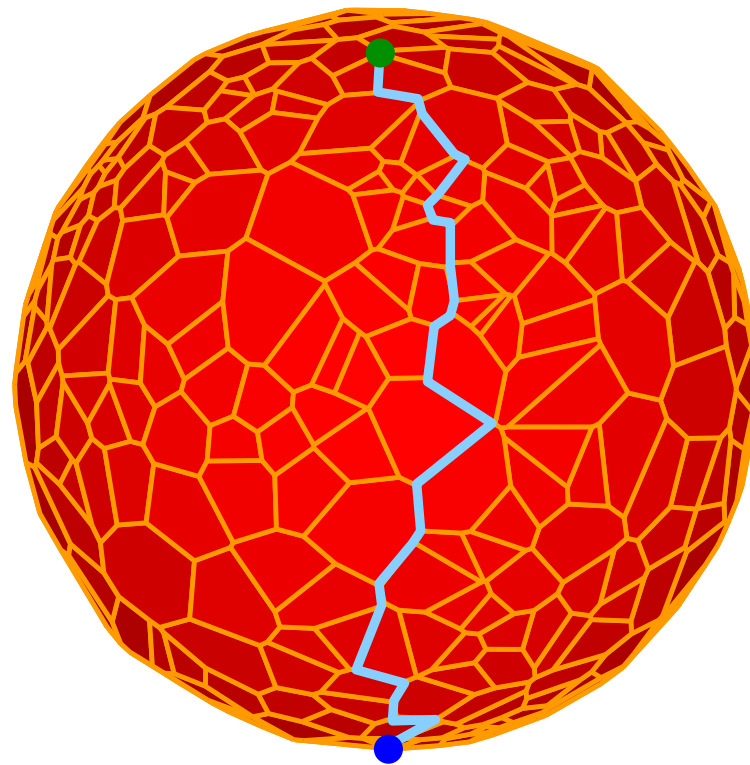


Random Monotone Paths on Polyhedra

Günter M. Ziegler

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Joint work with Volker Kaibel, Rafael Mechtel, and Micha Sharir

arXiv:math.CO/0309351

Given $A \in \mathbb{R}^{n \times d}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^d$, solve

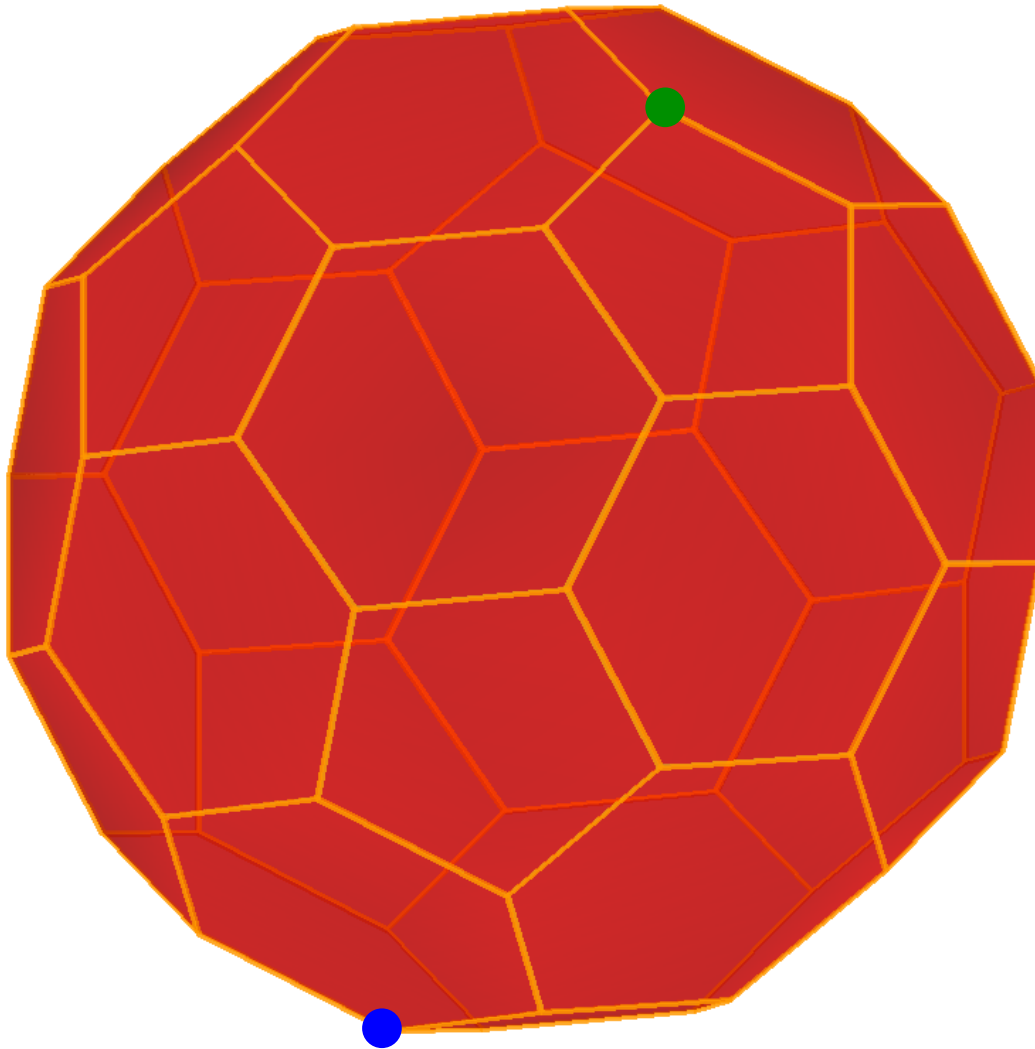
$$\min \{c^t x : Ax \leq b\}$$

Given a linear inequality system (a polytope)
and a linear objective function (“height”),
find the “lowest” vertex

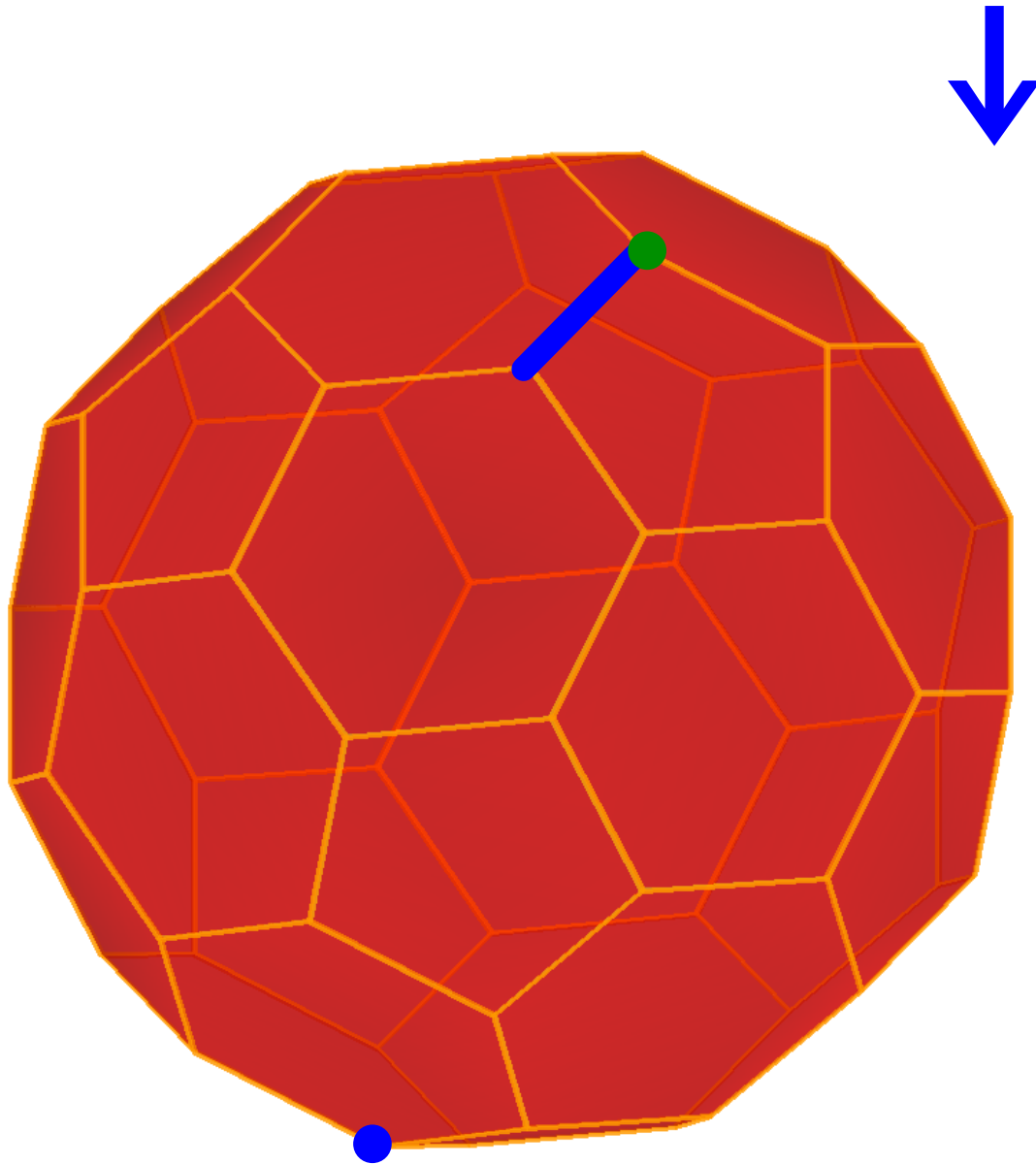
Given a linear inequality system (a polytope) and a linear objective function (“height”), find the “lowest” vertex

- ▷ Simplex Algorithm DANTZIG 1947
- ▷ Ellipsoid Method: **POLYNOMIAL** KHACHIYAN 1979
- ▷ Interior-Point Method: **POLYNOMIAL** KARMAKAR 1984
- ▷ Algorithm in fixed dimension: **LINEAR** MEGIDDO 1984

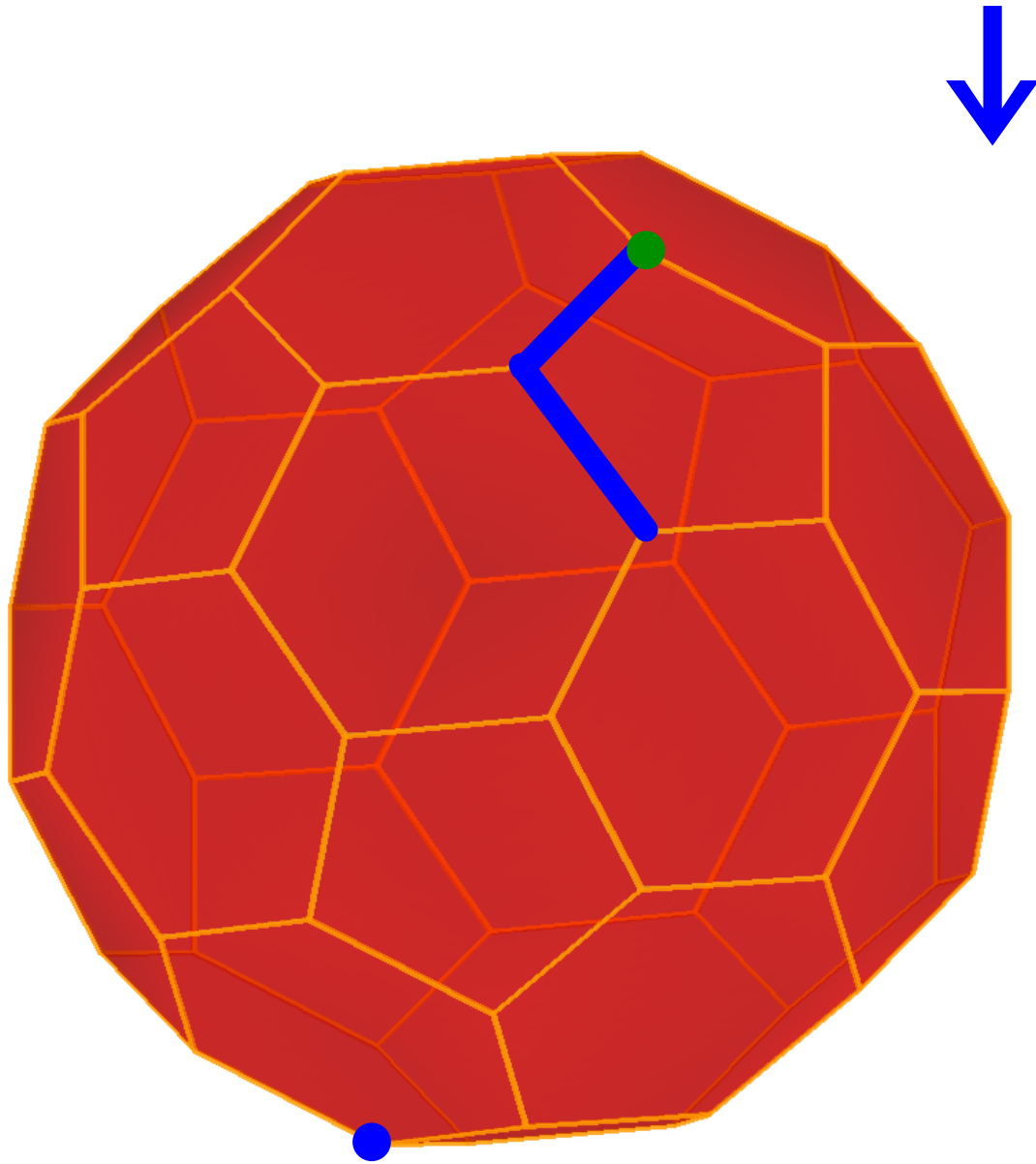
The Simplex Algorithm



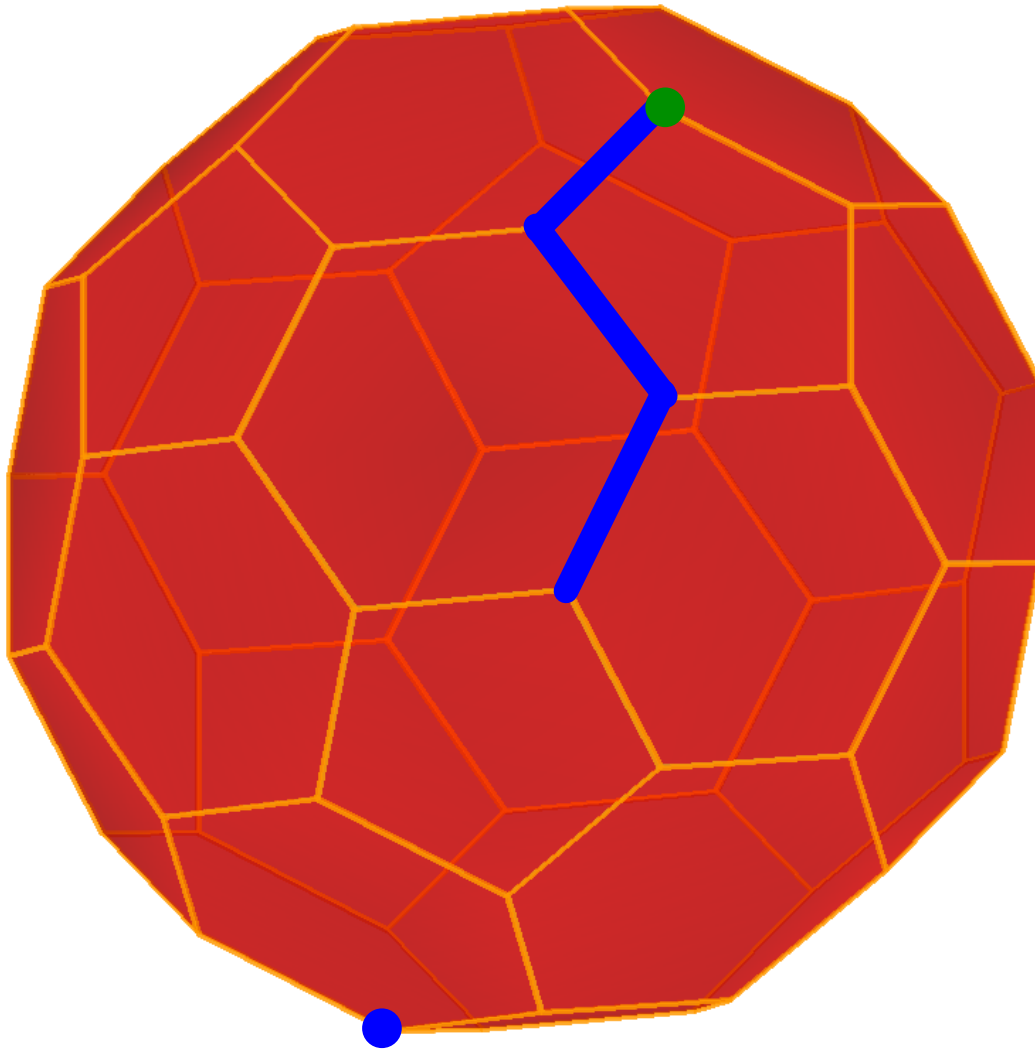
The Simplex Algorithm



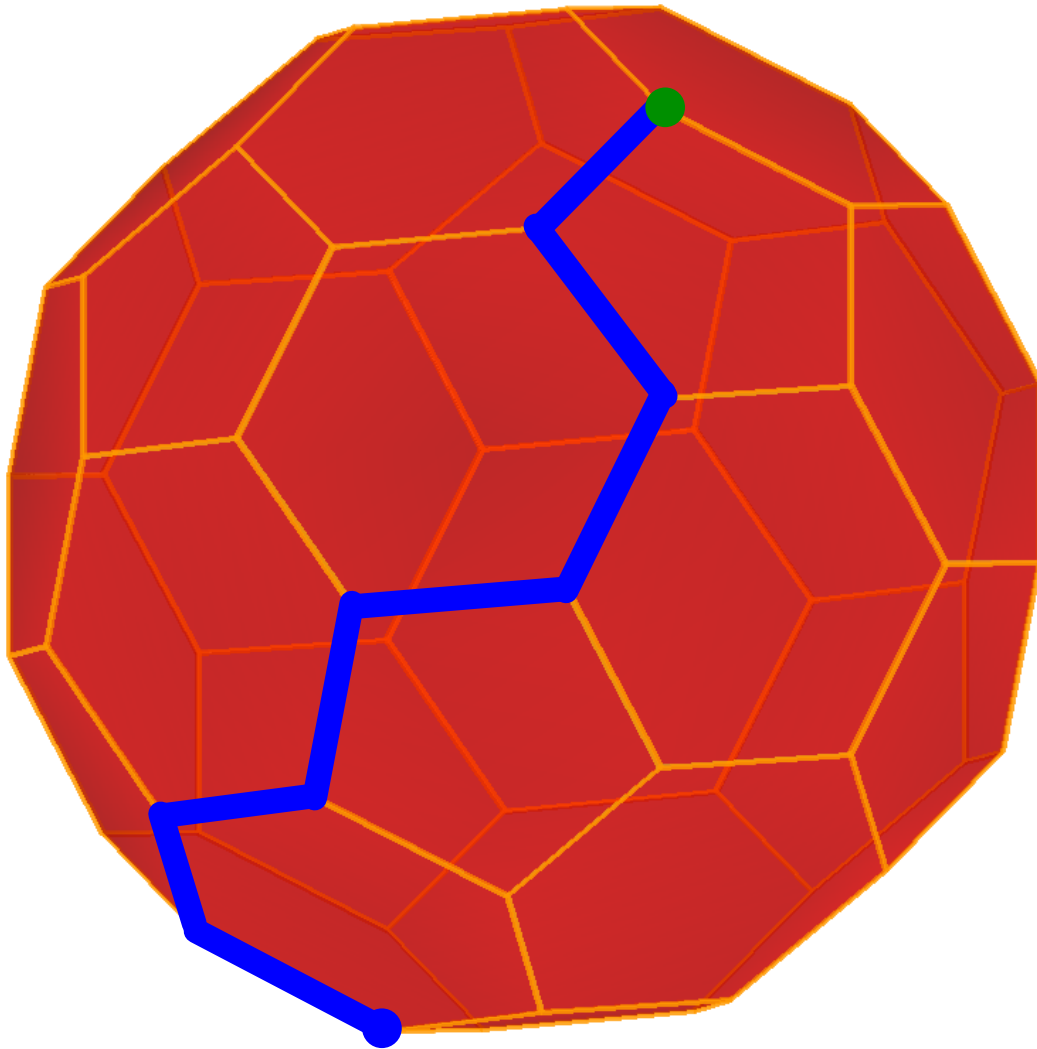
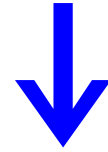
The Simplex Algorithm



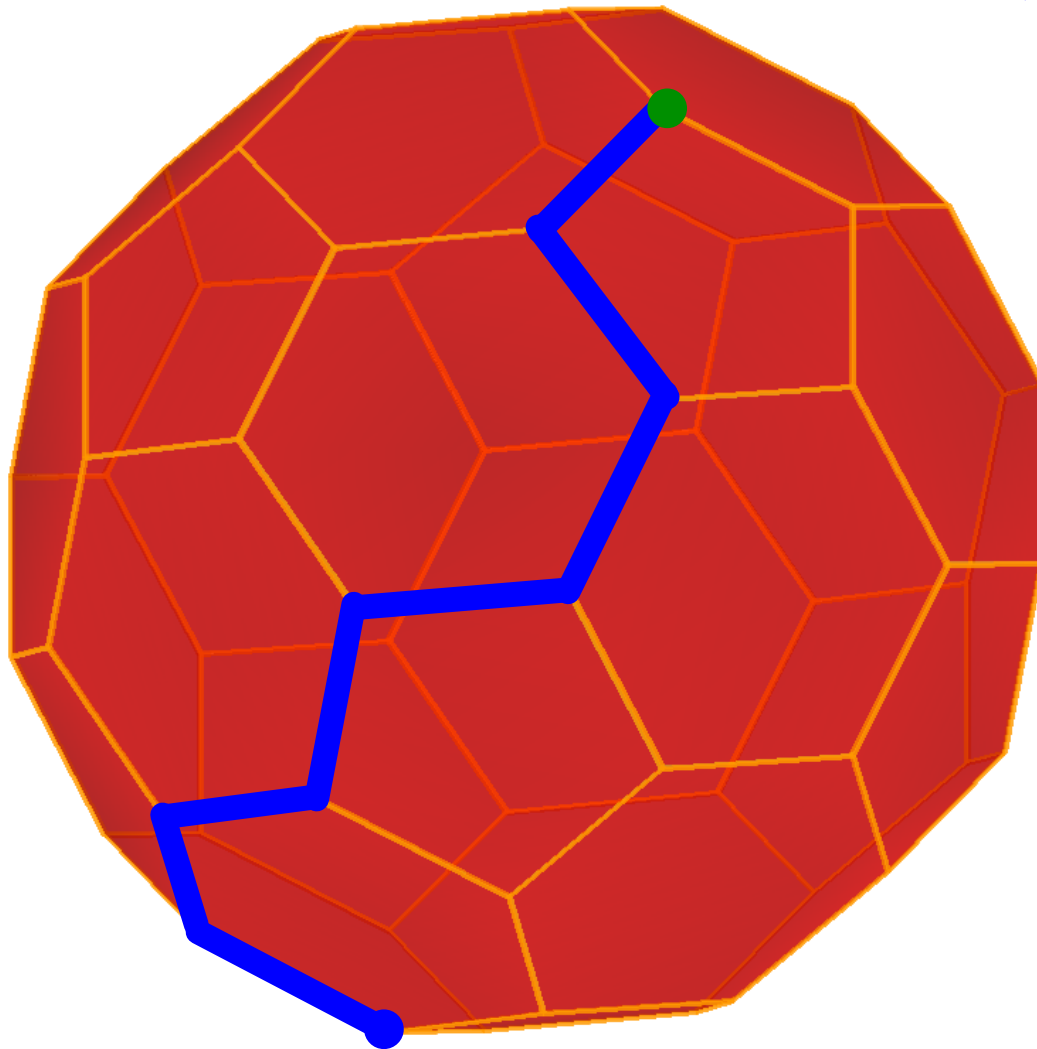
The Simplex Algorithm



The Simplex Algorithm



The Simplex Algorithm



- ▷ simple polytope, generic linear objective function
- ▷ *pivoting in a tableau*
- ▷ which path: *pivot rules*
- ▷ successful in practice
- ▷ (*strongly*) polynomial?

▷ **Maximal number $f(d, n)$ of vertices**
for a d -polytope with n facets?

MOTZKIN 1957

- ▶ **Maximal number $f(d, n)$ of vertices**
for a d -polytope with n facets?

MOTZKIN 1957

- ▶ **Upper Bound Theorem:**

$$\begin{aligned} f(d, n) &= \binom{n - \lceil \frac{d}{2} \rceil}{\lfloor \frac{d}{2} \rfloor} + \binom{n - 1 - \lceil \frac{d-1}{2} \rceil}{\lfloor \frac{d-1}{2} \rfloor} \\ &= O(n^{\lfloor d/2 \rfloor}) \text{ for fixed } d \end{aligned}$$

McMULLEN 1970

- ▶ Equality achieved e. g. by duals of cyclic polytopes

- ▶ **Maximal number $M(d, n)$ of vertices on a monotone path**
for a d -polytope with n facets?

KLEE 1966

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for a d -polytope with n facets? KLEE 1966
- ▶ The duals of cyclic polytopes have Hamilton paths. KLEE 1965

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- ▶ The duals of cyclic polytopes have Hamilton paths. KLEE 1965
- ▶ *Monotone upper bounds:*
 - $$M(3, n) = f(3, n)$$
 - $$M(4, n) = f(4, n)$$
 PFEIFLE 2003
 - but $M(6, 9) < f(6, 9)$ PFEIFLE & Z. 2003
- ▶ Indeed, $27 \leq M(6, 9) < f(6, 9) = 30$.

- ▷ **Is there always a short path to the minimal vertex?** HIRSCH 1959
- ▷ *(Monotone) Hirsch Conjecture:* (monotone) diameter $\leq n - d$?

- ▷ **Is there always a short path to the minimal vertex?** HIRSCH 1959
- ▷ *(Monotone) Hirsch Conjecture:* (monotone) diameter $\leq n - d$?
- ▷ $d = 3$: monotone Hirsch Conjecture **holds** KLEE 1965
 $d > 3$: monotone Hirsch Conjecture **false** TODD 1980
- ▷ $d > 3$: strict monotone Hirsch Conjecture **open**
 (start from highest vertex) Z. 1995

▷ **Is there a pivot rule that always finds a short path?**

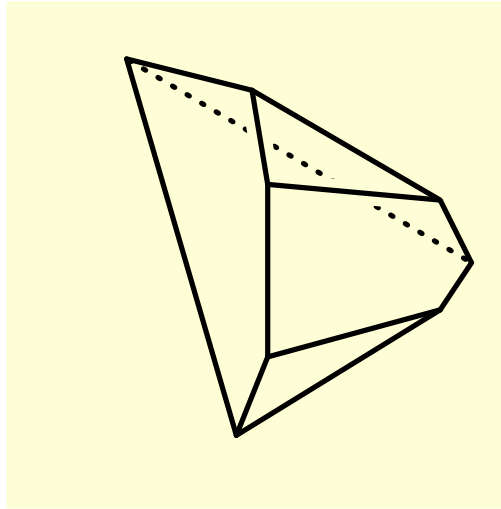
▷ **Is there a pivot rule that always finds a short path?**

▷ Worst-case analysis:

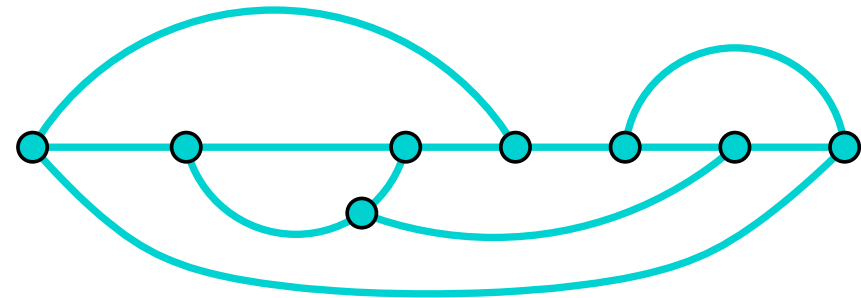
Dantzig's rule	exponential	KLEE & MINTY 1970
Greatest decrease	exponential	
Bland's Least index	exponential	
Steepest decrease	exponential	
Shadow vertex	exponential	
Smallest decrease	exponential	
<i>etc.</i>	exponential	AMENTA & Z. 1998
Zadeh's Least entered	?	
Random facet	subexponential	KALAI 1992
Random edge	???	

Random Monotone Paths on Polyhedra

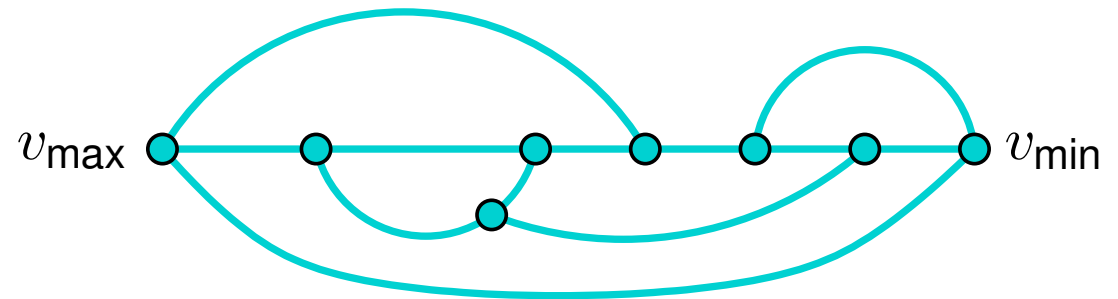
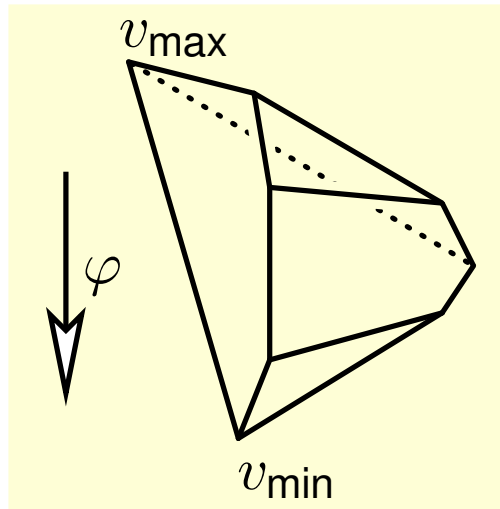
- ▷ A combinatorial model (for dimension three)
- ▷ Worst-case analysis: “coefficient of linearity”
 - > **Random edge** — lower and upper bound
 - > Other pivot rules:
 - Greatest decrease
 - Random facet
 - Least entered
 - Bland, Dantzig
 - Steepest decrease
 - Shadow vertex
- ▷ More remarks on Random edge



▷ 3-polytope P

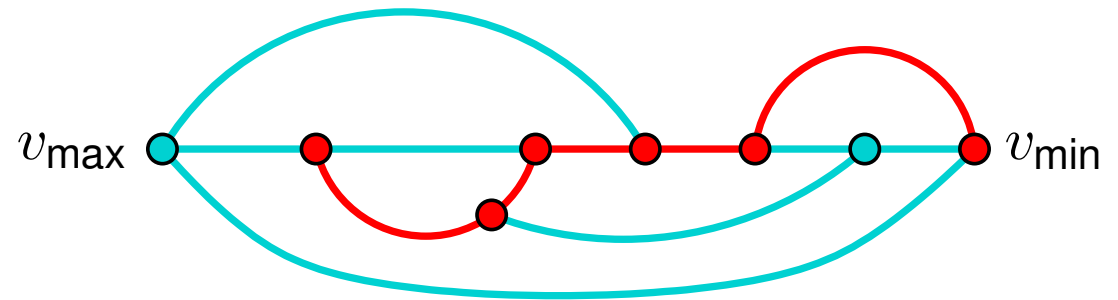
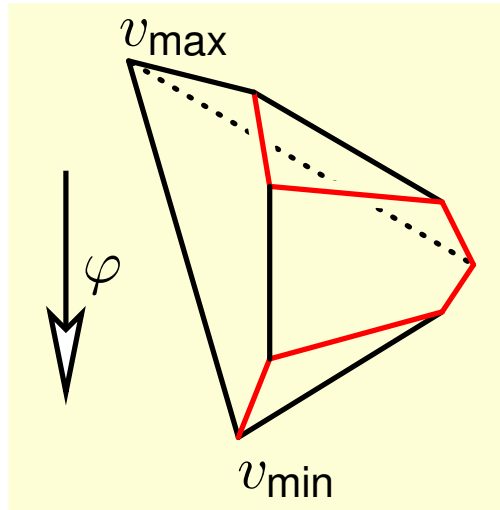


▷ planar, 3-connected graph G
(no loops, no parallel edges)



- ▷ 3-polytope P
- ▷ generic linear function $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}$

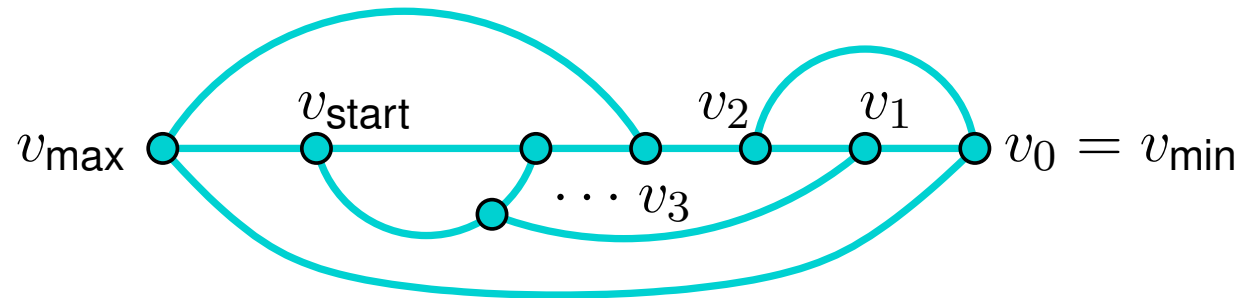
- ▷ 3-polytopal graph G STEINITZ
- ▷ acyclic orientation with unique sink, source in every face (AOF)
- ▷ three vertex-disjoint monotone paths from v_{\max} to v_{\min}



- ▷ 3-polytope P
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- ▷ 3-polytopal graph G STEINITZ
- ▷ acyclic orientation with unique sink, source in every face (AOF)
- ▷ three vertex-disjoint monotone paths from v_{\max} to v_{\min}

$P \subset \mathbb{R}^3$ simple polytope, n facets, $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}$ generic linear function



embedded planar graph, 3-regular, n faces
 vertices decreasingly ordered from left to right
 unique sinks, three disjoint v_{\max} - v_{\min} paths

- ▷ $2n - 4$ vertices (Euler's formula)
- ▷ # (out-degree-) 1-vertices = $n - 3$
- ▷ # (out-degree-) 2-vertices = $n - 3$

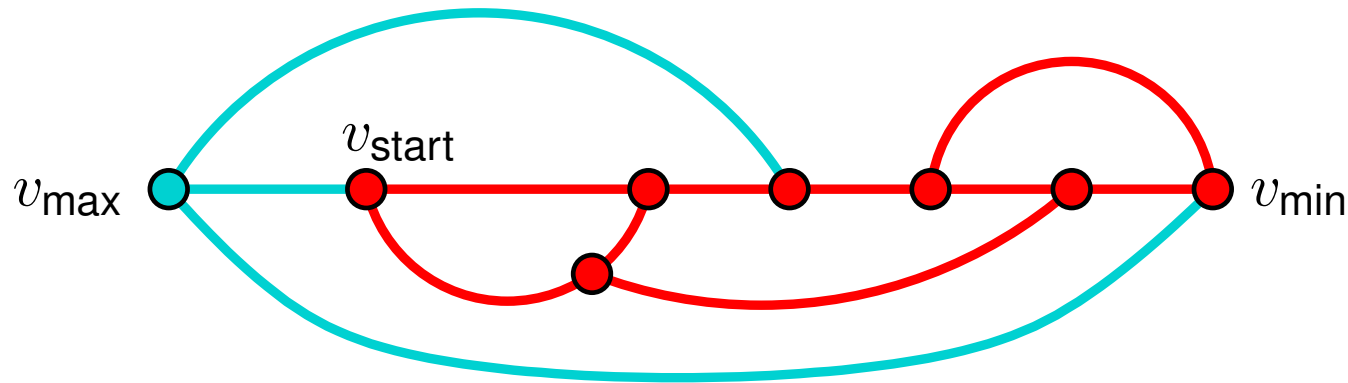
For a pivot rule \mathcal{R} :

$$\begin{aligned}\Lambda(\mathcal{R}) &:= \limsup_{n \rightarrow \infty} \frac{\text{worst-case running time on polyhedron with } n \text{ facets}}{n} \\ &= \min\{\Lambda : \mathcal{R} \text{ needs } \Lambda n + \text{const. steps in worst case}\} \\ &=: \text{the } \textit{linearity coefficient} \text{ of } \mathcal{R}\end{aligned}$$

$$1 \leq \Lambda(\mathcal{R}) \leq 2 \quad \text{for every pivot rule}$$

pivot rule \mathcal{R}	$\Lambda(\mathcal{R})$	dimension d	
Random edge	≥ 1.3473 ≤ 1.4943	unknown, could be quadratic!	
Greatest decrease	1.5	exponential	
Random facet	2	subexponential	KALAI 1992
Least entered	2	unknown: \$1,000 reward	ZADEH 1980
Bland's Least index	2	exponential	
Dantzig's rule	2	exponential	
Steepest decrease	2	exponential	
Shadow vertex	2	exponential	
Smallest decrease	2	exponential: long paths!	

At any non-optimal vertex, the *random edge* pivot rule takes a step to one of its improving neighbors, chosen uniformly at random.



THEOREM

$$1.3473 \leq \Lambda(\text{RE}) \leq 1.4943$$

Lower bound Construction of a family of polytopes

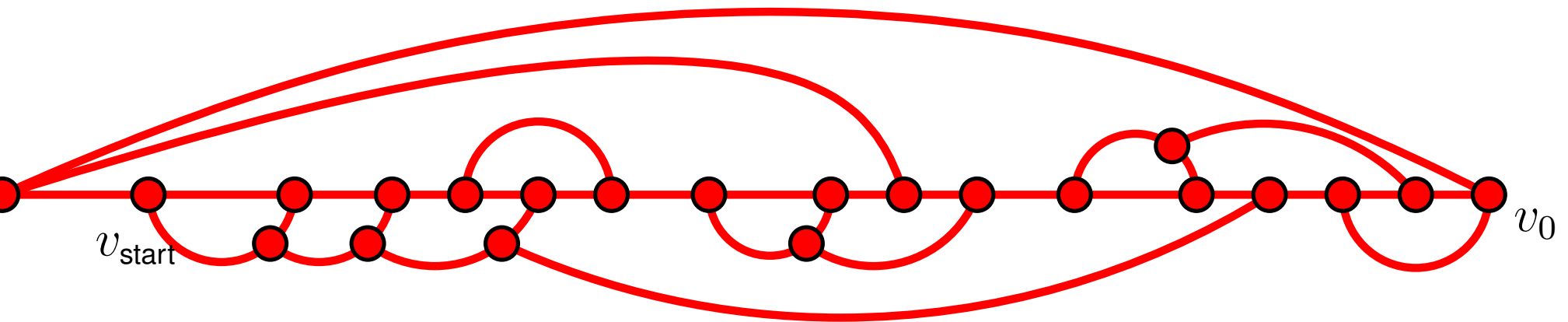
Upper bound Induction on the number of 1- and 2-vertices “ahead”

For $n \leq 12$, we enumerated all the 3-connected cubic graphs with n faces (using `plantri`) ...

BRINKMANN & MCKAY

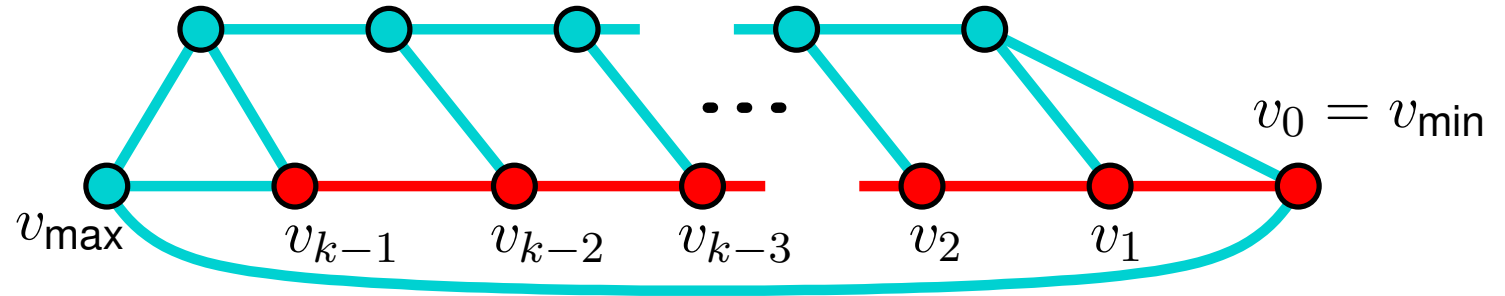
... and all the “abstract objective functions” on each of these ...

... to see what worst-case examples *look like*:

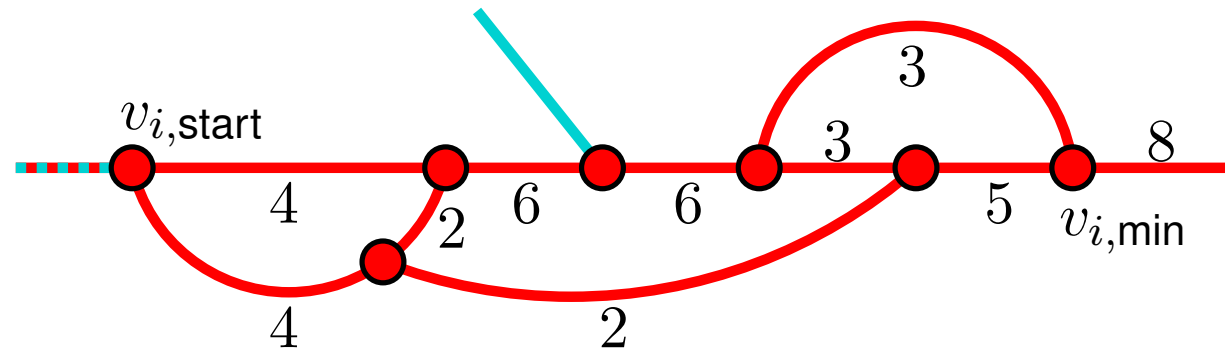


Random edge: lower bounds II

the backbone:



a configuration:

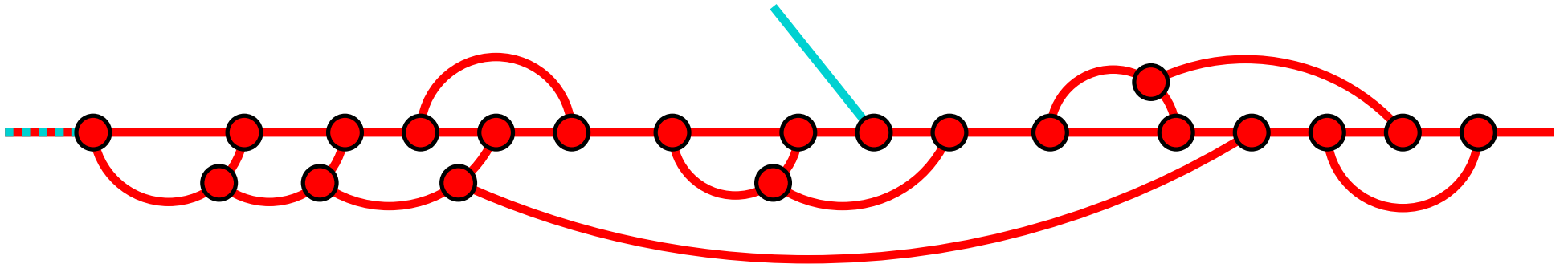


flow costs per configuration: $\frac{43}{8}$

facets per configuration: $3 + 1$

$$\Rightarrow \Lambda(\text{RE}) \geq \frac{43}{8 \cdot 4} = \frac{43}{32} \approx 1.3437$$

a worse configuration:



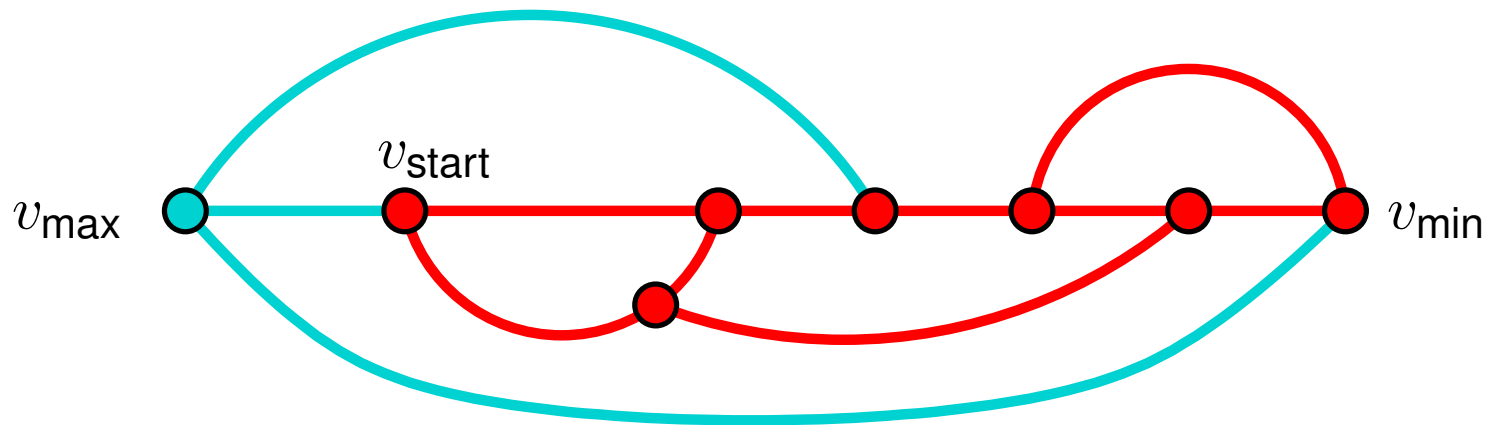
flow costs per configuration: $\frac{1897}{128}$

facets per configuration: $10 + 1$

$$\implies \Lambda(\text{RE}) \geq \frac{1897}{1408} \approx 1.3473$$

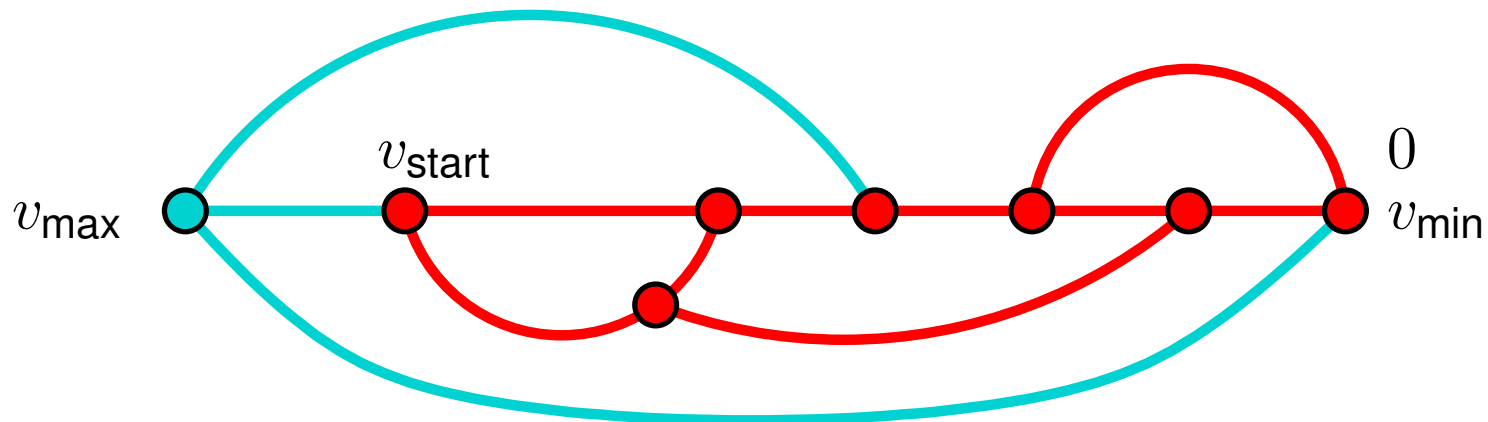
Recursion formula for the expected number of pivot steps $\mathbf{E}(v)$
“starting from v ”:

$$\mathbf{E}(v) = 1 + \frac{1}{|\delta^+(v)|} \sum_{u:(v,u) \in \delta^+(v)} \mathbf{E}(u),$$



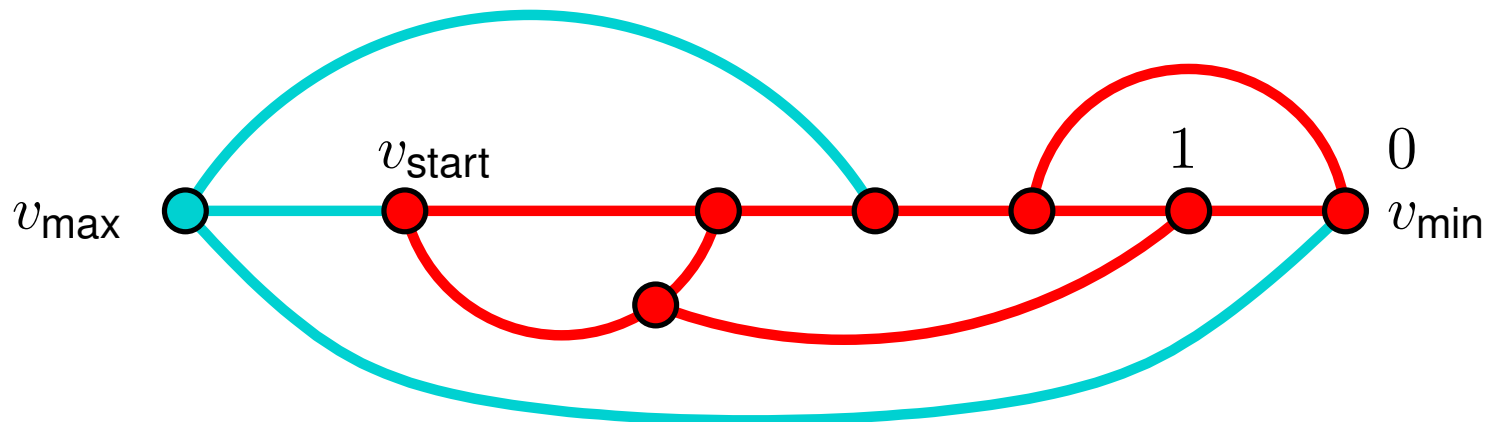
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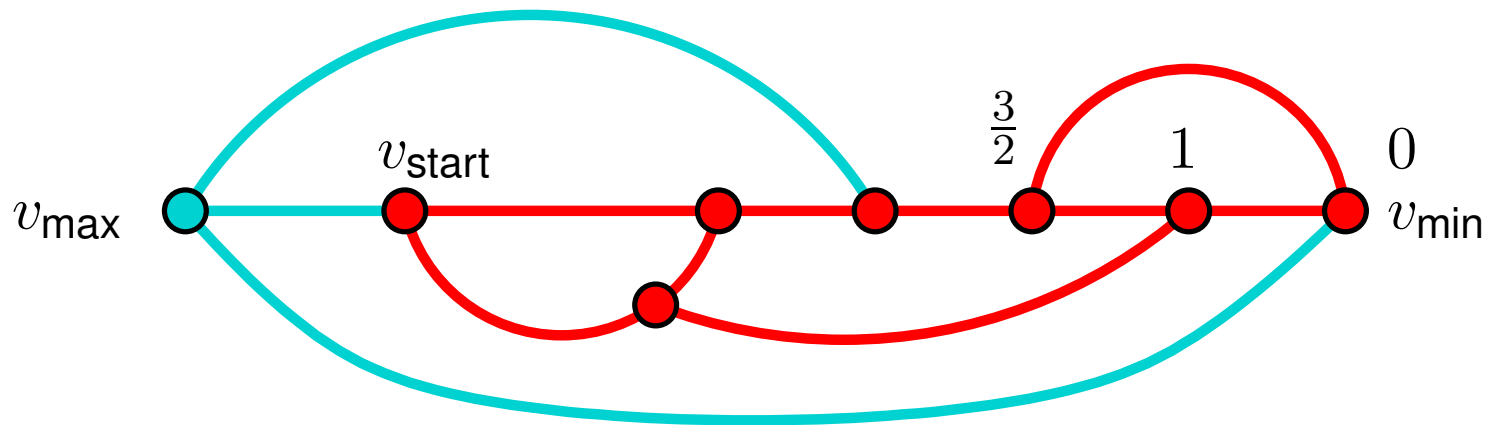
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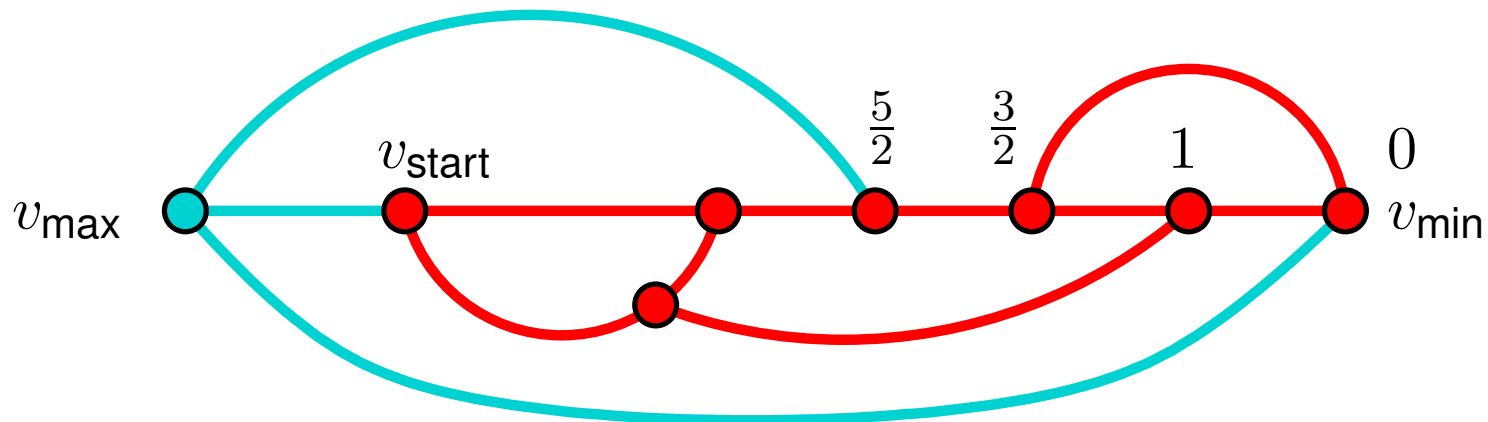
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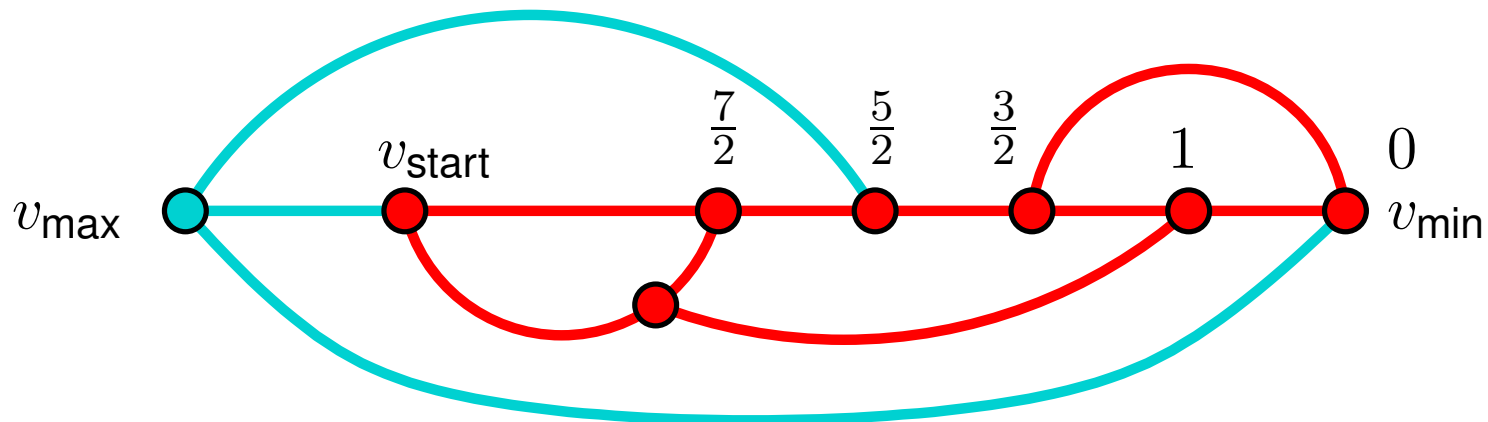
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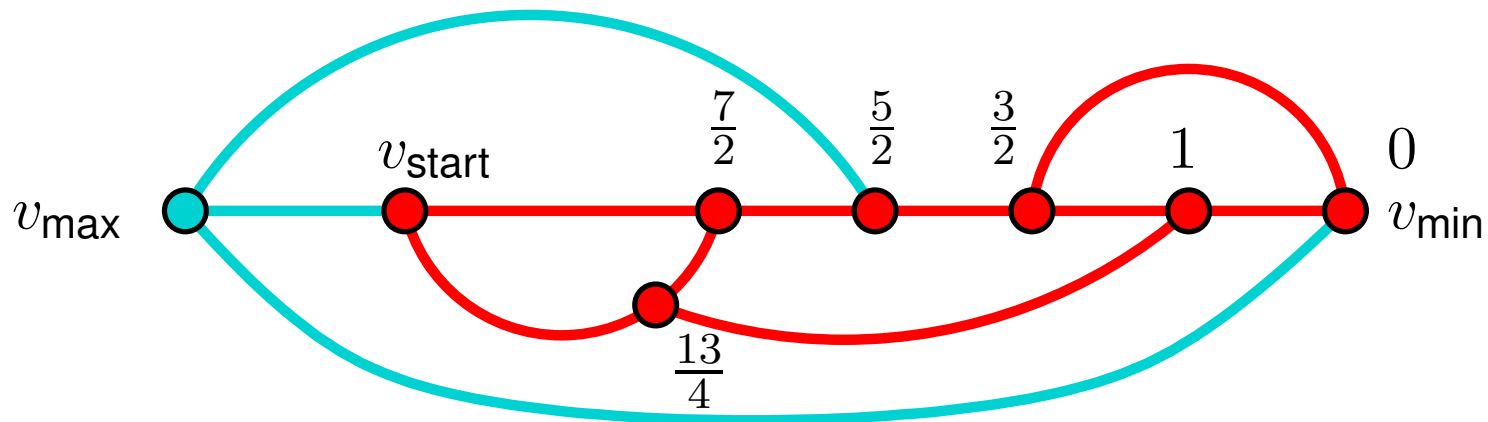
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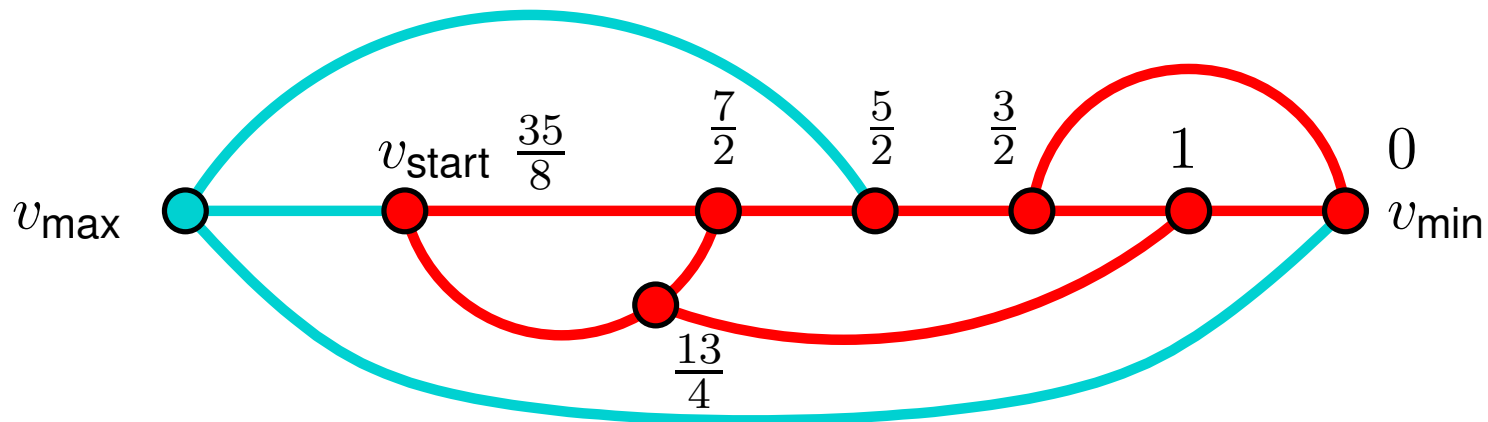
Recursion formula for the expected number of pivot steps $\mathbf{E}(v)$
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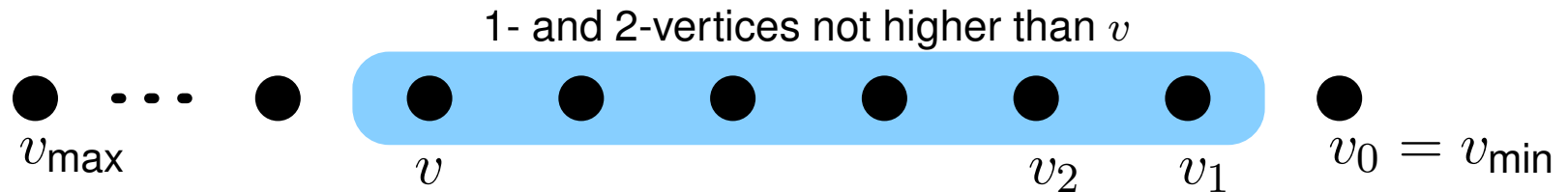
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Recursion formula for the expected number of pivot steps $\mathbf{E}(v)$
 “starting from v ”:

$$\mathbf{E}(v) = 1 + \frac{1}{|\delta^+(v)|} \sum_{u:(v,u) \in \delta^+(v)} \mathbf{E}(u),$$





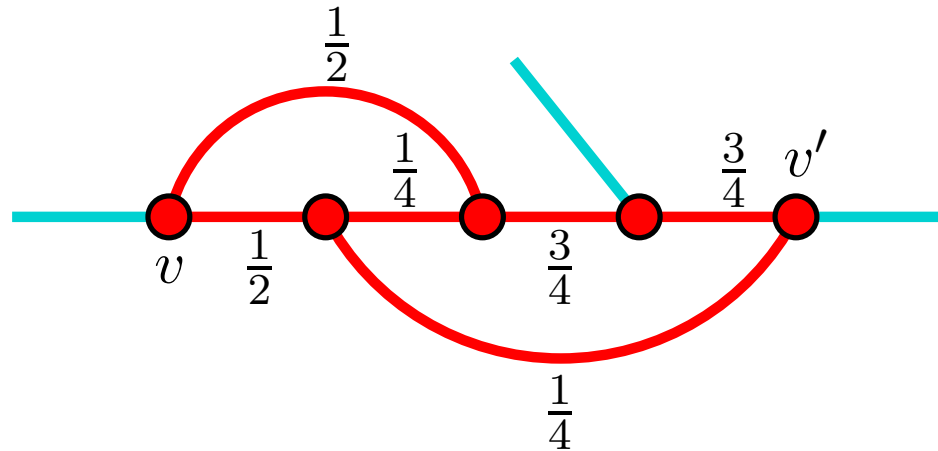
$$N_1(v) = \# \{1\text{-vertices not higher than } v\} \leq n - 3$$

$$N_2(v) = \#\{2\text{-vertices not higher than } v\} \leq n - 3$$

$$N(v) = N_1(v) + N_2(v) = \#\{\text{vertices below } v\} \leq 2n - 6$$

- ▶ Find feasible values for α and β such that $\mathbf{E}(v) \leq \alpha N_1(v) + \beta N(v)$ holds and thus $\Lambda(\text{RE}) \leq \alpha + 2\beta$.
- ▶ Induction on $N(v)$ rests on linear inequalities on α and β .
- ▶ Solve LP minimizing $\alpha + 2\beta$ on feasible region. The optimal solution is $\alpha = \frac{46}{87}$, $\beta = \frac{42}{87}$ and thus

$$\Lambda(\text{RE}) \leq \frac{46}{87} + 2 \frac{42}{87} \leq 1.4943$$

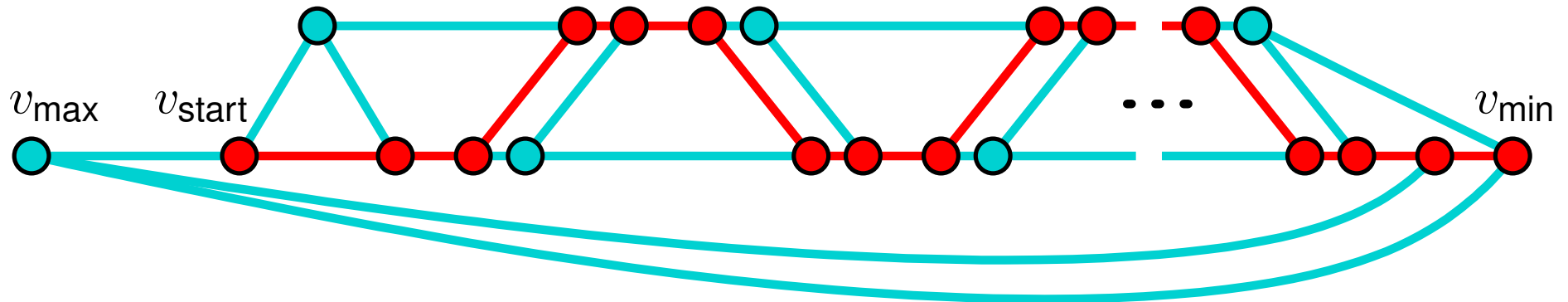


$$\begin{aligned}
 E(v) &= \frac{5}{2} + E(v') \\
 &\leq \frac{5}{2} + \alpha N_1(v') + \beta N(v') \stackrel{!}{\leq} \alpha N_1(v) + \beta N(v) \\
 \iff \frac{5}{2} &\leq (N_1(v) - N_1(v')) \alpha + (N(v) - N(v')) \beta
 \end{aligned}$$

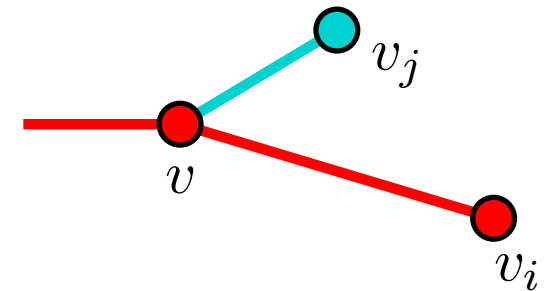
We have $N_1(v) - N_1(v') = 2$ and $N(v) - N(v') = 3$, thus

this case is o.k. if $\frac{5}{2} \leq 2\alpha + 3\beta$.

Move to the neighbor with the smallest objective function value.



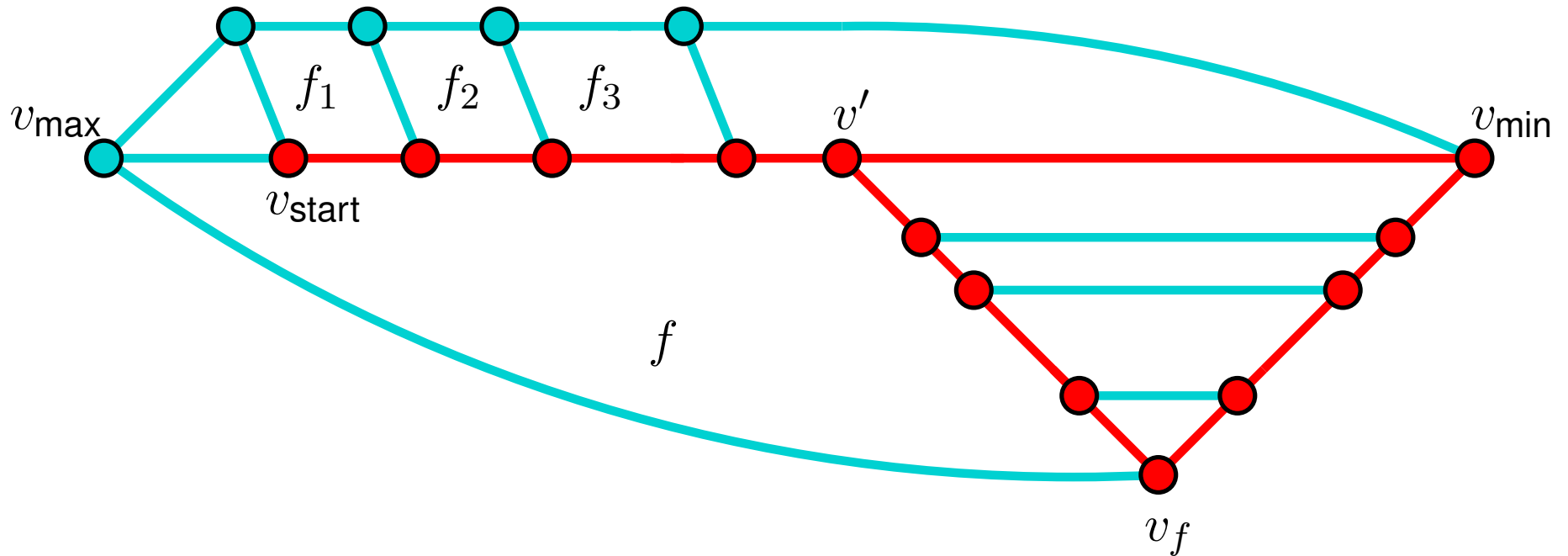
- ▶ Example using all $(n - 3)$ 1-vertices and half of the $(n - 3)$ 2-vertices
- ▶ For every 2-vertex, one vertex is skipped



THEOREM

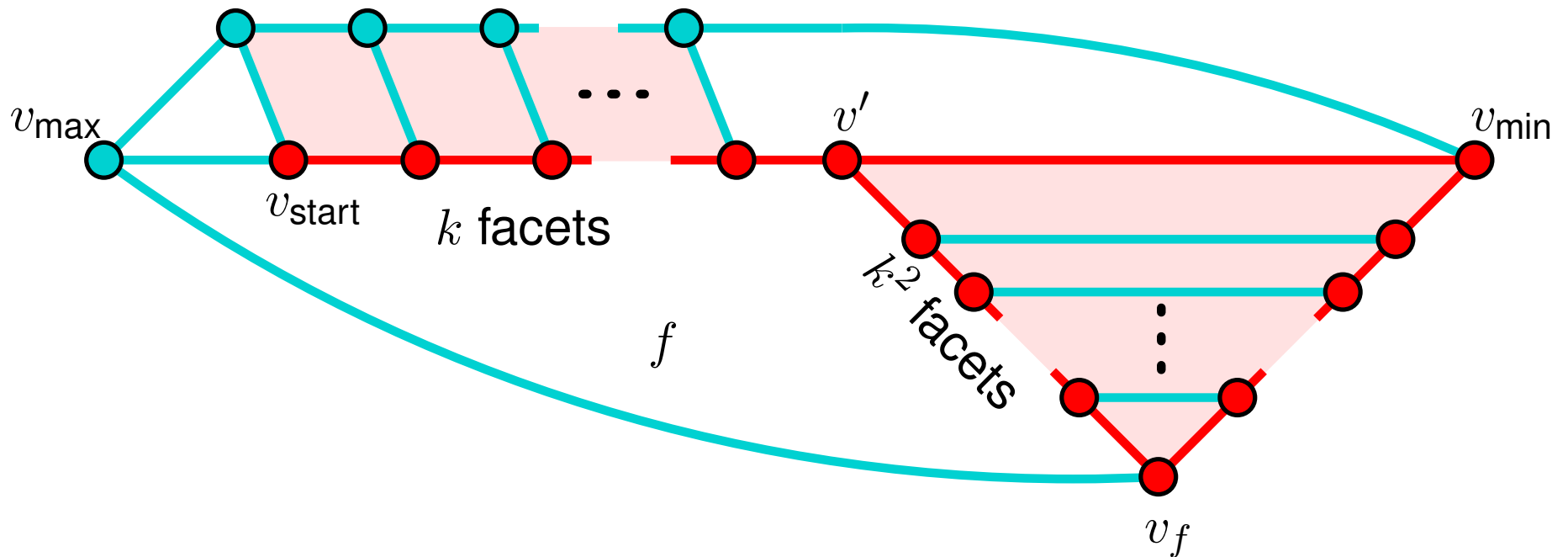
$$\Lambda(\text{GD}) = 1.5$$

Choose a facet that contains v uniformly at random, and solve by applying (RF) recursively.



THEOREM

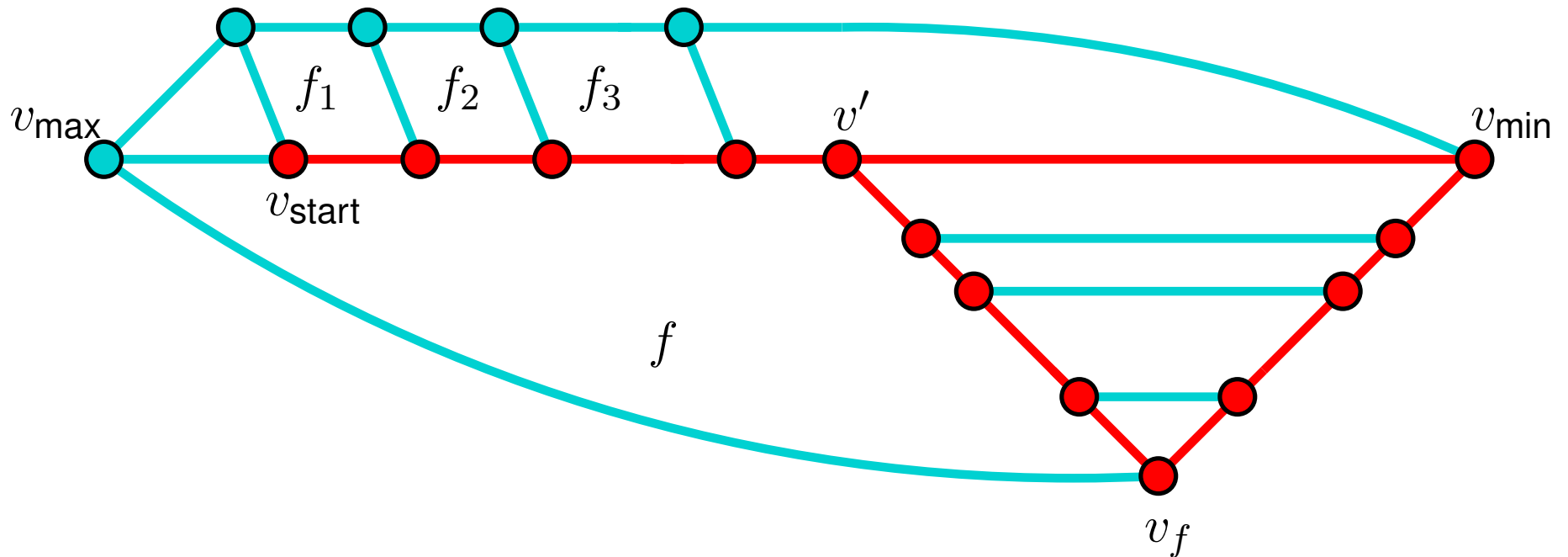
$$\Lambda(\text{RF}) = 2$$



$$\mathbf{E}(v_{\text{start}}) \geq \left(1 - \left(\frac{1}{2}\right)^k\right) (2k^2 + k + 1) \quad \text{and} \quad n = k^2 + k + 4$$

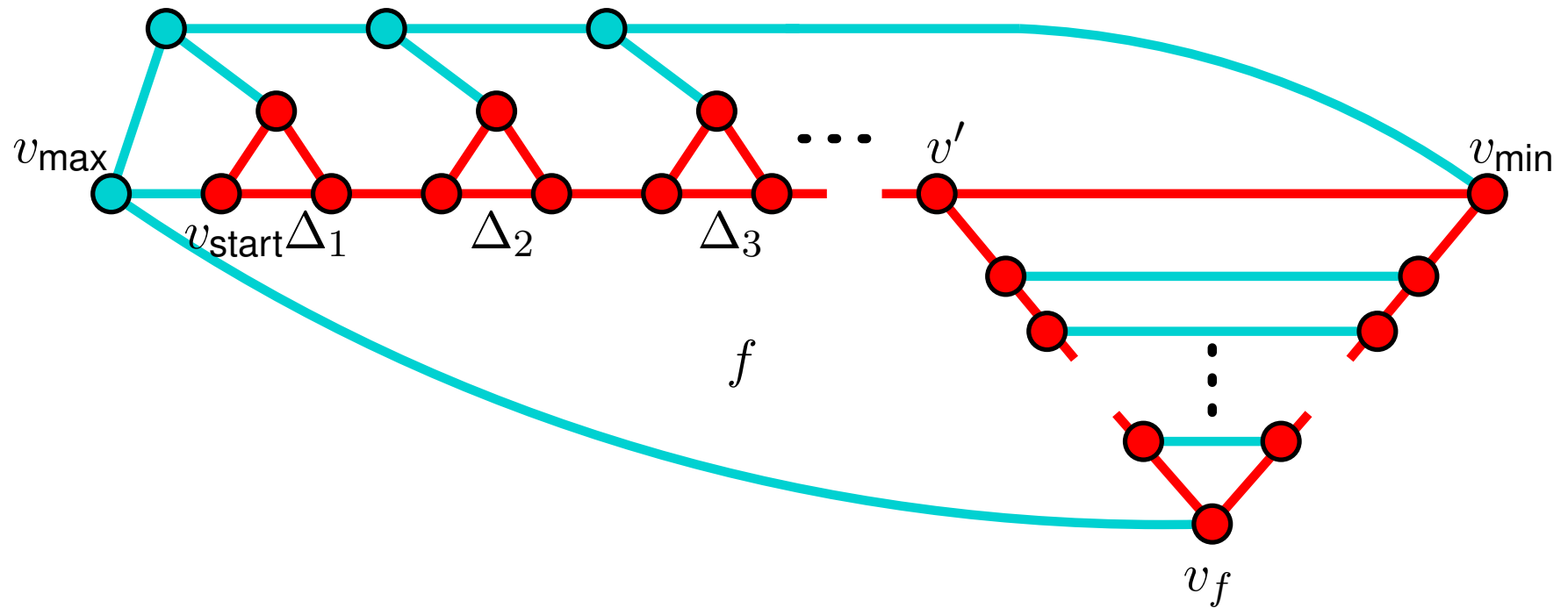
$$\implies \Lambda(\text{RF}) \geq 2$$

At a 1-vertex, follow the outgoing edge.
 Otherwise choose a facet that contains v uniformly at random, and solve by applying (RF-B) recursively.



- ▷ follows path of 1-vertices deterministically
- ▷ cut off the first k 1-vertices

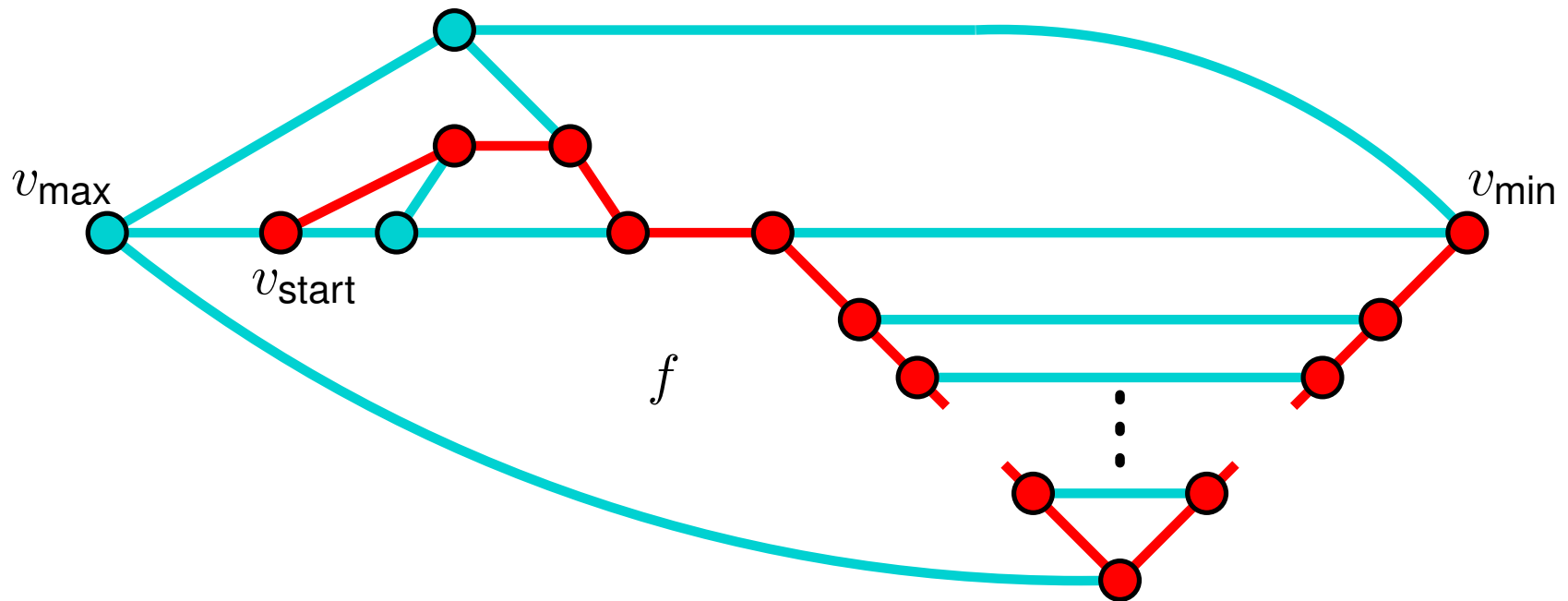
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 Otherwise choose a facet that contains v uniformly at random, and solve by applying (RF-B) recursively.



Zadeh's Least entered (with Greatest decrease)

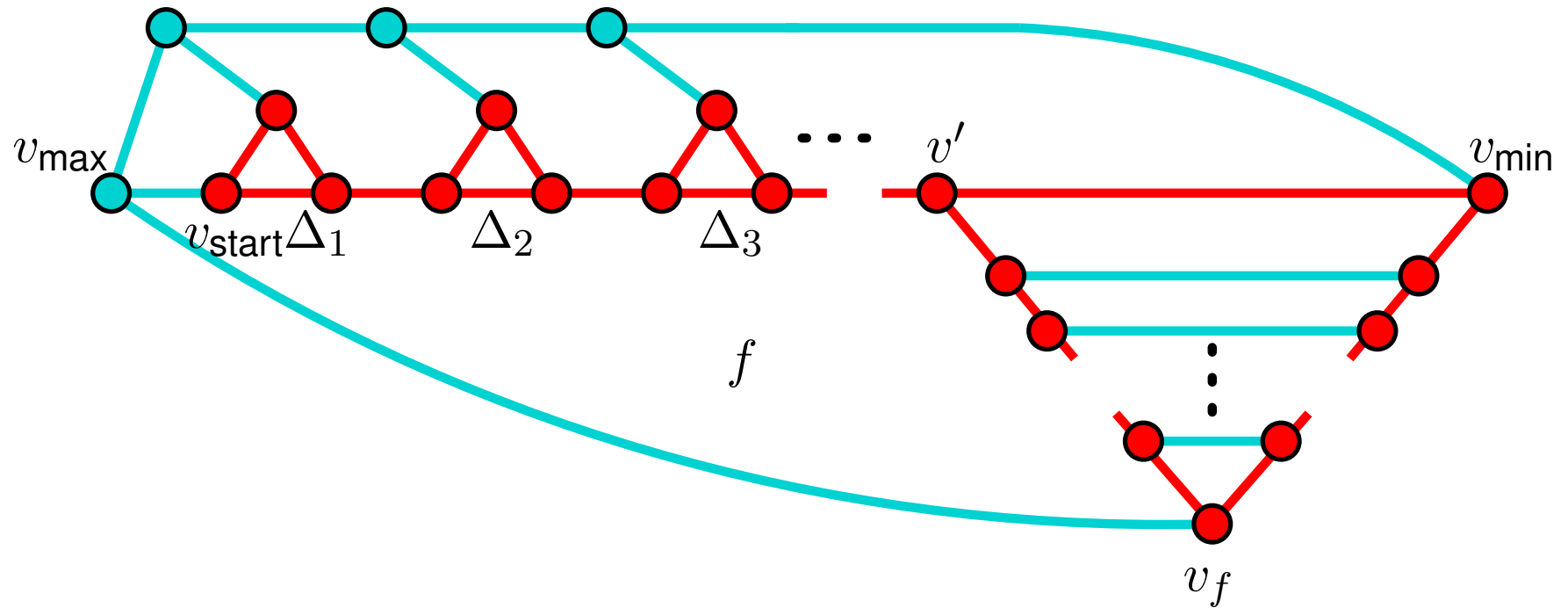
Choose an edge that leaves a facet which was entered least often in previous moves. (A tie-breaking rule is needed.)

- ▷ Greatest decrease as tie-breaking rule

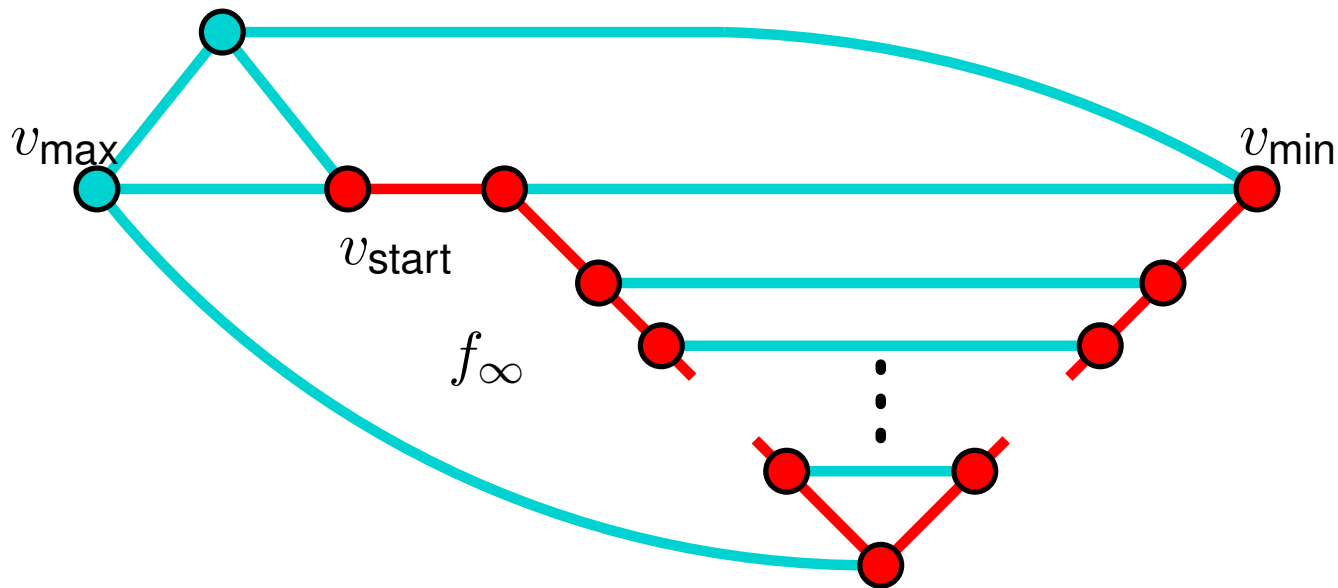


ZADEH's Least entered (with Random edge)

▷ Random edge as tie-breaking rule



Choose the edge that leaves the facet with the smallest number.

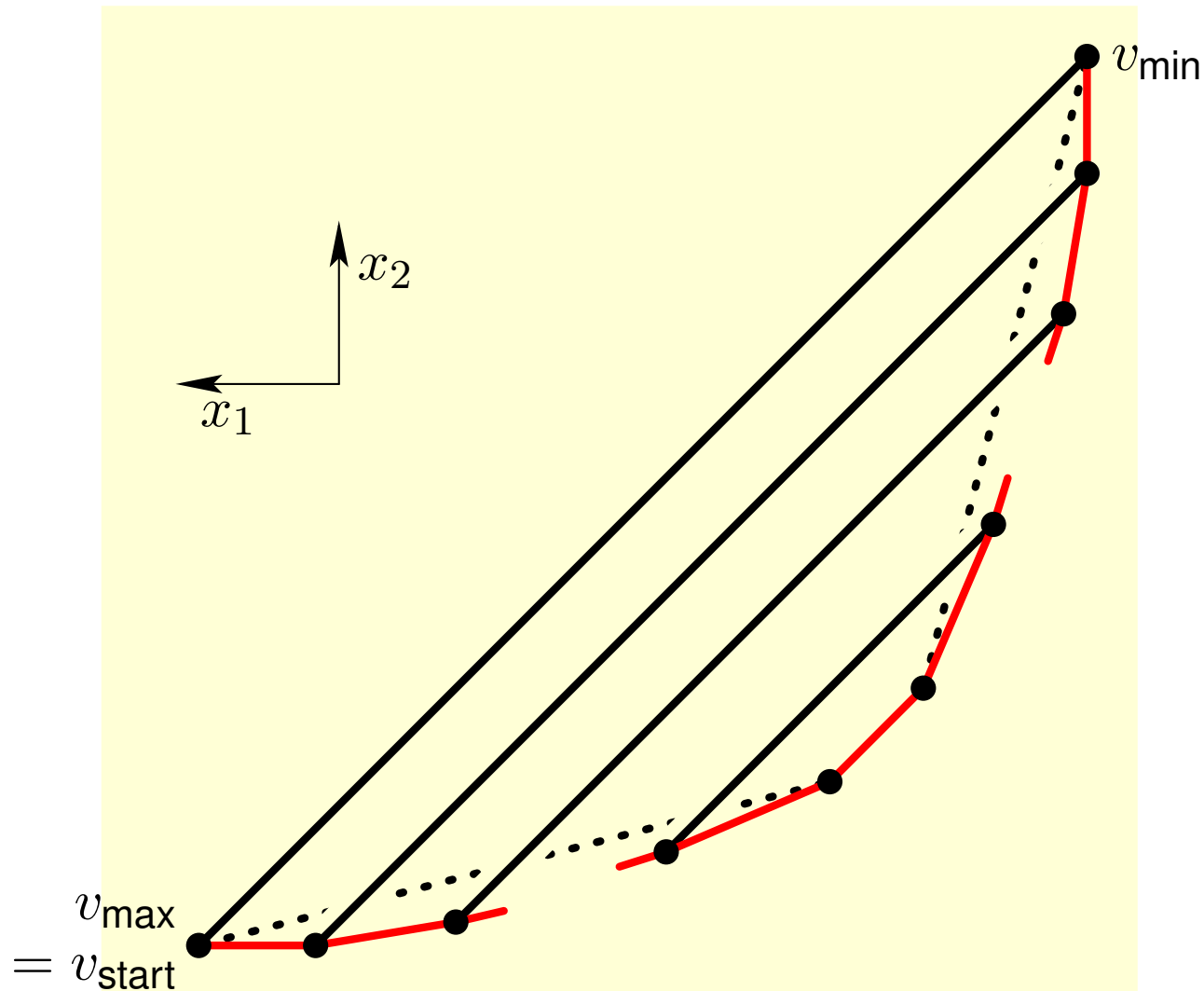


The original rule proposed by Dantzig:

Select the edge according to the best “reduced cost coefficient” in the simplex tableau.

Observation: DANTZIG's rule isn't better than BLAND's rule.

Move along steepest decreasing edge vw , with $\frac{\langle c, w-v \rangle}{\|w-v\| \|c\|}$ minimal



- ▶ Best general upper bound: $O\left(\frac{f_0}{\sqrt{d}}\right)$ KAIBEL 2004
- ▶ Klee–Minty cubes: $\Theta(d^2)$ GÄRTNER, HENK, Z. 1998
PEMANTLE & BALOGH 2004
- ▶ Linear assignment: polynomial TOVEY 1986
- ▶ $n = d + 1$: $\Theta(\log d)$
 $n = d + 2$: $\Theta(\log^2 d)$ GÄRTNER ET AL. 2001
- ▶ d -cube AOF's: $\Omega(c^{\sqrt[3]{d}})$ MATOUŠEK & SZABO 2004

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- ▶ **Conjecture:** On polytopes $O(nd)$???