The Random_Edge rule on three-dimensional linear programs

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Abstract

The worst-case expected running time of the simplex algorithm with the Random_Edge pivot rule on a 3-dimensional linear program with n constraints is $\frac{4}{3}n + o(n)$. This bound is sharp.

1 Introduction

The Random_Edge pivot rule is undoubtedly the most natural, and simplest (randomized) pivot rule for linear programming. Furthermore, it still (until proven otherwise) has the potential to be polynomial (even quadratic?!) — thus solving one of the core open problems in mathematical optimization, the quest for a strongly polynomial algorithms for linear programming. In particular, the Random_Edge rule is not tricked into exponential running times by the "deformed products" on which all the classical deterministic pivot rules show exponential running times (see Klee & Minty [4], … , Amenta & Ziegler [1]).

However, it has turned out that the analysis of the simplex algorithm with the Random_Edge pivot rule is (perhaps unexpectedly) difficult and subtle even on simple examples. The main/only two cases that have been analyzed successfully up to now are the d-dimensional linear programs with $d + 2$ constraints (Gärtner, Solymosi, Tschirschnitz, Valtr & Welzl [3]), and the d-dimensional Klee-Minty cubes (Gärtner, Henk & Ziegler [2]).

Here we treat the case of 3-dimensional linear programs (the situation for dimensions $d \leq 2$ not being interesting). With the usual techniques of perturbation (see, e. g., Ziegler [6, Lect. 3]), we may assume that our linear program is

$$\min\ x_3\ :\ x \in P,$$

where P is a simple, 3-dimensional polytope with exactly n facets, and hence $2d-4$ vertices and $3d-6$ edges, with the additional assumption that none of the edges is "horizontal." In fact, we may impose the slightly stronger condition that no two vertices have the same objective function value x_3). Thus we have a unique ordering of the vertices $v_{2n-5}, v_{2n-6}, \ldots, v_1, v_0$ by decreasing objective function. Here $v_0 = v_{\min}$ is the unique minimal (optimal) vertex of the linear program, while $v_{2n-5} = v_{\max}$ is the unique maximal vertex.

The expected running time of the simplex algorithm on the linear program, with the Random_Edge rule, is then given by
$E(v_0) = 0$,
$E(v_i) = 1 + E(v_j)$, if v_j is the only "lower neighbor" of v_i,
$E(v_i) = 1 + \frac{1}{2}(E(v_j) + E(v_k))$, if v_j, v_k are the "lower neighbors" of v_i, and
$E(v_{2n-5}) = 1 + \frac{1}{3}(E(v_j) + E(v_k) + E(v_l))$, if v_j, v_k, v_l are the neighbors of v_{2n-5}.

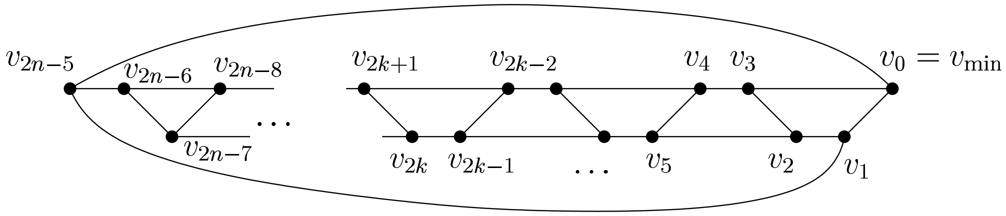
2 Lower bounds

It will be sufficient to give combinatorial descriptions of examples where the Random_Edge rule is "slow," due to the following characterization theorem. (However, for both the explicit classes of examples given below it is not too difficult to explicitly construct the corresponding convex polytopes "by hand."

Theorem 1 (Mihalisin & Klee [5]). *Let D be an acyclic, simple, cubic, planar graph D that is 3-connected (as an undirected graph) and that has three vertex-disjoint directed paths from the unique source to the unique sink.*

Then there is a simple 3-polytope $P \subset \mathbb{R}^3$ in general position such that the graph of P, with the edges directed according to the objective function x_3, is isomorphic to D.

Example 2. In the first type of linear programs there is a monotone Hamiltonian path, that is, every vertex v_i is adjacent to v_{i-1}, for $0 < i \leq 2n-5$. Additionally, v_{2k+1} is adjacent to v_{2k-2}, for $0 < k \leq n-4$. Finally, there are three edges $v_{2n-6} \to v_{2n-8}$, $v_{2n-5} \to v_0$, and $v_{2n-5} \to v_1$ that that will not enter our calculations. This yields a directed graph that satisfies all the assumptions of the Mihalisin–Klee theorem, as one sees by inspection from the figure.



For the expected running times $E(v_i)$, we then have the starting values $E(v_0) = 0$ and $E(v_1) = 1$, and the recursions

$$E(v_{2k}) \;=\; E(v_{2k-1}) + 1, \qquad E(v_{2k+1}) = \tfrac{1}{2}\big(E(v_{2k}) + E(v_{2k-2})\big) + 1$$

for $0 < k \le n-4$. Thus by induction we obtain

$$E(v_{2k}) + 2E(v_{2k+1}) \;=\; 4k + 2$$

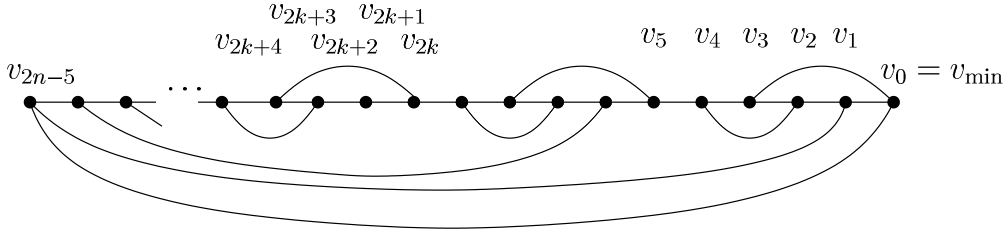
for $0 \le k \le n-4$. In particular, for $k = n-4$ this yields

$$\max\Big\{E(v_{2n-8}), E(v_{2n-7})\Big\} \;\ge\; \tfrac{4n}{3} - 4.$$

If one is interested in the additive constant, then with a bit more work the linear recursions can be evaluated exactly, as $E(v_{2k}) = \frac{4}{3}k + \frac{1}{9}[4 - 4(-\frac{1}{2})^k]$ and $E(v_{2k+1}) = \frac{4}{3}k + \frac{1}{9}[7 + 2(-\frac{1}{2})^k]$, which yields

$$E(v_{2n-6}) \;=\; \tfrac{4}{3}n - \tfrac{67}{18} - \tfrac{16}{9}(-\tfrac{1}{2})^n.$$

Example 3. Our second example will be even easier to evaluate; its additive constant is a bit worse. We assume that $n = 3m + 2$, that is, $2n - 4 = 6m$. Again it has the special feature that v_i and v_{i-1} are adjacent for all i. The first $5m$ vertices come in groups of five, as indicated in the figure.



That is, there are the "short edges" $v_{5k-1} \to v_{5k-3}$ and $v_{5k-2} \to v_{5k-5}$ as well as the "long edges" $v_{2n-4-k} \longrightarrow v_{5k-4}$ for $0 < k \leq m$. Finally, there is one additional long edge from $v_{\max} \longrightarrow v_{\min}$.

A straightforward induction argument yields

$$E(v_i) \;=\; \left\lfloor \tfrac{4i+2}{5} \right\rfloor \qquad \text{for } i \leq 5m-1.$$

(It is an intriguing feature of this example that all these vertices have integral expected running times.) In particular, this means that

$$E(v_{5m-1}) \;=\; 4m-1 \;=\; \tfrac{4n}{3} - \tfrac{11}{3}.$$

Remark. By splitting the maximal vertex, one can also construct examples where the expected number of steps *starting at the maximal vertex* is at least $(\frac{4}{3} - \varepsilon)n$.

(*Was this an observation of Günter Rote? Should I insert the argument?*)

3 Upper bounds

References

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