A non-abelian version of Deligne's Fix part Theorem

Joint work with Moritz Kerz

Für Gerd Faltings, in Freundschaft

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Bonn, July 25, 2024

Abstract

We prove a non-abelian version of Deligne's Fix Part Theorem. It is a statement which is purely anchored in complex geometry. The reason for the consideration is a vaster program which aims at understanding some aspects of the *monodromy-weight* conjecture in unequal characteristic by 'tilting it' to a complex situation for which we have the tools developed notably by Morihiko Saito and Takuro Mochizuki. This lecture focuses on a small part of it. In progress.

Deligne's Fix Part Theorem

Assumption

$f : X \to S$ smooth projective, S smooth quasi-projective $/\mathbb{C} \Longrightarrow f$ is a fibration $\Longrightarrow$ natural action of $\pi_1(S, s)$ on the cohomology $H^i(X_s, \mathbb{Q})$.

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Hodge class

$\xi \in H^{2i}(Z, \mathbb{Q})$, Z smooth quasi-projective over $\mathbb{C}$, is a Hodge class if

$$\xi \in \mathrm{im}(H^{2i}(Z, \mathbb{Z})) \cap F^i H^{2i}(Z, \mathbb{C}) \cap W_i H^{2i}(Z, \mathbb{C}).$$

Deligne's Fix Part Theorem

Theorem (Deligne '71)

$f : X \to S$ smooth projective, S smooth quasi-projective $/\mathbb{C}$, S is an Artin $K\pi1 \implies$ the following conditions are equivalent

(1) the orbit of $\pi_1(S, s) \cdot \xi$ in $H^i(X_s, \mathbb{Q})$ is finite;

(2) ξ essentially i.e after a finite étale cover of S extends to a Hodge class in $H^{2i}(X, \mathbb{Q})$;

(3) the Gauß-Manin flat deformation of ξ to $H^{2i}_{\mathrm{d}R}(\widehat{X}/\widehat{S})$ lies in $F^i H^{2i}_{\mathrm{d}R}(\widehat{X}/\widehat{S})$.

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Proof $\mathbb{Q}$-version (1) (2): Deligne; (3) Griffiths

$\mathbb{Z}$-version: apply the Leray spectral sequence to $H^{2i}(X, \frac{1}{N}\mathbb{Z}/\mathbb{Z})$ and the following observation to $\mathbb{L} = R^i f_*(\frac{1}{N}\mathbb{Z}/\mathbb{Z})$.

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$\mathbb{L}$ be a local system on S with finite fibers, S is an Artin $K\pi 1$, $j>0$ is an integer $\implies$ $\exists$ a finite étale cover $S'\to S$ such that the map $H^i(S,\mathbb{L})\to H^i(S',\mathbb{L})$ vanishes.

Non-Abelian Version (E-Kerz '24)

Definitions

A *good holomorphic map* $f : X \to S$ is the complement of a relative normal crossings divisor in $\bar{f} : \bar{X} \to S$ smooth projective with S a complex holomorphic manifold.

For K a field $\in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$ a $(\mathbb{Z})K\text{-PVHS}$ is a polarized K-variation $(\mathbb{L}_s, F_s, Q_s)$ of Hodge structure on a K- local system $\mathbb{L}_s$ on X_s *definable* over $\mathbb{Z}$.

Non-Abelian Version (E-Kerz '24)

Theorem 1

$f : X \to S$ good holomorphic, S quasi-projective, $s \in S$, $(\mathbb{L}_s, F_s, Q_s)$ $(\mathbb{Z})K$-PVHS; S Artin $K\pi 1$ or X_s hyperbolic curve. Then the following conditions are equivalent:

(1) the orbit $\pi_1(S, s) \cdot [\mathbb{L}_s]$ is finite;

(2) $(\mathbb{L}_s, F_s, Q_s)$ essentially extends to a $(\mathbb{Z})K$-PVHS on X;

(3) $(\mathbb{L}_s, F_s, Q_s)$ extends to a $(\mathbb{Z})K$-PVHS on X_Δ, Δ ball $\ni s$.

Remarks

- Earlier work by Jost-Zuo, Katzarkov-Pantev, Landesman-Litt.
- We see the parallel between 1) 2) in the abelian and non-abelian versions; as for 3) the non-abelian version over Δ [F absolute] is 'weaker' than the abelian formal version [F relative]. We do not know (yet) how to prove a formal non-abelian version.

Algebraically isomonodromic deformation

Definition

An *algebraically isomonodromic deformation* of $\mathbb{L}_s$ on X_s for f good holomorphic is an *extension* $\mathbb{L}$ on X such that

$$\mathrm{Mo}^0(\mathbb{L}_s, x) \cong \mathrm{Mo}^0(\mathbb{L}, x)$$

where $\mathrm{Mo}(\mathbb{L}, x) \subset \mathrm{GL}(\mathbb{L}_x)$ is the Zariski closure of the monodromy group of $\mathbb{L}$ (i.e. the Tannaka group of $\mathbb{L}$ in $\mathrm{GL}(\mathbb{L}_x)$ based at $x \in X_s$ in the category of K-local systems) and ^{0} is the 1-component.

Algebraically Isomonodromic Deformations

Theorem 2

With $(f, (\mathbb{L}_s, F_s, Q_s))$ as in Theorem 1, then (1) is equivalent to

(2') $\mathbb{L}_s$ on X_s essentially extends to an algebraically isomonodromic $\mathbb{L}$ on X.

Then $\mathbb{L}$ in (2') extends to a $(\mathbb{Z})K\text{-PVHS}$ on X (Theorem 1 (2)).

Moreover

a) $\mathbb{L}$ in (2') essentially unique;
b) if $\mathbb{L}_s$ is absolutely simple, then *any* extension $\mathbb{L}$ to X which has finite determinant is algebraically isomonodromic.

Non-abelian Hodge Locus

Simpson's non-Abelian Hodge Locus

For $f: X \to S$ a good *proper* holomorphic map, and $r \in \mathbb{N}_{\geq 1}$, Simpson defines the *Hodge locus* $\mathrm{NL}(f, r)$ (in fact on the de Rham side, here we present it on the Betti side) as the *set* of point $\mathbb{M}$ in the total étale space

$$T = \left(\tilde{S} \times H^1(X_s, \mathrm{GL}(r, \mathbb{C}))\right)/\pi_1(S, s)$$

to $R^1 f_* \mathrm{GL}(r, \mathbb{C})$ such that $\mathbb{M}$ underlies a $(\mathbb{Z})\mathbb{C}$-PVHS.

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Theorem (Simpson '97)

$\mathrm{NL}(f, r) \subset T$ is closed analytic and finite over S.

He *conjectures* that $\mathrm{NL}(f, r) \subset T$ is *algebraic* when S is quasi-projective.

Non-abelian Hodge Locus

Finiteness

Simpson's main tool is *Deligne's finiteness theorem* '87: there are only finitely many local systems $\mathbb{L}$ of bounded rank on a given Z quasi-projective complex smooth $/\mathbb{C}$ which lift to a $(\mathbb{Z})\mathbb{C}$-PVHS.
Deligne: *The present work is the result of an effort to try to understand Faltings' finiteness theorem for abelian varieties* '83.

Theorem 3

Simpson's theorem is true for f a good (*not necessarily proper*) holomorphic map.

In addition to Simpson's proof, one needs a form of the nilpotent orbit theorem which holds in families near the special fibre, which enables one to recognize on a mixed Hodge structure when it arises as a limiting mixed Hodge structure of a $(\mathbb{Z})\mathbb{C}$-PVHS.

The Mumford-Tate group in a algebraically isomonodromic deformation

Definition

For $(\mathbb{L}_x, F_x, Q_x)$ a $(\mathbb{Z})\mathbb{Q}$ PHS on $x = \mathrm{Spec}\ \mathbb{C}$, its Mumford-Tate group $\mathrm{MT}(\mathbb{L}_x, F_x)$ is the *smallest* linear $\mathbb{Q}$-algebraic subgroup of $\mathrm{GL}(\mathbb{L}_x)$ such that its complex points comprise the maps $\mathbb{L}_{x,\mathbb{C}} \to \mathbb{L}_{x,\mathbb{C}}$ sending λ to $\lambda^i\bar{\lambda}^j$ for all $(i,j) \in (\mathbb{N})^2$ with $i + j =$weight $\mathbb{L}_x$. It does not depend on Q_x.

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Proposition 4

X smooth quasi-projective $/\mathbb{C}$, $(\mathbb{L}, F, Q)$ $(\mathbb{Z})\mathbb{Q}$-PVHS on X, $x \in X$. Then $\mathrm{MT}(\mathbb{L}_x, F_x)$ normalises $\mathrm{Mo}(\mathbb{L}, x)$ and $\mathrm{Mo}^0(\mathbb{L}, x)$ in $\mathrm{GL}(\mathbb{L}_x)$.

It yields our Definition

The Mumford-Tate group of the $(\mathbb{Z})\mathbb{Q}$-PVHS $(\mathbb{L}, F, Q)$:

$$\mathrm{MT}((\mathbb{L}, F), x) = \mathrm{Mo}^0(\mathbb{L}, x) \cdot \mathrm{MT}(\mathbb{L}_x, F_x) \subset \mathrm{GL}(\mathbb{L}_x).$$

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Corollary

1) $\mathrm{MT}((\mathbb{L}, F), x)$ has parallel transport as $x \in X$;
2) in an algebraically isomonodromic situation, for $x \in X_s$ has

$$\mathrm{MT}((\mathbb{L}_s, F_s), x) \xrightarrow{\cong} \mathrm{MT}((\mathbb{L}, F), x).$$

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Remark

Deligne: outside of a thin set of points $x \in X$, iso-class of $\mathrm{MT}(\mathbb{L}_x, F_x)$ is 'constant'; 1) defines $\mathrm{MT}((\mathbb{L}, F), x)$ as a local system across this thin subset.

Sketch of Proof of Theorem 2

Go to profinite completions: group theory

Main point is to construct $\mathbb{L}$ out of the finiteness of the orbit of $\mathbb{L}_s$. Assumption $K\pi 1$ or family of hyperbolic curves $\Longrightarrow$ $\widehat{\pi_1(X_s)} \hookrightarrow \widehat{\pi_1(X)}$. Pass to ℓ-adic local systems. It becomes a purely group theoretic statement. Assume for simplicity $\mathbb{L}_{s,\ell}$ absolutely irreducible. Then it lifts uniquely to X as a projective ℓ-adic system which we may essentially lift as an ℓ-adic system $\mathbb{L}_\ell$ with finite determinant. Restrict to the topological fundamental group $\Longrightarrow \mathbb{L}$. A priori $\mathrm{Mo}^0(\mathbb{L})$ is semi-simple. Prove it is simple.

Sketch of Proof of Theorem 2 II

Simpson-T. Mochizuki correspondence

Assume $\mathbb{L}_s$ absolutely irreducible for simplicity $\Longrightarrow$ essential unique $\mathbb{L}$ with finite determinant. h "unique metric which is fiber-wise harmonic" on $\mathbb{L} \otimes \mathcal{O}_X$, so agrees with h_s on $\mathbb{L}_s \otimes \mathcal{O}_{X_s}$. It defines Higgs which is log at infinity. Eigenvalues of $\mathbb{C}^\times$ are constant (log compactification) so recognizable on X_s where they are cyclic, so defines a sum of Hodge bundles with a nilpotent Higgs operator. So the same on X. Has to go back: to such a nilpotent Higgs the corresponding $\mathbb{L}$ is a $\mathbb{C}$-PVHS. This is due to T. Mochizuki and does not request the full package of the Simpson correspondence.

Sketch of Proof of Proposition 4

Exact sequence

Point is the exact sequence (d'Addezio-E, weaker version by André) on X

$$1 \to \mathrm{Mo}(\mathbb{L}, x) \to \pi(\langle(\mathbb{L}, F)\rangle, x) \to \pi(\mathcal{C}, x) \to 1$$

of $\mathbb{Q}$-algebraic groups. π for Tannaka group. $\mathcal{C} \subset \langle(\mathbb{L}, F)\rangle$ full sub of those objects with trivial local system.

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Then

$$\mathrm{MT}((\mathbb{L}, F), x) = \pi(\langle(\mathbb{L}, F)\rangle, x).$$