ARITHMETIC PROPERTIES OF LOCAL SYSTEMS

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Abstract. A building block of homotopy theory is the fundamental group of varieties, in topology and in arithmetic geometry. We know very little on it. One way to approach it is via local systems, that is linear representations modulo conjugation. In doing so we lose a lot of information. Still it yields obstructions for a finitely presented group to be the fundamental group of a smooth complex quasi-projective variety. Among those there are the ones coming from geometry, that is the motivic ones. Deep conjectures predict over various fields when local systems should but motivic.

1. Lecture 1

1.1. **Geometric groups.** For X a smooth connected complex quasi-projective variety, we denote by $\pi_1(X, x)$ its group of homotopy classes of continuous loops based at x. Up to isomorphism, the group does not depend on the choice of x. At path γ_{xy} from x to y induces an isomorphism $\gamma_{xy}^{-1} \circ \pi_1(X, y) \circ \gamma_{xy} = \pi_1(X, x)$ and another path is of the shape $\gamma_{xy} \circ \gamma_{xx}$. We write $\pi_1(X)$ for its isomorphism class.

$X(\mathbb{C})$ has the homotpy type of a finite CW complex, so is $\pi_1(X, x)$ finitely presented (fp). Any fp is the fundamental group of a finite CW complex (the 1-cells are the generators, the 2 cells the relations), vice-versa the fundamental group of a finite CW complex is fp.

We say that a fp group π is *geometric* if there is a smooth connected complex quasi-projective variety X such that $\pi = \pi_1(X, x)$.

Problem 1 (General Problem)**.** How to recognize the geometric groups among all fp groups?

There is no general answer to this problem, which is of the same magnitude as the Hodge conjecture in complex geometry, the Tate conjecture in arithmetic geometry etc...we do not even have a conjectural answer (unlike for those 'abelian' cases). So we look for obstructions for fp groups to be geometric.

1.2. **Examples.**

1.2.1. *Algebraic Geometry.* Any finite group is geometric (Serre [Ser58]): the group acts freely on a complete intersection of very high dimension and Lefschetz theory implies that those complete intersections as simply connected, i.e. $\pi_1 = \{1\}$.

1.2.2. So in particular the abelianization of any fp group $\mathbb{Z}^b \oplus \Gamma$, with Γ finite (abelian) is geometric, as π_1 is compatible with products and $\mathbb{Z}^b = \pi_1(\mathbb{C}^{\times b})$.

1.2.3. *Abelian Harmonic Analysis.* The Malčev completion of $\pi_1(X, x)$ admits a mixed Hodge structure.

1.3. **Local Systems.** This is a representation $\rho\colon \pi_1(X, x) \to \mathsf{GL}_r(\mathbb{C})$ modulo conjugacy by $\mathsf{GL}_r(\mathbb{C})$ (so may be written $\rho\colon \pi_1(X) \to \mathsf{GL}_r(\mathbb{C})$). We write $\mathbb{L}_\rho$ for this conjugacy class. Can be thought of as a topological fibration $\mathbb{L}_\rho \to X$ in $\mathbb{C}^r$ vectorspaces, with transition functions defined by ρ, or simply $(X_x \times \mathbb{C}^r)/\pi_1(X, x)$ where $X_x \to X$ is the universal covering based at x and $\pi_1(X, x)$ acts diagonally.

We can make this definition *abstract* by replacing $\pi_1(X, x)$ by any fp group.

Since we are disarmed to study $\pi_1(X, x)$ directly, we study local systems, from a categorial and / or geometric viewpoint.

Warning 1. The study of local systems reflects only a small ? part of $\pi_1(X, x)$.

Indeed: As $\pi_1(X, x)$ is fp, so in particular fg (finitely generated), a representation ρ has values in $\mathsf{GL}_r(A)$ for $A \subset \mathbb{C}$ a $\mathbb{Z}$-algebra of finite type, so $\rho = \rho_A \otimes_A \mathbb{C}$. But any such A possesses a $\mathcal{O}_v$ point, so $A \hookrightarrow \mathcal{O}_v$, where $\mathcal{O}_v$ is an ℓ-adic ring, so with finite residue field. But $\mathsf{GL}_r(\mathcal{O}_v)$ is profinite, thus $\rho_{\mathcal{O}_v} := \rho_A \otimes_A \mathcal{O}_v$ factors through $\hat{\rho}_{\mathcal{O}_v} : \widehat{\pi_1(X, x)} \to \mathsf{GL}_r(\mathcal{O}_v)$, where the profinite completion $\widehat{\pi_1(X, x)}$ is by the Riemann existence theorem the étale fundamental group pf X (based at x), and $\hat{\rho}_{\mathcal{O}_v}$ is continuous for the profinite topology.

But (Toledo [Tol93]): the kernel $K := \mathrm{Ker}\big(\pi_1(X, x) \to \widehat{\pi_1(X, x)}\big)$ of the profinite completion may be non-trivial.

Problem 2. We know nothing on K.

Similarly we define ℓ-adic local systems as continuous representations $\hat{\rho}\colon \widehat{\pi_1(X)} \to \mathsf{GL}_r(\mathcal{O}_v)$ where $\mathcal{O}_v$ is an ℓ-adic ring. Absolute irreducibility means that $\hat{\rho} \otimes_{\mathcal{O}_v} \bar{\mathbb{Q}}_\ell$ is irreducible.

1.4. **Weakly integral fp groups.**

Definition 1 ([dJE23])**.** A fp group π is *weakly integral* if whenever there is an irreducible representation $\rho\colon \pi \to \mathsf{GL}_r(\mathbb{C})$ with determinant of finite order dividing $\delta \in \mathbb{N}_{\geq 1}$, then for all prime numbers ℓ, there is an absolutely irreducible

$$\hat{\rho}_v\colon \hat{\pi} \to \mathsf{GL}_r(\mathcal{O}_v)$$

where $\mathcal{O}_v$ is some ℓ-adic ring, with determinant of order dividing $\delta \in \mathbb{N}_{\geq 1}$.

Theorem 2 (Main Group Theoretic Theorem, [dJE23])**.**

$$\pi \textit{ geometric} \Longrightarrow \pi \textit{ weakly integral}.$$

1.5. **Obstruction.**

Corollary 3. *Being weakly integral is an obstruction for a fp π to be geometric.*

1.6. **Breuillard's examples.** $\Gamma_w = \langle a, b | w = 1\rangle$ with $w = a^2ba^{-2}b^{-2}, \delta = 1, r = 2$. Let

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \; B = \begin{pmatrix} j & 0 \\ 0 & j^2 \end{pmatrix}$$

where $j = e^{2i\pi/3}$ is a primitive cube root of unity. Then irreducible, conjugate over $\mathbb{Q}(j)$ (quadratic over $\mathbb{Q}$) yield the only 2 irreducible $SL_2(\mathbb{C})$ local systems. However

$$AB = \frac{j}{\sqrt{2}}\begin{pmatrix} 1 & j \\ -1 & j \end{pmatrix}$$

so $\mathrm{Tr}(AB) = -\frac{1}{\sqrt{2}} \notin \bar{\mathbb{Z}}_2$ so Γ_w is not weakly integral as it is not integral by $\ell = 2$!

Here the second example which I haven't checked: choose $w = a^2ba^{-2}ba^{-4}b^{-3}$ and A, B replace by C, D:

$$C = -\frac{1}{2}\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix} \; D = \begin{pmatrix} a & \frac{\phi}{\sqrt{3}} \\ 0 & a^{-1} \end{pmatrix}$$

where $\phi = (1+\sqrt{5})/2$ is the golden mean, and a is a solution to the quadratic equation $a^2 + a\phi + 1 = 0$ (so that $a + a^{-1} = -\phi = 2\cos(\frac{4\pi}{5})$). Note that C has order 12 and D order 5. We can also check (say using Mathematica) that:

$$C^2DC^{-2} = D^3C^4D^{-1}.$$

Furthermore we can compute (again with Mathematica) equations for the character variety of Γ_w, and check that (C, D) is an isolated point of it, and that the subgroup it generates is indeed Zariski-dense (and thus irreducible). Finally we readily check that

$$\mathrm{Tr}(CD) = \frac{\phi}{\sqrt{3}},$$

which is non-integral at the prime $\ell = 3$ (it is a root of the non-monic $\mathbb{Q}$-irreducible polynomial $9X^4 - 9X^2 + 1$); in particular CD has infinite order. So it does $\ell = 3$!

Problem 3. Can we find for each separate ℓ a Breuillard type example? (It is interesting first to justify Breuillard's two statements, thus for $\ell = 2$ and for $\ell = 3$, above!)

2. Lecture 2

2.1. **Non-Abelian Hamonic Analysis; Simpson's theory on X smooth connected projective over $\mathbb{C}$.** In [Sim92] Simpson constructed via GIT moduli spaces $M_B(X, r)$, $M_{dR}(X, r)$, $M_{Dol}(X, r)$ of rank r semi-simple local systems $\mathbb{L}$, semi-simple flat connections (E, ∇), semi-stable Higgs bundles (V, θ) with vanishing Chern classes.

$M_B(X,r)$ depends only on $\pi_1(X)$, while $M_{dR}(X,r)$, $M_{Dol}(X,r)$ depend on the analytic structure. He extended the complex analytic Riemann-Hilbert isomorphism between $M_B(X,r)$ and $M_{dR}(X,r)$ to a real analytic isomorphism between $M_{dR}(X,r)$ and $M_{Dol}(X,r)$. The Hitchin map $h : M_{Dol}(X,r) \to \mathbb{A}^N$, where

$$N = \dim_{\mathbb{C}} H^0(X, \oplus_{i=1}^r \mathrm{Sym}^i \Omega^1_X)$$

is proper, $M_B(X,r)$ is affine. It has tremendous consequences. e.g. if $N = 0$ then the 3 spaces are 0-dimensional. This is the case e.g. if X is a Shimura variety of exceptional type (Margulis superrigidity).

These isomorphisms extend to the stable locus on all sides, which are then fine moduli. For B those points are the *irreducible* local systems, so dR they are *simple* flat connections, for Dol there are stable Higgs bundles with vanishing Chern classes.

$M_{Dol}(X,r)$ is endowed with the *Higgs flow*

$$\mathbb{G}_m \times M_{Dol}(X,r) \to M_{Dol}(X,r),\ (t,(V,\theta)) \mapsto (V, t\cdot\theta).$$

Its fix points correspond via Simpson's correspondence of $\mathbb{C}$ PVHS (complex polarizable variations of Hodge structure).

It yields an obstruction for a fp group π to be geometric: its (affine) character variety $M_B(\pi,r)$ should be real analytically endowed with a proper map h to an affine space.

The correspondence holds if we fix a order δ of the determinant as above. The theory does not quite extend in this shape to the non-proper case.

2.2. **Betti moduli with boundary and determinant conditions.** Fix $X \subset \bar{X}$ a good compatification, with $\bar{X} \setminus X = \cup D_i$ a strict normal crossings divisor. We fix δ so $\det(\mathbb{L})^\delta = \mathbb{I}$. We also fix the eigenvalues $\lambda_{ij} \in \mu_\infty \subset \mathbb{C}^\times$ of the monodromies at infinity (so they are quasi-unipotent with a fix set of eigenvalues). We could also fix the Jordan decomposition type. We don't in the sequel. The resulting Betti moduli, which is denoted by $M_B(X,r;\delta,\lambda_{ij})$, is easily constructed: fix small loops γ_i around the D_i and complete with c_j to span $\pi_1(X,x)$. Then consider for any ring R matrices $\Gamma_i, C_j \in \mathsf{GL}_r(R)$ with the relations: those coming from the relations of $\pi_1(X,x)$ and $\det(\Gamma_i)^\delta = \det(C_j)^\delta = \mathbb{I}$ and symmetric functions of the eigenvalues of the Γ_i describing $(\lambda_{i1},\dots,\lambda_{ir})$. It is an affine finite type variety defined over $\mathbb{Z}$, on which GL_r acts by conjugacy. Then $M_B^{\mathrm{all}}(X,r;\delta,\lambda_{ij})$ is the GIT quotient. It is an affine finite type variety defined over $\mathbb{Z}$. Its stable points over $\mathbb{C}$ correspond to irreducible local systems with the (δ,λ_{ij}) conditions. Its points over $\mathbb{C}$ correspond to semi-simple local systems with the (δ,λ_{ij}) conditions. We then consider the open $M_B(X,r;\delta,\lambda_{ij})$ in $M_B^{\mathrm{all}}(X,r;\delta,\lambda_{ij})$ which is the complement in $M_B^{\mathrm{all}}(X,r;\delta,\lambda_{ij})$ of the closure of the Zariski closure of the non-absolutely irreducible points in characteristic 0. We denote by

$$\epsilon\colon M_B(X,r;\delta,\lambda_{ij}) \to \mathrm{Spec}(\mathbb{Z}),\ \epsilon^{\mathrm{all}}\colon M_B^{\mathrm{all}}(X,r;\delta,\lambda_{ij}) \to \mathrm{Spec}(\mathbb{Z})$$

the structure morphism.

Theorem 4 (Main Geometric Theorem, [dJE23]). *ϵ dominant $\Longrightarrow$ for all prime numbers ℓ, ϵ^{all} admits a $\bar{\mathbb{Z}}_\ell$-point hitting $M_B(X, r; \delta, \lambda_{ij}) \otimes_{\mathbb{Z}} \mathbb{Q}$ non-trivially.*

In fact more is true, but under certain irreducibility assumptions, the stronger form is equivalent to this one.

Corollary 5 (Diophantine Corollary: Rumely's theorem [Rum86], see [MB89], Théorème 1.6 for this version). *If*

1) *$M_B^{\mathrm{all}}(X, r; \delta, \lambda_{ij})(\mathbb{C})$ is non-empty and consists of irreducible local systems;*
2) *$M_B^{\mathrm{all}}(X, r; \delta, \lambda_{ij})$ and $M_B^{\mathrm{all}}(X, r; \delta, \lambda_{ij}) \otimes_{\mathbb{Z}} \mathbb{C}$ are irreducible;*

then $M_B^{\mathrm{all}}(X, r; \delta, \lambda_{ij})$ possesses an integral (i.e. $\bar{\mathbb{Z}}$) point meeting $M_B(X, r; \delta, \lambda_{ij}) \otimes_{\mathcal{O}} \mathbb{C}$ non-trivially.

Problem 4. The conditions 1) and 2) in the Diophantine Corollary are difficult to meet. *We thank Peter Sarnak for his interest and questions on the diophantine consequences of our Main Geometric Theorem.* Initial dream with J. de Jong: prove or disprove that $M_B^{\mathrm{all}}(X, r; \delta) \to \mathrm{Spec}(\mathbb{Z})$ is pure in the sense of Gruson-Peskine (would be sufficient for Diophantine Applications). Impossible in the present state of knowledge (the problem being the use of companions).

2.3. **de Jong's conjecture,** [dJ01]**.** It predicts that an arithmetic $\mathsf{GL}_r(\overline{k_v[[t]]})$ representation, where k_v is a finite char. ℓ field and X is defined over $\mathbb{F}_q$ of char. prime to ℓ, is geometrically finite. From the conjecture, Johan deduces the following: let $\bar{\rho}\colon \pi_1^t(X_{\bar{\mathbb{F}}_p}) \to \mathsf{GL}_r(k_v)$ be an absolutely irreducible representation of the tame fundamental group (i.e. tame on all curves, i.e. the Galois cover defined by $\bar{\rho}$ is such that in restriction to all curves, the wild ramification index at punctures is trivial and the residual field extension is separable), $\mathrm{Def}(\bar{\rho})$ be Mazur's deformation's space which represents the functor assigning to a formal local $W(\mathbb{F}_v)$-algebra $\hat{R}$ with residue field k_v the set of equivalence classes of representations $\pi_1(X_{\bar{\mathbb{F}}_p}) \to \mathsf{GL}_r(\hat{R})$ lifting $\bar{\rho}$. It is a formal scheme formally of finite type over $W(\mathbb{F}_v)$. We can also decorate the functor with a determinant condition and conditions at infinity etc.

$\bar{\mathbb{Q}}_\ell$-points of $\mathrm{Def}(\bar{\rho})$ are precisely ℓ-adic local systems with residual local system equals to $\bar{\rho}$. The group $\pi_1^t(X_{\bar{\mathbb{F}}_p})$ is topologically finitely generated and $\mathsf{GL}_r(k_v)$ is finite, thus there are only finitely many such $\bar{\rho}$, thus a power m say of Frobenius Φ fixes $\bar{\rho}$ thus $\mathrm{Def}(\bar{\rho})$. Replacing $\mathbb{F}_q$ by $\mathbb{F}_{q^m}$, Φ acts $\mathrm{Def}(\bar{\rho})$ and we can study its fix points. Those are precisely the ℓ-adic local systems which lift $\bar{\rho}$ and are *arithmetic*, that is descend to $X/\mathbb{F}_q$ once we have assumed that $X(\mathbb{F}_q) \neq \varnothing$ to represent $\pi_1(X)$ as an extension of $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_q) = \hat{\mathbb{Z}}$ with $\pi_1(X_{\bar{\mathbb{F}}_p})$.

Theorem 6 (deJong, [dJ01])**.** *If $\mathrm{Def}(\bar{\rho})_{\mathrm{red}}$ is formally smooth, then $\mathrm{Def}(\bar{\rho})_{\mathrm{red}}^{\Phi} \to \mathrm{Spf}(W(\mathbb{F}_v)))$ is finite flat. Moreover $\cup_{n \in \mathbb{N}_{\geq 1}} \mathrm{Def}(\bar{\rho})_{\mathrm{red}}^{\Phi^n} \subset \mathrm{Def}(\bar{\rho})$ is Zariski dense.*

Similarly with decorations.

2.4. **de Jong's conjecture implies that the Main Geometric Theorem implies the Main Group Theoretic Theorem.** See [dJE23], Sections 3 and 4.

First let us remark that there is a big difference between the two theorems. In the geometric one, we do have the loops γ_i at infinity while in the group theoretic one, π is abstract, so there are no privileged γ_i. Should it be geometric, it means in particular that in it, there are hidden some γ_i with Γ_i quasi-unipotent etc.

We use the notation $M_B(X, r; \delta)$ dropping the λ_{ij} conditions at infinity. This defines a forgetful morphism $M_B(X, r; \delta, \lambda_{ij}) \hookrightarrow M_B(X, r; \delta)$ which is a closed embedding. Note again the left hand side sees the geometry as we need the boundary divisor to define it, the right hand side depends solely on $\pi_1(X)$. As a application of de Jong's conjecture proved by Gaitsgory [Gai07] in general (for $p \geq 3$) one obtains:

Proposition 7 ([EK23])**.**

$$\cup_{\lambda_{ij} \in \mu_\infty} M_B(X, r; \delta, \lambda_{ij}) \subset M_B(X, r; \delta)$$

is Zariski dense.

Thus if ϵ on $M_B(X, r; \delta)$ is dominant, so is it on some $M_B(X, r; \delta, \lambda_{ij})$ and we can apply the geometric theorem.

3. Lecture 3

The goal of this Lecture is to sketch the proof of the Main Geometric Theorem, see [dJE23, Section 2].

3.1. **Companions.** We refer to [Esn23, Sections 7.1,7.2,7.3] for the motivation on the complex side of the existence of companions, the meaning in rank one and some remarks on geometricity.

3.2. **Algebraic Geometry Facts.** Set $\epsilon \colon M := M_B(X, r; \delta, \lambda_{ij}) \to \mathrm{Spec}(\mathbb{Z})$.

• M of finite type $\Longrightarrow$ ϵ generically smooth on M_{red}. Let $z \in M$ be a closed point, of residue field $\mathbb{F}_v = \mathbb{F}_{\ell^m}$. It corresponds to an absolutely irreducible

$$\bar{\rho} \colon \pi_1(X) \to \mathsf{GL}_r(\mathbb{F}_{\ell^m}).$$

• For a good model of X, Grothendieck's specialization homomorphism $\Longrightarrow$ factorization

$$\bar{\rho} : \pi_1(X) \xrightarrow{\mathrm{sp}} \pi_1^t(X_{\bar{\mathbb{F}}_p}) \xrightarrow{\bar{\rho}_{\bar{\mathbb{F}}_p}} \mathsf{GL}_r(\mathbb{F}_{\ell^m}).$$

• On $X_{\bar{\mathbb{F}}_p}$ has Mazur's deformation's space $\mathrm{Def}(X_{\bar{\mathbb{F}}_p}, \bar{\rho}_{\bar{\mathbb{F}}_p}; \delta, \lambda_{ij})$

$$\widehat{M_z} \cong \mathrm{Def}(X_{\bar{\mathbb{F}}_p}, \bar{\rho}_{\bar{\mathbb{F}}_p}; \delta, \lambda_{ij})$$

as $W(\bar{\mathbb{F}}_{\ell^m})$-local formal schemes.

• De Jong's conjecture: there is a ℓ-adic local system $\mathbb{L}$ on $X_{\mathbb{F}_q}$ with residual local system $\bar{\rho}_{\bar{\mathbb{F}}_p}$.

• Argument stemming from [EG18]: for any algebraic isomorphism $\sigma : \bar{\mathbb{Q}}_\ell \cong \bar{\mathbb{Q}}_{\ell'}$ construct the companion (which respects after Deligne the boundary conditions and the determinant) and consider sp^{-1} of it. It yields the theorem for all $\ell \neq p$.

• Redo for a larger p.

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