(H^*_G(\mathrm{pt}) \xrightarrow{p^*} H^*_G(X)\right) \;\subseteq\; H^*_G(\mathrm{pt}).$$

The cohomological index is monotone in the sense that if there is a G-map $X \longrightarrow_G Y$ then

$$\mathrm{Ind}_G(X) \supseteq \mathrm{Ind}_G(Y). \tag{4}$$

If r is odd then the cohomology of $\mathbb{Z}_r$ with $\mathbb{F}_r$-coefficients as an $\mathbb{F}_r$-algebra is

$$H^*(\mathbb{Z}_r) = H^*(B\mathbb{Z}_r) \cong \mathbb{F}_r[x, y]/(y^2),$$

where $\deg(x) = 2$ and $\deg(y) = 1$. If r is even, then $r = 2$ and $H^*(\mathbb{Z}_r) \cong \mathbb{F}_2[t]$, $\deg t = 1$.

The index of the complex K was computed in [5, Corollary 2.6]:

Theorem 2.2. $\mathrm{Ind}_{\mathbb{Z}_r}(K) = H^{*\geq N+1}(B\mathbb{Z}_r)$.

Therefore in the proof of Theorem 1.2 it remains to show that $\mathrm{Ind}_{\mathbb{Z}_r}(M^{*r}_{\Delta(r)})$ contains a non-zero element in dimension less or equal to N. Indeed, the monotonicity of the index (4) implies the non-existence of a test map (3), which in turn implies the existence of a colored Tverberg partition.

Let us remark that the index of K becomes larger as an ideal than in Theorem 2.2 if just one color class C_i has more than $r - 1$ elements. That is, in this case our proof of Theorem 1.2 does not work anymore. In fact, for any r and d there exist $N + 1$ colored points in $\mathbb{R}^d$ such that one color class is of size r and all other color classes are singletons that admit no colored Tverberg partition.

2.4 The index of the deleted join of the manifold

We have inclusions

$$T_{M^{*r}} \;\hookrightarrow\; \left\{\sum \lambda_i x \in M^{*r} \;:\; \lambda_i > 0, \sum \lambda_i = 1, x \in M\right\} \cong M \times \Delta^\circ_{r-1} \;\hookrightarrow\; M^{*r},$$

where Δ°_{r-1} denotes the open $(r-1)$-simplex. Since M is a smooth $\mathbb{Z}_r$-invariant manifold, $T_{M^{*r}}$ has a $\mathbb{Z}_r$-equivariant tubular neighborhood in M^{*r}; see [6]. Its closure can be described as the disk bundle $D(\xi)$ of an equivariant vector bundle ξ over M. We denote its sphere bundle by $S(\xi)$. The fiber F of ξ is as a $\mathbb{Z}_r$-representation the $(d+1)$-fold sum of W_r, where $W_r = \{x \in \mathbb{R}[\mathbb{Z}_r] \;:\; x_1 + \ldots + x_r = 0\}$ is the augmentation ideal of $\mathbb{R}[\mathbb{Z}_r]$.

The representation sphere $S(F)$ is of dimension $N - 1$. It is a free $\mathbb{Z}_r$-space, hence its index is

$$\mathrm{Ind}_{\mathbb{Z}_r}(S(F)) = H^{*\geq N}(B\mathbb{Z}_r). \tag{5}$$

This can be directly deduced from the Leray–Serre spectral sequence associated to the Borel construction $E\mathbb{Z}_r \times_{\mathbb{Z}_r} S(F) \to B\mathbb{Z}_r$, noting that the images of the differentials to the bottom row give precisely the index of $S(F)$, which can be seen from the edge-homomorphism. For background on Leray–Serre spectral sequences we refer to [12, Chapter 5, 6].

The Leray–Serre spectral sequence associated to the fibration $S(\xi) \to M$ collapses at E_2, since $N = (r-1)(d+1) \geq d+1$ and hence there is no differential between non-zero entries. Thus the map $i^* : H^{N-1}(S(\xi)) \to H^{N-1}(S(F))$ induced by inclusion is surjective.

The Mayer–Vietoris sequence associated to the triple $(D(\xi), M^{*r}_{\Delta(r)}, M^{*r})$ contains the subsequence

$$H^{N-1}(M^{*r}_{\Delta(r)}) \oplus H^{N-1}(D(\xi)) \xrightarrow{j^*+k^*} H^{N-1}(S(\xi)) \xrightarrow{\delta} H^N(M^{*r}).$$

We see that $H^N(M^{*r})$ is zero: This follows from the formula

$$\widetilde{H}^{*+(r-1)}(M^{*r}) \cong \left(\widetilde{H}^*(M)\right)^{\otimes r},$$

as long as $N-(r-1) > re$, where e is the cohomological dimension of M. This inequality is equivalent to $d > \frac{r}{r-1}e$, which can be assumed by applying the reduction from Section 2.1 at least $\lfloor 1 + \frac{e}{r-1} \rfloor$ times. Hence we can assume that $H^N(M^{*r}) = 0$.

Furthermore inequality (1) implies that $N-1 \geq d > \operatorname{cohdim}(M)$. Hence the term $H^{N-1}(D(\xi)) = H^{N-1}(M)$ of the sequence is zero as well.

Thus the map $j^* : H^{N-1}(M^{*r}_{\Delta(r)}) \to H^{N-1}(S(\xi))$ is surjective. Therefore the composition $(j \circ i)^* : H^{N-1}(M^{*r}_{\Delta(r)}) \to H^{N-1}(S(F))$ is surjective as well. We apply the Borel construction functor $E\mathbb{Z}_r \times_{\mathbb{Z}_r} (\_) \to B\mathbb{Z}_r$ to this map and apply Leray–Serre spectral sequences; see Figure 2.

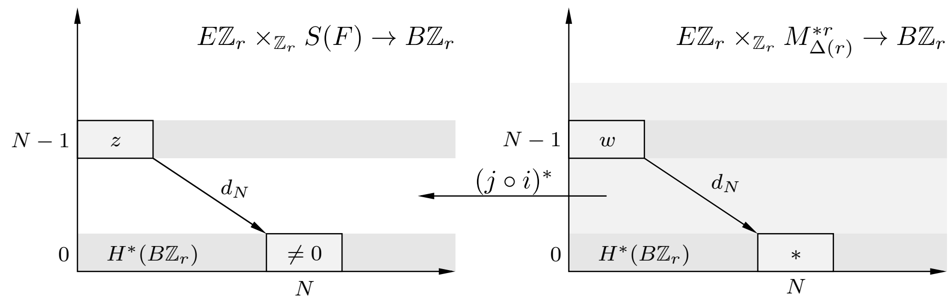

Figure 2: We associate to the map $S(F) \xrightarrow{j \circ i} M^{*r}_{\Delta(r)}$ the Borel constructions and spectral sequences to deduce that $M^{*r}_{\Delta(r)}$ contains a non-zero element in dimension N.

At the E_2-pages, the generator z of $H^{N-1}(S(F))$ has a preimage w since $(j \circ i)^*$ is surjective. At the E_N-pages $(j \circ i)^*(d_N(w)) = d_N(z)$, which is non-zero by (5). Hence $d_N(w) \neq 0$, which is an element in the kernel of the edge-homomorphism $H^*(B\mathbb{Z}_r) \to H^*_{\mathbb{Z}_r}(M^*_{\Delta(r)})$.

Therefore, the index of $M^{*r}_{\Delta(r)}$ contains a non-zero element in dimension N. This completes the proof of Theorem 1.2. □

3 Remarks

3.1 Theorem 1.2 strictly generalizes Theorem 1.1

One may ask whether Theorem 1.2 can be reduced to Theorem 1.1 by factorizing the given map $f : \Delta_N \to M$ over $\mathbb{R}^d$,

$$f : \Delta_N \xrightarrow{f'} \mathbb{R}^d \to M.$$

In this case Theorem 1.1 immediately implies Theorem 1.2. However this is not always possible.

Proposition 3.1. *Let f be the composed map $\Delta_3 \to S^3 \to S^2$ that first quotients out the boundary of Δ_3 and then sends S^3 to S^2 via the Hopf map. Then f does not factor over $\mathbb{R}^d$.*

The following proof is due to Elmar Vogt.

Proof. Suppose that f factors as $f : \Delta_N \xrightarrow{f'} \mathbb{R}^d \xrightarrow{g} M$. Let $h : S^3 \to S^2$ denote the Hopf map. Let $z := h^{-1}(n) \subset S^3$, where n is the north pole of S^2. We think of z being the closure of the z-axis in the stereographic projection of S^3 to $\mathbb{R}^3$ in the one-point compactification of $\mathbb{R}^3$. Let D be the halfspace $\{(x, y, z) \ : \ x > 0, \ y = 0\}$, which is a disc in S^3 whose boundary is z. Then all fibers $h^{-1}(x)$ other than z intersect D transversally. In particular, h maps D homeomorphically to $S^2 \backslash \{n\}$. Then f' also maps D (regarded as a disc in Δ_3) homeomorphically to a set $D' \subset \mathbb{R}^2$. Moreover, for every $x \neq n$, $f'(h^{-1}(x))$ is a singleton in D'. Further, D' is bounded, since Δ_3 is compact. Let $p \in z$ and let (p_i) be a sequence in D converging to p. Then the fibers $h^{-1}(h(p_i))$ contain sequences of points that come close to any other point of z. By continuity, $f'(z)$ must be a singleton in $\mathbb{R}^2$ as well. But $f'(z)$ must also be the boundary of D' which is bounded and homeomorphic to an open disc, which gives a contradiction. □

3.2 The case without color constraints

Suppose we color the vertices of Δ_N in Theorem 1.2 with pairwise distinct colors. Then all faces of Δ_N are rainbow faces, hence the condition of being a rainbow face is empty. This case was already treated by Volovikov, in a slightly stronger version.

Theorem 3.2 (Volovikov [14]). *Let $d \geq 1$, let $r = p^k$ be a prime power, $N := (d + 1)(r - 1)$, and $f : \partial\Delta_N \to M$ be a continuous map from the boundary N-simplex to a d-dimensional topological manifold. If $p = 2$ then we further assume that the degree of f is even. Then Δ_N has r disjoint rainbow faces whose images under f intersect.*

Our proof given in this paper works also for prime powers $r = p^k$ in the case without color constraints, since then

- the configuration space is the join $[r]^{*(N+1)}$, which is $(N - 1)$-connected and $(\mathbb{Z}_p)^k$-free, hence its index is $H^{*\geq N+1}(B((\mathbb{Z}_p)^k))$, and
- the group $(\mathbb{Z}_p)^k$ acts fixed point freely on the sphere $S(F)$ and $\mathrm{Ind}_{(\mathbb{Z}_p)^k}(S(F))$ consequently contains an element of degree N, particularly $d_N(z)$ in the notation of Section 2.4.

3.3 Reduction to the subcomplex of rainbow faces

One could ask whether Δ_N in Theorem 1.2 can be replaced by the subcomplex R that consists of all rainbow faces. The methods of this paper seem to establish this only if we assume that sufficiently many colors are used. (The assumptions of Theorem 1.2 imply that the $N + 1$ vertices of Δ_N are colored with at least $\lceil \frac{N+1}{r-1} \rceil = d + 2$ colors.)

Corollary 3.3. *Let $d \geq 1$, $r \geq 2$ prime, and $N := (d+1)(r-1)$. Let the vertices of Δ_N be colored with at least $d+3+\lfloor \frac{d}{r-1} \rfloor = d+2+g$ colors such that all color classes C_i are of size $|C_i| \leq r - 1$. Let R be the subcomplex of Δ_N consisting of all rainbow faces. Let $f : R \to M$ be a continuous map from R to a d-dimensional manifold M. Then R has r disjoint faces whose images under f intersect.*

The proof of Corollary 3.3 is analogous to that of Theorem 1.2. The main change occurs in the reduction to the case where the manifold is $M' = M \times I^g$, see Section 2.1. Here one needs to be a bit more careful, because the reduction might not be possible. Instead of letting f send the $r - 1$ new vertices of $\Delta_{N'}$ to points above $f(v_N)$ and giving them a new color, we send them above the images of possibly different vertices and color them with the same color as the vertex below. This has to be done in such a way that all new color classes are still of size less than r. This is possible since the number of used colors is at least $d + 2 + g$.

Using more advanced machinery we can even prove Theorem 1.2 even for maps $f : R \to M$, without the constraint on the number of color classes from Corollary 3.3.

Corollary 3.4. *Let $d \geq 1$, $r \geq 2$ prime, and $N := (d+1)(r-1)$. Let the vertices of Δ_N be colored such that all color classes C_i are of size $|C_i| \leq r - 1$. Let R be the subcomplex of Δ_N consisting of all rainbow faces. Let $f : R \to M$ be a continuous map from R to a d-dimensional manifold M. Then R has r disjoint faces whose images under f intersect.*

To prove this, we use the deleted-product scheme. The deleted-join scheme amounts to show that the test map (3) does not exist. In the corresponding deleted-product scheme one instead has to show that the test map

$$f^{\times} : R^r_{\Delta(2)} \to M^r_{\Delta_r} \tag{6}$$

does not exist. For this to do, we calculate the index of $R^r_{\Delta(2)}$,

$$\mathrm{Ind}_{\mathbb{Z}_r}(R^r_{\Delta(2)}) = H^{*\geq N-r}(B\mathbb{Z}_r).$$

This follows from Theorem 2.2, which gives the index of the corresponding deleted join $R^{*r}_{\Delta(2)}$, and a reduction lemma due to Karasev [10, Lemma 3.2] and independently Carsten Schultz (unpublished). Then one can proceed exactly as in the proof of Volovikov's Theorem 3.2, see [14], which is based on Theorem 1 of his paper [13].

3.4 Deleted joins versus deleted products

An interesting question is which test map scheme is more powerful, the one coming from the deleted-product construction or the one from the deleted-join construction?

Let us first define general deleted products and deleted joins for a *simplicial complex* X. We define the *r-fold ℓ-wise deleted product* $X^r_{\Delta(\ell)}$ of the simplicial complex X to be the subcomplex of the cell complex X^r containing only cells $F_1 \times \ldots \times F_r$ such that the F_i are ℓ-wise disjoint, that is, no ℓ of them have a point in common. Analogously, we define the *r-fold ℓ-wise deleted join* $X^{*r}_{\Delta(\ell)}$ of the simplicial complex X as the subcomplex of the simplicial complex X^{*r} that contains only those faces $F_1 * \ldots * F_r$ such that the F_i are ℓ-wise disjoint. Compare with [11, Definition 6.3.1].

Now we introduce general deleted products and deleted joins for a *topological space* Y. The *r-fold ℓ-wise deleted product* of the space Y is

$$Y^r_{\Delta(\ell)} := \{(y_1, \ldots, y_r) \in Y^r \ : \ \text{no } \ell \text{ of the } y_i \text{ are equal}\},$$

while the *r-fold ℓ-wise deleted join* of Y we define as

$$Y^{*r}_{\Delta(\ell)} := \{\lambda_1 y_1 + \ldots + \lambda_r y_r \in Y^{*r} \ : \ \text{if } \lambda_1 = \ldots = \lambda_r = \tfrac{1}{r} \text{ then no } \ell \text{ of the } y_i \text{ are equal}\}.$$

In many applications we investigate the existence of a Σ_r-equivariant test map of deleted products

$$f^{\times} : X^r_{\Delta(\ell)} \longrightarrow_{\Sigma_r} Y^r_{\Delta(k)}, \tag{7}$$

where X is a simplicial complex and Y is a space. Here Σ_r stands for the group of permutations on r letters. The corresponding Σ_r-equivariant test map for deleted joins would be

$$f^{*} : X^{*r}_{\Delta(\ell)} \longrightarrow_{\Sigma_r} Y^{*r}_{\Delta(k)}. \tag{8}$$

In the case when $Y = \mathbb{R}^d$, for some d, the existence of the Σ_r-equivariant map $f^{\times}$ implies the existence of the Σ_r-equivariant map f^*. Indeed, if $f^{\times} : X^r_{\Delta(\ell)} \longrightarrow_{\Sigma_r} (\mathbb{R}^d)^r_{\Delta(k)}$ is given, then we can define

$$f^*(\lambda_1 x_1 + \ldots + \lambda_r x_r) := \sum_{i=1}^{r} \lambda_i y_i,$$

where

$$y_i := \left(\prod_{j=1}^{r} r\lambda_j\right) \cdot f_i^{\times}(x_1, \ldots, x_r) \in \mathbb{R}^d.$$

The constant factor r^r is included in the definition of y_i because $X^r_{\Delta(\ell)}$ can be seen as a subspace of $X^{*r}_{\Delta(\ell)}$ where all join coefficients are $\frac{1}{r}$. The cell $F_1 \times \ldots \times F_r$ of the deleted product complex can be identified with the subspace $\{\frac{1}{r} \cdot x_1 + \ldots + \frac{1}{r} \cdot x_r \mid x_i \in F_i\}$ of the simplex $F_1 * \ldots * F_r$ of the deleted join complex.

Therefore, in the case when $Y = \mathbb{R}^d$, the deleted-product scheme (7) is stronger than the deleted-join scheme (8).

Nevertheless, *proving* the non-existence of f^* might be easier than proving the non-existence of $f^\times$. For instance, if one only wants to argue with the high connectivity of the domain, then this is usually easier for f^*, see e.g. [11, Sections 5.5–5.8].

Also the monotonicity of the Fadell–Husseini index sometimes puts a stronger condition on f^* than on $f^\times$. In particular, this affects Theorem 1.2. The range of $f^\times$ is $M^r_{\Delta(r)}$. If $M = \mathbb{R}^d$ then the corresponding index is

$$\mathrm{Ind}_{\mathbb{Z}_r}((\mathbb{R}^d)^r_{\Delta(r)}) = H^{*\geq d(r-1)}(B\mathbb{Z}_r),$$

since $(\mathbb{R}^d)^r_{\Delta(r)}$ deformation retracts equivariantly to a fixed-point free sphere whose dimension is $d(r-1)-1$. Hence we can show the non-existence of $f^\times$ using the monotonicity of the index. However, for $M = S^d$ the index is smaller with respect to inclusion by the following proposition, so the monotonicity of the index alone is not enough to prove Theorem 1.2 in the deleted-product scheme.

Proposition 3.5. *If $d \geq 2$, then*

$$\mathrm{Ind}_{\mathbb{Z}_r}((S^d)^r_{\Delta(r)}) \subseteq H^{*\geq d(r-1)+1}(B\mathbb{Z}_r).$$

Proof. We have to show that in the Leray–Serre spectral sequence associated to $EG \times_G (S^d)^r_{\Delta(r)} \to BG$, no non-zero differential hits the bottom row in filtration degree smaller or equal to $d(r-1)$. The E_2 entries are $H^*(BG, H^*((S^d)^r \backslash \Delta))$, where Δ is the thin diagonal in $(S^d)^r$. Now,

$$H^i((S^d)^r \backslash \Delta) \cong H_{dr-i}((S^d)^r, \Delta) \cong H^{dr-i}((S^d)^r, \Delta).$$

From the long exact sequence in cohomology of the pair $((S^d)^r, \Delta)$,

$$\ldots \to H^*((S^d)^r, \Delta) \to H^*((S^d)^r) \to H^*(\Delta) \to \ldots,$$

we see that $H^{dr}((S^d)^r, \Delta) = \mathbb{F}_r$, $H^d((S^d)^r, \Delta) \cong \mathbb{F}_r[\mathbb{Z}_r]/(1+t+\ldots+t^r)\mathbb{F}_r$, and for $d < j < dr$ we have $H^j((S^d)^r, \Delta) \cong \mathbb{F}_r[\mathbb{Z}_r]^{\oplus \alpha_j}$, where $\alpha_j \geq 0$ depends on j. Therefore the first non-zero row (up to the 0-column entries) in the spectral sequence above the bottom row is the $d(r-1)$-row. Thus the first element in the bottom row that is hit by a differential has degree at least $d(r-1)+1$. □

On the other hand, the monotonicity of the Fadell–Husseini index proves the non-existence of f^* for $M = S^d$, since $(S^d)^{*r}_{\Delta(r)}$ deformation retracts equivariantly to an $(N-1)$-dimensional fixed-point free sphere, whose index is equal to $H^{*\geq N}(B\mathbb{Z}_r)$.
So, in this context, the deleted-join scheme is stronger than the deleted-product scheme.

Acknowledgements. We are grateful to the referee for critical comments that improved differen aspects of the paper, and to Aleksandra, Julia, and Torsten for constant support.

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