IMPORTANT: Solutions to exercises marked with ‘⊿’ have to be given until 27.04.2012.

NOTATION
Let $N$ be a finite set and $S \subseteq N$. Then, if $x: N \to \mathbb{R}$ is a function, we write

$$x(S) = \sum_{a \in S} x(a).$$

We sometimes introduce a function $x: N \to \mathbb{R}$ as being a ‘cost function’ or a ‘weight function’, for example. The adjectives ‘cost’ or ‘weight’ do not carry any mathematical content, they just allow us to talk of the ‘cost’ or ‘weight’ $x(a)$ of an element $a \in N$, or of the ‘cost’ or ‘weight’ $x(S)$ of a set $S \subseteq N$.

EXERCISE 1. Consider the Boolean variables $x_1, \ldots, x_n$. A clause is a disjunction of some of the variables and their negations, so

$$x_1 \lor x_3 \lor \neg x_5,$$
$$x_2 \lor x_4 \lor x_5,$$
$$\neg x_1 \lor x_3 \lor x_6$$

are examples of clauses.

Given a collection of clauses on variables $x_1, \ldots, x_n$, the maximum satisfiability problem asks for an assignment of true/false values to the variables $x_1, \ldots, x_n$ that maximizes the number of clauses that evaluate to true. Give an integer programming formulation for this problem and its corresponding linear programming relaxation.

⊿ EXERCISE 2. Let $N$ be a finite set of items. Each item $k$ has a weight $w(k) \geq 0$ and a cost $c(k)$. Given a number $M > 0$, the knapsack problem asks for a set $S \subseteq N$ that maximizes $c(S)$ subject to $w(S) \leq M$ (we want to fill our knapsack of capacity $M$ with a set of items of maximum cost). Give an integer programming formulation for this problem.

EXERCISE 3. Let $G = (V, E)$ be a graph. Consider the polytope

$$P(G) = \{ x \in \mathbb{R}^E : x(\delta(v)) \leq 1 \text{ for all } v \in V \text{ and } x \geq 0 \}. $$

If $M \subseteq E$ is a matching, then its characteristic vector $x: E \to \{0, 1\}$, which is such that $x(e) = 1$ if and only if $e \in M$, belongs to $P(G)$.

Recall that a graph $G$ is bipartite if it does not contain circuits of odd length or, equivalently, if its vertex set can be partitioned into sets $A$ and $B$ such that all edges of $G$ have one endpoint in $A$ and the other in $B$.

(a) Show that the characteristic vector of any matching in $G$ is a vertex of $P(G)$.
(b) Give an example to show that when $G$ is not bipartite, then $P(G)$ might have nonintegral vertices.
(c) Suppose $G$ is bipartite. Let $x$ be a vertex of $P(G)$. Show that the set $\{ e \in E : x(e) > 0 \}$ does not contain a circuit.
(d) Use the above to show that, when $G$ is bipartite, the vertices of $P(G)$ are precisely the characteristic vectors of matchings in $G$. 
Exercise 4. Show how the linear programming problem
\[
\begin{align*}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b
\end{align*}
\]
can be expressed in the standard form.

Exercise 5. Recall that the dual of the standard form problem
\[
\begin{align*}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad Ax = b, \\
& \quad x \geq 0
\end{align*}
\]
is the problem
\[
\begin{align*}
\text{minimize} & \quad y^T b \\
\text{subject to} & \quad y^T A \geq c.
\end{align*}
\]
Express the dual problem in the standard form and determine its dual, showing finally that the dual of the dual problem is the primal problem again.

Exercise 6. Compute the dual of the following problems, and in each case prove weak duality.

(a) \[
\begin{align*}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b
\end{align*}
\]

(b) Here \(A_{ij} \in \mathbb{R}^{m_i \times n_j}\). Also \(l, u \in \mathbb{R}^{n_1}, c_i \in \mathbb{R}^{n_i}, \) and \(b_i \in \mathbb{R}^{m_i}\).
\[
\begin{align*}
\text{maximize} & \quad c_1^T x_1 + c_2^T x_2 \\
& \quad A_{11} x_1 + A_{12} x_2 = b_1, \\
& \quad A_{21} x_1 + A_{22} x_2 \leq b_2, \\
& \quad l \leq x_1 \leq u
\end{align*}
\]

Exercise 7. Let \(x_1, \ldots, x_d \in \mathbb{R}^n\). Prove Carathéodory’s theorem: If \(z\) is a conic combination of vectors \(x_1, \ldots, x_d\), then it is a conic combination of at most \(n\) of these vectors.

Use this to prove the following alternative version of the theorem: If \(z\) is a convex combination of vectors \(x_1, \ldots, x_d\), then it is a convex combination of at most \(n + 1\) of these vectors.

Exercise 8. Write a SAGE function that, given a SAGE graph \(G = (V, E)\), solves the linear programming problem
\[
\begin{align*}
\text{maximize} & \quad x(E) \\
& \quad x(\delta(v)) \leq 1 \quad \text{for all} \ v \in V, \\
& \quad x \geq 0 \quad \text{and} \ x \in \mathbb{R}^E.
\end{align*}
\]
Then, by looking at the optimal solution, your function should either return a maximum matching, or conclude that the graph is not bipartite.

Observation: Check out the SAGE Graph class, and also about how to solve linear programs in SAGE. This exercise will be discussed in the classroom.