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Many Triangulated 3-Spheres

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Abstract We construct $2^{\Omega(n^{5/4})}$ combinatorial types of triangulated 3-spheres on n vertices. Since by a result of Goodman and Pollack (1986) there are no more than $2^{O(n \log n)}$ combinatorial types of simplicial 4-polytopes, this proves that asymptotically, there are far more combinatorial types of triangulated 3-spheres than of simplicial 4-polytopes on n vertices. This complements results of Kalai (1988), who had proved a similar statement about d-spheres and $(d+1)$-polytopes for fixed $d \geq 4$.

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1 Introduction

In 1988, Kalai [16] proved a lower bound of

$$\log s(d,n) \;=\; \Omega(n^{\lfloor d/2 \rfloor}) \qquad \text{for fixed } d \geq 3$$

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for the number $s(d,n)$ of distinct combinatorial types of simplicial PL d-spheres on n vertices.[1] Combining this with Goodman and Pollack's [10] [11] upper bound

$$\log p(d,n) \ \leq \ d(d+1)n\log n$$

for the number $p(d,n)$ of combinatorial types of simplicial d-polytopes on n vertices, he derived that for $d \geq 4$, there are far more simplicial d-spheres than simplicial $(d+1)$-polytopes. In particular, *most* of these spheres, in a very strong sense, are not polytopal, i.e. there is no convex polytope with the same face lattice. On the other hand, we proved in earlier work [18] that in dimension $d = 3$, Kalai's construction produces only polytopal spheres, and up to now only few families of non-polytopal 3-spheres were known.

In this paper, we combine two constructions from a recent paper by Eppstein, Kuperberg & Ziegler [7] to show for the first time that for n large enough, there are far more simplicial 3-spheres than 4-polytopes on n vertices.

Theorem 1 *There are at least*

$$s(3,n) \ = \ 2^{\Omega(n^{5/4})}$$

combinatorially non-isomorphic simplicial 3*-spheres on* n *vertices.*

In brief, we prove Theorem 1 by producing a cellular decomposition $\mathcal{S}$ of S^3 with n vertices and $\Theta(n^{5/4})$ octahedral facets, and triangulating each octahedron independently. The cellulation $\mathcal{S}$ is constructed from a Heegaard splitting $S^3 = H_1 \cup H_2$ of S^3 of high genus by appropriately subdividing the thickened boundary surface $(H_1 \cap H_2) \times [0,1]$.

Because of their sheer number, *most* of the spheres we construct are combinatorially distinct: There can be at most $n!$ spheres combinatorially isomorphic to any given one, where $n! < n^n = 2^{n\log n}$. Also note that the only currently known upper bound for $s(3,n)$ is the rather crude estimate $\log s(3,n) = O(n^2 \log n)$ obtained from Stanley's proof of the Upper Bound Theorem for spheres [19].

[1] Here and in the following, we use Landau's O-notation for positive functions: $f = \Omega(g)$ denotes that there is a positive constant c such that $f(n) \geq cg(n)$ holds for all sufficiently large n. Similarly $f = O(g)$ if there is a $c' > 0$ such that $f(n) \leq c'g(n)$ for all large n, and $f = \Theta(g)$ denotes that both conditions hold.

2 Background

For $d \leq 2$ all simplicial d-spheres are realizable as polytopes: 1-dimensional spheres are trivial to realize, and Steinitz' famous theorem [20], [21] from the beginning of the 20th century asserts that *all* 2-spheres, including the non-simplicial ones, are polytopal (i.e., they arise as boundary complexes of 3-dimensional polytopes). Tutte [23] showed in 1980 that the number of combinatorially distinct rooted simplicial 3-polytopes with n vertices is asymptotically

$$\frac{3}{16\sqrt{6\pi n^5}}\left(\frac{256}{27}\right)^{n-2},$$

and Bender [2] established sharp asymptotic formulas counting the number of unrooted 3-dimensional polytopes.

The first example—the so-called *Brückner sphere*—of a simplicial sphere that is *not* the boundary complex of a polytope was inadvertedly found by Brückner [5] in 1910 in an attempt to enumerate all combinatorial types of 4-polytopes with 8 facets. As noted in 1967 by Grünbaum and Sreedharan [13], one of the 3-dimensional complexes that Brückner thought to represent a polytope is in fact *not* realizable in a convex way in $\mathbb{R}^4$. As the (polytopal) complex Brückner considered is simple (any vertex is contained in exactly 4 facets), its combinatorial dual is a simplicial 3-sphere.

Another known 'sporadic' example of a non-polytopal simplicial sphere is *Barnette's sphere* [1], which is nicely explained in [9, Chapter III.4]. From these two examples one can build infinite series, but apart from such sporadic families, no substantial number of non-polytopal spheres on a fixed number of vertices was known until Kalai's 1988 construction.

The related problem of estimating the number $t(3,m)$ of combinatorial types of simplicial 3-spheres with m *facets* has attracted attention in gravitational quantum physics [6]. Gromov [12] has asked whether there exists a constant $c > 0$ such that $t(3,m) \leq 2^{cm}$. By duality, this is equivalent to bounding the number of *simple* 3-spheres on m vertices. The problem is "dual" to the one we treat here, but seems to require different methods.

3 Definitions and notation

A *cellulation* $\mathcal{C}$ of a manifold X is a finite CW complex whose underlying space is X. $\mathcal{C}$ is *regular* if all closed cells are embedded, and *strongly regular* if in addition the intersection of any two cells is a cell.

The *star* of a cell $\sigma \in \mathcal{C}$ is the union of the closure of all cells containing σ, and the *link* of σ consists of all cells of star σ not incident to σ. The entry f_i of the *f-vector* $f(\mathcal{C}) = (f_0, f_1, \dots)$ of a cellulation counts the number of i-dimensional cells. The d-dimensional cells are called *facets*, and $(d-1)$-dimensional ones *ridges*.

4 The ingredients for the construction

4.1 Heffter's embedding of the complete graph

In 1898, Heffter [14] constructed remarkable cellulations of closed orientable surfaces:

Proposition 1 *Let $q = 4k+1$ be a prime power, and α be any generator of the cyclic group $\mathbb{F}_q^*$ of invertible elements of the finite field $\mathbb{F}_q$ on q elements. Then there exists a regular but not strongly regular cellulation $\mathcal{C}_q^\alpha$ of the closed orientable surface S_g of genus $g = q(q-5)/4+1$ with f-vector $\big(q, \binom{q}{2}, q\big)$, all of whose 2-cells are $(q-1)$-gons. $\mathcal{C}_q^\alpha$ can be refined to a strongly regular triangulation $\mathcal{T}_q^\alpha$ of S_g with f-vector $\big(2q, \binom{q}{2} + q(q-1), q(q-1)\big)$.*

Proof There exist infinitely many *prime* numbers q of the form $q = 4k+1$; see [8]. For any prime *power* q of this form, take as vertices of the cellulation $\mathcal{C}_q^\alpha$ the elements of $\mathbb{F}_q$, and as 2-cells the $(q-1)$-gons (compare Figure 1, left)

$$F^\alpha(s) \;=\; \Big\{ v^\alpha(s,k) = s + \frac{\alpha^k - 1}{\alpha - 1} : 0 \le k \le q-2 \Big\} \qquad \text{for } s \in \mathbb{F}_q.$$

It is straightforward to check (see [14] and [7, Lemma 12]) that this cellulation is regular (all vertices in each $F(s)$ are distinct), neighborly (any two vertices are connected by an edge), and closed (any edge is shared by exactly two polygons), but not strongly regular (any two polygons share $q-2$ vertices). An Euler characteristic calculation yields the genus of the underlying surface S_g of $\mathcal{C}_q^\alpha$. By subdividing each polygon as in Figure 1 (bottom), the cellulation becomes strongly regular with the stated f-vector.

Remark 1 This cellulation was independently obtained in [7] as an abelian covering of the canonical one-vertex cellulation of S_g.

Remark 2 Heffter's original construction involved only prime numbers. As it turns out, allowing prime powers becomes necessary for symmetric embeddings: According to Biggs [3], if the complete graph K_n

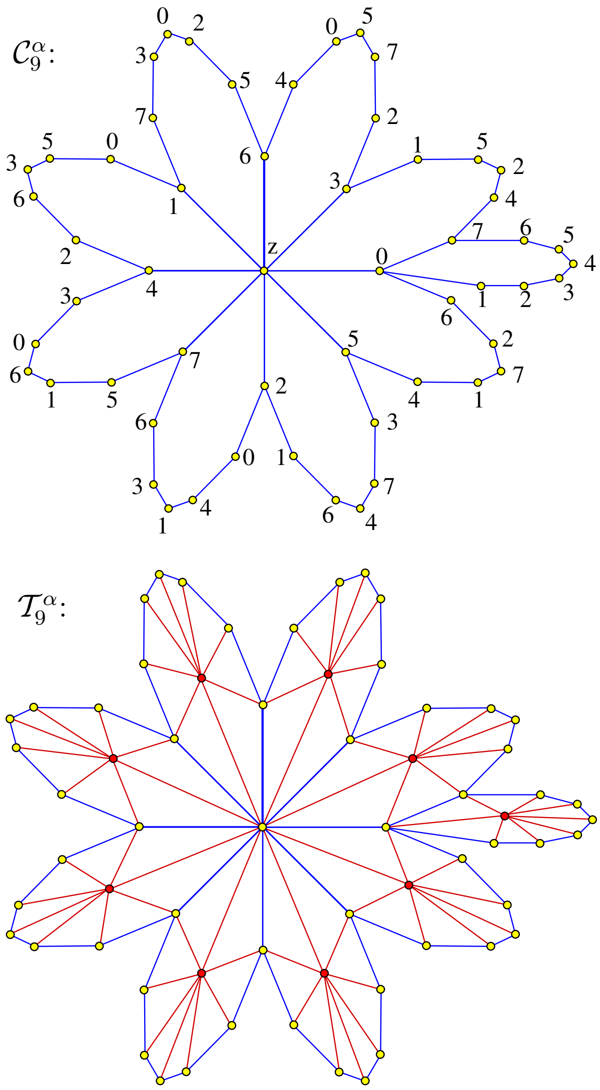

Fig. 1 *Top:* The Heffter cellulation $\mathcal{C}_9^\alpha$ of a surface S_g of genus $g = 10$ for $\alpha = 2x + 2 \in \mathbb{F}_9 \cong \mathbb{F}_3[x]/\langle x^2 + x + 2\rangle$. The vertex z corresponds to $0 \in \mathbb{F}_9$, and the vertices labeled i to the element α^i. Note that any two of the $q = 9$ vertices are adjacent, and that all vertices in any given one of the 9 polygons are distinct. However, the link of every vertex contains identified vertices, and so the vertex stars are not embedded. *Bottom:* After subdividing the surface to the triangulation $\mathcal{T}_q^\alpha$ using q new vertices, all stars are embedded disks.

embeds into a closed orientable surface in a symmetric way (i.e. there exists a "rotary" or "chiral" combinatorial automorphism, see [24]), then n is the power of a prime number, and James & Jones [15] showed that *any* such embedding of K_n is actually one from Heffter's family.

Remark 3 Two cellulations $\mathcal{C}_q^\alpha$ and $\mathcal{C}_q^\beta$ are combinatorially distinct for $\beta \neq \alpha, 1/\alpha \in \mathbb{F}_q$: By [14], the only automorphisms of $\mathcal{C}_q^\alpha$ are induced by affine maps $\varphi : \mathbb{F}_q \to \mathbb{F}_q$, $x \mapsto ax + b$ with $a \in \mathbb{F}_q^*, b \in \mathbb{F}_q$. An easy calculation shows that requiring $\varphi(v^\alpha(s, k+i)) = v^\beta(t, \ell + i)$ resp. $\varphi(v^\alpha(s, k+i)) = v^\beta(t, \ell - i)$ for $t \in \mathbb{F}_q$, $0 \leq \ell \leq q-2$ and $i = 0, 1, 2$ already implies $\beta = \alpha$ resp. $\beta = 1/\alpha$.

4.2 The E-construction

Proposition 2 [7] *Given a cellulation $\mathcal{C}$ of a d-dimensional manifold M with boundary with f-vector $(f_0, f_1, \ldots, f_d)$ and f_{d-1}^{in} interior ridges, there exists a cellulation $E(\mathcal{C})$ of M with $f_0 + f_d$ vertices consisting of f_{d-1}^{in} bipyramids and $f_{d-1} - f_{d-1}^{\text{in}}$ pyramids.*

Proof Cone a new vertex to the inside of each d-cell F of M to create $f_{d-1} + f_{d-1}^{\text{in}}$ pyramids, then combine each pair of pyramids over the same interior ridge to a bipyramid.

Example 1 Let $\mathcal{C}$ be a cellulation of a closed orientable surface S with f-vector $(f_0,\ f_1,\ f_2)$. Then applying Proposition 2 to $\mathcal{C} \times [0,1]$ yields a cellulation of the prism $S \times [0,1]$ with $2f_0 + f_2$ vertices consisting of f_1 octahedra and $2f_2$ pyramids; see Figure 2.

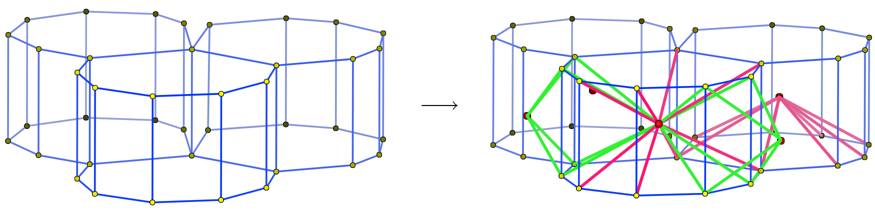

Fig. 2 The E-construction applied to $\Pi = \mathcal{C}_9^\alpha \times [0,1]$. *Left:* Three of the nine prisms of Π. *Right:* Three of the 18 octagonal pyramids and two out of $\binom{9}{2}$ octahedra of $E(\Pi)$.

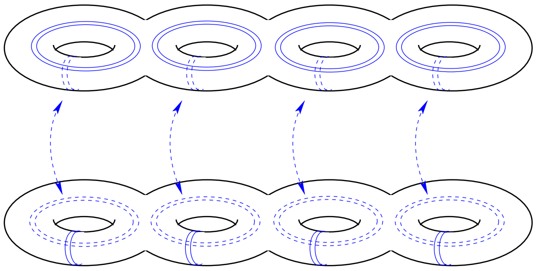

Fig. 3 Heegaard splitting of S^3 of genus $g = 4$. The complement of one solid handlebody in the 3-sphere is the other solid handlebody of the same genus. One copy of each doubled solid (resp. dashed) homology generator on the upper handlebody H_1 is identified with one copy of the solid (resp. dashed) one on the lower handlebody H_2 as indicated. The union of all copies of the generators induces a cellulation of H_1 (resp. H_2) into one 3-ball and g solid cylinders.

4.3 Heegaard splittings

Proposition 3 (see [22, Section 8.3.2]) *For any $g \geq 1$, the 3-sphere may be decomposed into two solid handlebodies $S^3 = H_1 \cup H_2$ that are identified along a surface $S_g = H_1 \cap H_2$ of genus g. Conversely, any 3-manifold can be split into handlebodies H_1, H_2 and is determined up to homeomorphism by the images $h(m_1), \ldots, h(m_{2g})$ on H_2 of the canonical meridians $m_1, \ldots, m_{2g}$ of H_1 under the identification map $h : \partial H_1 \to \partial H_2$.* □

Theorem 2 (Lazarus et al. [17, Theorem 1]) *Any triangulation $\mathcal{T}$ of a closed orientable surface of genus g with a total of $f = f_0 + f_1 + f_2$ cells can be refined to a triangulation $\tilde{\mathcal{T}}$ with $O(fg)$ vertices that contains representatives of the canonical homology generators in its 1-skeleton. These representatives only intersect in a single vertex, and each one uses $O(f)$ vertices and edges.* □

Proof (Idea of proof.) Lazarus et al. present two algorithms that actually *compute* the canonical homology generators, and from which the subdivision is easy to derive. Both algorithms are "optimal" from a worst-case complexity point of view.

The first algorithm is inductive, removing one triangle at a time from the surface in question and maintaining information about the still unvisited part of the surface and its collared boundary.

The second algorithm (based on Brahana [4]) starts with a maximal subgraph G of the vertex-edge graph of the surface that has a connected complement $\mathcal{T} \backslash G$, which is thus an open disk. One derives generators for the fundamental group of G, which also generate the fundamental group of $\mathcal{T}$. These generators are then modified to yield canonical generators for the fundamental group of $\mathcal{T}$.

5 Many triangulated 3-spheres

Proof (Proof of Theorem 1:) We build a cellular decomposition $\mathcal{S}$ of S^3 with n vertices and $\Theta(n^{5/4})$ octahedral facets from two triangulated handlebodies and a stack of prisms over a Heffter surface. The theorem then follows by independently triangulating the octahedra.

The construction begins with a Heegaard splitting $S^3 = H_1 \cup H_2$ of S^3 of genus $g = q(q-5)/4 + 1$ as in Proposition 3, for any prime power q of the form $q = 4k+1$ for $k \geq 1$. We replace the boundary $S_g = H_1 \cap H_2$ of the handlebodies by the prism $\Pi_g = S_g \times [0,1]$, pick a generator α of $\mathbb{F}_q^*$, and embed a copy of the Heffter triangulation $\mathcal{T}_q^\alpha$ on $S_g \times \{0\}$ and $S_g \times \{1\}$.

- *The triangulated handlebodies.* Use Theorem 2 to refine each copy of $\mathcal{T}_q^\alpha$ to a triangulation of S_g that contains representatives of the canonical homology generators $\{a_i, b_i : 1 \leq i \leq g\}$ in its 1-skeleton, such that the a_i's span meridian disks in H_1 and the b_i's do the same in H_2. This introduces $O(q^2 g) = O(q^4)$ new vertices. Double all generators as in Figure 3 using another $O(q^4)$ vertices to obtain a triangulation $\mathcal{T}'$ of S_g, and in each handlebody triangulate the meridian disks spanned by all these polygonal curves (using a total of $O(q^2 g)$ triangles, but no new vertices). Then cone the boundary of each of the $2g$ solid cylinders bounded by the meridian disks to a new vertex (introducing a total of $O(q^2 g)$ tetrahedra), and cone the triangulated boundary of each of the remaining two 3-balls to another new vertex. This last step uses $2g+2$ new vertices and $O(q^4)$ tetrahedra. The total f-vector of this triangulation $\mathcal{T}''$ of $H_1 \cup H_2$ is
$$\big(O(q^4),\, O(q^4),\, O(q^4),\, O(q^4)\big).$$
- *The stack of prisms.* Let $\mathcal{C}_q^\alpha \times I_m$ cellulate the manifold with boundary $\Pi_g = S_g \times [0,1]$, where I_m is the subdivision of $[0,1]$ into $m = \Theta(q^3)$ closed intervals, and refine each of $\mathcal{C}_q^\alpha \times \{0\}$ and $\mathcal{C}_q^\alpha \times \{1\}$ into the triangulation $\mathcal{T}'$. This refined cellulation $\mathcal{C}$ is composed of $\Theta(q^4)$ prisms over $(q-1)$-gons and $2q$ 3-cells whose boundary consists of $q-1$ 4-gons, one $(q-1)$-gon, and on average $O(q^3)$ triangles that together triangulate another $(q-1)$-gon. The boundary of $\mathcal{C}$ consists of the union of these $O(q^4)$ triangles, and its total f-vector is
$$\big(\Theta(q^4),\, \Theta(q^5),\, \Theta(q^5),\, \Theta(q^4)\big).$$

  Apply the E-construction (Proposition 2) to $\mathcal{C}$, using $\Theta(q^4)$ new vertices, to arrive at a cellulation $E(\mathcal{C})$ of Π_g into $\Theta(q)$ simplices (pyramids over the boundary triangles), $\Theta(q^4)$ bipyramids over $(q-1)$-gons, and $\Theta(q^5)$ octahedra. Now triangulate the bipyramids by joining each main diagonal to each edge of the base $(q-1)$-gon. This cellulation $\mathcal{C}'$ of Π_g consists of $\Theta(q^5)$ simplices and $\Theta(q^5)$ octahedra (Figure 4). Its total f-vector is

$$\big(\Theta(q^4),\, \Theta(q^5),\, \Theta(q^5),\, \Theta(q^5)\big).$$

The desired cellulation of S^3 is $\mathcal{S} = \mathcal{T}'' \cup \mathcal{C}'$.

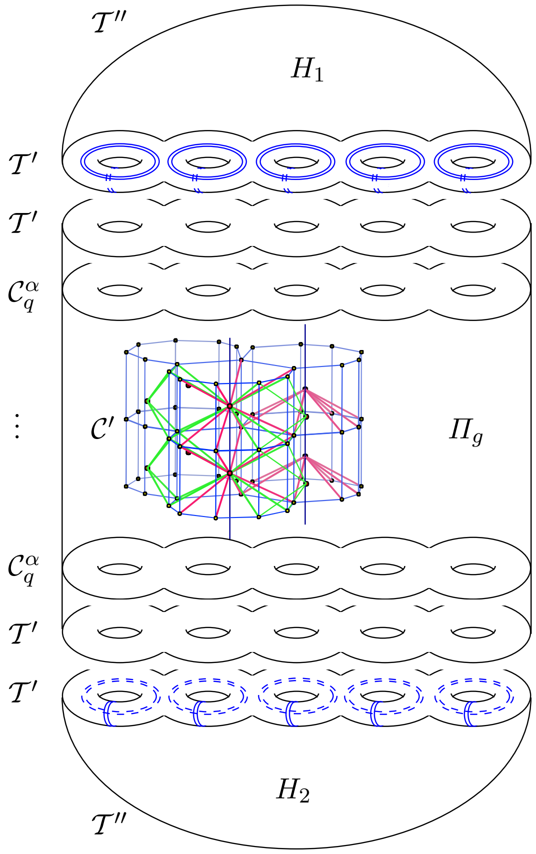

Fig. 4 The thickened Heegaard splitting $S^3 = H_1 \cup \mathcal{C}' \cup H_2$ of S^3. Not shown is the triangulation of the handlebodies H_1 and H_2. Independently triangulating the $\Theta(n^{5/4})$ octahedral 3-cells of $\mathcal{C}'$ in different ways yields "many triangulated 3-spheres".

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