# Polyhedral surfaces of high genus

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**Abstract.** The construction of the *combinatorial* data for a surface with $n$ vertices of maximal genus is a classical problem: The maximal genus $g = \lfloor \frac{1}{12}(n-3)(n-4) \rfloor$ was achieved in the famous "Map Color Theorem" by Ringel et al. (1968). We present the nicest one of Ringel's constructions, for the case $n \equiv 7 \bmod 12$, but also an alternative construction, essentially due to Heffter (1898), which easily and explicitly yields surfaces of genus $g \sim \frac{1}{16}n^2$.

For *geometric* (polyhedral) surfaces with $n$ vertices the maximal genus is not known. The current record is $g \sim \frac{1}{8} n \log_2 n$, due to McMullen, Schulz & Wills (1983). We present these surfaces with a new construction: We find them in Schlegel diagrams of "neighborly cubical 4-polytopes," as constructed by Joswig & Ziegler (2000).

## 0. Introduction

In the following we present constructions for surfaces that have extremely and perhaps surprisingly high topological complexity (genus, Euler characteristic) compared to their number of vertices. We believe that not only the resulting surfaces, but also the constructions themselves are interesting and worth studying — also in the hope that they can be substantially improved.

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**0.1. What is a surface?**

What do we mean by "a surface"? This is not a stupid question, since combinatorialists, geometers, and topologists work with quite different frameworks, definitions and concepts of surfaces. As we will see, in the high-genus case it is not clear that the same results are valid for the different concepts.

A *topological* surface may be defined as a closed (compact, without boundary), connected, orientable, Hausdorff, 2-dimensional manifold. By adopting this model, we already indicate that one could have worked in much greater generality: Here we do not consider the non-orientable case, we do not worry about manifolds with boundary, etc.

The *combinatorial* version of a surface may be presented by listing the *faces* (vertices, edges, and 2-cells), and giving the necessary incidence information (for example, by specifying for each face the vertices in its boundary, in clockwise order according to the orientation). Such combinatorial data must, of course, satisfy some consistency conditions if we are to be guaranteed that they do correspond to a surface. Such conditions are easy to derive.

The vertices and edges of a combinatorial surface together make up its *graph*.

In the following, we will insist throughout that the combinatorial surface data we look at are regular (no identifications on the boundaries of the cells), and they must satisfy the *intersection condition*: The intersection of any two faces is again a face (which may be empty). This condition implies that any two vertices are connected by at most one edge, and that any two 2-faces have at most two vertices in common (which must then be connected by an edge).[1]

*Geometric* surfaces consist of flat convex polygons. We also require that all these faces are simultaneously realized in $\mathbb{R}^3$, without intersections. Any such geometric surface yields a combinatorial surface, which in turn determines a topological manifold.

**0.2. The $f$-vector**

The $f$-vector of a combinatorial or geometric surface $S$ is the triple

$$f(S) := (f_0, f_1, f_2),$$

where $f_0$ denotes the number of vertices, $f_1$ is the number of edges, and $f_2$ is the number of 2-dimensional cells.

The $f$-vector contains a lot of information. For example, we can tell from the $f$-vector whether the surface is simplicial. Indeed, one always has $3f_2 \leq 2f_1$, by double-counting: Every 2-face has at least three edges, every edge lies in two 2-faces. Equality $3f_2 = 2f_1$ holds if and only if every face is bounded by exactly three edges, that is, for a triangulated (simplicial) surface.

Similarly, we have $f_1 \leq \binom{f_0}{2}$, with equality for a neighborly surface, whose 1-skeleton is a complete graph. A neighborly surface is necessarily simplicial.

[1] A combinatorial surface with the intersection condition is called a "polyhedral map" in some of the discrete geometry literature; see Brehm & Wills [9].

### 0.3. The genus

The classification of (orientable, closed, connected — the generality outlined above) surfaces up to homeomorphism is well-known: For each integer $g \geq 0$, there is exactly one topological type, "the surface of genus $g$," which may be obtained by attaching $g$ handles to the 2-sphere $S^2$.

The genus of a surface may be defined, viewed, and computed in various different ways, also depending on the model in which the surface is presented.

*Topologically*, the genus may for example be obtained from a homology group, as $g = \frac{1}{2} \dim H_1(S_g; \mathbb{Q})$. Alternatively, the genus may be expressed as the maximal number of disjoint, non-separating, closed loops (this is the definition given by Heffter [15]). It is also half the maximal number of non-separating loops that are disjoint except for a common basepoint.

*Combinatorially*, we can compute the genus in terms of the Euler characteristic, $\chi(S_g) = 2 - 2g = f_0 - f_1 + f_2$. So combinatorially the genus is given by $g = 1 - \frac{1}{2}(f_0 - f_1 + f_2) \geq 0$.

### 0.4. The construction and realization problems

Any combinatorial surface describes a topological space. Conversely, any 2-manifold can be triangulated, but it is not at all clear a priori how many vertices would be needed for that. Thus we have the construction problem for combinatorial surfaces:

**Combinatorial construction problem:** For which parameters $(f_0, f_1, f_2)$ are there combinatorial surfaces?

This is not an easy problem; in the triangulated case of $2f_1 = 3f_2$ it is solved by Ringel's Map Color Theorem, discussed below.

Any geometric surface yields a combinatorial surface, but in the passage from combinatorial to geometric surfaces, there are substantial open problems:

**Geometric construction problem:** For which parameters $(f_0, f_1, f_2)$ are there geometric surfaces?

This problem is much harder. It may be factored into two steps, where the first one asks for a classification or enumeration of the combinatorial surfaces with the given parameters, and the second one tries to solve the following realization problem for all the combinatorial types:

**Realization problem:** Which combinatorially given surfaces have geometric realizations?

In general the answer to the Geometric construction problem does *not* coincide with the answer for the Combinatorial construction problem, that is, the second step may fail even if the first one succeeds. Let's look at some special cases:

- In the case of genus 0, that is $f_1 = f_0 + f_2 - 2$, the construction problem was solved by Steinitz [32]: The necessary and sufficient conditions both for combinatorial and for geometric surfaces are $3f_2 \leq 2f_1$ (as discussed before) and $3f_1 \leq 2f_0$, or equivalently $f_2 \leq 2f_0 - 4$ and $f_0 \leq 2f_2 - 4$.
  By a second, much deeper, theorem by Steinitz [33] [34] [36, Lect. 4], every

combinatorial surface of genus 0 has a geometric realization in $\mathbb{R}^3$, as the boundary of a convex polytope. This solves the realization problem for the case $g = 0$.

- In the case of a simplicial torus, the case of genus 1, the combinatorially possible $f$-vectors are easily seen to be $(n, 3n, 2n)$, for $n \geq 7$. A still pending, old conjecture of Grünbaum [14, Exercise 13.2.3, p. 253] states that *every* triangulated torus (surface of genus 1, with $f$-vector $(n, 3n, 2n)$) has a geometric realization in $\mathbb{R}^3$. Certainly for each $n \geq 7$ there is at least one simplicial torus with $f$-vector $(n, 3n, 2n)$ that is realizable in $\mathbb{R}^3$, so for simplicial tori the set of $f$-vectors for the combinatorial and for the geometric model coincide.
- On the other hand, there are combinatorial tori with $f$-vector $(2n, 3n, n)$, but *none* of them has a geometric realization. Indeed, the condition $3f_0 = 2f_1$ means that the surface in question has a cubic graph (all vertices have degree 3). Thus we are looking at the dual cell decompositions of the simplicial tori. But any geometric surface with a cubic graph is necessarily convex — that is, a 2-sphere (cf. [14, Exercise 11.1.7, p.206]).
- Rather little is known about geometric surfaces of genus $g \geq 2$: Lutz [22] enumerated that there are 865 triangulated surfaces of genus 2 and 20 surfaces of genus 3 on 10 vertices. Together with J. Bokowski he showed that all of these have geometric realizations; Hougardy, Lutz & Zelke [18] even obtained small integer vertex coordinates for all of these. Specific examples of geometric surfaces of genus $g = 3$ and $g = 4$ with a minimal number of vertices were constructed by Bokowski & Brehm [6] [7].
- There are also triangulated combinatorial surfaces that have no geometric realization: Let's look at the $f$-vector $(12, 66, 44)$, which corresponds to a neighborly surface of genus 6 with 12 vertices. Altshuler [2] has enumerated that there are exactly 59 types of such triangulations. One single one, number 54, which is particularly symmetric, was shown not to be geometrically realizable by Bokowski & Guedes de Oliveira [8]. Thus, 58 possible types remain, and we do not know for any single one whether it can be realized or not.

In general, it seems difficult to show for any given triangulated surface that no geometric realization exists. Besides the oriented matroid methods of Bokowski and Guedes de Oliveira, the obstruction theory criteria of Novik [26] and a linking-number approach of Timmreck [35] have been developed in an attempt to do such non-realizability proofs.

In these lectures we look at families of combinatorial surfaces whose genus grows quadratically in the number of vertices, such as the neighborly triangulated surfaces on $n \gg 7$ vertices. We *think* that no geometric realizations exist for these, but no methods to prove such a general result seem to be available yet. On the geometric side we present a construction for surfaces of genus $n \log n$, which may be considered "high genus" in the category of geometric surfaces. We hope that someone will be able to show that this is good, or even best possible, or to improve upon it.

## 1. Two combinatorial constructions

Let us now look at a combinatorial surface with $f_0 = n$ vertices. The following upper bound is quite elementary — the challenge is in the construction of examples that meet it, or at least get close.

**Lemma 1.1.** *A combinatorial surface with $n$ vertices has genus at most*

$$g \;\leq\; \tfrac{1}{12}(n-3)(n-4). \tag{1}$$

*Equality can hold only if $n$ is congruent to $0, 3, 4$ or $7 \bmod 12$, for a triangulated surface that is neighborly.*

*Proof.* Due to the intersection condition, any two vertices are connected by at most one edge, and thus $f_1 \leq \binom{n}{2}$.

In the case of a triangulated/simplicial surface, we have $3f_2 = 2f_1$. With this, a simple calculation yields

$$g \;=\; 1 - \tfrac{1}{2}(f_0 - f_1 + f_2) \;=\; 1 - \tfrac{1}{2}f_0 + \tfrac{1}{6}f_1 \;\leq\; 1 - \tfrac{1}{2}n + \tfrac{1}{6}\binom{n}{2} \;=\; \tfrac{1}{12}(n-3)(n-4).$$

This holds with equality only if the surface is neighborly. The genus $\frac{1}{12}(n-3)(n-4)$ then is an integer, that is, $n \equiv 0, 3, 4$ or $7 \bmod 12$.

If the surfaces is not simplicial, then it can be triangulated by introducing diagonals, without new vertices, and without changing the genus. However, this always results in triangulated surfaces with missing edges (diagonals that have not been chosen), and thus in surfaces that do not achieve equality in (1). □

The case of neighborly surfaces is indeed very interesting, and has received a lot of attention. In particular, it occurred first in connection with (a generalization of) the four color problem: Its analog on surfaces of genus $g > 0$, known as the "Problem der Nachbargebiete," the problem of neighboring countries, is solved by constructing a maximal configuration of "countries" that are pairwise adjacent. If one draws the dual graph to such a configuration, then this will yield a triangulation of the surface (Kempe 1879 [21]; Heffter 1891 [15]). As the "thread problem" ("Fadenproblem") the question was presented in the famous book by Hilbert & Cohn-Vossen [17].

The case $n = 4$ is trivial (realized by the tetrahedron); the first interesting case is $n = 7$, where a combinatorially-unique configuration exists, the simplicial "Möbius Torus" on 7 vertices [25]. We will look at it below. Möbius' triangulation was rediscovered a number of times, realized by Császár [11], and finally exhibited in the Schlegel diagram of a cyclic 4-polytope on 7 vertices, by Duke [12] and Altshuler [1]. For the other neighborly cases, $n \geq 12$, no realizations are known.

When $n$ is not congruent to $0, 3, 4$, or $7$ the maximal genus of a surface on $n$ vertices is of course smaller than the bound given above, but it could be just the bound rounded down, and indeed it is.

**Theorem 1.2 (Ringel et al. (1968); see** [30]**).** *For each $n \geq 4$, $n \neq 9$, there is a (combinatorial) $n$-vertex surface of genus*

$$g_{\max} \;=\; \left\lfloor \tfrac{(n-3)(n-4)}{12} \right\rfloor.$$

In his 1891 paper, Heffter [15, §3] proved this theorem for $n \leq 12$; in particular, in doing this he introduced some of the basic concepts and notation, and thus "set the stage." From then, it needed another 77 years to complete the proof of Theorem 1.2. The full proof is complicated, with intricate combinatorial arguments divided into twelve cases (according to $n$ mod 12) and a number of ad-hoc constructions needed for sporadic cases of "small $n$." In the following we will sketch Ringel's construction for the nicest of the twelve cases, the case of $n \equiv 7$ mod 12. This is the only case where we can get a surface with a cyclic symmetry, according to Heffter, and in fact we do! (This special case was first solved by Ringel in 1961, but our exposition follows his book from 1973, to which we also refer for the other eleven cases.) Then we also present a second construction, based on a paper by Heffter [16] from 1898: This produces surfaces that are not quite neighborly, but they still do have genus that grows quadratically with the number of vertices. Moreover, this construction is very conceptual and explicit. For simplicity we will give a simple combinatorial description. However, the surface has a $\mathbb{Z}_q$-action, whose quotient is the "perfect" cellulation of the surface of genus $g$ with just one vertex and one 2-cell that arises if one identifies the opposite edges of a $4g$-gon. Heffter's surface thus arises as an abelian covering of this perfect cellulation.

### 1.1. A neighborly triangulation for $n \equiv 7$ mod 12

It was observed already by Heffter that a combinatorial surface is completely determined if we label the vertices and for each vertex describe the cycle of its neighbors (in counter-clockwise/orientation order).

For example, Figure 1 shows a "square pyramid" in top view (a 2-sphere with 5 vertices, consisting of one quadrilateral and four triangles). It is given by a *rotation scheme* of the form

$$\begin{array}{rcl} 0 & : & (1,2,3,4) \\ 1 & : & (0,4,2) \\ 2 & : & (0,1,3) \\ 3 & : & (0,2,4) \\ 4 & : & (0,3,1) \end{array}$$

which says that $1,2,3,4$ are the neighboring vertices, in cyclic order, for vertex 0, etc. In particular, we could have written $(2,3,4,1)$ instead of $(1,2,3,4)$, since this denotes the same cyclic permutation. Some checking is needed, of course, to see whether a scheme of this form actually describes a surface that satisfies the intersection condition.

In the case of a triangulated surface, the corresponding consistency conditions are rather easy to describe. Indeed, if $j,k$ appear adjacent in the cyclic list of neighbors of a vertex $i$, then this means that $[i,j,k]$ is an oriented triangle of the

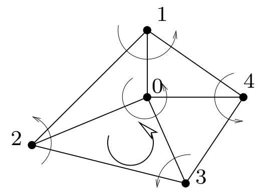



FIGURE 1. Top view of a polyhedral surface (boundary of a square pyramid), with the data that yield the rotation scheme.

surface — and thus $k, i$ have to be adjacent in this order in the cycle of neighbors of $j$, and similarly $i, j$ have to appear in the list for $k$.

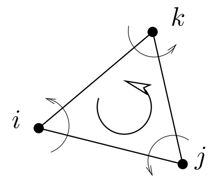



FIGURE 2. Reading off data of the rotation scheme from a triangle in an oriented surface.

Thus in terms of the rotation scheme, the *triangulation condition* (which Ringel calls the "rule $\Delta^*$") says that if the row for vertex $i$ reads

$$i \;:\; (\ldots\;\ldots\;\ldots\;, j, k,\;\ldots\;\ldots\;\ldots\;)$$

then in the rows for $j$ and $k$ we have to get

$$\begin{array}{ccl} j & : & (\ldots\;\ldots\;, k, i,\;\ldots\;\ldots\;\ldots\;\ldots\;) \\ k & : & (\ldots\;\ldots\;\ldots\;\ldots\;, i, j,\;\ldots\;\ldots\;). \end{array}$$

We want to construct triangulated surfaces with a cyclic automorphism group $\mathbb{Z}_n$ — so the scheme for one vertex should yield each of the others by addition of a constant, modulo $n$. Unfortunately, this is possible *only* for $n = 4, 5, 6$ and for $n \equiv 7 \bmod 12$, according to Heffter [15, §4].

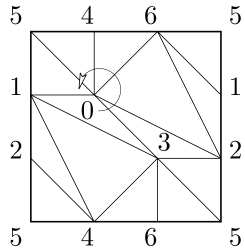
For example, for $n = 7$ there is such a surface, the Möbius torus [25] of Figure 3.

Now let's assume we have a rotation scheme for a triangulated surface with automorphism group $\mathbb{Z}_n$. If the row for vertex 0 reads

$$0 \;:\; (\ldots\;\ldots\;\ldots\;, j, k,\;\ldots\;\ldots\;\ldots\;)$$

then the triangulation condition, rule $\Delta^*$, yields that

$$\begin{array}{ccl} j & : & (\ldots\;\ldots\;, k, 0,\;\ldots\;\ldots\;\ldots\;\ldots\;) \\ k & : & (\ldots\;\ldots\;\ldots\;\ldots\;, 0, j,\;\ldots\;\ldots\;) \end{array}$$




$$
\begin{array}{rcl}
0 & : & (1,3,2,6,4,5) \\
1 & : & (2,4,3,0,5,6) \\
2 & : & (3,5,4,1,6,0) \\
3 & : & (4,6,5,2,0,1) \\
4 & : & (5,0,6,3,1,2) \\
5 & : & (6,1,0,4,2,3) \\
6 & : & (0,2,1,5,3,4).
\end{array}
$$

Figure 3. The Möbius torus, and its rotation scheme. Read off the rotation scheme from the sketch! Observe how the first row determines all others by addition modulo 7.

and then the $\mathbb{Z}_n$-automorphism implies (subtracting $j$ resp. $k$) that

$$
\begin{array}{rcl}
0 & : & ( \dots \dots , k-j, -j, \dots \dots \dots ) \\
0 & : & ( \dots \dots \dots , -k, j-k, \dots \dots ).
\end{array}
$$

In other words, if in the neighborhood of 0, we have that "$k$ follows $j$," then also "$-j$ follows $k-j$," and "$j-k$ follows $-k$" (where all vertex labels are interpreted in $\mathbb{Z}_n$, that is, modulo $n$).

The condition that we have thus obtained can be viewed as a flow condition (a "Kirchhoff law") in a cubic graph: The cyclic order in the neighborhood of 0 can be derived from a walk in an edge-labelled graph, whose edge labels satisfy a flow condition — see Figure 4.

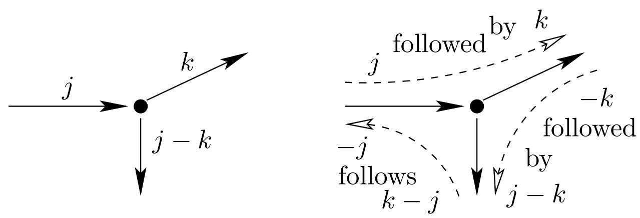



Figure 4. The flow condition, "Kirchhoff's law." The left figure shows how a flow of size $j$ is split into two parts. The right figure shows how to read the labels when passing through the vertex, turning left in each case.

Thus in order to obtain a valid "row 0" we have to produce a cyclic permutation of $1, 2, \dots, n-1$ that can be read off from a flow (circulation) in a cubic graph. Ringel's solution for this in the case $n \equiv 7 \bmod 12$ is given by Figure 5.

It is based on writing $\mathbb{Z}_n = \{0, \pm 1, \pm 2, \dots, \pm(6s+3)\}$. The figure encodes the full construction: It describes a cubic graph with $2s+4$ vertices and $6s+3$ edges, where

– each edge label from $\{1, 2, \dots, 6s+3\}$ occurs exactly once,

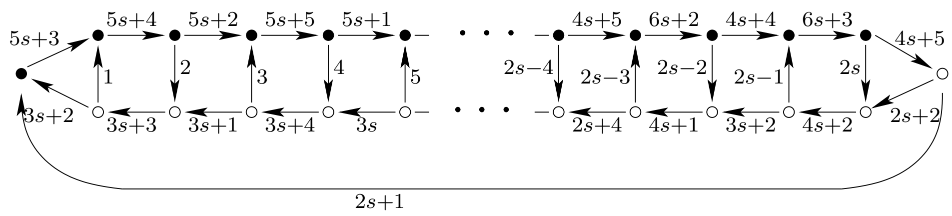



FIGURE 5. A network for a neighborly surface with $n = 12s + 7$ vertices.

– at each vertex, the flow condition is satisfied (modulo $n$).

Now the construction rule is the following: Travel on this graph,

– at each black vertex $\bullet$ turn left, to the next arc in clockwise ordering of the arcs that are adjacent to the vertex, at each white vertex $\circ$ turn right, and
– record the label of each edge traversed in arrow direction, resp. the negative of the label if traversed against arrow direction,

The main claim to be checked is that this prescription leads to a single cycle in which each edge is traversed in each direction exactly once, so each value in $\{\pm1, \pm2, \dots, \pm(6s+3)\}$ occurs exactly once. For example, if we start at the arrow labelled "1," then the sequence we follow will start

$$1,\ -(5s+3),\ -(3s+2),\ -(3s+3),\ -(3s+1),\ -(3s+4),\ -3s,\ -(3s+5),\ \dots$$

This is the first line of the rotation diagram for Ringel's neighborly surface with $n = 12s + 7$ vertices.

Note: *any* cyclic order yields a surface, but we need to control the intersection property, and the genus, e.g. by enforcing the triangle condition. On the other side, if we just take a random permutation (cyclic order), then this yields a very interesting model of a random surface of random genus. See Pippenger & Schleich [28] for a current discussion of such models.

### 1.2. Heffter's surface and a triangulation

Here is a much simpler construction, which yields a not-quite neighborly surface. The underlying remarkable cellular surface was first discovered by Heffter [16], much later rediscovered by Eppstein et al. [13]. (See also Pfeifle & Ziegler [27].)

Let $q = 4g+1$ be a prime power with $g \geq 1$ (one can find suitable primes $q$, or simply take $q = 5^r$). The one algebraic fact we need is that there is a finite field $\mathbb{F}_q$ with $q$ elements, and that the multiplicative group $\mathbb{F}_q^* = \mathbb{F}_q \backslash \{0\}$ is cyclic (of order $q-1 = 4g$), that is, there is a generator $\alpha \in \mathbb{F}_q^*$ such that $\mathbb{F}_q^* = \{\alpha, \alpha^2, \dots, \alpha^{q-1}\}$, with $\alpha^{q-1} = \alpha^{4g} = 1$, and $\alpha^{2g} = -1$. For example, for $g = 3$ and $q = 13$ we may take $\alpha = 2$, with $(1, \alpha, \alpha^2, \dots) = (1, 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7)$.

For any $g \geq 1$, a *perfect* cellulation of $S_g$ is obtained from a $4g$-gon by identifying opposite edges in parallel. In the prime power case, a combinatorial description for this is as follows: Label the directed edges of the $4g$-gon by $1, \alpha, \alpha^2, \alpha^3, \dots$

in cyclic order, and identify the antiparallel edges labelled $s$ and $-s$. (Compare Figure 6.)

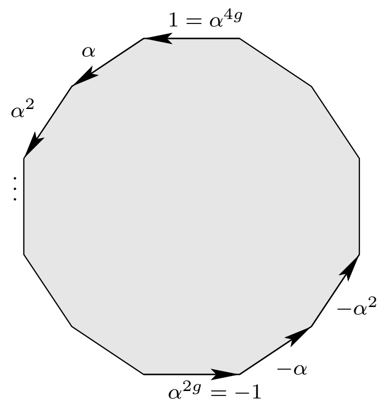



Figure 6. Identifying the opposite edges of a $4g$-gon we obtain a perfect cellulation of $S_g$: All vertices are identified.

The resulting cell complex has the $f$-vector $f = (1, 2g, 1)$. It is *perfect* in the view of Morse theory, since this is also the sequence of Betti numbers (ranks of the homology groups). However, this cell decomposition is not "regular" in the terminology of cell complexes, since there are identifications on the boundaries of cells: All the ends of the edges are identified, so all the edges are loops, and there are lots of identifications on the boundary of the 2-cell.

Now we explicitly write down a $q$-fold "abelian covering" of this perfect cellulation. It has both its vertices and its 2-cells indexed by $\mathbb{F}_q$: For simplicity we identify the vertices with the $q$ elements of $F_q$. The surface has $q$ 2-faces $F_s$, indexed by $s \in \mathbb{F}_q$. The face $F_s$ has the vertices

$$s,\ s+1,\ s+1+\alpha,\ s+1+\alpha+\alpha^2,\ \ \ldots\ \ , s+1+\alpha+\cdots+\alpha^{4g-2},$$

in cyclic order (as indicated by Figure 7), that is,

$$v_s^k \ := \ s + \sum_{i=0}^{k-1} \alpha^i \ = \ s + \frac{\alpha^k - 1}{\alpha - 1} \qquad \text{for } 0 \le k < 4g-1.$$

For each face $F_s$ this yields $q-1$ distinct values/vertices: $\alpha^k$ takes on every value except for 0, and thus $s + \frac{\alpha^k - 1}{\alpha - 1}$ yields all elements of $\mathbb{F}_q$ except for $s + \frac{-1}{\alpha - 1}$. (See Figure 8 for the case $q = 5$. The case $q = 9$ appears in [27].)

Now we have to verify that this prescription does indeed give a surface: For this, check that each vertex comes to lie in a cyclic family of $q-1$ faces.

We thus have a quite remarkable combinatorial structure: The cellular surface $\widetilde{S}_g$ has $q$ vertices and $q$ faces; the vertices have degree $q-1$, the faces have $q-1$ neighbors. Thus the 1-skeleton of the surface is a complete graph $K_q$ (each vertex

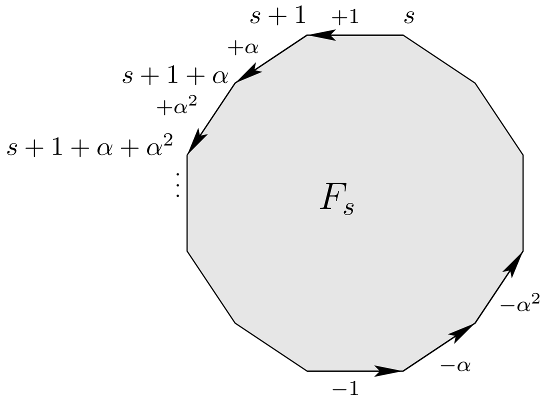



FIGURE 7. One of the $q$ 2-cells, and the labelling of its $q-1=4g$ vertices (by elements of $\mathbb{F}_q$).

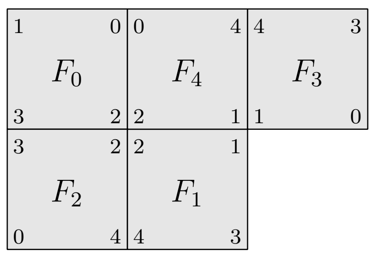



FIGURE 8. The Heffter surface for $q=5$, $g=1$, $\mathbb{F}_5=\mathbb{Z}_5$, and $\alpha=5$ is glued from five squares, as indicated.

is adjacent to every other vertex), and so is the dual graph (each 2-face is adjacent to every other one). So the surface is neighborly and dual neighborly! Moreover, the surface is self-dual, that is, isomorphic to its dual cell decomposition.

With all the combinatorial facts just mentioned, we have in particular computed the $f$-vector of the surface: It is

$$f(\widetilde{S}_g) \;=\; (q, 2gq, q) \;=\; (q, \tbinom{q}{2}, q).$$

Thus we have an orientable surface with Euler characteristic $2q-\binom{q}{2}=2-2g$, and genus $g=\frac{1}{2}\binom{q}{2}-q+1$.

Moreover, "by construction" the surface is very symmetric: First, there clearly is an $\mathbb{F}_q$-action by addition; and if we mod out by this action, we recover the original, "perfect" cell decomposition $\widetilde{S}_g/\mathbb{Z}_q \cong S_g$ with one face. But also the multiplicative group $\mathbb{F}_q^*$ acts by multiplication, with $\alpha\cdot(s+1+\cdots+\alpha^{k-1}) = \alpha s+\alpha+\cdots+\alpha^k = (\alpha s-1)+1+\alpha+\cdots+\alpha^k$. Thus the action on the faces is given by

$$F_s \mapsto F_{\alpha s-1}, \qquad v_s^k \mapsto v_{\alpha s-1}^{k+1}.$$

The full symmetry group of $S_g$ is a "metacyclic group" with $q(q-1)$ elements.

The surface $\widetilde{S}_g$ is a regular cell complex, in that its 2-cells have no identifications on the boundary, but it does not satisfy the intersection condition: Any two 2-cells intersect in $q-2$ vertices (since each facet includes all but one of the $q$ vertices).

Thus we triangulate $\widetilde{S}_g$, by stellar subdivision of the 2-cells: Then we have $n = 2q$ vertices, $q$ of degree $q-1$, and $q$ of degree $2q-2$. Furthermore, there are $f_1 = 3\binom{n}{2}$ edges, namely $\binom{q}{2}$ "old" ones and $q(q-1)$ "new ones" introduced by the $q$ stellar subdivisions. Furthermore, we have now $q(q-1)$ triangle faces, which yields an $f$-vector

$$f \;=\; \big(2q,\, 3\tbinom{q}{2},\, 2\tbinom{q}{2}\,\big),$$

and hence

$$g \;=\; 1 - \tfrac{1}{2}(2q - 3\tbinom{q}{2} + 2\tbinom{q}{2}) \;=\; \tfrac{1}{16}(n^2 - 10n + 16).$$

So for these simplicial surfaces, for which we have a completely explicit and very simple combinatorial description, the genus is quadratically large in the number of vertices, $g \sim \frac{n^2}{16}$, but they don't quite reach the value $g \sim \frac{n^2}{12}$ of neighborly surfaces.

**Conclusion**

So what is the moral? The moral is that using combinatorial constructions, we do obtain triangulated surfaces whose genus grows quadratically with the number of vertices. To find the constructions for surfaces with the exact maximal genus is very tricky, and certainly one would hope for simpler and more conceptual descriptions/constructions, but combinatorial surfaces whose genus grows quadratically with the number of vertices are quite easy to get.

## 2. A geometric construction

If a surface is smoothly embedded or immersed in $\mathbb{R}^3$, then any generic linear function on $\mathbb{R}^3$ is a Morse function for the surface: All critical points are isolated, and a small neighborhood looks like a quadratic surface, a minimum, a maximum, or a saddle point. Morse theory then tells us that the topological complexity of the surface is bounded by the number of critical points. In particular, we get a chain of inequalities

$$g = \dim H_1(S) < \dim H_*(S) \leq \#\text{ critical points}.$$

If we think of a simplicial/polyhedral surface in $\mathbb{R}^3$ as an approximation to a smooth surface, then we might use a general-position linear function as a Morse function. We might say that all the critical points should certainly be at the vertices, and thus the genus $g$ cannot be larger than the number of vertices for an embedded (or immersed) surface.

*However*, in the case of high genus the approximation of a smooth surface by a simplicial surface is not good, it is very coarse, and the critical points induced by a linear function on a simplicial surface certainly do not satisfy the Morse condition of looking like quadratic surfaces. (Barnette, Gritzmann & Höhne [5] analyzed the local combinatorics of the critical points for a linear function.) And indeed, the result suggested by our argument is far from being true. It was disproved by McMullen, Schulz & Wills [24], who in 1983 presented "polyhedral 2-manifolds in $E^3$ with unusually high genus": They produced sequences both of simplicial and of quad-surfaces on $n$ vertices whose genus grows like $n \log n$.

In the following we will give a simple combinatorial description of their quad-surfaces $Q_m$, and describe an explicit, new geometric construction for them, which is non-inductive, yields explicit coordinates, and "for free" even yields a cubification of the convex hull of the surface without additional vertices. We obtain this by putting together (simplified versions of) several recent constructions: Based on intuition from Amenta & Ziegler [3], a simplified construction of the neighborly cubical polytopes of Joswig & Ziegler [20], which are connected to the construction of high genus surfaces via Babson, Billera & Chan [4] and observations by Joswig & Schröder [19]. The constructions as presented here can be generalized and extended quite a bit, which constitutes both recent work as well as promising and exciting directions for further research. See e.g. [37].

The construction in the following will be in five parts:

1. combinatorial description of the surface as the mirror-surface of the $n$-gon, embedded into the $n$-dimensional standard cube,
2. construction of a deformed $n$-cube,
3. general definition and characterization of faces that are "strictly preserved" under a polytope projection,
4. identification of some faces of our deformed $n$-cube that are strictly preserved under projection to $\mathbb{R}^4$, and
5. realizing the desired surfaces in $\mathbb{R}^3$ via Schlegel diagrams.

### 2.1. Combinatorial description

The surface $Q_m$ is most easily described as a subcomplex of the $m$-dimensional cube $C_m = [0,1]^m$.

Any nonempty face of $C_m$ consists of those points in $C_m$ for which some coordinates are fixed to be 0, others are fixed to be 1, and the rest are left free to vary in $[0,1]$. Thus there is a bijection of the non-empty faces with $\{0,1,*\}^m$.

**Definition 2.1.** For $m \geq 3$, the quad-surface $Q_m$ is given by all the faces of $C_m$ for which only two, cyclically-successive coordinates may be left free.

Thus the subset $|Q_m| \subset [0,1]^m$ consists of all points that have at most two fractional coordinates — and if there are two, they have to be either adjacent, or they have to be the first and the last coordinate. (This description perhaps first appeared in Ringel [29]. Compare Coxeter [10, p. 57].) In particular, $Q_3$ is just the boundary of the unit 3-cube.

Let's list the faces of $Q_m$: These are *all* the $f_0(Q_m) = 2^m$ vertices of the 0/1-cube, encoded by $\{0,1\}^m$; then $Q_m$ contains *all* the $f_1(Q_m) = m2^{m-1}$ edges of the $m$-cube, corresponding to strings with exactly one $*$ and 0/1-entries otherwise. And finally we have $f_2(Q_m) = m2^{m-2}$ quad faces, corresponding to strings with two cyclically-adjacent $*$s and 0/1s otherwise.

Why is this a surface? This is since all the vertex links are circles. Indeed, if we look at any vertex, then we see in its star the $m$ edges emanating from the vertex, and the $m$ square faces between them, which connect the edges in the cyclic order, as in Figure 9.

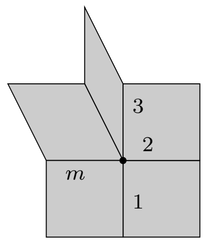



FIGURE 9. The star of a vertex in $Q_m$

The surface is indeed orientable: An explicit orientation is obtained by dictating that in the boundary of any 2-face for which the fractional coordinates are $k-1$ and $k$ (modulo $m$), the edges with a fractional $(k-1)$-coordinate should be oriented from the even-sum vertex to the odd-sum vertex, while the edges corresponding to a fractional $k$-coordinate are oriented from the odd-sum vertex to the even-sum vertex (cf. Figure 10).

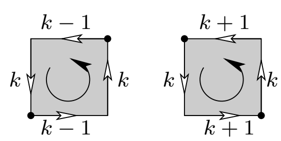



FIGURE 10. The orientation of $Q_m$ described in the text; here the black vertices are the ones with an even sum of coordinates.

Thus, $Q_m$ is an orientable polyhedral surface, realized geometrically in $\mathbb{R}^m$ as a subcomplex of $C_m$. Its Euler characteristic is

$$\chi(Q_m) \;=\; 2^m - m2^{m-1} + m2^{m-2} \;=\; (4-2m+m)2^{m-2}$$

and thus with $g = 1 - \frac{1}{2}\chi$ and $n := f_0(Q_m) = 2^m$ the genus is

$$g(Q_m) \;=\; 1 + (m-4)2^{m-3} \;=\; 1 + \tfrac{n}{8}\log_2\tfrac{n}{16} \;=\; \Theta(n\log n).$$

So we are dealing with a 2-sphere for $m = 3$, with a torus for $m = 4$, while for $m = 5$ we already get a surface of genus 5. There are also simple recursive ways to describe the surface $Q_m$, as given by McMullen, Schulz & Wills in their original paper [24]. The combinatorial description here is a special case of the "mirror complex" construction of Babson, Billera & Chan [4], which from any simplicial $d$-complex on $n$ vertices produces a cubical $(d+1)$-dimensional subcomplex of the $n$-cube on $2^n$ vertices with the given complex as the link of each vertex: Here we are dealing with the case of $d = 1$, where the simplicial complex is a cycle on $m$ vertices.

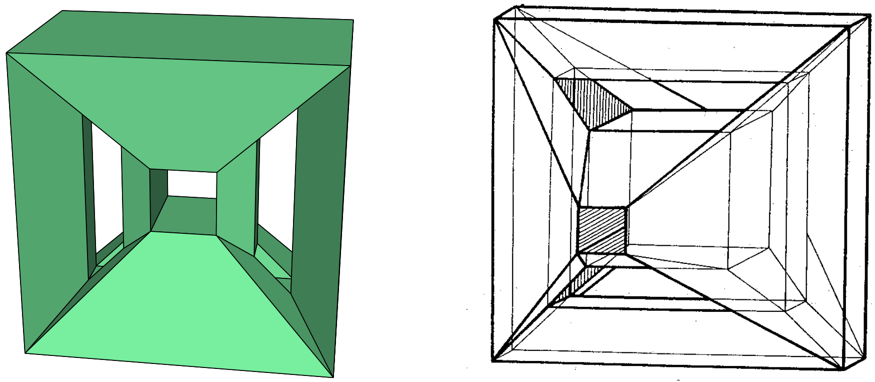


FIGURE 11. Realizations of the surface $Q_5$ of genus 5 in $\mathbb{R}^3$. On the left we show `polymake/javaview` graphics by Thilo Schröder); on the right a figure from McMullen, Schulte & Wills [23, p. 300].

### 2.2. Construction of a deformed $m$-cube

In the last section, we have described a surface $Q_m$ as a subcomplex of the standard orthogonal $m$-cube $C_m = [0,1]^m$, and thus as a polyhedral complex in $\mathbb{R}^m$. If we take any other realization of the $m$-cube, then this yields a corresponding realization of our surface as a subcomplex. The object of this section is to describe an entirely explicit "deformed" cube realization $D_m^\varepsilon$, which contains the surface $Q_m^\varepsilon$ as a subcomplex. Here it is.

**Definition 2.2.** For $m \geq 4$ and $\varepsilon > 0$ let $D_m^\varepsilon$ be the set of all points $x \in \mathbb{R}^m$ that satisfy the linear system of $2m$ linear inequalities

$$\begin{pmatrix} \pm\varepsilon &&&&&&&&& \\ 2 & \pm\varepsilon &&&&&&&& \\ -7 & 2 & \pm\varepsilon &&&&&&& \\ 7 & -7 & 2 & \pm\varepsilon &&&&&& \\ -2 & 7 & -7 & 2 & \pm\varepsilon &&&&& \\ & -2 & 7 & -7 & \cdot & \multicolumn{1}{c|}{\cdot} &&&& \\ && -2 & \cdot & -7 & \multicolumn{1}{c|}{2} & \pm\varepsilon &&& \\ &&& \cdot & 7 & \multicolumn{1}{c|}{-7} & 2 & \pm\varepsilon && \\ &&&& -2 & \multicolumn{1}{c|}{7} & -7 & 2 & \pm\varepsilon & \\ &&&&& \multicolumn{1}{c|}{-2} & 7 & -7 & 2 & \pm\varepsilon \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \cdot \\ \cdot \\ \cdot \\ x_{m-2} \\ x_{m-1} \\ x_m \end{pmatrix} \leq \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ \cdot \\ \cdot \\ \cdot \\ b_{m-2} \\ b_{m-1} \\ b_m \end{pmatrix} \tag{2}$$

This defines a polytope with the combinatorics of $C_m$ if $\varepsilon > 0$ is small enough and if the sequence of right-hand side entries $b_1, b_2, b_3, \dots$ grows fast enough. The following lemma gives concrete values "that work."

**Lemma 2.3.** *For $0 < \varepsilon < \frac{1}{2}$ and $b_k = (\frac{6}{\varepsilon})^{k-1}$, the set $D_m^\varepsilon$ is combinatorially equivalent to the $m$-cube.*

*Proof.* The $k$-th pair of inequalities from (2) may be written as

$$|x_k| \;\le\; \frac{1}{\varepsilon}(b_k - 2x_{k-1} + 7x_{k-2} - 7x_{k-3} + 2x_{k-4}), \tag{3}$$

with $x_0 \equiv x_{-1} \equiv x_{-2} \equiv x_{-3} :\equiv 0$. So if the $x_{k-1}, x_{k-2}, \dots$ are bounded, and if $b_k$ is guaranteed to be larger than

$$L_k \;:=\; 2|x_{k-1}| + 7|x_{k-2}| + 7|x_{k-3}| + 2|x_{k-4}|,$$

then the right-hand side of (3) is strictly positive, and the $x_k$ is bounded again. In this situation, we find that the two inequalities represented by (3) cannot be simultaneously satisfied with equality, but any one of them can. Thus, by induction we get that the first $2k$ inequalities of the system (2) define a $k$-cube (in the first $k$ variables).

With the concrete values suggested by the lemma, we verify by induction that $|x_k| < \frac{1}{3}(\frac{6}{\varepsilon})^k$. Indeed, this is certainly true for $k \le 0$, where we have $x_k \equiv 0$. Thus, with $\varepsilon < \frac{1}{2}$ for $k \ge 1$ and induction on $k$ we get

$$\begin{aligned} L_k &= 2|x_{k-1}| + 7|x_{k-2}| + 7|x_{k-3}| + 2|x_{k-4}| \\ &< (\tfrac{6}{\varepsilon})^{k-1}[\tfrac{2}{3} + \tfrac{7}{3}\tfrac{\varepsilon}{6} + \tfrac{7}{3}(\tfrac{\varepsilon}{6})^2 + \tfrac{2}{3}(\tfrac{\varepsilon}{6})^3] \;<\; (\tfrac{6}{\varepsilon})^{k-1} \;=\; b_k \end{aligned}$$

and thus the right-hand side in (3) is always strictly positive, and we also get the inequality $|x_k| < \frac{1}{\varepsilon}(b_k + L_k) = \frac{2}{\varepsilon}(\frac{6}{\varepsilon})^{k-1} < \frac{1}{3}(\frac{6}{\varepsilon})^k$. □

### 2.3. Strictly preserved faces

In the following, we are considering an arbitrary $m$-dimensional polytope $P \subset \mathbb{R}^m$, but of course you should think of $P = D_m^\varepsilon$, the polytope that we will want to apply this to.

The nontrivial faces $G \subset P$ of such a polytope are defined by linear functions: A nonzero linear function $x \mapsto c^t x$ *defines* the face $G \subset P$ if $G$ consists of the points of $P$ for which the value $c^t x$ is maximal, that is, if

$$G \;=\; \{x \in P : c^t x = c_0\} \;=\; P \cap H,$$

where $c_0 = \max\{c^t x : x \in P\}$, and where $H = \{x \in \mathbb{R}^m : c^t x = c_0\}$ is a hyperplane.

Given a face $G$, how do we find a linear functional $c^t x$ that defines it? For *facets* (faces of dimension $d-1$) $F \subset P$ the linear functional is unique up to taking positive multiples: $F = \{x \in P : {n_F}^t x = c_F\}$, where $n_F$ is a normal vector to $F$, and $c_F$ is the maximal value that the linear functional $x \mapsto {n_F}^t x$ takes on $P$. From this it easy to check (see [36, Lect. 2] for proofs, and Figure 12 for intuition) that

$c$ defines $G$ if and only if it is a linear combination, with positive coefficients, of normal vectors $n_F$ of the facets $F \subset P$ that contain $G$.

In particular, the affine hull of $G$, $\operatorname{aff} G$, is the intersection of all the hyperplanes spanned by the facets $F$ that contain $G$:

$$\operatorname{aff} G \;=\; \{x \in \mathbb{R}^m : {n_F}^t x = \max \text{ for all facets } F \supseteq G\}.$$

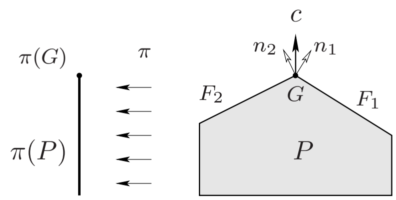



FIGURE 12. The normal vector $c$ for the face $G$ can be written as a positive combination of the normal vectors of the facets $F_1, F_2$ that contain $G$. The face $G \subset P$ is strictly preserved by the projection of $P$ to the last coordinate.

Now we look at a projection of $P$, that is, we look at a surjective linear map $\pi : \mathbb{R}^m \to \mathbb{R}^d$. The image $\pi(P)$ is then a $d$-dimensional polytope, and the faces of $\pi(P)$ are all induced by faces of $P$: If $\bar{G} \subset \pi(P)$ is a face of $\pi(P)$, then $\pi^{-1}(\bar{G})$ is a unique face of $P$.

The faces $\bar{G}$ of the projected polytope $\pi(P)$ thus correspond those faces of $P$ that can be defined by hyperplanes that are parallel to the kernel of the projection. Equivalently, the faces of $\pi(P)$ correspond to those faces of $P$ that can be defined by normal vectors that are orthogonal to the projection.

However, in general the face $\pi^{-1}(\bar{G})$ is not the only face that projects to $\bar{G}$, and in general it will have a higher dimension than $\bar{G}$, and it will have faces that do not project to faces of $\pi(P)$. (See Figure 12 for easy examples.) Thus, we single out a very specific, nice situation, where this does not happen: $G$ will map to a face $\pi(G) \subseteq \pi(P)$ of the same dimension as $G$, and the faces of $G$ map to the faces of $\pi(G)$.

**Definition 2.4 (Strictly preserved faces).** Let $\pi : P \to \pi(P)$ be a polytope projection. A nontrivial face $G \subset P$ is *strictly preserved* by the projection if $\pi(G)$ is a face of $\pi(P)$, with $G = \pi^{-1}(\pi(G))$, and such that the map $G \to \pi(G)$ is injective.

One can work out linear algebra conditions that characterize faces $G$ that are strictly preserved by a projection (see [37]): We need that the normal vectors $n_F$ to the facets $F$ that contain $G$, after projection to the kernel (or to a fiber) of the projection do span this fiber positively, that is, the projected vectors have to span the fiber, and they have to be linearly dependent with positive coefficients.

Here we want to apply this only in a very specific situation, namely for an orthogonal projection to the last $k$ coordinates, that is, for the projection $\pi : \mathbb{R}^m \to$

$\mathbb{R}^k$ given by $x = (x', x'') \mapsto x''$, where $x''$ denotes the last $k$ coordinates of $x$, and $x'$ denotes the first $m - k$ coordinates. For this situation, the characterization of strictly preserved faces boils down to the following.

**Lemma 2.5.** *Let $P \subset \mathbb{R}^m$ be an $m$-dimensional polytope, and let $\pi : \mathbb{R}^m \to \mathbb{R}^k$, $(x', x'') \mapsto x''$ be the projection to the last $k$ coordinates, which maps $P$ to the $k$-polytope $\pi(P)$.*

*Then a nontrivial face $G \subset P$ is strictly preserved by the projection if and only if the facet normals $n_F$ to the facets $F \subset P$ that contain $G$ satisfy the following two conditions: Their restrictions $n'_F \in \mathbb{R}^{m-k}$ to the first $m - k$ coordinates*

- *must be positively dependent, that is, they must satisfy a linear relation of the form $\sum_{F \supset G} \lambda_F n'_F = 0$ with real coefficients $\lambda_F > 0$, and*
- *they have full rank, that is, the vectors $n'_F$ span $\mathbb{R}^{m-k}$.*

### 2.4. Positive row dependencies for the matrix $A'_m$

Our aim is to prove that lots of faces of the deformed cube $D^\varepsilon_m$ constructed in Section 2.2 survive the projection $\pi : \mathbb{R}^m \to \mathbb{R}^4$ to the last four coordinates; in particular, we want to see that all the 2-faces of the surface $Q^\varepsilon_m$ survive the projection.

In view of the criteria just discussed, we have to verify that the corresponding rows of the matrix from (2), after deletion of the last four components, are positively dependent and spanning. This may seem a bit tricky because of the $\varepsilon$ coordinates around, and because we have to treat lots of different faces, and thus choices of rows. However, it turns out to be surprisingly easy.

We start with the matrix $A'_m \in \mathbb{R}^{m\times(m-4)}$,

$$A'_m \;:=\; \begin{pmatrix} 0 & & & & & \\ 2 & 0 & & & & \\ -7 & 2 & 0 & & & \\ 7 & -7 & 2 & 0 & & \\ -2 & 7 & -7 & 2 & \cdot & \\ & -2 & 7 & -7 & \cdot & 0 \\ & & -2 & \cdot & -7 & 2 \\ & & & \cdot & 7 & -7 \\ & & & & -2 & 7 \\ & & & & & -2 \end{pmatrix}.$$

This is the matrix that you get from the left-hand side matrix of (2) if you put $\varepsilon$ to zero, and if you delete the last four coordinates in each row.

The vectors

$$\begin{array}{l} (1,0,0,\;\ldots\;,0) \\ (1,1,1,\;\ldots\;,1) \\ (1,2,4,\;\ldots\;,2^n) \\ (1,\tfrac{1}{2},\tfrac{1}{4},\;\ldots\;,\tfrac{1}{2^n}) \end{array}$$

lie in the kernel of this matrix, that is, they describe row dependencies. Indeed, the coefficients $(2, -7, 7, -2)$ that appear in the columns of $A^{\varepsilon}_m$, and hence of $A'_m$, have been chosen exactly to make this true.

In particular, for any $t \in \mathbb{Z}$ the rows of $A'_m$ are dependent with the coefficient 0 for the first row, and coefficients

$$(2^{i-t} - 1)(1 - 2^{t+1-i}) \;=\; 2^{-t}2^i + 2^{t+1}\tfrac{1}{2^i} - 3$$

for the $i$-th row, $2 \le i \le m-4$. These coefficients are positive, except for the coefficients for $i = 0, t, t{+}1$, which are zero. Thus, if the first, $t$-th and $(t{+}1)$-st row are deleted from $A'_m$, the remaining $m{-}3$ rows are positively dependent. Moreover, the remaining $m-3$ rows span $\mathbb{R}^{m-3}$, as one sees by inspection of $A'_m$: The rows $2, \dots, t-1$ have the same span as the first $t-2$ unit vectors $e_1, \dots, e_{t-2}$, since the corresponding submatrix has lower-triangular form with diagonal entries $+2$, and the rows numbered $t+2, \dots, m$ together have the same span as $e_{t-2}, \dots, e_{m-4}$, due to a corresponding upper-triangular submatrix with diagonal entries $-2$.

So the $m-3$ rows from $A'_m$ corresponding to the index set $[n] \setminus \{1, t, t+1\}$ are positively dependent and spanning, for $1 < t < n$. In particular, this is true for the rows with index set $[n] \setminus \{t, t+1\}$ for $1 \le t < n$ as well as for the rows given by $[n] \setminus \{1, n\}$. That is, if we delete any two cyclically-adjacent rows from $A'_m$, then the remaining rows are positively dependent and spanning. Moreover, the property of a vector configuration to be "positively dependent and spanning" is stable under sufficiently small perturbations: Thus if we delete the last four columns, the first row, and any two adjacent rows from $A^{\varepsilon}_m$, then the rows of the resulting matrix will be positively dependent, and spanning. Thus we have proved the following result.

**Proposition 2.6.** *For sufficiently small $\varepsilon > 0$, the projection $\pi : \mathbb{R}^m \to \mathbb{R}^4$ yields a polyhedral realization $\pi(D^{\varepsilon}_m)$ of the surface $Q^{\varepsilon}_m$ in $\mathbb{R}^4$, as part of the boundary complex of the polytope $\pi(D^{\varepsilon}_m)$.*

### 2.5. Completion of the construction, via Schlegel diagrams

In the last section, we have constructed a 4-dimensional polytope

$$\bar{P}_m \;:=\; \pi(D^{\varepsilon}_m) \subset \mathbb{R}^4$$

as the projection of an $m$-cube. One can quite easily prove that the projection is in sufficiently general position with respect to the $D^{\varepsilon}_m$, so the resulting 4-polytope is *cubical*: All its facets are combinatorial cubes.

Moreover, the 1-skeleton of this polytope is exactly that of the $m$-cube: We have constructed *neighborly cubical* 4-polytopes. (Indeed, our construction is very closely related to the original one by Joswig & Ziegler [20].)

The boundary complex of any 4-polytope may be visualized in terms of a Schlegel diagram (see [36, Lect. 5]): By radial projection from a point that is very close to a facet $F_0 \subset \bar{P}_m$, we obtain a polytopal complex $\mathcal{D}(\bar{P}_m, F_0)$ that faithfully represents all the faces of $\bar{P}_m$, except for $F_0$ and $\bar{P}_m$ itself. Hence we have arrived at the goal of our construction.

**Theorem 2.7.** *For $m \geq 3$, there is a polyhedral realization of the surface $Q_m$, the "mirror complex of an m-gon," in $\mathbb{R}^3$.*

*For $m \geq 4$ such a realization may be found as a subcomplex of*

$$\mathcal{D}(\pi(D_m^\varepsilon), F_0),$$

*the Schlegel diagram (with respect to an arbitrary facet $F_0$) of a projection of the deformed m-cube $D_m^\varepsilon \subset \mathbb{R}^m$ (with sufficiently small $\varepsilon$) to the last 4 coordinates.*

Thus we have obtained quadrilateral surfaces, polyhedrally realized in $\mathbb{R}^3$, of remarkably high genus. If you prefer to have triangulated surfaces, you may of course further triangulate the surfaces just obtained, without introduction of new vertices. This yields a simplicial surface embedded in $\mathbb{R}^3$, with $f$-vector

$$(2^m, 3m2^{m-2}, m2^{m-1}).$$

For even $m \geq 4$ this may be done in such a way that the resulting surface has all vertex degrees equal (to $\frac{3}{2}m$): To achieve this, triangulate the faces with fractional coordinates $k-1$ and $k$ by using the diagonal between the even-sum vertices if $k$ is even, and the diagonal between the odd-sum vertices if $k$ is odd. (Figure 13 indicates how two adjacent quadrilateral faces are triangulated by this rule.)

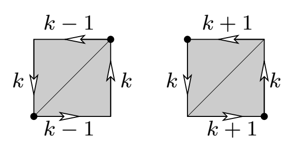



FIGURE 13. The triangulation of $Q_m$ described above. Here we assume that $k$ is even. The black vertices are the ones with an even sum of coordinates.

In other words, this yields *equivelar* triangulated surfaces of high genus, which is what McMullen et al. were after in [24].

Let's finally note that this construction has lots of interesting components that may be further analyzed, varied, and extended. Thus a lot remains to be done, and further questions abound. To note just a few aspects briefly:

- Give explicit bounds for some $\varepsilon > 0$ that is "small enough" for Proposition 2.6.
- There are higher-dimensional analogues of this: So, extend the construction as given here in order to get neighborly cubical $d$-polytopes, with the $(\frac{d}{2}-1)$-skeleton of the $N$-cube, for $N \geq d \geq 2$. (Compare [20].)
- Extend this to surfaces that you get as "mirror complexes" in products of polygons, rather than just $m$-cubes (which are products of quadrilaterals, for even $m$).

See Ziegler [37], Joswig & Schröder [19] and Sanyal et al. [31] for further work and ideas related to these questions.

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## References

[1] A. Altshuler, *Polyhedral realizations in* $\mathbb{R}^3$ *of triangulations of the torus and* 2*-manifolds in cyclic* 4*-polytopes*, Discrete Math., 1 (1971), pp. 211–238.

[2] A. Altshuler, J. Bokowski, and P. Schuchert, *Neighborly* 2*-manifolds with* 12 *vertices*, J. Combin. Theory Ser. A, 75 (1996), pp. 148–162.

[3] N. Amenta and G. M. Ziegler, *Deformed products and maximal shadows*, in Advances in Discrete and Computational Geometry (South Hadley, MA, 1996), B. Chazelle, J. E. Goodman, and R. Pollack, eds., vol. 223 of Contemporary Mathematics, Providence RI, 1998, Amer. Math. Soc., pp. 57–90.

[4] E. K. Babson, L. J. Billera, and C. S. Chan, *Neighborly cubical spheres and a cubical lower bound conjecture*, Israel J. Math., 102 (1997), pp. 297–315.

[5] D. W. Barnette, P. Gritzmann, and R. Höhne, *On valances of polyhedra*, J. Combinatorial Theory, Ser. A, 58 (1991), pp. 279–300.

[6] J. Bokowski and U. Brehm, *A new polyhedron of genus* 3 *with* 10 *vertices*, in Intuitive geometry (Siófok, 1985), vol. 48 of Colloq. Math. Soc. János Bolyai, North-Holland, Amsterdam, 1987, pp. 105–116.

[7] ———, *A polyhedron of genus* 4 *with minimal number of vertices and maximal symmetry*, Geom. Dedicata, 29 (1989), pp. 53–64.

[8] J. Bokowski and A. Guedes de Oliveira, *On the generation of oriented matroids*, Discrete Comput. Geometry, 24 (2000), pp. 197–208.

[9] U. Brehm and J. M. Wills, *Polyhedral manifolds*, in Handbook of Convex Geometry, P. Gruber and J. Wills, eds., North-Holland, Amsterdam, 1993, pp. 535–554.

[10] H. S. M. Coxeter, *Regular skew polyhedra in* 3 *and* 4 *dimensions and their topological analogues*, Proc. London Math. Soc. (2), 43 (1937), pp. 33–62. Reprinted in "Twelve Geometric Essays," Southern Illinois Univ. Press, Carbondale, 1968, 75-105.

[11] Á. Császár, *A polyhedron without diagonals*, Acta Sci. Math. (Szeged), 13 (1949/50), pp. 140–142.

[12] R. A. Duke, *Geometric embeddings of complexes*, Amer. Math. Monthly, 77 (1970), pp. 597–603.

[13] D. Eppstein, G. Kuperberg, and G. M. Ziegler, *Fat* 4*-polytopes and fatter* 3*-spheres*, in Discrete Geometry: In honor of W. Kuperberg's 60th birthday, A. Bezdek, ed., vol. 253 of Pure and Applied Mathematics, Marcel Dekker Inc., New York, 2003, pp. 239–265. `http://www.arXiv.org/math.CO/0204007`.

[14] B. Grünbaum, *Convex Polytopes*, vol. 221 of Graduate Texts in Math., Springer-Verlag, New York, 2003. Second edition prepared by V. Kaibel, V. Klee and G. M. Ziegler (original edition: Interscience, London 1967).

[15] L. Heffter, *Ueber das Problem der Nachbargebiete*, Math. Annalen, 38 (1891), pp. 477–508. Facsimile available at JFM, http://www-gdz.sub.uni-goettingen.de/cgi-bin/digbib.cgi?PPN235181684_00%38.

[16] ———, *Ueber metacyklische Gruppen und Nachbarconfigurationen*, Math. Annalen, 50 (1898), pp. 261–268. Facsimile available at JFM, http://www-gdz.sub.uni-goettingen.de/cgi-bin/digbib.cgi?PPN235181684_0050.

[17] D. Hilbert and S. Cohn-Vossen, *Anschauliche Geometrie*, Springer-Verlag, Berlin Heidelberg, 1932. Second edition 1996. English translation: Geometry and the Imagination, Chelsea Publ., 1952.

[18] S. Hougardy, F. H. Lutz, and M. Zelke, *Polyhedra of genus* 2 *with* 10 *vertices and minimal coordinates*. Preprint, 3 pages, 2005; arXiv:math.MG/0507592.

[19] M. Joswig and T. Schröder, *Neighborly cubical polytopes and spheres*. Preprint, March 2005, 17 pages; http://www.arxiv.org/math/0503213.

[20] M. Joswig and G. M. Ziegler, *Neighborly cubical polytopes*, Discrete & Computational Geometry (Grünbaum Festschrift: G. Kalai, V. Klee, eds.), (2-3)24 (2000), pp. 325–344. http://www.arXiv.org/math.CO/9812033.

[21] A. B. Kempe, *On the geographical problem of the four colours*, American J. Math., 2 (1879), pp. 193–200.

[22] F. H. Lutz, *Enumeration and random realization of triangulated surfaces*. Preprint, TU Berlin, June 2005, 15 pages; www.arXiv.org/math.CO/0506316.

[23] P. McMullen, E. Schulte, and J. M. Wills, *Infinite series of combinatorially regular polyhedra in three-space*, Geometriae Dedicata, 26 (1988), pp. 299–307.

[24] P. McMullen, C. Schulz, and J. M. Wills, *Polyhedral* 2*-manifolds in* $E^3$ *with unusually large genus*, Israel J. Math., 46 (1983), pp. 127–144.

[25] A. F. Möbius, *Mittheilungen aus Möbius' Nachlass: I. Zur Theorie der Polyëder und der Elementarverwandtschaft*, in Gesammelte Werke II, F. Klein, ed., Verlag von S. Hirzel, Leipzig, 1886, pp. 515–559.

[26] I. Novik, *Upper bound theorems for simplicial manifolds*, Israel J. Math., 108 (1998), pp. 45–82.

[27] J. Pfeifle and G. M. Ziegler, *Many triangulated* 3*-spheres*, Mathematische Annalen, 330 (2004), pp. 829–837. http://www.arXiv.org/math.MG/0304492.

[28] N. Pippenger and K. Schleich, *Topological characteristics of random surfaces generated by cubic interactions*. Preprint, 58 pages, June 2003; arXiv:gr-qc/0306049; Random Structures & Algorithms, to appear.

[29] G. Ringel, *Über drei kombinatorische Probleme am n-dimensionalen Würfel und Würfelgitter*, Abh. Math. Sem. Univ. Hamburg, 20 (1955), pp. 10–19.

[30] ———, *Map Color Theorem*, vol. 234 of Grundlehren Series, Springer-Verlag, New York, 1974.

[31] R. Sanyal, T. Schröder, and G. M. Ziegler, *Polytopes and polyhedral surfaces via projection*. in preparation, 2005.

[32] E. Steinitz, *Über die Eulerschen Polyederrelationen*, Archiv für Mathematik und Physik, 11 (1906), pp. 86–88.

[33] ———, *Polyeder und Raumeinteilungen*, in Encyklopädie der mathematischen Wissenschaften, Geometrie, III.1.2., Heft 9, Kapitel III A B 12, W. F. Meyer and H. Mohrmann, eds., B. G. Teubner, Leipzig, 1922, pp. 1–139.

[34] E. Steinitz and H. Rademacher, *Vorlesungen über die Theorie der Polyeder*, Springer-Verlag, Berlin, 1934. Reprint, Springer-Verlag 1976.

[35] D. Timmreck, *Necessary conditions for geometric realizability of simplicial complexes*. Preprint in preparation, TU Berlin 2005.

[36] G. M. Ziegler, *Lectures on Polytopes*, vol. 152 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1995. Revised edition, 1998; "Updates, corrections, and more" at `http://www.math.tu-berlin.de/~ziegler`.

[37] ———, *Projected products of polygons*, Electronic Research Announcements AMS, 10 (2004), pp. 122–134. `http://www.arXiv.org/math.MG/0407042`.

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