LECTURES ON LOCAL SYSTEMS IN ALGEBRAIC-ARITHMETIC GEOMETRY

HÉLÈNE ESNAULT

Abstract. The topological fundamental group of a smooth complex algebraic variety is poorly understood. One way to approach it is to consider its complex linear representations modulo conjugation, that is complex local systems. One fundamental problem is to recognize those coming from geometry, and more generally subloci of the moduli space of local systems with special arithmetic properties. This is the object of deep conjectures. We'll study some consequences of those, notably integrality and crystallinity properties.

1. Lecture 1: General Introduction

The *topological fundamental group* $\pi_1^{\rm top}(X,x)$ of a finite CW-complex X based at a point x in topology, as defined by Poincaré, is a finitely presented group. In turn, any finitely presented group is the fundamental group of a finite CW complex. The finite generation enables one to define a 'moduli' (parameter) space $M_B(X,r)$ of all its semi-simple complex linear representations $\rho : \pi_1^{\rm top}(X,x) \to GL_r(\mathbb{C})$ in a given rank r, modulo conjugation, or equivalently, of all its rank r semi-simple complex local systems $\mathbb{L}$. It is called the character variety, also the *Betti moduli space* of X in rank r, and is a scheme of finite type defined over the ring of integers $\mathbb{Z}$.

We are interested in the case when X consists of the complex points of an algebraic manifold (smooth algebraic variety of finite type over the complex numbers $\mathbb{C}$). We know extremely little, if not nothing on the restrictions it imposes on $\pi_1^{\rm top}(X,x)$. On the other hand, there are naturally defined local systems $\mathbb{L}$: those which in restriction to some Zariski dense open $U \hookrightarrow X$ are subquotients (equivalently, summands by Deligne's semi-simplicity theorem) of a local system on U which comes from the variation of the cohomology of the fibers of a smooth projective morphism $g : Y \to U$. Such $\mathbb{L}$ are called *geometric*. For example, $U = X$ and g *finite* (equivalently, $\mathbb{L}$ has finite ${\rm Im}(\rho)$, which is called monodromy), which by the Riemann existence theorem is equivalent to g being finite étale, and thus relates $\pi_1^{\rm top}(X,x)$ to its profinite completion $\pi_1(X_{\mathbb{C}},x)$, the étale fundamental group, defined by Grothendieck, itself related to the Galois group of the field of functions of $X_{\mathbb{C}}$.

So it is very natural to try to single out geometric or even finite complex points of $M_B(X,r)$. More generally, it is natural to try to define a notion of geometric subloci of higher dimension. It is clearly an inaccessible task, which is reminiscent of the Hodge and the Tate conjectures: how can we construct g out of $\mathbb{L}$? There are several *conjectures* relying on various aspects of $M_B(X,r)$.

Grothendieck's p-curvature conjecture: It relies on the *Riemann-Hilbert correspondence* which equates the complex points $\mathbb{L}$ of $M_B(X,r)$ with algebraic integrable connections (E,∇) on X (say X projective for simplicity to avoid boundary growth conditions): we consider (E,∇) mod p for all large primes p and request this characteristic $p>0$ connection to be generated by flat sections. This is the original formulation and should characterize finite local systems $\mathbb{L}$. More generally, to characterize geometric local systems $\mathbb{L}$, we request (E,∇) mod p for all large p to be filtered so that the associated graded is spanned by flat sections. Since the work by Katz [Kat72] which roughly (a bit less) shows that on a geometric $\mathbb{L}$ we can characterize its finiteness like this, the ones by Chudnosvky [Chu85], Bost [Bos01] and André [And04] which handle the solvable case, and some remarks like [EK18], there is essentially no big progress on this viewpoint.

Gieseker-de Jong conjecture: It relies simply on the *finite generation* of $\pi_1^{\rm top}(X,x)$ which implies the theorem of Malčev-Grothendieck [Mal40], [Gro70] saying that $\pi_1(X_{\mathbb{C}},x)$ controls the size of $M_B(X,r)$: if $\pi_1(X_{\mathbb{C}},x)=\{1\}$ then $M_B(X,r)$ consists of one point, the trivial $\mathbb{L}$ of rank r (in fact there are no extensions as well). Gieseker's conjecture [Gie75] solved in [EM10] asserts an analog in characteristic $p>0$ for infinitesimal crystals, while de Jong's conjecture, which is still unsolved in its generality (see [ES18] for small steps) predicts an analog for isocrystals. It is also related to the Langlands program: if the ground field is $\bar{\mathbb{F}}_p$ and the isocrystal is endowed with a Frobenius structure, then the existence of ℓ-adic companions ([AE19], [Kel22], initially predicted by Deligne in Weil II [Del80]) proves the conjecture. It would be of interest to understand a generalization of de Jong's conjecture on prismatic crystals which encompasses his initial formulation.

Simpson's motivicity conjecture: Rigid Local Systems. Those are the 0-dimensional components of $M_B(X,r)$. Simpson [Sim92] predicted that they are all geometric. It relies on the corresponding theorem by Katz [Kat96] when X has dimension 1, in which case X has to be an open in $\mathbb{P}^1$ (so in the definition of $M_B(X,r)$ one has to fix conjugacy classes of quasi-unipotent monodromy at infinity). And it relies on the so-called Simpson's correspondence, which when X is projective equates real analytically $M_B(X,r)$ with the moduli space of semi-stable Higgs bundles with vanishing Chern classes. Those are endowed with a $\mathbb{C}^\times$-flow, thus rigid local systems, viewed on the Higgs side, are fixed by it, so by Simpson's theorem, underly a polarizable complex variation of Hodge structures. From there it is one short step to dream of geometricity.

Relative Fontaine-Mazur conjecture: Revisited by Petrov [Pet22], relying on p-adic geometry through the work by Scholze [Sch13] and Liu-Zhu [LZ17], it predicts that an irreducible $\mathbb{L}$, viewed ℓ-adically, comes from geometry if and only if it is defined over a form X_F of X where $F\subset\mathbb{C}$ is a subfield of finite type.

Higher dimensional special subloci: It is possible to define *arithmetic* subloci of $M_B(X,r)$ and ℓ-adic analogs. One predicts that they themselves come from geometry in a specific sense. This program is carried out in rank one in [EK20] over $\mathbb{C}$ and in [EK21] in characteristic $p>0$. Furthermore, over $\bar{\mathbb{F}}_p$ a special case of a general density conjecture of arithmetic local systems is proved in [EK22, EK22], while over $\mathbb{C}$, this density initially predicted in [EK23] can not be true for varieties defined over characteristic zero fields fields ([LL22a], [LL22b], [Lam22]). That one may pose the *density conjecture* relies on de Jong's theorem [dJ01] on the structure of Mazur's deformation spaces, which are the complete local analogs of $M_B(X,r)$ over $\bar{\mathbb{F}}_p$, and on Drinfeld's use of it [Dri01] to prove Kashiwara's conjecture to the effect the direct image of (regular singular) D-modules by a proper morphism preserves semi-simplicity.

Short of being able to prove such so general conjectures, one can consider corollaries of them. May be the bigger progress in the last years has been reached for rigid local systems. Simpson had proven that if $\mathbb{L}$ is rigid, then an ℓ-adic completion of it for ℓ large is defined over a form X_F for F a field of finite type. Given Petrov's formulation of the relative Fontaine-Mazur conjecture, one can now say that Simpson's geometricity conjecture, which comes from complex geometry, is a special case of Fontaine-Mazur's one, which comes from geometry over a number field. If true, rigid local systems should be integral, that is stemming from a representation $\rho : \pi_1^{\rm top}(X,x) \to GL_r(\bar{\mathbb{Z}})$. This is called the *integrality conjecture*, formulated by Simpson himself in [Sim92]. It is proven in [EG18] under the extra condition that $\mathbb{L}$ as a complex point of $M_B(X,r)$ is smooth, that is has no multiplicity. It is a cohomological condition and for this reason one says that $\mathbb{L}$ is *cohomologically rigid*. The proof uses X modulo $p > 0$ for p large, and on it the ℓ-adic companions mentioned above (the existence of which, in the form used, has been proved by L. Lafforgue for curves as a consequence of the Langlands program [Laf02], and by Drinfeld in higher dimension [Dri12]). For example, Katz proves in [Kat96] that in dimension 1 rigid local systems are cohomologically rigid. Also, by Margulis superrigidity, on Shimura varieties of real rank ≥ 2, all local systems are semi-simple and cohomologically rigid.

Another consequence of Simpson's geometricity conjecture can be drawn: at a place of good reduction of residual characteristic $p > 0$ where X descends to $X_{W(\mathbb{F}_q)}$, and p is large, the induced p-adic local system on the underlying geometric p-adic variety $X_{\overline{{\rm Frac}(W(\mathbb{F}_q))}}$ descends to a *crystalline* p-adic local system on $X_{{\rm Frac}(W(\mathbb{F}_q))}$. We prove this fact in [EG20] for X smooth projective. We also prove this in [EG21] assuming X is a Shimura variety of real rank ≥ 2. This result *loc. cit.* is the building block of the recent proof of the André-Oort conjecture on such Shimura varieties [PST21]. One dream would be to be able to understand whether given a rigid local system in $M_B(X,r)$ for X projective, we can assign to it for p large a prismatic F-crystal in the sense of Bhatt-Scholze [BS21]. This would be the best possible generalization of [EG20].

However, as shown very recently in [dJEG22], it is *not* the case (unlike in dimension 1 and on Shimura varieties of real rank ≥ 2) that all rigid local systems are cohomolopgically rigid. So a new proof of Simpson's integrality conjecture has to be found.

2. Lecture 2: Kronecker's Rationality Criteria and Grothendieck's p-Curvature Conjecture

Abstract. We recall two of Kronecker's criteria, say an analytic one and an algebraic one, for an integral number to be root of unity and an algebraic number to be rational. We recall Grothendieck's p-curvature conjecture and its generalization. We show how the two are related and mention Katz's proof using (a generalization of) the analytic criterion.

2.1. **Kronecker's criteria.** Let $a \in \mathbb{C}$ be a complex number, and write it as $a = \exp(2\pi\sqrt{-1}b)$ for $b \in \mathbb{C}$ defined modulo the integers $\mathbb{Z}$. We list Kronecker's criteria for $a \in \mu_\infty \subset \mathbb{C}$, that is for a to be a root of unity, or equivalently for $b \in \mathbb{Q}$, that is for b to be a rational number.

2.1.1. *Kronecker's analytic criterion,* [Kro57]. Recall that the subring $\bar{Z} \subset \mathbb{C}$ of algebraic integers of the complex numbers consists of those complex numbers $a \in \mathbb{C}$ satisfying an equation $f(a) = a^d + c_1 a^{d-1} + \ldots + c_d = 0$ with $c_i \in \mathbb{Z}$, and the subfield $\bar{\mathbb{Q}} \subset \mathbb{C}$ of algebraic numbers consists of those a as before but with $c_i \in \mathbb{Q}$. Any field automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$ of $\mathbb{C}$ leaves $\mathbb{Z} \subset \mathbb{Q}$ invariant.

Proposition 2.1. *Assuming $a \in \bar{\mathbb{Z}}$, then $a \in \mu_\infty$ if and only if for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, the complex absolute value of $\sigma(a)$ is equal to* 1.

Proof. See https://mathoverflow.net/questions/10911/english-reference-for-a-result-of-kronecker For $f = f_1$ as above, we write $f_n(X) = \prod_{i=1}^d (X - \alpha_i^n) \in \mathbb{C}[X]$ for all $n \in \mathbb{N}_{>0}$, with $\alpha_1 = a$. The coefficients of f_n are symmetric functions in the α_i, thus are expressable as polynomials with rational coefficients in the c_i, and on the other hand they are in $\bar{\mathbb{Z}}$, thus they lie in $\mathbb{Z}$, and have bounded norms. Thus there are finitely many such f_n, thus the set $\{a^n, n \in \mathbb{N}_{\geq 1}\}$ is finite, thus lies in μ_∞. □

2.1.2. *Kronecker's algebraic criterion.* Assume $b \in \bar{\mathbb{Q}}$, so b lies in the number field $\mathbb{Q}(b)$, of degree d say over $\mathbb{Q}$, and is integral at almost all (that is at all but finitely many) primes $\mathfrak{p}$ of $\mathcal{O}_{\mathbb{Q}(b)}$, so we can take its reduction mod $\mathfrak{p}$ for almost all primes $\mathfrak{p}$.

Proposition 2.2. *If for almost all primes $\mathfrak{p}$ of $\mathcal{O}_{\mathbb{Q}(b)}$, $(b \bmod \mathfrak{p}) \in \mathbb{F}_p \subset \mathcal{O}_{\mathbb{Q}(b)}/\mathfrak{p}$, then $b \in \mathbb{Q}$, that is $d = 1$. In words: if b lies in the prime field in characteristic p for almost all p, then it does in characteristic* 0.

Proof. b defines a finite ring extension $\mathbb{Z} \hookrightarrow \mathcal{O}_{\mathbb{Q}(b)}$ of degree d. This degree is constant mod $\mathfrak{p}$ for all primes $\mathfrak{p}$ of $\mathcal{O}_{\mathbb{Q}(b)}$, and b is integral over $\mathbb{Z}$ with finitely many primes $\{q\}$ removed. So for $\mathfrak{p}$ of characteristic not one of those q, $\mathcal{O}_{\mathbb{Q}(b)}/\mathfrak{p}$ is spanned by b mod $\mathfrak{p}$ over $\mathbb{F}_p$. So the condition $(b \bmod \mathfrak{p}) \in \mathbb{F}_p \subset \mathcal{O}_{\mathbb{Q}(b)}/\mathfrak{p}$ is equivalent to b mod $\mathfrak{p}$ being completely split. This can not happen for all $\mathfrak{p}$ except finitely many. □

2.1.3. *Translation of Kronecker's algebraic criterion in terms of differential equations.* We consider the complex algebraic variety $X = \mathrm{Spec}\mathcal{O}(X)$, $\mathcal{O}(X) = \mathbb{C}[t, t^{-1}]$ and on it the linear differential equation

$$(\star)\ \frac{df}{f} = b\frac{dt}{t}$$

for some $b \in \mathbb{C}$ with analytic solution

$$f_\lambda(t) = \lambda t^b, \ \lambda \in \mathbb{C}.$$

Then

Lemma 2.3. *For $\lambda \neq 0$, f_λ is integral over $\mathcal{O}(X)$ if and only if $b \in \mathbb{Q}$.*

Proof. If $\mathbb{Q} \ni b = \frac{m}{n}, \ m, n \in \mathbb{Z}$ then $f_\lambda^n \in \mathcal{O}(X)$. Vice-versa, assume f_λ is integral over $\mathcal{O}(X)$, so in particular f_λ is algebraic over the Laurent power series field $\mathbb{C}((t))$ containing $\mathcal{O}(X)$. So f_λ defines a finite field extension $\mathbb{C}((t)) \hookrightarrow \mathbb{C}((t))(f_\lambda)$ of degree n say. By Kummer theory for $\mathbb{C}((t))$, the field extension has to be defined by a polynomial $U^n - \nu t^m$ for some $m \in \mathbb{N}_{>0}$ and $\lambda \in \mathbb{C}$, thus $f_\lambda = \lambda t^{\frac{m}{n}}$ for some $\lambda \in \mathbb{C}$.

□

Remark 2.4. The following conditions are equivalent
(i) f is algebraic over the field of fractions $\mathrm{Frac}(\mathcal{O}(X)) = \mathbb{C}(t)$;
(ii) f is integral over $\mathcal{O}(X)$;
(iii) the monodromy of $(\star)$ is finite.

Proof. (i) $\Longrightarrow$ (ii): f_λ as a solution of $(\star)$ is analytic on X, and by (i) lies in $\bar{\mathbb{C}}(t)$, thus lies in the integral closure $\overline{\mathcal{O}(X)} \subset \overline{\mathbb{C}(t)}$.
(ii) $\Longrightarrow$ (iii): By Lemma 2.3, $f_\lambda = \lambda t^b$. So $\gamma^* f_\lambda$, the restriction of f_λ to the path $\big(\gamma : [01] \to, \tau \mapsto \exp(2\pi\sqrt{-1}\tau)\big)$ is the function $\big([01] \to \mathbb{C}, \ \tau \mapsto \lambda\exp(2\pi\sqrt{-1}b\tau)\big)$. So the monodromy $\gamma^* f_\lambda(\tau = 1)/\gamma^* f_\lambda(\tau = 0) = \exp(2\pi\sqrt{-1}b) =: a$ is a root of unity by (ii).
(iii) $\Longrightarrow$ (i): If a is a root of unity, then b is a rational number so (i) holds.

□

We now assume $b \in \bar{\mathbb{Q}}$. For almost all primes $\mathfrak{p}$ of $\mathbb{Q}(b)$ where it makes sense, we consider the differential equation

$$(\star)_{\mathfrak{p}} \ \frac{df}{f} = (b \bmod \mathfrak{p})\frac{dt}{t}$$

which is simply $(\star)$ by viewed over $\mathcal{O}_{\mathbb{Q}(b)}/\mathfrak{p}$. We write $\mathcal{O}_{\mathbb{Z}}(X) = \mathbb{Z}[t, t^{-1}]$ so $\mathcal{O}(X) = \mathcal{O}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$. The only way to make sense of the solutions λt^b for $\lambda \in \mathcal{O}_{\mathbb{Q}(b)}/\mathfrak{p}$ is to request them to lie in $\mathcal{O}_{\mathbb{Z}}(X) \otimes_{\mathbb{Z}} (\mathcal{O}_{\mathbb{Q}(b)}/\mathfrak{p})$. For $\lambda \neq 0$ in $\mathcal{O}_{\mathbb{Q}(b)}/\mathfrak{p}$, λt^b lies in $\mathcal{O}_{\mathbb{Z}}(X) \otimes_{\mathbb{Z}} (\mathcal{O}_{\mathbb{Q}(b)}/\mathfrak{p})$ if and only if $(b \bmod \mathfrak{p})$ lies in $\mathbb{F}_p \subset \mathcal{O}_{\mathbb{Q}(b)}/\mathfrak{p}$. We now summarize the discussion:

Theorem 2.5. *If $b \in \bar{\mathbb{Q}}$, the differential equation $(\star)$ has a full set of algebraic solutions over $\mathcal{O}(X)$ if and only if for almost all prime $\mathfrak{p} \in \mathcal{O}_{\mathbb{Q}(b)}$, $(\star)_{\mathfrak{p}}$ has a full set of solutions in $\mathcal{O}_{\mathbb{Z}}(X) \otimes_{\mathbb{Z}} (\mathcal{O}_{\mathbb{Q}(b)}/\mathfrak{p})$ (that is its p-curvature vanishes).*

Proof. $(\star)$ has a full set of algebraic solutions over $\mathcal{O}(X)$ if and only if $b \in\in \mathbb{Q}$ (Lemma 2.3) if and only if $(b \bmod \mathfrak{p}) \in \mathbb{F}_p \subset (\mathcal{O}_{\mathbb{Q}(b)}/\mathfrak{p})$ for almost all p (Proposition 2.2) if and only if $(\star)_{\mathfrak{p}}$ has a full set of solutions in $\mathcal{O}_{\mathbb{Z}}(X) \otimes_{\mathbb{Z}} (\mathcal{O}_{\mathbb{Q}(b)}/\mathfrak{p})$ (previous discussion).

□

The content of Grothendieck's p-curvature conjecture is to predict that Theorem 2.5 extends to any X and any $(\star)$. We shall explain the formulation and a generalization of it without ever mentioning the definition of the p-curvature.

2.2. **Grothendieck's p-curvature conjecture.** Let us first formulate what a posteriori is equivalent to the p-curvature conjecture. We write $X = \mathrm{Spec}\mathcal{O}(X),\ X = \mathrm{Spec}\mathbb{C}[t, t^{-1}, (t-1)^{-1}]$. Then we pose the linear differential equation

$$(\star)\ \frac{df}{f} = b\frac{dt}{t} + c\frac{dt}{t-1}$$

where now $b = (b_{ij}),\ c = (c_{ij}) \in M_r(\bar{\mathbb{Q}})$. For almost all primes $\mathfrak{p}$ of $\mathbb{Q}(b_{ij}, c_{ij})$ for $1 \le i, j \le r$ where it makes sense, we consider the differential equation

$$(\star)_{\mathfrak{p}}\ \frac{df}{f} = (b \bmod \mathfrak{p})\frac{dt}{t} + (c \bmod \mathfrak{p})\frac{dt}{t-1}$$

which is simply $(\star)$ by viewed over $\mathcal{O}_{\mathbb{Q}(b_{ij})}/\mathfrak{p}$. We write $\mathcal{O}_{\mathbb{Z}}(X) = \mathbb{Z}[t, t^{-1}, (t-1)^{-1}]$ so $\mathcal{O}(X) = \mathcal{O}_{\mathbb{Z}}(X) \otimes_{\mathbb{Z}} \mathbb{C}$.

Conjecture 2.6. *The differential equation $(\star)$ has a full set of algebraic solutions over $\mathcal{O}(X)$ if and only if $(\star)_{\mathfrak{p}}$ has a full set of solutions in $\mathcal{O}_{\mathbb{Z}}(X) \otimes_{\mathbb{Z}} (\mathcal{O}_{\mathbb{Q}(b_{ij})})/\mathfrak{p})$ (that is its p-curvature vanishes).*

The initial Grothendieck's conjecture is documented in Katz's article [Kat72, Introduction]: let X be a smooth quasi-projective variety over $\mathbb{C}$, (E, ∇) be an integrable algebraic connection. Let S be a scheme of finite type over $\mathbb{Z}$ so $(X, (E, \nabla))$ descends to $(X_S, (E_S, \nabla_S))$ and has good reduction at all closed points $s \in S$. Then the prediction is

Conjecture 2.7 (Grothendieck's p-curvature conjecture)**.** *(E, ∇) has a full set of algebraic solutions if and only there is a dense open $S^\circ \subset S$ such that for all $s \in S^\circ$ the reduction (E_s, ∇_s) has a full set of solutions (that is its p-curvature vanishes).*

We explain briefly why Conjecture 2.7 is equivalent to its special case Conjecture 2.6.

Claim 2.8 ([Kat70], Theorem 13.0)**.** If there is a dense open $S^\circ \subset S$ such that for all $s \in S^\circ$ the reduction (E_s, ∇_s) has a full set of solutions, then (E, ∇) is regular singular and has finite monodromies at infinity.

Claim 2.9. (E, ∇) has a full set of algebraic solutions if and only if there is a finite cover $h' : Y' \to X$ such that $h^{'*}(E, \nabla)$ has a full set of solutions if and only if there is a finite unramified cover $h : Y \to X$ such that $h^*(E, \nabla)$ has a full set of solutions.

Proof. If h exists, the solutions of (E, ∇) in $\mathcal{O}(Y)$ are algebraic solutions over $\mathcal{O}(X)$. Vice-versa, assume given a basis over $\mathcal{O}(X)$ of algebraic solutions f_i say. The ring $\mathcal{O}(X) \hookrightarrow \mathcal{O}(X)[f_i]$ is then finite, thus defines a finite possibly ramified finite cover $h' : Y' \to X$ so $h^{'*}(E, \nabla)$ is trivial. So the underlying monodromy representation $\pi_1(X) \to GL_r(\mathbb{C})$ of (E, ∇) trivializes on $h'_*\pi_1(Y')$ in $\pi_1(X)$, which has finite index. This defines a finite unramified cover $h : Y \to X$ with $\pi_1(Y) = h'_*\pi_1(Y')$ such that $h^*(E, \nabla)$ is trivial.

□

Claim 2.10. Conjecture 2.7 in general is equivalent to its special case when X is a smooth projective curve.

Proof. By Claim 2.9, Conjecture 2.7 is true for (E, ∇) on X if and only if it is for $h^{'*}(E, \nabla)$ for any $h' : Y' \to X$ finite cover. So by Claim 2.8 we may assume that

(E, ∇) extends to a smooth projective compactification of X. Then the Lefschetz hyperplane theorem reduces it to the case of a smooth projective curve.

□

Claim 2.11 ([And04],Théorème 0.6.1)**.** Conjecture 2.7 is equivalent to its special case when X is a smooth projective curve defined over a number field.

So applying Belyi's theorem [Bel80], we formulate

Claim 2.12. Conjecture 2.7 is equivalent to its special case when X is $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, that is Conjecture 2.6. defined over a number field.

2.3. **Back to Kronecker's analytic criterion: Gauß-Manin connections.** Let X be a quasi-projective smooth variety over $\mathbb{C}$, $g : Y \to X$ be a smooth projective morphism, $\mathbb{L}$ being the local system $R^i g_* \mathbb{C}$ for some i. Then

Theorem 2.13 ([Kat72])**.** *Conjecture 2.7 holds for* $\mathbb{L}$.

Proof. The proof consists of two parts. First, the assumption on the full set of solutions for $(\star)_{\mathfrak{p}}$ implies that the F-filtration $R^i g_* \Omega^{\geq j}_{Y/X} \hookrightarrow \mathbb{L}$ is stabilized by the Gauß-Manin connection. Then this implies that the monodromy is an automorphism of the integral polarized Hodge structure of the fibre, which is the intesection of the integral group $\mathrm{Aut}(H^i(Y_x, \mathbb{Z}))$ with $\mathrm{Aut}(H^i(Y_x, \mathbb{R}), (2\pi\sqrt{-1}^i(x, Cy))$ inside of $\mathrm{Aut}(H^i(Y_x, \mathbb{R}))$, where C is the Weil operator. A variant of Kronecker's analytic criterion says that this intersection is finite.

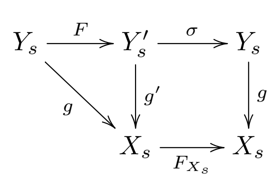
So the stabilization of the F-filtration by the connection is the main point. We give a presentation which slightly differs from *loc. cit.* To this, over our model X_S, we take a closed point s of large characteristic p so the coherent sheaves $R^{i-j} g_* \Omega^j_{Y/X}$ remain locally free, and the map

$$(*) \quad R^i g_* \Omega^{\geq j}_{Y/X} \to R^i g_* \Omega^{\bullet}_{Y/X},$$

which by Hodge theory is injective, remains inective by base change. We want to prove that the condition on the existence of a full set of solutions implies that the F-filtration is stabilized by the connection, that is that the $\mathcal{O}_X$-linear Kodaira-Spencer class

$$KS : R^{i-j} g_* \Omega^j_{Y/X} \xrightarrow{\nabla} \Omega^1_X \otimes R^{i+1-j} g_* (\Omega^{j-1}_{Y/X} \to \Omega^j_{Y/X}) \to \Omega^1_X \otimes R^{i+1-j} g_* \Omega^{j-1}_{Y/X}$$

dies. The important diagram is then



where $F : Y_s \to Y'_s$ is the Frobenius on Y_s relative to X_s, that is on local functions $\mathcal{O}(Y_s) = \mathcal{O}(X_s)[y_1, \ldots, y_n]/I$ it sends the class of $f(y_1, \ldots, y_n)$ to the class of $f(y_1^p, \ldots, y_n^p)$, and $F_{X_s} : X_s \to X_s$ is the absolute Frobenius, so it sends $f \in \mathcal{O}(X_s)$ to $f^p \in \mathcal{O}(X_s)$. By computing the dimension of de Rham cohomology $H^i_{dR}(Y_x)$ of the fibres of g, the injectivity of (*) implies that

$$(**) \; R^i g_* \tau_{\leq j} \Omega^{\bullet}_{Y_s/X_s} = R^i g'_* \tau_{\leq j} F_* \Omega^{\bullet}_{Y_s/X_s} \to R^i g'_* F_* \Omega^{\bullet}_{Y_s/X_s} = R^i g_* \Omega^{\bullet}_{Y_s/X_s}$$

is injective as well. By the Cartier isomorphism and base change for σ, it holds

$$R^{i-j}g_*\mathcal{H}^j = F^*_{X_s}R^{i-j}g'_*\Omega^j_{Y'_s/X_s}$$

where $\mathcal{H}^j$ is the j-th cohomology sheaf of $F_*\Omega^\bullet_{Y_s/X_s}$. This yields the exact sequence

$$(1)\quad 0 \to F^*_{X_s}R^{i-j}g'_*\Omega^j_{Y'_s/X_s} \to R^i g_*(\tau_{\le j+1}F_*\Omega^\bullet_{Y_s/X_s}/\tau_{\le j-1}F_*\Omega^\bullet_{Y_s/X_s}) \to F^*_{X_s}R^{i-j-1}\Omega^{j+1}_{Y'_s/X_s} \to 0$$

of connections, and defines a de Rham extension

$$\epsilon \in H^1_{dR}(X_s, F^*(R^{i-j-1}(\Omega^{j+1}_{Y'_s/X_s})^\vee \otimes R^{i-j}g'_*\Omega^j_{Y'_s/X_s})).$$

Now we use the assumption which is equivalent to saying that (1) is $F^*_{X_s}$ of an exact sequence

$$0 \to R^{i-j}g'_*\Omega^j_{Y'_s/X_s} \to V \to R^{i-j-1}\Omega^{j+1}_{Y'_s/X_s} \to 0$$

for some vector bundle V on X_s. So in particular this sequence splits on a non-empty affine open of X_s thus ϵ dies as well on a non-empty affine open. On the other hand,

$$H^1_{dR}(X_s, F^*(R^{i-j-1}g'_*(\Omega^{j+1}_{Y'_s/X_s})^\vee \otimes R^{i-j}g'_*\Omega^j_{Y'_s/X_s})) = H^1_{dR}(X_s, (R^{i-j-1}g'_*(\Omega^{j+1}_{Y'_s/X_s})^\vee \otimes R^{i-j}g'_*\Omega^j_{Y'_s/X_s}) \otimes F_{X_s *}(\Omega^\bullet_{X_s}))$$

maps, again via the Cartier isomorphism, to

$$H^0(X_s, (R^{i-j-1}g'_*(\Omega^{j+1}_{Y'_s/X_s})^\vee \otimes R^{i-j}g'_*\Omega^j_{Y'_s/X_s}) \otimes \Omega^1_{X_s}).$$

Katz [Kat72, Theorem 3.2] (in a slightly different presentation) computes that the image of ϵ

$$(R^{i-j-1}g'_*(\Omega^{j+1}_{Y'_s/X_s}) \to \Omega^1_{X_s} \otimes R^{i-j}g'_*\Omega^j_{Y'_s/X_s})$$

is precsiely the Kodaira-Spencer class for g', which thus dies. As the Kodaira-Spencer class is a $\mathcal{O}_{X_s}$-linear between vector bundles, it dies on X_s. Thus KS dies in restriction to all closed points $s \in |S|$ of a non-empty open. Thus it dies.

□

As mentioned in Lecture 0, the works by Chudnosvky [Chu85], Bost [Bos01] and André [And04] handle the solvable case. There are also some remarks like [EK18] under a stronger condition than just generation of the differential equation on characteristic p. Else there is essentially no big progress on this viewpoint.

However, there are two more points. The proof above using the stabilization of the τ- filtration (also called *conjugate filtration*) suggests the generalization mentioned in Lecture 0:

Conjecture 2.14 (Generalized Grothendieck's p-curvature conjecture, [And89], Appendix to Ch. V)**.** *To characterize geometric local systems $\mathbb{L}$, we request that the underlying vectore bundle with a connection (E, ∇) has the property that mod p for all large p it is filtered so that the associated graded is spanned by flat sections (that is its p-curvature is nilpotent).*

Said differently, Katz proves in a particular situation where the condition that mod p for all large p, (E, ∇) is filtered so that the associated graded is spanned by flat sections, and he knows (E, ∇) comes from geometry, that in fact if this filtration is trivial, then (E, ∇) is finite. There is one other instance of a similar situation. Let X be a smooth projective variety defined over $\mathbb{C}$ and let us assume that $H^0(X, \mathrm{Sym}^\bullet \Omega^1_X) = 0$. I had conjectured that then all local systems have finite monodromy. This has been proved in [BKT13] by *analytic methods* as a corollary of Zuo's theorem [Zuo00] to the effect that if the derivative of the period map associated to an integral variation of Hodge structure is injective, then Ω^1_X is big. In [BKT13] one finds a large list of examples of such varieties. Algebraically, the vanishing $H^0(X, \mathrm{Sym}^\bullet \Omega^1_X) = 0$ implies that the underlying bundles with connection (E, ∇) have the property that mod p for all large p they are filtered so that the associated graded is spanned by flat sections. So according to the generalized Grothendieck's p-curvature conjecture they should be geometric. But here unlike for Katz, we do not know a priori the geometricity, it comes as a consequence of the more difficult finiteness result.

3. Lecture 3: Malčev-Grothendieck's Theorem, its Variants in Characteristic $p > 0$, Gieseker's Conjecture, de Jong's Conjecture, and the One to Come

Abstract. We recall Malčev's theorem to the effect that the triviality of the profinite completion of a finitely generated group implies the triviality its algebraic completion, we recall Grothendieck's version of it formulated with $\mathcal{D}$-modules using the Riemann-Hilbert correspondence, then the Gieseker conjecture, its pendant in characteristic $p > 0$, its solution and generalizations.

3.1. **Over $\mathbb{C}$.** Let Γ be a finitely generated group, $\hat{\Gamma}$ be its profinite completion, $\Gamma^{\rm alg}$ be its proalgebraic completion. So $\hat{\Gamma} = \varprojlim_{\{G\to Q\}} Q$ where Q runs through all finite quotients, $\Gamma^{\rm alg} = \varprojlim_{\{G\to Q\}} Q$ where Q runs through all subgroups of $GL_r(\mathbb{C})$ for some $r \in \mathbb{N}_{>0}$.

Theorem 3.1 (Malčev [Mal40])**.** $\hat{\Gamma} = \{1\} \Longrightarrow \Gamma^{\rm alg} = \{1\}$.

Proof. Indeed, if $\rho : \Gamma \to GL_r(\mathbb{C})$ is a representation, then $Q = {\rm Im}(\rho)$ lies in $GL_r(A)$ where A is a ring of finite type over $\mathbb{Z}$. Let $\mathfrak{m} \subset A$ be a maximal ideal, so with finite residue field $A/\mathfrak{m}$, and define $\iota : A \hookrightarrow \hat{A} = \varprojlim_{n\in\mathbb{N}} A/\mathfrak{m}^n$ be the completion of A at $\mathfrak{m}$. Then ρ is non-trivial if and only if $\iota\circ\rho : \Gamma \to GL_r(\hat{A})$ is non-trivial. As $\iota\circ\rho$ factors through $\hat{\rho} : \hat{\Gamma} \to GL_r(\hat{A})$, then $\hat{\Gamma} = \{1\} \Longrightarrow \rho$ is trivial, i.e. has image equal to $1 \in GL_r(A)$. □

Remark 3.2. If Γ is not finitely generated, the implication of the theorem might be wrong. For example, let $\Gamma = \mathbb{Q}$, then $\hat{\Gamma} = \{1\}$, but e.g. the representation $\rho : \mathbb{Q} \to GL_2(\mathbb{C}),\ n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ is non-trivial.

We now assume that X is a smooth quasi-projective variety defined over $\mathbb{C}$. Then the Riemann-Hilbert correspondence [Del70] is an equivalence between complex local systems and algebraic integrable connections on X which are regular singular at infinity ([Del70]), or equivalently $\mathcal{O}_X$-coherent regular singular $\mathcal{D}_X$-modules. On the other hand, the topological fundamental group $\pi_1(X(\mathbb{C}), x)$ based at some complex point x is finitely generated, even finitely presented but this is not used here in the discussion, and by the Riemann Existence Theorem, its profinite completion is Grothendieck's étale fundamental group $\pi_1(X_{\mathbb{C}}, x)$. So Malčev's theorem can be rephrased in the context of complex algebraic geometry by saying

Theorem 3.3 (Grothendieck [Gro70])**.** $\pi_1(X_{\mathbb{C}}, x) = \{1\} \Longrightarrow$ *there are no non-trivial $\mathcal{O}_X$-coherent regular singular $\mathcal{D}_X$-modules on X.*

3.2. **Over an algebraically closed field of characteristic $p > 0$.** I am not aware of a direct proof of Grothendieck's theorem without passing through the Riemann-Hilbert correspondence. Gieseker [Gie75, p.8] conjectured the 'same' theorem over an algebraically closed field of characteristic $p > 0$.

Theorem 3.4 ([EM10], Theorem 1.1)**.** *Let X be a smooth projective variety defined over an algebraically closed field k of characteristic $p > 0$. If $\pi_1(X) = \{1\}$, there are no non-trivial $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules.*

Proof. Following Katz [Gie75, Theorem 1.3], we first identify $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules with Frobenius divided sheaves. Then the strategy consists, starting from a non-trivial Frob-divided sheaf $E = E_0 = \mathrm{Frob}^* E_1, E_1 = \mathrm{Frob}^* E_2, \ldots$ of rank r, to find a non-trivial vector bundle M of rank r with the property that $(\mathrm{Frob}^n)^*(M) \cong M$ for some $n \in \mathbb{N}_{>0}$. Indeed the underlying Lang torsor under $GL_r(\mathbb{F}_{p^n})$ has to be trivial by assumption, so M has to be trivial, a contradiction. To this aim, we observe that the Frob divisibility forces all kinds of Chern classes of E to be trivial and E_i itself to be stable for i large if the object $(E_0, E_1, E_2, \ldots)$ itself is simple. So we consider in the moduli of vector bundles of rank r with vanishing Chern classes the Zariski closure of the locus of the E_n for n large, on which Frob is a rational dominant map. We can then specialize the situation to X over $\bar{\mathbb{F}}_p$ and apply Hrushovski's theorem to the effect that there are (many) preperiodic points. By the above discussion, those yield non-trivial Lang torsors of $X_{\bar{\mathbb{F}}_p}$, thus by Grothendieck's specialization homomorphism, also a non-trivial finite étale cover of X, a contradiction. □

Remarks 3.5. 1) Hrushovski's proof [Hru04], which relies on model theory, has been meanwhile proven purely in the framework of arithmetic geometry by Varshavski, see [Var18].

2) The core of the proof is to find such an M with $(\mathrm{Frob}^n)^*(M) \cong M$ for some $n \in \mathbb{N}_{>0}$. This unfortunately does not seem to be deductable from Katz' 'Riemann-Hilbert' theorem in characteristic $p > 0$ or its (so far) ultimate generalization [BL17]. So in the present state of understanding, we can not draw a parallel with the complex proof.

$\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules are the same as crystals in the infinitesimal site, as was developed by Grothendieck. This led J. de Jong to pose the following vast strengthening of Gieseker conjecture (Theorem 3.4):

Conjecture 3.6 (de Jong 2010, see [ES18])**.** *Let X be a smooth projective variety defined over an algebraically closed field k of characteristic $p > 0$. If $\pi_1(X) = \{1\}$, there are no non-trivial isocrystals.*

There are (very) partial results on the conjecture, see notably [ES18, Theorem 1.2]. The main obstruction to our understanding lies in the prefix 'iso'. All approaches I know start with the mod p reduction of a crystal associated to an isocrystal and try

- to use again an argument as in the proof of Theorem 3.4 on the existence of M, with some sharpening (see [ES18, Proposition 3.2]), so ultimately it relies on Hrushovski's theorem; this also requests vanishing results for Chern classes of the mod p reduction of crystals, see [ES19], which have been generalized by Bhatt-Lurie (unpublished);

- so to trivialize the crystal itself (see e.g. [ES18, Corollary 3.8]), not only the isocrystal.

In rank 1 it is irrelevant (see [ES18, Lemma 2.12]), not however in higher rank. One would wish to find a Frob invariant isocrystal on the scratch, without passing through randomly chosen underlying crystals. However Hrushovski's theorem then no longer holds.

3.3. **Over a p-adic field.** The wish would be to have a generalization of de Jong's conjecture for prismatic 'iso'-crystals. Here one problem is that the notion of prismatic crystals is documented ([BS21]), not however the one of prismatic isocrystals. The question is what one has to invert to make the problem meaningful. The formulation should also be compatible with a complex embedding of the p-adic field and the initial Grothendieck formulation over $\mathbb{C}$.

4. Lecture 4: Interlude on some Similarity between the Fundamental Groups in Characteristic 0 and $p > 0$

Abstract. We describe two theorems concerning the structure of the (tame) fundamental group of a smooth (quasi)-projective variety in characteristic $p > 0$. The first one [ESS22] shows a similarity with the complex situation concerning the finite presentation and ultimately relies on a theorem of Lubotzky [Lub01], the second one [ESS22b] yields a new obstruction for a smooth projective variety in characteristic $p > 0$ to lift to characteristic 0 and ultimately relies on Grothendieck's specialization homomorphism. In particular, and interestingly, the second one shows that we can not explain the finite presentation of the first one by a lifting argument. In Lecture 4 we treat the finite presentation, in Lecture 5 the obstruction.

4.1. **Lubotzky's theorem.**

Definition 4.1 ([Lub01], p.1 of the Introduction)**.** *A profinite group π is said to be*

1) *finitely generated if there is a free profinite group F on finitely many generators and a continuous surjection $F \xrightarrow{\epsilon} \pi$;*
2) *finitely presented if it is finitely generated and* $\mathrm{Ker}(\epsilon)$ *is finitely generated as a normal subgroup of F.*

Concretely, in 1) there are finitely many elements $f_i, i = 1, \ldots, N$ in F such that the abstract subgroup $\langle f_1, \ldots, f_N\rangle \subset F$ surjects to any finite quotient $\pi \to G$ in the profinite structure of F. For 2) there are finitely many elements $r_j, j = 1, \ldots, M$ such that $\mathrm{Ker}(F \to \pi)$ is spanned by $\langle f r_j f^{-1},\ j = 1, \ldots, M, f \in F\rangle$. For 1) it is very classical, for 2) a slick way is simply to apply

Theorem 4.2 (Lubotzky [Lub01], Theorem 0.3)**.** *π is finitely presented if and only if it is finitely generated and in addition there is a constant $C \in \mathbb{R}_{>0}$, such that for any $r \in \mathbb{N}_{>0}$, for any prime number ℓ, for any continuous linear representation $\rho : \pi \to GL_r(\mathbb{F}_\ell)$, it holds $\dim_{\mathbb{F}_\ell} H^2(\pi, \rho) \le C \cdot r$.*

Example 4.3. The standard example is given by π being the profinite completion of an abstract finitely presented group, in the classical sense, so with the same definition without topology.

4.2. **Tame fundamental group.** Let X be a regular connected scheme. For x a closed point of the normal compactification $\bar X$ of X, we denote by $k(X)_x$ the completion of the field of functions $k(X)$ at x. A connected finite étale cover $\pi : Y \to X$ defines for any closed point x of the normal compactification $\bar X$ of X a finite field extension $k(X)_x \hookrightarrow k(Y)_y$ for any closed point y above x in the normalization $\bar Y$ of $\bar X$ in the field of functions $k(Y)$ of Y, which is the same as the normal compactification $\bar Y$ of Y.

Definition 4.4 ([KS10], Introduction and Theorem 1.1)**.** *1) If X has dimension* 1*, a finite étale cover $\pi : Y \to X$ is tame if and only if all the field extensions $k(X)_x \hookrightarrow k(Y)_y$ are tame, that is the index of ramification are prime to p and the residual extension $k(x) \hookrightarrow k(y)$ are tame.*
2) In higher dimension, π is tame if and only if for any morphism of a normal dimension 1 *scheme $C \to X$, the induced morphism $C \times_X Y \to C$ is tame on all the irreducible components.*

4.3. **Grothendieck's specialization homomorphism,** [SGA1]**, Exposé X, Exposé XIII.** Let $X_S \to S$ be a smooth morphism, where S is any scheme. We consider two field value points $\mathrm{Spec}(F) \to S$ and $\mathrm{Spec}(k) \xrightarrow{s} S$ with the property that $\mathrm{Spec}(k)$ lies in the Zariski closure of $\mathrm{Spec}(F)$. If X is not proper, we assume in addition that it admits a compactification $X_S \hookrightarrow \bar{X}_S$ so $\bar{X}_S \to S$ is smooth proper, and $\bar{X}_S \setminus X_S \to S$ is a relative normal crossings divisor with smooth components. We call it a *good* compactification (over S). So we have a diagram

$$\mathrm{Spec}(F) \longrightarrow \widehat{S_s} \longleftarrow s$$

where $\widehat{S_s}$ is the completion of S at s, together with the scheme over it

$$\begin{array}{ccccc} X_F & \longrightarrow & X_{\widehat{S_s}} & \longleftarrow & X_s \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(F) & \longrightarrow & \widehat{S_s} & \longleftarrow & s \end{array}$$

We denote by $\bar{F} \supset F$ and $\bar{k} \supset k$ algebraic closures defining $\bar{s} \to s$. Then, upon choosing an S-point $x_S : S \to X_S$, one defines [SGA1, Exp. XIII 2.10] a specialization homomorphism

$$sp : \pi_1^t(X_{\bar{F}}, x_{\bar{F}}) \to \pi_1^t(X_{\bar{s}}, x_{\bar{s}})$$

which is the composite of the functoriality homomorphism

$$\pi_1^t(X_{\bar{F}}, x_{\bar{F}}) \to \pi_1^t(X_{S_{\widehat{\bar{s}}}}, x_{\bar{s}})$$

with the inverse of the base change homomorphism

$$\pi_1^t(X_{\bar{s}}) \xrightarrow{\cong} \pi_1^t(X_{\widehat{S_{\bar{s}}}}).$$

The specialization homomorphism is surjective and induces an isomorphism on the pro-p'-completion [SGA1, Exposé XIII 2.10, Corollaire 2.12].

4.4. **Finite generation.** We apply Grothendieck's specialization homomorphism in the following situation. Let X be smooth projective defined over a characteristic 0 field together with a good compactification $X \hookrightarrow \bar{X}$, that is $\bar{X}$ is smooth projective and $\bar{X} \setminus X$ is a strict normal crossings divisor. Let S be a scheme of finite type over $\mathbb{Z}$ over which this compactification $X_S \hookrightarrow \bar{X}_S$ is defined and is a relative normal crossing divisor. Then, upon choosing a S-point $x_S : S \to X_S$, we have the specialization homomorphism

$$sp : \pi_1(X_F, x_F) \to \pi_1^t(X_s, x_s)$$

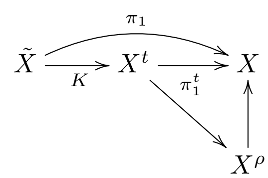
where F is a geometric point of $X_{k(S)}$ which specializes to a $\bar{\mathbb{F}}_p$-point of S. sp is surjective and is an isomorphism on the pro-p'-completion. Taking $F = \mathbb{C}$ then $\pi_1(X_F, x_F)$ is by Riemann existence theorem [SGA1, Exp. XII, Thm 5.1] the profinite completion of $\pi_1(X(\mathbb{C}), x_{\mathbb{C}})$, is itself is an *abstract finitely presented group*, see [Esn17, Theorem 1.1]. Thus $\pi_1^t(X_s, x_s)$ is *finitely generated.*

Theorem 4.5 ([ESS22], Theorem 1.1)**.** *Let X be a smooth quasi-projective variety defined over an algebraically closed field k of characteristic $p > 0$, with a good compactification $X \hookrightarrow \bar{X}$. Then $\pi^t(X)$ (based at any geometric point) is finitely presented.*

Remark 4.6. 1) The tameness assumption is necessary. If X is not proper, 'in the rule' $\pi_1(X)$ is not finitely generated. For example, for each natural number s, and a choice of s natural numbers m_i prime to p and pairwise different, the Artin-Schreier covers $y^p - y = z^{m_i}$ of the affine line $\mathbb{A}^1_k = \mathrm{Spec}(k[z])$ yield a surjection $\pi_1(\mathbb{A}^1_k) \twoheadrightarrow \oplus_{i=1}^s \mathbb{Z}/p\mathbb{Z}$.

2) We need the good compactification assumption in the argument but it should not be necessary as the result does not see the good compactification and in addition we (perhaps ?) believe in resolution of singularities. We'll mention in the proof which replacement for resolution of singularities we need, which might be called a 'numerical resolution'. Hübner and Temkin announced a solution to the problem.

4.5. **Proof of Theorem 4.5, the $\ell \neq p$ part.** We could call this part of the proof a 'SGA type proof'. All the ingredients are contained there. We have to prove the linear growth in r of $H^2(\pi_1^t, \rho)$ for a representation $\rho : \pi_1^t \to GL_r(\mathbb{F}_\ell)$ (which is necessarily continuous). Here we simplify the notation $\pi_1^t(X_s, x_s)$ to π_1^t and $\pi_1(X_s, x_s)$ to π_1. We denote by $X^\rho \to X$ the cover defined by $\mathrm{Ker}(\rho)$ which by assumption factors the universal tame cover $X^t \to X$. We split the universal cover based at x



We denote by M the underslying vector space $\oplus_1^r \mathbb{F}_\ell$ of the ρ-representation. The Hochshild-Serre spectral sequence yields the exact sequence

$$(H^1(K, \mathbb{F}_\ell) \otimes M)^{\pi_1^t} \to H^2(\pi_1^t, M) \to H^2(\pi_1, M)$$

By definition there is no $\mathbb{F}_\ell$-abelian quotient of K, thus

$$(\star)\ 0 = H^1(K, \mathbb{F}_\ell) = H^1(K^{ab}, \mathbb{F}_\ell)$$

from which we derive $H^2(\pi_1^t, M) \hookrightarrow H^1(\pi_1, M)$. On the other hand, the Hochshild-Serre spectral sequence induces the exact sequence

$$(H^1(\tilde{X}, \mathbb{F}_\ell) \otimes M)^{\pi_1} \to H^2(\pi_1, M) \to H^2(X, M)$$

and $H^1(\tilde{X}, \mathbb{F}_\ell) = 0$. So $H^2(\pi_1, M) \hookrightarrow H^1(X, M)$ is injective as well. We show the existence of the linear bound on the a priori larger group $H^2(X, M)$. By the Lefschetz theorem [EK16, Theorem 1.1 b)] we may assume that X is a surface. Using an 'ample' curve $C \to X$ in good position so the compactification of C and the boundary of X form a strict normal crossings divisor and $X \setminus C$ is affine, purity yields the exact sequence $H^0(C, M) \to H^2(X, M) \to H^2(X \setminus C, M)$. As $\dim_{\mathbb{F}_\ell} H^0(C, M) \leq r$ we may assume that X is affine. Then Artin vanishing and the equation $\chi(X, M) = r\chi(X, \mathbb{I})$, which comes from tameness, the dimension of which is purely topological and does not depend on ℓ, reduce the problem to bounding linearly the growth of $H^1(X, M)$. From the exact sequence $0 \to H^1(\pi_1^t, M) \to H^1(\pi, M) \to (H^1(K, \mathbb{F}_\ell) \otimes M)^{\pi_1^t}$, again using $(\star)$, we see that $H^1(X, M)$, which is equal to $H^1(\pi_1, M)$, is equal to $H^1(\pi_1^t, M)$. As a H^1-cocycle $\varphi : \pi_1^t \to M$ fulfills $\varphi(ab) = a\varphi(b) + \varphi(a)$, it holds $\dim_{\mathbb{F}_\ell} H^1(\pi_1^t, M) \leq \delta r$ where δ is the number of topological generators of π_1^t. This finishes this computation, which in fact can be performed even if we do not have a good model, using alteration à la de Jong-Gabber-Temkin.

4.6. **Proof of Theorem 4.5, the p part.** We denote by $j : X \hookrightarrow \bar{X}$. We denote by $\underline{M}$ the local system associated to M. The key point is to relate $H^2(\pi_1^t, M)$ to a cohomology group which comes from cohomology of some extension of a local system. Being able to use local systems rather than abstract representations yields more flexibility as we have at disposal the cohomology of all kinds of constructible extensions of the local systems as opposed to just group cohomology.

Theorem 4.7. *There is an injective $\mathbb{F}_p$-linear map $H^2(\pi_1^t(X), M) \to H^2(\bar{X}, j_*\underline{M})$.*

Proof. First let us fix what we have to prove: it holds

$$H^2(\pi_1^t, M) = \varinjlim_{U \subset \pi^t \text{open normal}} H^2(G_U, M^U),\ G_U = \pi_1^t/U.$$

For each such U we define the fibre square

$$\begin{array}{ccc} X_U & \xrightarrow{j_U} & \bar{X}_U \\ {\scriptstyle h_U}\downarrow & & \downarrow{\scriptstyle \bar{h}_U} \\ X & \xrightarrow{j} & \bar{X} \end{array}$$

where h_U is the tame Galois cover defined by G_U and $\bar{X}_U$ is the normalization of $\bar{X}$ in the field of functions $k(X_U)$ of X_U. As j is a normal crossings compactification, $\bar{h}_U$ is numerically tame in the sense of [KS10, Section 5], see [KS10, Theorem 5.4 (a)]. Thus by [Gro57, Corollaire p.204] one has

$$R^{>0}\bar{h}_{U*}^{G_U} j_{U*}\underline{M}^U = 0,$$
$$H^n(\bar{X}_U, G_U, j_{U*}\underline{M}^U) = H^n(\bar{X}, j_*\underline{M}).$$

We conclude that the spectral sequence converging to equivariant cohomology

$$E_2^{ab} = H^a(G_U, H^b(\bar{X}_U, j_{U*}\underline{M}^U)) \Rightarrow H^{a+b}(\bar{X}_U, G_U, j_{U*}\underline{M}^U),$$

for each $U \subset \mathrm{Ker}(\rho)$ yields a short exact sequence

$$H^0(G_U, H^1(\bar{X}_U, j_{U*}\underline{M}^U)) \to H^2(G_U, H^0(\bar{X}_U, j_{U*}\underline{M}^U)) \to H^2(\bar{X}, j_*\underline{M})$$

where

$$H^1(\bar{X}_U, j_{U*}\underline{M}^U)^{G_U} = (H^1(\bar{X}_U, \mathbb{F}_p) \otimes_{\mathbb{F}_p} M)^{G_U}.$$

On the other hand, $H^1(\bar{X}_U, \mathbb{F}_p) \to H^1(X_U, \mathbb{F}_p)$ is injective and again by $(\star)$ it holds $\lim_U H^1(\bar{X}_U, \mathbb{F}_p) = 0$.

□

Remark 4.8. We see that the main point is that a good normal crossings compactification implies that the tower h_U is numerically tame. The problem is then how, out of a random normal compactification, to construct one which in the tame tower is numerically tame.

Once there, we can cut down again by Lefschetz [EK16, Theorem 1.1 b)] to surfaces and on them, using Artin-Schreier exact sequence

$$0 \to j_*\underline{M} \to \mathcal{M} \xrightarrow{1-F} \mathcal{M} \to 0$$

where $\mathcal{M}$ is a locally free sheaf on $\bar{X}$ with restriction to X equal to $\underline{M} \otimes_{\mathbb{F}_p} \mathcal{O}_X$, and the classical fact that this sequence remains exact on cohomology, we are reduced to bounding above $\dim_k H^2(\bar{X}, \mathcal{M})$ linearly in r, which is the same as $\dim_k H^0(\bar{X}, \mathcal{M}^\vee \otimes \omega_{\bar{X}})$. Now the main point is the following. If $h : Y \to X$ is the Galois cover defined by the monodromy representation of M, with Galois group G, thus $h^*\underline{M}$ is trivial, and $\bar{h} : \bar{Y} \to \bar{X}$ is its compactification as above, then $\mathcal{M} = (\bar{h}_*\mathcal{L})^G$, so $\bar{h}^*\mathcal{M} \subset \mathcal{L} := H^0(Y, h^*\underline{M}) \otimes_{\mathbb{F}_p} \mathcal{O}_{\bar{Y}}$ and as a consequence of Abhyankar's lemma, tameness implies $\mathcal{L} \otimes \bar{h}^*\mathcal{O}_{\bar{X}}(-D) \subset \bar{h}^*\mathcal{M}$ for $D = (\bar{X} \setminus X)_{\mathrm{red}}$. This enables us to conclude that the restriction map $H^0(\bar{X}, \mathcal{M}^\vee \otimes \omega_{\bar{X}}) \to H^0(C \cap C', \mathcal{M}^\vee \otimes \omega_{\bar{X}})$ is injective for C, C' generic curves in the linear system of $\mathcal{H}' = \mathcal{H} \otimes \omega_{\bar{X}}(D)$ where $\mathcal{A}$ is chosen to be very ample and we request $\mathcal{H}'$ to be very ample as well. Note $\mathcal{H}$ and $\mathcal{H}'$ depend only on X and r. On the other hand, $C \cap C'$ is the union of $c_2(\mathcal{H}')$-points so the right hand side of the inequality is equal to $c_1(\mathcal{H}')^2 \cdot r$.

5. Lecture 5: Interlude on some Difference between the Fundamental Groups in Characteristic 0 and $p > 0$

Abstract. See the Abstract of Lecture 4, we show here the existence of an obstruction to lift a smooth quasi-projective variety defined over an algebraically closed field k of characteristic $p > 0$ to characteristic 0 which relies purely on the shape of its fundamental group.

5.1. **Serre's construction.** This problem has been addressed for the first time by Serre [Ser61]. We use the notation of Section 4.2 but assume more concretely that k is an algebraically closed field of characteristic $p > 0$, $X_s = X_k =: X$ is smooth projective and obtained as follows. There is a finite Galois étale cover $Y \to X$ such that $Y \hookrightarrow \mathbb{P}^n$ is a smooth complete intersection of dimension ≥ 3. So in particular $\pi_1(Y) = \{1\}$. In addition, the Galois group G is the restriction of a linear action $\rho : G \to GL_{n+1}(k)$. Then [Ser61, Lemma] shows that if X lifts to X_R with $S = \mathrm{Spec}(R)$, R noetherian local complete ring, then ρ lifts to $\rho_R : G \to PGL_{n+1}(R)$, which is not possible if the p-Sylow subgroup of G is large.

5.2. **Various obstructions.** We mention three major directions of obstructions which have been settled since Serre's work. Clearly this list is not exhaustive.

Deligne-Illusie in [DI87] proved that X smooth proper, lifting to $W_2(k)$, k perfect of characteristic $p > 0$, has the property that its Hodge to de Rham spectral sequence degenerates in E_1. This yields an obstruction to lift to $W_2(k)$ as examples for which the spectral sequence does not degenerate were previously known [Ray78]. This has been the basis of vast further developments.

Achinger-Zdanowicz construct in [AZ17] specific varieties non-liftable to characteristic 0 by blowing up the graph of Frobenius which is assumed to be non-liftable in a rigid variety with no corner piece of the F-filtration. It is remarkable that their example has cohomology of Tate type.

van Dobben de Bruyn proved that if $X \subset C^3$ is a smooth ample divisor where C is a supersingular genus ≥ 2 curve over $\bar{\mathbb{F}}_p$, then X does not lift to characteristic 0, nor does any smooth proper variety which is rationally dominant over X, [vDdB21, Theorem. 1]. The main property [vDdB21, Theorem. 2] is that if $X_S \to S$ lifts X, and X admits a morphism to a smooth projective genus ≥ 2 curve C, then the morphism lifts up to base change and Frobenius twists of X.

5.3. **An abstract obstruction to lift to characteristic 0, based on the structure of the fundamental group.** The Hodge-de Rham obstruction singled out by Deligne-Illusie is of theoretical nature, that is does not depend on a concrete way to construct the variety. However it is *not* an obstruction to lift to characteristic 0; there are schemes which lift to characteristic 0, yet in a ramified way, not over W_2. A classical example is a supersingular Enriques surface over k algebraically closed of characteristic 2, see [Ill79, Proposition II.7.3.8].

The other obstructions to lift to charactaristtic 0 rely on the construction of the variety.

The aim of the remaining part of Chapter 5 is to show that there is an essential difference between the prime to p quotient of the fundamental group of varieties in characteristic $p > 0$ and the one in characteristic 0. It provides a *conceptua*l obstruction. It is in contrast with the similarity we explained in the previous section, where we showed that the (tame) fundamental group of a smooth projective variety defined over an algebraically closed field of characteristic $p > 0$ (admitting a good compactification) is finitely presented, as it is in characteristic 0. It is also in *contrast* with the foundational theorem by Achinger [Ach17, Theorem 1.1.1] after which every connected affine scheme of positive characteristic is a $K(\pi, 1)$ space for the étale topology. His theorem notably says that affine varieties over an algebraically closed field in characteristic $p > 0$ are analog to Artin neighborhoods in characterisitc 0, see [SGA4, Exposé XI]. The theorem means precisely that for any locally constant étale sheaf of finite abelian groups $\mathcal{F}$ on X, the homomorphisms $H^i(\pi_1(X, x), \mathcal{F}_x) \to H^i(X, \mathcal{F})$ coming from the Hochshild-serre spectral sequence are isomorphisms for all i. Here $x \to X$ is a geometric point. It has not really been documented in the literature, but we could think of Achinger's theorem as the building block of the theory of étale cohomology.

5.4. **Main definition.**

Definition 5.1 (See [ESS22b], Definition A)**.** *A profinite group π is said to be p'-discretely finitely generated (resp. p'-discretely finitely presented) if there is a finitely generated (resp. presented) discrete group Γ together with a group homomorphism $\gamma : \Gamma \to \pi$ such that*

(1) *the profinite completion $\hat{\gamma} : \hat{\Gamma} \to \pi$ is surjective;*
(2) *for any open subgroup $U \subset \pi$ with $\Gamma_U := \gamma^{-1}(U)$ the restriction $\gamma_U : \Gamma_U \to U$ induces a continuous group isomorphism on pro-p'-completions*

$$\gamma_U^{(p')} : \Gamma_U^{(p')} \to U^{(p')}.$$

Grothendieck's specialization homomorphism 4.3 together with the lifting property [EGAIV$_4$, Théorème 18.1.2] imply that if $\pi = \pi_1(X, x)$ is the fundamental group of X smooth proper, then it is p'-discretely finitely presented (in particular it is p'-discretely finitely generated), see [ESS22b, Proposition 2.7]. Property (1) is also true for $\pi_1^t(X, x)$ when there is a good compactification which comes from characteristic 0. But to check it in the tower as requested in (2) is more subtle. We thank the referee of [ESS22b] for kindly noticing this. Nonetheless the property is true, see [ESS22b, Example 2.8].

Example 5.2. Of course if as in Example 4.3, π is the profinite completion of a finitely presented (resp. generated) group, then π is p'-discretely finitely presented (resp. generated).

Theorem 5.3 ([ESS22b], Theorem C)**.** *There are smooth projective varieties X defined over an algebraically closed field k of characteristic $p > 0$ such that $\pi_1(X, x)$ is not p'-finitely generated. In particular this notion is an obstruction to liftability to characteristic* 0.

Remark 5.4. We remark that in view of Example 5.2 and of the finite presentation of Theorem 4.5, Theorem 5.3 implies in particular that there are smooth projective varieties X defined over an algebraically closed field k of characteristic $p > 0$ such that $\pi_1(X, x)$ is not the profinite completion of a finitely presented group. In fact, this property is easier to see than Theorem 5.3 itself.

5.5. **Independence of ℓ and Schur rationality.** Let π be a profinite group, and let $\varphi : \pi \twoheadrightarrow G$ be a continuous finite quotient with kernel $U_\varphi = \mathrm{Ker}(\varphi)$. We denote by U_φ^{ab} its abelianization. Then conjugation induces a commutative diagram

$$\begin{array}{ccccc} \pi & \longrightarrow & \mathrm{Aut}(U_\varphi) & \longrightarrow & \mathrm{Aut}(U_\varphi^{ab}) \\ \downarrow{\scriptstyle \varphi} & & \downarrow & \nearrow & \\ G & \longrightarrow & \mathrm{Out}(U_\varphi) & & \end{array}$$

If π is finitely generated, then U_φ is finitely generated so U_φ^{ab} is a finitely generated $\hat{\mathbb{Z}}$-module. We set

$$\rho_{\varphi,\ell} : G \to \mathrm{GL}(U_\varphi^{ab} \otimes \mathbb{Q}_\ell),$$

for the induced representation, with character

$$\chi_{\varphi,\ell} = \mathrm{Tr}(\rho_{\varphi,\ell}) : G \to \mathbb{Q}_\ell.$$

The first main point is the following

Proposition 5.5 (See [ESS22b], Propositions 3.4, 3.5)**.** *1) If π is p' discretly finitely generated, then for all $\ell \neq p$, $\chi_{\varphi,\ell}$ has values in $\mathbb{Z}$ and is independent of ℓ.*
2) If X is a smooth projective variety defined over an algebraically closed characteristic $p > 0$ field, and $\varphi : \pi \twoheadrightarrow G$ is a finite quotient (thus in particular continuous), then for all $\ell \neq p$, $\chi_{\varphi,\ell}$ has values in $\mathbb{Z}$ and is independent of ℓ.

Proof. The property 1) comes essentially the definition: for $\Gamma_\varphi = \mathrm{Ker}(\Gamma \to \pi \to G)$ we have for $\ell \neq p$ the relation $\Gamma_\varphi^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell = U_\varphi^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$. The property 2) is more interesting in view of the final result Theorem 5.10. Let $Y \to X$ be the Galois cover with group G. Then $U_\varphi^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell = H^1(Y, \mathbb{Q}_\ell)^\vee$. As a consequence of the Weil conjectures, see [KM74], the characteristic polynomial of $g \in G$ acting on $H^1(Y, \mathbb{Q}_\ell)$ lies in $\mathbb{Z}[T]$ and does not depend on ℓ.

□

Remark 5.6. In particular 2) tells us that this independence of $\ell \neq p$ property can not be our sought obstruction. On the contrary, we shall use it now in order to define a rationality obstruction.

The second main point is the following

Proposition 5.7 (See [ESS22b], Proposition 3.8)**.** *If π is p'-finitely presented, then for any continuous finite quotient $\varphi : \pi \twoheadrightarrow G$, there is a $\mathbb{Q}$-vector space V_φ and a representation $\rho_\varphi : G \to GL(V_\varphi)$ such that for every $\ell \neq p$, the relation $\rho_{\varphi,\ell} = \rho_\varphi \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ holds.*

Proof. Indeed, $V_\varphi = \Gamma_\varphi^{ab}$ and $\rho_\varphi : G \to GL(\Gamma_\varphi^{ab} \otimes \mathbb{Q})$.

□

Definition 5.8. *We say that for $\ell \neq p$, $\rho_{\varphi,\ell}$ is Schur rational.*

Remark 5.9. In fact, as we see in the proof, $(V_\varphi, \rho_\varphi)$ even has an integral structure, so is *Schur integral*, but our obstruction shall disregard this integrality property.

Theorem 5.10. *The Schur rationality is an obstruction for a smooth projective variety defined over an algebraically closed field of characteristic $p > 0$ to lift to characteristic* 0.

So we have to exhibit an example of a smooth projective variety X defined over an algebraically closed field k of characteristic $p > 0$ and a quotient $\varphi : \pi \twoheadrightarrow G$ such that $\rho_{\varphi,\ell}$ is not Schur rational.

5.6. **The Roquette curve, combined with Serre's construction.** The Roquette curve is the smooth projective curve C over $\mathbb{F}_p,\ p \geq 3$ which is the normal compactification of the affine curve with equation

$$y^2 = x^p - x.$$

It is defined in [Roq70], has genus $g = (p-1)/2$, is supersingular, and is the only curve for $p \geq 5$, (so $g \geq 2$ and ρ_ℓ is faithful), with the property that the cardinality of its group of automorphisms G is larger than the Hurwicz bound $84(g-1)$. Precisely it is $2p(p^2-1)$ and all automorphisms are defined over $\mathbb{F}_{p^2}$. The equation of C presents it as an Artin-Schreier cover of $\mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_p)$. This realizes $\mathbb{Z}/p =: N$ as a subgroup of G, which thus is a p-Sylow which is in fact normal. It is not difficult to compute (see [ESS22b, Appendix A])) that $\rho_\ell|_N$ is non-trivial, thus ρ_ℓ is absolutely irreducible. The action of $\mathbb{Q}_\ell[G] \to \mathrm{End}(H^1(C, \mathbb{Q}_\ell))$ has values in $\mathbb{Q}_\ell[G] \to \mathrm{End}_{\mathrm{Frob}}(H^1(C, \mathbb{Q}_\ell))$, and is surjective by the absolute irreducibility. The Tate conjecture identifies it with the the action $\mathbb{Q}_\ell[G] \to \mathrm{End}^0(C) \otimes \mathbb{Q}_\ell$. The quaternion algebra $\mathrm{End}^0(C)$ is ramified at p and ∞, which prevents $\rho_{\varphi,\ell}$ to be rational, see [ESS22b, Proposition 4.6].

We now take a smooth projective variety P defined over $\mathbb{F}_p$ wich is simply connected over $\bar{\mathbb{F}}_p$ and on which G acts without fixpoints. We do Serre's construction setting $X = (C \times_{\mathbb{F}_p} P)/G$ where G acts diagonally. It yields an exact sequence $1 \to \pi_1(C) \to \pi_1(X) \xrightarrow{\varphi} G \to 1$ which can be understood as the Galois sequence for $C \times_{\mathbb{F}_p} P \to X$ or equivalently as the homotopy exact sequence for $X \to P/G$. The action of G on $H^1(C \times_{\bar{\mathbb{F}}_p} P, \mathbb{Q}_\ell) = H^1(C_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ is the same as the outer action studied above. So it is not Schur rational and X does not lift to characteristic 0.

6. Lecture 6: Quasi-Unipotent Monodromies at Infinity and Density

Abstract. By a theorem of Clemens and Landman, see [Gri70, Thm. 3.1] in complex geometry and Grothendieck [SGA7.2, XIV 1.1.10] in arithmetic geometry, geometric local (complex or ℓ-adic) local systems have quasi-unipotent monodromies at infinity. We explain in this section why this property is good for going from complex models to models over finite fields, and why in the Betti moduli space the complex local systems with unipotent monodromy at infinity are Zariski dense. So for certain problems (to be defined) one would then wish to follow Drinfeld's method [Dri01] to conclude that it is enough to check them on local systems with quasi-unipotent monodromy at infinity.

6.1. **Definitions.** For the definition we only need X to be a normal variety and $X \hookrightarrow \bar{X}$ to be a normal compactification. This defines the codimension 1 components D_i in $\bar{X} \setminus X$ and small loops T_i around there. A representation $\rho : \pi_1(X,x) \to GL_r(\mathbb{C})$ has *quasi-unipotent monodromies at infinity* if $\rho(T_i)$ is quasi-unipotent. This property does not depend on the choice of the compactification [Kas81, Thm.3.1]. See [EG21, Section 3] where the concept is used also for GL_r replaced by any linear algebraic group as well.

Set $\pi = \pi_1(X(\mathbb{C}), x(\mathbb{C}))$ for the topological fundamental group based at some complex point. The *framed character variety* $Ch(\pi, r)^{\square}$ is defined to be the affine variety defined over $\mathbb{Z}$ by the moduli functor which takes affine rings R to the set $\mathrm{Hom}(\pi, GL_r(R))$. It is a fine moduli functor and the resulting scheme is also called the *framed Betti moduli* $M_B(X,r)^{\square}$, also defined over $\mathbb{Z}$. The group GL_r acts by conjugation (gauge transformations) on $Ch(\pi,r)^{\square}$. Its categorial quotient $Ch(\pi,r) = Ch(\pi,r) /\!\!/ GL_{r,\mathbb{C}}$ defined by $\mathcal{O}(Ch(\pi,r)) = \mathcal{O}(Ch(\pi,r)^{\square})^{GL_r}$ is the *character variety*, also called the *Betti moduli space* $M_B(X,r)$. Its complex points are semi-simple local systems of rank r. The fibres of $M_B(X,r)^{\square} \to M_B(X,r)$ are the closures of GL_r-orbits. Such an orbit is closed over an irreducible complex local system.

6.2. **Why quasi-unipotent monodromies at infinity.** The first reason is that *geometric* local systems (see Lecture 1) have quasi-unipotent monodromies at infinity. We indicate Brieskorn's complex proof [Del70, p.125]: by base change we may assume that $Y \xrightarrow{g} U \hookrightarrow X$ is defined over a number field. So the eigenvalues λ_i of the residues of the Gauß-Manin connections lie in $\bar{\mathbb{Q}}$. On the other hand, the Gauß-Manin local system is defined over $\mathbb{Z}$, as this is the variation of the Betti cohomology of g, so the eigenvalues of the monodromy at infinity lie in $\bar{\mathbb{Q}}$. By [Del70, Corollaire 5.6, p.96] $\mu_i = \exp(2\pi\sqrt{-1}\lambda_i)$. We conclude by Gelfond's theorem that $\lambda_i \in \mathbb{Q}$.

A second reason is as follows. Let S be an affine scheme of finite type over $\mathbb{Z}$ with $\mathcal{O}(S) \subset \mathbb{C}$, such that $X \hookrightarrow \bar{X}$ and a given complex point $x \in X$ have a model $X_S \hookrightarrow \bar{X}_S$ as a relative good compactification and x_S as an S-point of X_S, and the orders the eigenvalues of the T_i are invertible on S. For any closed point $s \in |S|$ of residue field $\mathbb{F}_q$ of characteristic $p > 0$, with a $\bar{\mathbb{F}}_p$-point $\bar{s}$ above it, we denote by

$$sp_{\mathbb{C},\bar{s}} : \pi_1(X_{\mathbb{C}}, x_{\mathbb{C}}) \to \pi_1^t(X_{\bar{s}}, x_{\bar{s}})$$

the continuous surjective specialization homomorphism to the tame fundamental group [SGA1, Exp. XIII 2.10, Cor.2.12]. Precomposing with the profinite completion homomorphism

$$\pi_1(X(\mathbb{C}), x(\mathbb{C})) \to \pi_1(X_{\mathbb{C}}, x_{\mathbb{C}})$$

yields

$$sp^{\rm top}_{\mathbb{C},\bar{s}} : \pi_1(X(\mathbb{C}), x(\mathbb{C})) \to \pi_1^t(X_{\bar{s}}, x_{\bar{s}})$$

which is compatible with the local fundamental groups, see [Del73, Section 1.1.10]. This enables one to transpose the quasi-unipotent monodromy condition to $X_{\bar{s}}$.

Finally we understand well how the Galois group Γ of $F = \mathrm{Frac}(\mathcal{O}(S))$ acts on the image $T_i^{\text{ét}}$ of the T_i in $\pi_1(X_{\mathbb{C}}, x_{\mathbb{C}})$, see [SGA7.2, XIV.1.1.10], [EK23, Lemma 2.1].

Lemma 6.1. *For each $1 \le i \le s$ the action of $\gamma \in \Gamma$ on π maps $T_i^{\text{ét}}$ to $(T_i^{\text{ét}})^{\chi(\gamma)}$, where $\chi\colon \Gamma \to \widehat{\mathbb{Z}}^\times$ is the cyclotomic character.*

6.3. **Density Theorem.**

Theorem 6.2 ([EK23], Theorem 3.2)**.** *The set of $\rho \in Ch(\pi, r)^\square(\mathbb{C})$ with quasi-unipotent monodromy at infinity is Zariski dense in $Ch(\pi, r)^\square(\mathbb{C})$.*

Proof. Let $Q \subset Ch(\pi, r)^\square(\pi)(\mathbb{C})$ be the Zariski closure of the set of quasi-unipotent representations. Assume $Q \ne Ch(\pi, r)^\square(\mathbb{C})$.

For each local monodromy at infinity $T_i \subset \pi$, choose $g_i \in T_i$. We have a morphism

$$\begin{array}{ccc} & & M = \mathbb{G}_m^s \\ & & \downarrow \varphi \\ Ch(\pi, r)^\square & \xrightarrow{\psi} & N = \prod_{i=1}^s (\mathbb{A}^{r-1} \times \mathbb{G}_m) \end{array}$$

of affine schemes of finite type over $\mathbb{C}$ defined for each $i = 1, \dots, s$ by the coefficients $(\sigma_1(\rho(g_i)), \dots, \sigma_r(\rho(g_i))) \in N$ of the characteristic polynomials

$$\det(T \cdot \mathbb{I}_r - \rho(g_i)) = T^r - \sigma_1(\rho(g_i))T^{r-1} + \dots + (-1)^r \sigma_r(\rho(g_i))$$

of a representation $\rho\colon \pi \to GL_r(\mathbb{C})$. (The morphism ψ factors through $Ch(\pi, r)$, but we do not use $Ch(\pi, r)$ in this lecture, even if we wrote its construction above to clarify). There is a scheme B of finite type over $\mathbb{Z}$ with factorization $\mathbb{Z} \to \mathcal{O}(B) \to \mathbb{C}$ over which the diagram (ψ, φ) and the inclusion $Q \hookrightarrow Ch(\pi, r)^\square$ are defined. We write $Q_B \hookrightarrow Ch(\pi, r)^\square_B$ and

$$\begin{array}{ccc} & & M_B = \mathbb{G}_{m,B}^s \\ & & \downarrow \varphi_B \\ Ch(\pi, r)^\square_B & \xrightarrow{\psi_B} & N_B = \prod_{i=1}^s (\mathbb{A}_B^{r-1} \times \mathbb{G}_{m,B}) \end{array}$$

Using the section $\Gamma \to \pi_1(X_F, x_{\mathbb{C}})$ given by x_F, the Galois group Γ acts by conjugacy on $\pi_1(X_{\mathbb{C}}, x_{\mathbb{C}})$, thus on the set of closed points $|Ch(\pi, r)^\square|$. Since a closed point $z \in |Ch(\pi, r)^\square|$ has finite monodromy, its stabilizer $\Gamma_z \subset \Gamma$ is an open subgroup. It acts on the completion $(\widehat{Ch(\pi, r)^\square})_z$ at z. Said differently, the action $\Gamma \to \mathrm{Aut}(|Ch(\pi, r)^\square|)$ is continuous. Furthermore, Lemma 6.1 enables one to extend the action of Γ on the diagram (ψ_B, φ_B) in a compatible way with the action on $|Ch(\pi, r)^\square|$.

Set T_B to be the reduced Zariski closure of $\mathrm{Im}(\psi_B)$ and $S_B = \varphi_B^{-1}(T_B)$. As $Ch(\pi, r)^\square(\mathbb{C}) \setminus Q \neq \emptyset$, $Ch(\pi, r)^\square_B \setminus Q_B$ dominates B. So in particular, T_B and thus S_B dominate B as well. By generic smoothness, the smooth locus S_B^{sm} over B dominates B. By generic flatness for ψ_B restricted to $Ch(\pi, r)^\square_B \setminus Q_B$, its image meets $\varphi_B(S_B^{\mathrm{sm}})$ (recall φ_B is finite). So there is a closed point $z \in |Ch(\pi, r)^\square_B \setminus Q_B|$ such that

- ψ is flat at z,
- $y = \psi_B(z) \in \varphi_B(S_B^{\mathrm{sm}})$.

We also fix a closed point $x \in S_B^{\mathrm{sm}} \cap \varphi_B^{-1}(y)$. Let Γ' be the intersection of stabilizers $\Gamma_x \cap \Gamma_z$, which is thus open in Γ, and let $b \in B$ be the image of the points x, y, z. So the closed subscheme $(\widehat{S_B})_x \hookrightarrow (\widehat{M_B})_x$ is Γ'-stable. We abuse notation and set $\Gamma' = \Gamma$.

We now take $s \in |S|$ a non-ramified closed point with residue field $\mathbb{F}_q$ of characteristic $p > 0$ different from the residual characteristic ℓ of x. Then the Frobenius $Frob_s$ lies in Γ and by Lemma 6.1 it acts on M_B and $(\widehat{M_B})_x$ by multiplication with q. By (as we are over $(\widehat{B})_b$, a variant of) de Jong's conjecture [dJ01, Conjecture 1.1] solved by Gaitsgory [Gai07], there are $Frob_s$-invariant points in $(\widehat{S_B})_x$, flat over $(\widehat{B})_b$, so there are points for which the coordinates in the group scheme M_B are $(q-1)$ roots of 1 which are flat over $(\widehat{B})_b$. By flatness of ψ_B restricted to $(\widehat{Ch(\pi, r)^\square_B \setminus Q_B})_z$, there is a point in $(\widehat{Ch(\pi, r)^\square_B \setminus Q_B})_z$, flat over $(\widehat{B})_b$. This yields a complex topological local system outside of Q with eigenvalues of the T_i being $(q-1)$ roots of 1, a contradiction. See [EK23] for more details.

□

6.4. **Remarks.** 1) When I lectured on zoom in December 2020, at the pic of corona, on our density theorem 6.2 with Moritz Kerz, Ben Bakker and Yohan Brunebarbe listened to the talk. They later on explained to us that they had a Hodge theoretical proof of the result. This would be nice, as it would add one stone to the line of similarities between complex and arithmetic methods.

2) It would also be nice to single out subloci of the one consisting of complex local systems with quasi-unipotent monodromies at infinity for which density is preserved. Of course, we should remark that if X was projective to start with, our theorem 6.2 would be void. Still if we think of varieties defined over number fields and Belyi's theorem, $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ is the key scheme for many problems and so clearly the monodromies at infinity span the whole fundamental group. In additon if we think of analogies with the number theory case, where we look at Galois representations, we have ramification at bad primes. Unlike what was hoped for in [EK23], the sublocus of arithmetic points is not dense, thanks to the work of Landesman-Litt [LL22a], [LL22b], see also [Lam22], while on the Mazur-Chenevier deformation formal schemes for smooth quasi-projective varieties over finite fields, it would truly be bad if the sublocus of arithmetic points was not dense. In rank one it is, see [EK21, Thm. 1.3]. Over $\mathbb{C}$, a notion of *weakly arithmetic complex local systems* is defined in [dJE22, Section 3]. It is shown that the locus of these is dense in the Betti moduli space.

7. Lecture 7: Companions and Integrality of Cohomologically Rigid Local Systems

Abstract. The notion of companions has been defined by Deligne in [Del80, Conjecture 1.2.10] who predicted its existence. We report on the (ℓ-adic version) of it (L. Lafforgue [Laf02, Théorème VII.6] in dimension 1, Drinfeld [Dri12, Theorem 1.1] in higher dimension in the smooth case), explain how we used Drinfeld's theorem in the proof of Simpson's integrality conjecture for cohomologically rigid complex local systems in [EG18, Theorem 1.1] and also why we proceeded like this.

7.1. **Motivation on the complex side.** Given a field automorphism τ of $\mathbb{C}$, we can postcompose the underlying monodromy representation of a complex local system $\mathbb{L}_\mathbb{C}$ by τ to define a *conjugate* complex local system $\mathbb{L}_\mathbb{C}^\tau$. Given a field automorphism σ of $\bar{\mathbb{Q}}_\ell$, which then can only be continuous if this is the identity, or more generally given a field isomorphism σ between $\bar{\mathbb{Q}}_\ell$ and $\bar{\mathbb{Q}}_{\ell'}$, the postcomposition of a *continuous* monodromy representation is not longer continuous, so *we can not define a conjugate* $\mathbb{L}_\ell^\sigma$ of an ℓ-adic local system.

Returning to the complex side, as a consequence of $\pi_1(X(\mathbb{C}), x(\mathbb{C}))$ being finitely generated (we do not even need the finite presentation here), there are finitely many elements $(\gamma_1, \ldots, \gamma_s)$ of $\pi_1(X(\mathbb{C}), x(\mathbb{C}))$ such that the characteristic polynomial map

$$M_B(X,r)^\square_\mathbb{C} \xrightarrow{\psi^\square} N_\mathbb{C} = \prod_{i=1}^s (\mathbb{A}^{r-1} \times \mathbb{G}_m)_\mathbb{C}$$
$$\rho \mapsto (\det(T-\rho(\gamma_1)), \ldots, \det(T-\rho(\gamma_s))),$$

defined in Section 6, which factors through

$$M_B(X,r)_\mathbb{C} \xrightarrow{\psi} N_\mathbb{C},$$

has the property that ψ is a *closed embedding.* (The $\psi^\square$ here was denoted by ψ there). The reason is that a semi-simple representation is determined uniquely up to conjugation by the characteristic polynomial function on all $\gamma \in \pi_1(X(\mathbb{C}), x(\mathbb{C}))$, so by finite generation of $\pi_1(X(\mathbb{C}), x(\mathbb{C}))$ on finitely many suitably chosen of them. So denoting by $\tau \circ \det(T-\rho(\gamma))$ by $\det(T-\rho(\gamma))^\tau$ to unify the notation we can summarize the discussion as follows:

An automorphism τ of $\mathbb{C}$ yields a commutative diagram

$$(\star) \quad \begin{array}{ccccc} M_B^{irr}(X,r)(\mathbb{C}) & \xrightarrow{\text{incl.}} & M_B(X,r)(\mathbb{C}) & \xrightarrow{\psi} & N(\mathbb{C}) \\ \downarrow{\scriptstyle \mathbb{L}_\mathbb{C}\mapsto\mathbb{L}_\mathbb{C}^\tau} & & \downarrow{\scriptstyle \mathbb{L}_\mathbb{C}\mapsto\mathbb{L}_\mathbb{C}^\tau} & & \downarrow{\scriptstyle (-)^\tau} \\ M_B^{irr}(X,r)(\mathbb{C}) & \xrightarrow{\text{incl.}} & M_B(X,r)(\mathbb{C}) & \xrightarrow[\psi]{} & N(\mathbb{C}) \end{array}$$

The upper script ^{irr} means the irreducible locus.

7.2. **Analogy over a finite field.** Let us now assume that X is smooth quasi-projective over a finite field $\mathbb{F}_q$. Let us denote by $M_\ell^{irr}(X, r, \mathcal{L})$ the *set* of all rank r ℓ-adic local systems defined over $X_{\bar{\mathbb{F}}_p}$ which are

- irreducible over $\bar{\mathbb{Q}}_\ell$;

- arithmetic;
- of torsion determinant $\mathcal{L}$ of order prime to p.

This is a replacement for $M_B^{irr}(X,r)(\mathbb{C})$ on the left side of $(\star)$. Given $\mathbb{L}_\ell \in M_\ell^{irr}(X,r,\mathcal{L})$, its arithmetic descent over $X_{\mathbb{F}_{q'}}$ for some finite extension $\mathbb{F}_q \subset \mathbb{F}_{q'}$ is, by Class Field Theory [Del80, Théorème 1.3.2], unique modulo twist by a character of $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_{q'})$ A replacement for $M_B(X,r)(\mathbb{C})$ should take into account this on each factor, we do not do it and restrict ourselves to $M_B^{irr}(X,r)(\mathbb{C})$.

We give ourselves an abstract field isomorphism $\sigma : \bar{\mathbb{Q}}_\ell \xrightarrow{\cong} \bar{\mathbb{Q}}_{\ell'}$. The only 'continuous' information it contains is that it sends a number field $K \subset \bar{\mathbb{Q}}_\ell$ to another one $K^\sigma \subset \bar{\mathbb{Q}}_{\ell'}$. So the right vertical arrow $(-)^\sigma$ makes sense on a $\gamma \in \pi_1(X_{\mathbb{F}_{q'}}, x_{\bar{\mathbb{F}}_p})$, doe $\mathbb{F}_q \subset \mathbb{F}_{q'} (\subset \bar{\mathbb{F}}_p)$ a finite extension, which has the property that the characteristic polynomial of γ has values in a number field. Furthermore, we wish the set of such γ to be able to recognize completely $M_\ell^{irr}(X,r,\mathcal{L})$ as ψ does over $\mathbb{C}$. The set of *conjugacy classes of the Frobenii at all closed points* $|X|$ *of* X has those two properties by [Laf02, Théorème 7.6] and Čebotarev's theorem. So we set $N^\infty = \prod_{|X|}(\mathbb{A}^{r-1} \times \mathbb{G}_m)$ and ψ^∞ for the characteristic polynomial map on those Frobenii of closed points. Then *Deligne's companion conjecture may be visualized on the diagram*

$$(\star\star) \qquad \begin{array}{ccc} M_\ell^{irr}(X,r,\mathcal{L}) & \xrightarrow{\psi^\infty} & N^\infty(\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_\ell) \\ {\scriptstyle ?\exists\ \mathbb{L}_\ell \mapsto \mathbb{L}_\ell^\sigma} \Big\downarrow & & \Big\downarrow {\scriptstyle (-)^\sigma} \\ M_{\ell'}^{irr}(X,r,\mathcal{L}) & \xrightarrow[\psi^\infty]{} & N^\infty(\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_{\ell'}) \end{array}$$

See the ℓ-adic part in [Del80, Conjecture 1.2.10] for Deligne's precise formulation.

7.3. **Geometricity.** The formulation $(\star\star)$ hides certain properties. For the fact that the eigenvalues of Frobenius on our irreducible ℓ-adic sheaves are algebraic numbers, we quote *loc. cit.* which postdates Deligne's conjecture. This is because if X is a smooth curve over $\mathbb{F}_q$, L. Lafforgue proves as a corollary of the Langlands program that an $\mathbb{L}_\ell$ as before is even *pure*. In fact Deligne in [Del80] and Deligne-Beilinson-Bernstein-Gabber in [BBDG82] prove that geometric local systems are pure or mixed, and Lafforgue proves geometricity on curves. Up to now, *we do not know geometricity* of arithmetic local systems over X smooth quasi-projective of higher dimension over $\mathbb{F}_q$. But we know thanks to Drinfeld ([Dri12]) how to extend the existence of companions from curves to such X. Let us remark that even on curves, *we can not* lift the geometricity property from $\bar{\mathbb{F}}_p$ to $\mathbb{C}$ given the examples in [LL22a], [LL22b], see also [Lam22], already mentioned.

Nonetheless, in view of Simpson's geometricity conjecture [Sim92, p.9], the fact that, in absence of geometricity, Drinfeld shows how to prove some integrality property for ℓ-adic local systems in the sense that once we have an irreducible ℓ-adic local system, we have all ℓ'-adic local systems with the 'same' characteristic polynomials at closed points, was the philosophical reason behind our proof in [EG18, Theorem 1.1].

7.4. **On Drinfeld's proof.** We sketch some aspects of it and refer to the original article [Dri12]. First let us comment that Drinfeld's proof unfortunately does not solve the conjecture on X normal over $\mathbb{F}_q$ as originally formulated by Deligne, but on X smooth. One problem is that the restriction of simple artihmetic local systems on a non-normal subvariety is not understood, in particular whether/ when it is semi-simple, not even for complex varieties, see on this [Moc07, Theorem 25.30], where there is a typo, the proof truly does assume that the domain and the target are smooth. See also [dJE22, Section 7.3] for an arithmetic proof of the semi-simplicity statement [Moc07, Theorem 25.30] for smooth varieties, and see [D'Ad20, Theorem 1.2.2] for further studies in the non-normal case.

We fix $\mathbb{L}_{\ell,X}$, an isomorphism $\sigma : \bar{\mathbb{Q}}_\ell \cong \bar{\mathbb{Q}}_{\ell'}$. By Deligne's Theorem [Del12, Théorème 3.1], $\psi^\infty(\mathbb{L}_{\ell,X}) \in N^\infty(\mathcal{O})$ where $\mathcal{O}$ is a number ring in $\bar{\mathbb{Q}}_\ell$. This fixes the number ring $\sigma(\mathcal{O}) \subset \bar{\mathbb{Q}}_{\ell'}$ and its extension to an ℓ'-adic ring $A \subset \bar{\mathbb{Q}}_{\ell'}$. We denote by $\mathfrak{m} \subset A$ the maximal ideal. The starting point of the proof is the datum (via [Laf02], *loc. cit.*) for any smooth curve $C/\mathbb{F}_q$, any morphism $f_C : C \to X$ which is generically an embedding, of an irreducible ℓ'-adic local system $\mathbb{L}_{A,C}$ defined over A on C with the property that on intersections points $C \times_X C'$ the pull-backs from C and C' are isomorphic. The goal is to construct an A-adic local system $\mathbb{L}_{A,X}$ on X with restrictions $\mathbb{L}_{A,C}$ on those curves. If we can do this, by Čebotarev's density theorem, $\mathbb{L}_{A,X}$ is unique. We assume below that $\mathbb{L}_{\ell,X}$ is tame. This is not a big restriction: the residual representation of an ℓ-adic local system defines a finite étale cover which tamifies it. The companion construction should preserve tameness. At the end one has to descend the companion from Y to X.

The first step ([Dri12, Proposition 2.12]) is to show that there is a dense open $U \subset X$ and a closed normal subgroup $H \subset \pi_1(U_{\mathbb{F}_q}, x_{\bar{\mathbb{F}}_p})$ such that $\pi_1(U_{\mathbb{F}_q}, x_{\bar{\mathbb{F}}_p})/H$ has a chance to be an upper bound for the monodromy group of the wished companion $\mathbb{L}^\sigma_{\ell,X}$. The properties singled out are

i) H is an extension of a finite group by a pro-ℓ-group;
ii) for any $n \in \mathbb{N}_{>0}$, there is an open subgroup $H \subset V_n \subset \pi_1(U_{\mathbb{F}_q}, x_{\bar{\mathbb{F}}_p})$ such that all $\psi^\infty(L_{\ell,C})^\sigma$ are trivial mod $\mathfrak{m}^n$ while pulled-back to the finite étale cover defined by V_n.

To this aim, we assume that U is an Artin neighborhood $U \to S$ with a section $S \to U$. The datum of $L^\sigma_{\ell,S}$ yields the tower $S_n \to S$ by taking the Galois étale cover associated to the residual representation mod $\mathfrak{m}^n$ of $L^\sigma_{\ell,S}$. Note here, as we have fixed A, this residual representation is well defined. On the other hand there a finite type parameter space T_n finite over S, parametrizing a universal $GL_r(A/\mathfrak{m}^n)$-torsor $\mathcal{T}_n$ on $T_n \times_S U$. Its Weil restriction down to X is denoted by Y_n. Then the tower defining H is built from the $Y_n \times_S S_n$. (Beware we might have to take one component when it splits).

The main point is then to prove that the preceding property suffices to glue the $\mathbb{L}_{\ell,C}$. The proof follows an idea of Kerz, see [Dri12, Section 4], Drinfeld himself initially had a more difficult argument. First we do a Lefschetz type argument to find a curve $\varphi : C \to U$ so $\varphi_* : \pi_1(C) \to \pi_1(U)/H$ is surjective. This is possible as the surjectivity is recognized on the finite quotient which is the extension of the residual quotient of $\pi_1(U)/H$ (if $\ell \geq 3$, if $\ell = 2$ we go mod $\mathfrak{m}^2$ here) by the maximal $\mathbb{F}_\ell$-vector space quotient, see [EK11, Lemma 8.2]. Essentially by construction the monodromy group of $\mathbb{L}^\sigma_{\ell,C}$ is a quotient of $\pi_1(C)/\mathrm{Ker}(\varphi_*) = \pi_1(U)/H$, and this *defines* a representation $\pi_1(U) \to GL_1(A)$ *which a priori depends on* C. Denote by $\mathbb{L}_{\ell',U}$ the underlying local system. By definition its restriction to C is equal to $\mathbb{L}^{\sigma}_{\ell,C}$. To compute the restriction of $\mathbb{L}_{\ell',U}$ to all closed points, we take other curves $\varphi_{C'} : C' \to U$ with the same property so they fill in all the closed points. If we make sure that C and C' meet in sufficiently many points, then $(\mathbb{L}_{\ell',U}|_C = \mathbb{L}^{\sigma}_{\ell,C}, \mathbb{L}_{\ell',U}|_{C'}, \mathbb{L}^{\sigma}_{\ell,C'})$ are recognized by their value on the intersection points by [Fal83, Satz 5].

To go from U to X is easy. This finishes the proof.

7.5. **Cohomologically rigid local systems are integral, see** [EG18]**, Theorem 1.1.** We sketch the proof with Michel Groechenig in *loc.cit.*. See 7.3 for the heuristic. Let X be a smooth quasi-projective variety over $\mathbb{C}$, fix a torsion character $\mathcal{L}$ and quasi-unipotent conjugacy classes $T_i \subset GL_r(\mathbb{C})$. Let $\mathbb{L}_{\mathbb{C}}$ be a point of $M_B(X, r, \mathcal{L}, T_i)(\mathbb{C})$ which is isolated. It is defined by a representation $\rho_{\mathbb{C}} : \pi_1(X(\mathbb{C}), x(\mathbb{C})) \to GL_r(A)$ where A is a ring of finite type over $\mathbb{Z}$. Let $a : A \to \bar{\mathbb{Z}}_\ell$ be a $\bar{\mathbb{Z}}_\ell$-point of A. It defines an ℓ-adic local system $\mathbb{L}_{\ell,\mathbb{C}}$ by the factorization

$$\begin{array}{ccc}
\pi_1(X(\mathbb{C}), x(\bar{\mathbb{C}})) & \longrightarrow & GL_r(A) \\
{\scriptstyle =}\downarrow & & \downarrow{\scriptstyle a} \\
\pi_1(X(\mathbb{C}), x(\bar{\mathbb{C}})) & \longrightarrow & GL_r(\bar{\mathbb{Z}}_\ell) \\
{\scriptstyle \text{profinite compl.}}\downarrow & \nearrow{\scriptstyle \mathbb{L}_{\ell,\mathbb{C}}} & \\
\pi_1(X_{\mathbb{C}}, x_{\mathbb{C}}) & &
\end{array}$$

Using the notation of 6.2, for a closed point $s \in |S|$ of characteristic prime to the order of the residual representation of $\mathbb{L}_{\ell,\mathbb{C}}$, the order of $\mathcal{L}$ and of the eigenvalues of the monodromies at infinity, and integral for the (finitely many) cohomologically rigid local systems, we have

1) $\mathbb{L}_{\ell,\mathbb{C}}$ descends to $\mathbb{L}_{\ell,s'}$ ([Sim92, Theorem 4]) for some $s' \to s$ finite below $\bar{s}$ (one has to complete the argument to take care of the conditions at infinity) keeping $\mathcal{L}$ and T_i ([Del73, Section 1.1.10]);
2) The companions $\mathbb{L}^{\sigma}_{\ell,s'}$ still have determinant $\mathcal{L}$ and the semi-simplification of the monodromies at infinity is the one of the T_i's ([Del72, Théorème 9.8]);
3) $IH(X, \mathcal{E}nd^0(\mathbb{L}_{\ell,\bar{s}})) = 0 = IH(X, \mathcal{E}nd^0(\mathbb{L}^{\sigma}_{\ell,\bar{s}}))$ by the weight argument [EG18, Proof of Theorem 1.1];
4) So $\mathbb{L}_{\ell,\mathbb{C}} \mapsto sp^{-1}_{\mathbb{C},\bar{s}}(\mathbb{L}^{\sigma}_{\ell,\bar{s}}))$ is a bijection on the set of cohomologically rigid ℓ-adic local systems to those of cohomologically rigid ℓ'-adic local systems.

So there can not be a non-integral place ℓ'. This finishes the proof.

8. Lecture 8: Rigid Local Systems and F-Isocrystals

Abstract. Rigid connections over $X_{\mathbb{C}}$ smooth projective $\mathbb{C}$, while restricted to the formal p-completion $\hat{X}_W$ a non-ramified projective p-adic model X_W of $X_{\mathbb{C}}$, yield F-isocrystals. This is proved in [EG20, Theorem 1.6], using the theory of Higgs-de Rham flows on the mod p reduction of X_W. We give here a p-adic proof of this theorem, obtained with Johan de Jong, which relies on the fact that for $p \geq 3$, the Frobenius pull-back of a connection on $\hat{X}_W$ is well defined, whether the p-curvature of the mod p reduction is nilpotent or not. However this proof so far does not give the crystallinity property proved in [EG20, Theorem 5.4] which can be expressed by saying that the p-adic local systems obtained on the p-adic variety $X_{\overline{\mathrm{Frac}(W)}}$ for p large descend to a crystalline local system over $X_{\mathrm{Frac}(W)}$. The version under the strong cohomological rigidity of the same theorem is worked out in [EG21]. We defer this discussion to Lecture 9.

8.1. **Crystalline site, crystals and isocrystals.** We refer to [EG20, Section 2.6] for the details of the definitions, and for the appropriate references. Given X smooth over a perfect characteristic $p > 0$ field k, we denote by $W = W(k)$ the ring of Witt vectors and by $W_n = W_n(k) = W/p^n$ its n-th truncation. We define the site (X/W_n) with objects the PD-thickenings (U, T, δ) over W_n, where $U \hookrightarrow T$ is a closed embedding defined by an ideal I on which δ defines a PD-structure over (W_n, pW_n). So for all $x_i,\ i = 1, \ldots, s$ over (W_n, pW_n) one has $mx_1^{[n_1]} \cdots x_s^{[n_s]} = 0$ for some powers $n_i, m \in \mathbb{N}$ (see [Ber74, I 1.3.1 ii), p.56]).

This yields functors $(X/W_n) \hookrightarrow (X/W_{n+1})$ and the *crystalline site* (X/W) as the colimit over n of the (X/W_n). The Homs are the obvious ones respecting the whole structure. A *crystal* $\mathcal{F}/W$ in coherent modules on X/W is the datum for all (U, T, δ) of a coherent sheaf $\mathcal{F}_T$ so the transition functions are isomorphisms. The category of *isocrystals* has the same objects and the Hom sets, which are abelian groups, are tensored over $\mathbb{Z}$ by $\mathbb{Q}$. For us the main concrete description is [EG20, Theorem 2.19, Corollary 2.20] according to which, *if* $\hat{X}_W$ *is an essentially smooth formal scheme over* W *lifting* X, a crystal is

i) a flat formal connection $(\hat{E}_W, \hat{\nabla}_W)$ on $\hat{X}_W$;
ii) such that $(\hat{E}_W, \hat{\nabla}_W) \otimes_W k$ is filtered by subconnections so the associated graded is spanned by a full set of algebraic solutions (that is its p-curvature is nilpotent, see Section 2).

By the classical theorem, i) implies that that $\hat{E}_K := \hat{E}_W \otimes_W K$ where $K = \mathrm{Frac}(W)$ is locally free. This is not the case for $\hat{E}_W$ even with the condition ii), see [ES15, 1.3]. It is also not known whether one can always find a locally free lattice in $\hat{E}_K$ which is stabilized by $\hat{\nabla}_K$.

8.2. **Nilpotent crystalline site, crystals and isocrystals.** We refer to [ES18, Section 1-2] and references in there. The *nilpotent crystalline site* has objects (U, T, δ) as the crystalline but in addition I itself is locally nilpotent. So for all $x_i,\ i = 1, \ldots, s$ over (W_n, pW_n) one has $x_1^{[n_1]} \cdots x_s^{[n_s]} = 0$ for some powers $n_i \in \mathbb{N}$. This yields a continuous map $(X/W)_{Ncrys} \to (X/W)_{crys}$ of topoi. Crystals and isocrystals are defined similarly but given $\hat{X}_W$ as above, then a crystal on the nilpotent crystalline site is

i) a flat formal connection $(\hat{E}_W, \hat{\nabla}_W)$ on $\hat{X}_W$.

(See [Ber74, I 3.1. 1 ii) p.56, III 1.3.1 p. 187, IV 1.6.6 p. 248].) There is *no* condition on the mod p reduction.

8.3. **The Frobenius action on the set of crystals on the nilpotent crystalline site.** As $\hat{X}_W$ is formally smooth over W, the Frobenius of X locally in the Zariski topology lifts to $\hat{X}_W$. The following proposition is due to J. de Jong. As far as I could see, it is not documented in the literature. However, the proof is exactly as the one in [SP, tag/07JH] where unfortunately there the condition ii) is assumed to be fulfilled, but .... not really really used.

Proposition 8.1. *Let $(\hat{E}_W,\ \hat{\nabla}_W)$ be a formal flat connection on $\hat{X}_W$. Assume $p \geq 3$. There is a unique formal flat connection denoted by $F^*(\hat{E}_W, \hat{\nabla}_W)$ on $\hat{X}_W$ with the property that if on an open $\hat{U}_W \subset \hat{X}_W$, the Frobenius F of U lifts to F_W, then*

$$F^*(\hat{E}_W, \hat{\nabla}_W)|_{\hat{U}_W} = F_W^*(\hat{E}_W,\ \hat{\nabla}_W).$$

Proof. (*loc. cit.*) The goal is to show that if F_W, G_W are two lifts to $\hat{U}_W$ of F on U, there is a commutative diagram

$$\begin{array}{ccc} F_W^*\hat{E}_W & \xrightarrow{\quad\psi\quad} & G_W^*\hat{E}_W \\ {\scriptstyle\hat{\nabla}}\downarrow & & \downarrow{\scriptstyle\hat{\nabla}} \\ \Omega^1_{\hat{U}_W}\hat{\otimes}_{\mathcal{O}_{\hat{U}_W}} F_W^*\hat{E}_W & \xrightarrow{1\otimes\psi} & \Omega^1_{\hat{U}_W}\hat{\otimes}_{\mathcal{O}_{\hat{U}_W}} G_W^*\hat{E}_W \end{array}$$

which is canonical, so it fulfils the cocycle condition. Here $\hat{\otimes}$ denotes the completed tensor product. So we may assume that $\hat{U}_W$ is étale, finite onto its image, an open of $\hat{\mathbb{A}}^d_W$, where $d = \dim X$, so as to have coordinates $(x_1, \ldots, x_d)$. This defines the derivation $\partial_i \in T_{\hat{U}_W}$ dual to dx_i, acting on $\hat{E}_W|_{\hat{U}_W}$, and the action of the differential operator $\partial_{\underline{k}} = \partial_1^{k_1} \cdots \partial_d^{k_d}$ on $\hat{E}_W|_{\hat{U}_W}$ for all multi-indices $\underline{k} = (k_1, \ldots, k_d)$. We write $F_W^*\hat{E}_W = F_W^{-1}\hat{E}_W \otimes_{F^{-1}\mathcal{O}_{\hat{U}_W}} \mathcal{O}_{\hat{U}_W}$. Then we define

$$\psi(e \otimes 1) = \sum_{\underline{k}} \partial_{\underline{k}}(e) \otimes_{G_W^{-1}\mathcal{O}_{\hat{U}_W}} \prod_{i=1}^{d} (F_W(x_i) - G_W(x_i))^{k_i}/(k_i)!.$$

By definition $(F_W(x_i) - G_W(x_i)) \in p\mathcal{O}(\hat{U}_W)$ so its p-adic valuation is ≥ 1. On the other hand, the p-adic exponential function converges on elements of p-adic valuation $> 1/(p-1)$. So ψ, thus F^*, is defined for $p \geq 3$. This finishes the proof. □

Remark 8.2. If the connection was defined on $X_{\mathcal{O}}$ where $\mathcal{O} \supset W$ is the ring of integers of a p-adic field of degree ≥ 2 over Frac(W), the same formula with the obvious change of notation would yield $(F_{\mathcal{O}}(x_i) - G_{\mathcal{O}}(x_i)) \in \pi\mathcal{O}(U_W)$ where π is a uniformizer of $\mathcal{O}$. Then the convergence would be violated and we could not write the diagram above. In this case we need condition ii) to define the Frobenius pullback of $(E_{\mathcal{O}}, \nabla_{\mathcal{O}})$ in this way when the index of ramification of $\mathcal{O}$ lies in $[2, p-1($.

8.4. **The Frobenius induces an isomorphism on cohomology in characteristic 0.** We set $K = \mathrm{Frac}(W)$. Notation as before. In addition we set $(\hat{E}_W, \hat{\nabla}_W) \otimes_W K = (\hat{E}_K, \hat{\nabla}_K)$ and abuse notation $F^*(\hat{E}_K, \hat{\nabla}_K) := F^*(\hat{E}_W, \hat{\nabla}_W) \otimes_W K$.

Proposition 8.3. *Let $(\hat{E}_W, \hat{\nabla}_W)$ be a flat formal connection on $\hat{X}_W$. Then for all cohomological degrees i,*

$$F^* : H^i(\hat{X}_W, (\hat{E}_W, \hat{\nabla}_W)) \otimes_W K \to H^i(\hat{X}_W, F^*(\hat{E}_W, \hat{\nabla}_W)) \otimes_W K$$

is an isomorphism and is compatible with cup-products.

Proof. By the Mayer-Vietoris spectral sequence it is enough to prove the first part of the proposition on an open $\hat{U}_W$ lifting U on which the Frobenius lifts to $\hat{F}_W$ so as to have an étale map $h : \hat{U}_W \to \hat{\mathbb{A}}^d_W$ finite onto its image, as in Section 8.3. On $\hat{\mathbb{A}}^d_W$ we choose the lift F_W which is defined by $F_W^*(x_i) = x_i^p$. We consider the cartesian diagram

$$\begin{array}{ccc} \hat{U}_W \times_{\hat{\mathbb{A}}^d, F_W} \hat{\mathbb{A}}^d_W & \xrightarrow{F'_W} & \hat{U}_W \\ \downarrow{\scriptstyle h\times 1} & & \downarrow{\scriptstyle h} \\ \hat{\mathbb{A}}^d_W & \xrightarrow{F_W} & \hat{\mathbb{A}}^d_W \end{array}$$

defining F'_W. We also have an isomorphism of formal schemes

$$\tau : \hat{U}_W \xrightarrow{\cong} \hat{U}_W \times_{\hat{\mathbb{A}}^d, F_W} \hat{\mathbb{A}}^d_W$$

over W, defining the lift of Frobenius $G_W = F'_W \circ \tau : \hat{U}_W \to \hat{U}_W$ and $g = (h \times 1) \circ \tau : \hat{U}_W \to \hat{\mathbb{A}}^d$. By base change, it holds

$$H^0(\hat{U}_W, G_K^*(\hat{E}_W, \hat{\nabla}_W)) = H^0(\hat{V}'_W, g_* G_K^*(\hat{E}_W, \hat{\nabla}_W)) = \\ H^0(\hat{V}'_W, F_W^* h_*(\hat{E}_W, \hat{\nabla}_W))$$

so we may assume

$$(G_W : \hat{U}_W \to \hat{U}_W) = (F_W : \hat{V}'_W \to \hat{V}_W) \text{ where } (\hat{V}_W = h(\hat{U}_W), \hat{V}'_W = F_W^{-1}\hat{V}_W).$$

Then

$$H^0(\hat{V}'_W, F_W^*(\hat{E}_W, \hat{\nabla}_W)) \otimes_W K = \oplus_{\chi_i, 0 \le n_{ij_i} \le p-1} H^0(\hat{V}_K, (\hat{E}_K, \hat{\nabla}_K) \otimes_{i=1}^d \chi^{n_{ij_i}})$$

where the χ_i are the characters defining the Kummer cover $x_i \mapsto x_i^p$. The only tensor product $\otimes_{i=1}^d \chi^{n_{ij_i}}$ which extends as a formal flat connection on $\hat{V}_W$ is the trivial one, that is $n_{ij_i} = 0$ for all $i = 1, \ldots, d$. Thus

$$H^i(\hat{V}'_W, F_W^*(\hat{E}_W, \hat{\nabla}_W)) \otimes_W K = H^i(\hat{V}_W, (\hat{E}_W, \hat{\nabla}_W)) \otimes_W K.$$

The second part of the proposition is also checked locally on $\hat{U}_W$ with Frobenius lift F_W, where it is trivial. This finishes the proof.

□

8.5. **Proof of** [EG20]**, Theorem 1.6.** Let $X_{\mathbb{C}}$ be a smooth projective variety. Let $(E_i, \nabla_i)_{\mathbb{C}},\ i = 1, \ldots, N$ be the finitely many irreducible rank r rigid local systems with a fixed torsion determinant $\mathcal{L}$. We denote by the same letter $\mathcal{L}$ the associated rank one connection $(\mathcal{L} \otimes_{\mathbb{C}} \mathcal{O}_{X_{\rm an}}, 1 \otimes d)$. Let $M_{dR}(X, r, \mathcal{L})$ be the de Rham moduli space of irreducible rank r flat connections on $X_{\mathbb{C}}$ with determinant $\mathcal{L}$. So the connections $(E_i, \nabla_i)_{\mathbb{C}}$ are the isolated complex points of $M_{dR}(X, r, \mathcal{L})$. Let S be a scheme such that

0) $S \to \mathrm{Spec}(\mathbb{Z})$ is smooth of finite type, 2 is not in the image;
1) $X_{\mathbb{C}}$ has a smooth projective model X_S;
2) $\mathcal{L}$ has a model $\mathcal{L}_S$ and $M_{dR}(X, r, \mathcal{L})$ has a flat model $M_{dR}(X, r, \mathcal{L})_S$ over S ([Lan14, Theorem 1.1]);
3) the connections $(E_i, \nabla_i)_{\mathbb{C}},\ i = 1, \ldots, N$ have a model $(E_i, \nabla_i)_S$ defining N-S-sections of $M_{dR}(X, r, \mathcal{L})_S$;
4) $M_{dR}(X, r, \mathcal{L})_{S,red}$ is étale over S at those N S-points;
5) the formal completion of $M_{dR}(X, r, \mathcal{L})_S$ at those sections is flat over S;
6) $K((E_i, \nabla_i)_S) = \mathrm{Ker}\big(H^1(X_S, \mathcal{E}nd(E_i, \nabla_i)_S) \to H^2(X_S, \mathcal{E}nd(E_i, \nabla_i)_S)\big)$ sending x to $x \cup x$ is a projective module over $S \otimes_{\mathbb{Z}} \mathbb{Q}$ satisfying base change, as well as $H^j(X_S, \mathcal{E}nd(E_i, \nabla_i)_S)$ for $j = 1, 2$ and $i = 1, \ldots, N$.

Theorem 8.4. *For any point* $\mathrm{Spec}W \to S$ *where* $W = W(k)$ *and* k *is a perfect characteristic* $p > 0$ *field, for any* $i = 1, \ldots, N$*, the* p*-completion* $(\hat{E}_i, \hat{\nabla}_i)_W$ *of* $(E_i, \nabla_i)_S|_{X_W}$ *is an isocrystal with a Frobenius structure. In particular it has nilpotent* p*-curvature.*

Proof. By [GM88, Proposition 4.4] the completion of $K((E, \nabla)_{\mathbb{C}})$ at 0 is isomorphic to the completion at the complex point $(E, \nabla)_{\mathbb{C}}$ of $M_{dR}(X, r, \mathcal{L})$. Grothendieck's comparison isomorphism

$$H^i(X_W, (E_i, \nabla_i)_S) \xrightarrow{\cong} H^i(\hat{X}_W, (\hat{E}_i, \hat{\nabla}_i)_W)$$

which implies the comparison isomorphism

$$H^i(X_W, (E_i, \nabla_i)_S)_{\mathbb{Q}} = H^i(X_K, (E_i, \nabla_i)_S) \xrightarrow{\cong} H^i(\hat{X}_W, (\hat{E}_i, \hat{\nabla}_i)_W)_{\mathbb{Q}} = H^i(\hat{X}_K, (\hat{E}_i, \hat{\nabla}_i)_K)$$

is compatible with cup-product. Here $K = \mathrm{Frac}(W)$ and $\hat{X}_K = \hat{X}_W \otimes_W K$. Proposition 8.3 then implies that F^* maps isomorphically the completion lof $M_{dR}(X, r, \mathcal{L})_S$ at $(\hat{E}_i, \hat{\nabla}_i)_K$ to the one at $F^*(\hat{E}_i, \hat{\nabla}_i)_K$. Thus the latter is 0-dimensional over K and $F^*(\hat{E}_i, \hat{\nabla}_i)_K = (\hat{E}_{\varphi(i)}, \hat{\nabla}_{\varphi(i)})$ for some $\varphi(i) \in \{1, \ldots, N\}$. Proposition 8.3 for H^0 implies then that $\varphi : \{1, \ldots, N\} \to \{1, \ldots, N\}$ is an injective map, thus is a bijection. Thus there is a natural number M, which divides $N!$, such that

$$\underbrace{\varphi \circ \ldots \circ \varphi}_{\text{M-times}} = \text{Identity}.$$

In other words

$$(F^M)^*(\hat{E}_i, \hat{\nabla}_i)_K = (\hat{E}_i, \hat{\nabla}_i)_K \ \forall i = 1, \ldots, N.$$

This finishes the first part of the proof. As for the second part, we apply [ES18, Proof of Proposition 3.1] which shows that the nilpotency of the p-curvature does not depend on the lattice chosen.

∎

Remark 8.5. As compared to the proof of [EG20, Theorem 1.6], this proof has the advantage that it is p-adic as opposed to a characteristic $p > 0$ proof, which is more natural as the result is p-adic. It also gives a larger S on which the result holds. It has the disadvantage to juggle with a cohomology which is nowhere studied in details (Proposition 8.3) and, what is the most important point, it does not yield on a nose a Fontaine-Lafaille module on X_W, and its pendant which is p-adic crystalline local systems on X_K. This is the topic of Lecture 9. The counting argument at the end is the same in the two versions.

9. Lecture 9: Rigid Local Systems, Fontaine-Laffaille modules and Crystalline Local Systems

Abstract. As seen in Lecture 8, originally proved with Michael Groechenig in [EG20, Theorem 1.6], rigid connections on $X_{\mathbb{C}}$ smooth projective $\mathbb{C}$, while restricted to the formal p-completion $\hat{X}_W$ a non-ramified projective p-adic model X_W of $X_{\mathbb{C}}$, yield F-isocrystals. More is true. By showing in [EG20] the existence of a *periodic* Higgs-de Rham flow on the formal connection $(\hat{E}_W,\ \hat{\nabla}_W)$ on $\hat{X}_W$, we prove the existence of a Fontaine-Laffaille module structure on $(\hat{E}_W,\ \hat{\nabla}_W)$ [EG20, Section 4], which, via Faltings' functor, eventually yields a *crystalline* $\mathbb{Z}_{p^f}$*-local system* on the algebraic scheme X_K, where f is the period of the Higgs-de Rham flow. This in turn implies that the rigid complex local systems on $X_{\mathbb{C}}$, for $p>0$ large so they are integral by p, the residual characteristic of such a good W, descend, as local systems on $X_{\bar{K}}$, to X_K as a crystalline $\mathbb{Z}_{p^f}$-local systems [EG20, Section 5]. This property remains true even if X is only quasi-projective under a strong cohomological rigidity assumption, which is fulfilled on Shimura varieties of real rank ≥ 2, see [EG21, Theorem A.4, Theorem A.22].

9.1. **The main theorems.** We summarize the theorems in the projective case and then in the quasi-projective case separately as the assumptions in the latter case are more technical.

9.1.1. *Good model in the projective case.* We use the notation of Lecture 8, Section 8.5. We denote by $\mathcal{M}$ the rank one Higgs bundle which corresponds to the torsion rank one connection $\mathcal{L}$. Explicitly, if $\mathcal{L}=(L,\nabla)$ then $\mathcal{M}=(L,0)$. Recall from Simpson's theory [Sim92, Lemma 4.5] that on the Dolbeault moduli space $M_{Dol}(X,r,\mathcal{M})$ of stable points over $\mathbb{C}$, we have the $\mathbb{C}^{\times}$-operation which acts as homotheties on the Higgs field. If $(V_i,\theta_i),\ i=1,\ldots,N$ are the N-Higgs bundles associated to the N rigid connections $(E_i,\nabla_i),\ i=1,\ldots,N$, then (V_i,θ_i) is stable under $\mathbb{C}^{\times}$, thus (E_i,∇_i) is a polarized complex variation of Hodge structure (E_i,Fil_i,∇_i) where $Fil_i\subset E_i$ is a locally split filtration which satisfies Griffiths transversality, and

$$(V_i,\theta_i)=(gr(E_i),gr(\nabla_i)),$$

see [Sim92, Lemma 4.5].

With reference to the conditions for the base S of a *good model* in Lecture 8, Section 8.5, we request now S to fulfil

1a) the same as 1) together with the existence of an S-point of X_S;
2a) the same as 2) for $M_{Dol}(X,r,\mathcal{M})$ in addition to $M_{Dol}(X,r)_S$;
3a) the same as 3) for $M_{Dol}(X,r,\mathcal{M})$ in addition to $M_{Dol}(X,r)_S$;
4a) the same as 4) for $M_{Dol}(X,r,\mathcal{M})$ in addition to $M_{Dol}(X,r)_S$;
5a) the same as 5) for $M_{Dol}(X,r,\mathcal{M})$ in addition to $M_{Dol}(X,r)_S$;
6a) the filtrations $Fil_i\subset E_i$ have a model $Fil_{i,S}\subset E_{iS}$ for $i=1,\ldots,N$, which is locally split;
7a) for all closed points $s\in|S|$, the characteristic of the residue field of s is $>r+1$ and $>\dim(X)$;
8a) the local systems $\mathbb{L}_i, i=1,\ldots,N$ associated by the Riemann-Hilbert correspondence to $(E_i,\nabla_i)_{\mathbb{C}}, i=1,\ldots,N$ are integral at all residue characteristics p of the closed points of S.

The condition 6a) can be fulfilled as there are finitely many filtrations Fil_i. The conditions 2a)3a)4a)5a) can be fulfilled applying the existence of the flat moduli spaces $M_{dR}(X, r, \mathcal{L})_S$ and $M_{Dol}(X, r, \mathcal{M})_S$ over S, due to Langer [Lan14, Theorem 1.1] and base change. Note that condition 8a) implies that the $\mathbb{L}_i$ come from étale $\bar{\mathbb{Z}}_p$- local systems on $X_{\mathbb{C}}$, which by base change, is the same as étale $\bar{\mathbb{Z}}_p$- local systems on $X_{\bar{K}}$, where $K = \mathrm{Frac}(W(s)$ and $\mathrm{Spec}(W(s)) \to S$ is a Witt-vector point with residual closed point $s \in |S|$. In addition they are irreducible over $\bar{\mathbb{Q}}_p$.

9.1.2. *Theorem in the projective case.*

Theorem 9.1 (See [EG20], Section 5)**.** *For any closed point $s \in |S|$ in the basis of a good model, the p-adic local systems $\mathbb{L}_i, i = 1, \dots, N$ on $X_{\bar{K}}$ descend to crystalline p-adic local systems on X_K.*

9.1.3. *Good model in the quasi-projective case.* We use the notation of Lecture 8, Section 8.5 and of 9.1.1. We fix a good compactification $X \hookrightarrow \bar{X}$ with smooth $\bar{X}$ projective such that $D := \bar{X} \setminus X$ is a strict normal crossings divisor. The de Rham moduli in rank r and fixed determinant $\mathcal{L}$ in the projective case is replaced by the de Rham moduli $M_{dR}(\bar{X}, r, \mathcal{L}, D)$ of connections (E, ∇) on $\bar{X}$ with log-poles along D and with nilpotent residues. We remark, even if we do not use it, that then $\mathcal{L}$ extends to a torsion rank one connection on $\bar{X}$. The nilpotency of the residues implies that the (Betti or de Rham) Chern classes of the underlying bundles E are 0, see [EV86, Proposition B.1]. We have N-complex points (E_i, ∇_i) of $M_{dR}(\bar{X}, r, \mathcal{L}, D)$ which describe the 0-dimensional components (rigid objects). Furthermore, [Sim92, Lemma 4.5] is replaced by [Moc06, Theorem 10.5] which guarantees that we have a locally split filtration $Fil_i \subset E_i$ as in the projective case which satisfied Griffiths transversality. Finally

$$(V_i, \theta_i) = (gr(E_i), gr(\nabla_i)),$$

is a stable Higgs bundle with logarithmic poles along D and nilpotent residues and determinant $\mathcal{M} = (L, 0)$. With reference to the conditions for the base S of a *good model* in Lecture 8, Section 8.5, and the ones in the projective case, we request now S to fulfil

1b) $X_S \hookrightarrow \bar{X}_S$ is a model of $X \hookrightarrow \bar{X}$ such that $\bar{X}_S/S$ is smooth projective and $D_S := \bar{X}_S \setminus X_S$ is a relative normal crossings divisor, and there is an S-point of X_S;
2b) the same as 2) with $M_{dR}(X, r, \mathcal{L})$ replaced by $M_{dR}(\bar{X}, r, \mathcal{L}, D)$;
3b) the same as 3) with $M_{dR}(X, r, \mathcal{L})$ replaced by $M_{dR}(\bar{X}, r, \mathcal{L}, D)$;
4b) the same as 4) with $M_{dR}(X, r, \mathcal{L})$ replaced by $M_{dR}(\bar{X}, r, \mathcal{L}, D)$;
5b) the same as 5) with $M_{dR}(X, r, \mathcal{L})$ replaced by $M_{dR}(\bar{X}, r, \mathcal{L}, D)$;
6b) the same as 6a) for the Mochizuki filtrations;
7b) for all closed points $s \in |S|$, the characteristic of the residue field of s in $> 2(r + 1)$ and $> \dim(X)$.

9.1.4. *Theorem in the quasi-projective case.* A log connection (E, ∇) which has nilpotent residues is in particular Deligne's extension of its restriction to X, so

$$H^j(X, (E, \nabla)|_X) = H^j(\bar{X}, (\Omega^\bullet_{\bar{X}}(\log D) \otimes_{\mathcal{O}_{\bar{X}}} E, \nabla)).$$

Note that

$$H^1(\bar{X}, j_{!*}\mathcal{E}nd^0(E, \nabla)) \hookrightarrow H^j(X, \mathcal{E}nd^0(E, \nabla)|_X)$$

where $\mathcal{E}nd^0$ denotes the trace free endomorphisms, so if the right hand side vanishes, so does the left hand side. We can then apply the integrality theorem [EG18, Theorem 1.1], see Lecture 7, so all prime numbers are integral for $\mathbb{L}_i, i = 1, \ldots, N$ on X. This explains why condition 8a) is automatically fulfilled under this cohomological assumption and does not appear in the list of conditions.

Theorem 9.2 (See [EG21], Theorem A.22)**.** *Assume that $H^1(X, \mathcal{E}nd(E_i, \nabla_i)) = 0$ for $i = 1, \ldots, N$. Then for all closed points $s \in |S|$ of a good model, and all Witt vector points $\mathrm{Spec}(W(s)) \to S$, the p-adic local systems $\mathbb{L}_i, i = 1, \ldots, N$ on $X_{\bar{K}}$ descend to a crystalline p-adic local systems on X_K where p is the residue characteristic of s.*

Theorem 9.2 applies for Shimura varieties of real rank ≥ 2. This the framework in which it is applied in the proof of the André-Oort conjecture in [PST21]. The aim of the rest of the Lecture is to formulate the main steps of the proof in the projective case.

9.2. **Simpson's versus Ogus-Vologodksy's correspondences in the projective case.** Recall first the the Ogus-Vologodsky correspondence [OV07] is a vast elaboration of Deligne-Illusie's splitting of the de Rham complex under the condition that X smooth over a perfect field k lifts to $W_2(k)$ ([DI87]): for example assume (E, ∇) has nilpotent p-curvature of level one, which means there is an exact sequence $0 \to (F^*S, \nabla_{can}) \to (E, \nabla) \to (F^*Q, \nabla_{can}) \to 0$ where S, Q are coherent sheaves on the Frobenius twist X' of X, $F : X \to X'$ is the relative Frobenius, and ∇_{can} is the canonical connection determined by its flat sections S, Q. Assume (S, Q) are vector bundles. This defines a class in $H^1_{dR}(X, (Q^{-1} \otimes S, \nabla_{can}))$ which by [DI87, Theorem 2.1] is equal to $H^1(X', Q^{-1} \otimes S) \oplus H^0(X', \Omega^1_{X'} \otimes Q^{-1} \otimes S)$. The class in $H^1(X', Q^{-1} \otimes S)$ yields a vector bundle extension $0 \to Q \to V' \to S \to 0$ on X' and the class in $H^0(X', \Omega^1_{X'} \otimes Q^{-1} \otimes S)$ yields a nilpotent Higgs field $\theta' : V' \to Q \to \Omega^1_{X'} \otimes S \to \Omega^1_{X'} \otimes V'$. So here we need $p > \dim(X)$ to apply [DI87] *loc. cit.*. Ogus-Vologodksy correspondence assigns $C^{-1}(V', \theta') := (E, \nabla)$ to (V', θ') under 7a). Assume now that $X = X_s$ for a closed point $s \in |S|$ as in Theorem 9.1, then starting with $(V_i, \theta_i) = (gr(E_i), gr(\nabla_i))$ on $X_{\mathbb{C}}$, we can restrict (V_i, θ_i) to $(V_i, \theta_i)_s$ on X_s, then take the pull-back $(V'_i, \theta'_i)_s$ under the arithmetic Frobenius on $\mathrm{Spec} k$ where $k = \kappa(s)$ is the residue field of s, and $C_s^{-1}(V'_i, \theta'_i)_s$. What is its relation to the restriction $(E_i, \nabla_i)_s$ of (E_i, ∇_i)? Here C_s is the Cartier-Ogus-Vologodksy functor on X_s. The miracle is the following theorem

Theorem 9.3 ([EG20], Proposition 3.5)**.** *$C_s^{-1}(V'_i, \theta'_i)_s$ is equal to one of the $(E_u, \nabla_u)_s$ where $u \in \{1, \ldots, N\}$.*

Ideal of proof. That C_s^{-1} preserves the stability, see [Lan14, Corollary 5.10], the key point is the so-called Beauville-Narasimhan-Ramanan correspondence [EG20, Theorem 2.17] as proved by Michael Groechenig in his PhD Thesis [Gro16, Proposition 3.15] which in a sense yields the Ogus-Vologodsky correspondence at the level of moduli spaces (on a curve) with a *scheme theoretic structure*. If $C_s^{-1}(V'_i, \theta'_i)$ was not rigid, there would be a deformation of its moduli point yielding a nilpotent structure around this point of order as high as we want. This nilpotent sructure transports on the side of the Dolbeault moduli space via the Ogus-Vologodsky correspondence, which by rigidity over $\mathbb{C}$ and the property 5a) yields a contradiction as over $\mathbb{C}$ in $M_{dR}(X, r, \mathcal{L})$ the multiplicities of the isolated points are bounded.

□

Remark 9.4. The proof above unlocked the program we had to prove those arithmetic crystalline properties in Theorem 9.1 associated to rigid connections. We relagated to [EG20, Corollary A.7] a proof based on the same principle showing nilpotence of the p-curvature (which we discussed via p-adic methods in Theorem 8.4), which, in the words of an anonymous referee, yields a canonical action of $\mathbb{G}_m^{\#}$ on the *category* of flat connections on X_s, where $\mathbb{G}_m^{\#}$, is the PD hull of the neutral element of $\mathbb{G}_m$.

9.3. **Periodic de Rham-Higgs flow on X_s and the $GL_r(\bar{\mathbb{F}}_p)$-local systems on X_K in the projective case.** As C_s^{-1} is an equivalence of categories (when it is defined), we see that in Theorem 9.3 the assignement $i \mapsto u$ is a bijection of $\{1, \ldots, N\}$.

Defining a *de Rham-Higgs flow* as a sequence on X_s of $\{(E_\iota, Fil_\iota, \nabla_\iota)\}$ with $(E_{\iota+1}, \nabla_{\iota+1}) = C_s^{-1}(grE_\iota, gr\nabla_\iota)$, we see that for any (E_ι, ∇_ι) with $\iota = 1 \ldots, N$, $(E_\iota, Fil_\iota, \nabla_\iota)$ defines a *periodic* de Rham-Higgs flow of period $f(\iota)|N!$.

By [OV07, Subsection 4.6], refined in [LSZ19, Corollary 3.10] to take into account the possibility of $f(\iota) > 1$, this defines a Fontaine-Lafaille module on X_s, which we do not define here as our emphasis is on the local system, see [FL82, Theorem 3.3]. By the enhancement of the Fontaine-Lafaille construction in *loc. cit.* due to Faltings [Theorem 2.6*][Fal88] (see [EG20, Proposition 4.3]), we obtain via Fontaine-Lafaille-Faltings functor

Claim 9.5. For $s \in |S|$ a good model we assign pairwise non-isomorphic local systems $\mathcal{L}_i(p)$ with values in $GL_r(\mathbb{F}_{p^{f(i)}})$ on X_K, for $i = 1, \ldots, N$.

Remark 9.6. The second miracle is provided by the theory of Fontaine-Lafaille-Faltings here: the local systems are defined over the *algebraic* X_K.

9.4. **Periodic de Rham-Higgs flow on $\hat{X}_W$ and the $GL_r(W(\bar{\mathbb{F}}_p))$-local systems on X_K in the projective case.** The functor C_s^{-1} of Ogus-Vologosdsky extends to C_n^{-1} on $X_W \otimes_W W_n$ for all n, defining $\hat{C}$ on $\hat{X}_W$ and the notion of de Rham-Higgs flow on the formal model $\hat{X}_W$ is competely similar to the one on X_s. See [LSZ19, Corollary 3.10, Theorem 4.1], [SYZ22, 1.2.1] and the work of Xu [Xu19] for very related methods and results. Then for any $(\hat{E}_{\iota,W}, \hat{\nabla}_{\iota,W})$, the restriction of (E_ι, ∇_ι) to $\hat{X}_W$, with $\iota = 1 \ldots, N$, $(E_\iota, Fil_\iota, \nabla_\iota)$, defines a *periodic* de Rham-Higgs flow of period $f(\iota)|N!$.

By [OV07, Subsection 4.6], refined in [LSZ19, Corollary 3.10] to take into account the possibility of $f(\iota) > 1$, this defines a Fontaine-Lafaille module on $\hat{X}_W$, see [FL82, Theorem 3.3]. By the enhancement of the Fontaine-Lafaille construction in *loc. cit.* due to Faltings [Theorem 2.6*][Fal88] (see [EG20, Proposition 4.3]), we obtain via Fontaine-Lafaille-Faltings functor

Claim 9.7. For $s \in |S|$ where S is a good model, we assign pairwise non-isomorphic p-adic local systems $\mathcal{L}_i(W)$ with values in $GL_r(W(\mathbb{F}_{p^{f(i)}}))$ on X_K, for $i = 1, \ldots, N$ which are crystalline.

Remark 9.8. This refinement of Faltings concerning crystallinity (also in the quesi-projective case) is precisely the important property used in [PST21].

9.5. **From crystalline p-adic local systems on X_K to p-adic local systems on $X_{\bar{K}}$ in the projective case.** The local systems $(\mathbb{L}_i, i = 1, \ldots, N)$ are integral at p, thus by the identification $\pi_1(X_{\mathbb{C}}, x_{\mathbb{C}})$ with $\pi_1(X_{\bar{K}}, x_{\bar{K}})$ become p-adic local systems on $X_{\bar{X}}$. We also have the base change $\mathcal{L}_i(W)|_{X_{\bar{K}}} =: \mathcal{L}_i(\bar{K})$ to $X_{\bar{K}}$. To conclude Theorem 9.1 we should make sure that there are not identified on $X_{\bar{K}}$ and really come from the topological rigid local systems.

Theorem 9.9. *The set of p-adic local systems $\{\mathcal{L}_1(\bar{K}), \ldots, \mathcal{L}_N(\bar{K})\}$ is, up to order, the set $\{\mathbb{L}_1, \ldots, \mathbb{L}_N\}$, viewed as p-adic local systems on $X_{\bar{K}}$. Expressed the other way around, the p-adic local systems $\mathbb{L}_i$ on $X_{\bar{K}}$ descend to crystalline p-adic local systems on X_K, for $i = 1, \ldots, N$ which in addition have values in $GL_r(W(\bar{\mathbb{F}}_p))$.*

Idea of proof. The construction of the Fontaine-Lafaille modules over $\hat{X}_W$ is based on the fact that $\hat{X}_W$ is a non-ramified lift of X_s. We could do it on $\hat{X}_{W(\bar{s})}$ and go up this way to local systems on $X_{\mathrm{Frac}(W(\bar{\mathbb{F}}_p))}$ while we have to go up to $X_{\bar{K}}$.

In order to go all the way up to $X_{\bar{K}}$, we use in [EG20, Section 5] Faltings' p-adic Simpson correspondence. For this we have to be sure that the Higgs bundles on $X_{\bar{K}}$ and the $\mathcal{L}_i(\bar{K})$ are small in the sense of Faltings. We can find infinitely many prime numbers $p > 0$ with the property that $\kappa(s) = \mathbb{F}_p$ and all the $f_i = 1$. This is the content of [EG20, Lemma 5.11] and yields a weaker form of Theorem 9.1. We do not get the result on all closed points of S but on an infinity of those with infinite different residual characteristics.

In general, we resort to [SYZ22], which relies on Faltings' p-adic Simpson correspondence, and does part of the Fontaine-Lafaille-Faltings program in the ramified case. We first argue that $\mathcal{L}_i(\bar{K}) \otimes_{\bar{Z}_p} \bar{\mathbb{F}}_p$ is irreducible. If not, using the W-point in 1a) and the consequence that $\pi_1(X_K)$ is then a semi-direct product of $\mathrm{Gal}(\bar{K}/K)$ with $\pi_1(X_{\bar{K}})$, we can kill on $\mathrm{Gal}(\bar{K}/K)$ the $GL_r(\bar{\mathbb{F}}_p)$-representation by a finite field extension K'/K and conclude that $\mathcal{L}_i(K')$ itself is reducible, which by [SYZ22, Theorem 5.15] violates the stability of the associated Higgs bundle on $X_{K'}$. We argue similarly to distinguish the $\mathcal{L}_i(\bar{K}) \otimes_{\bar{Z}_p} \bar{\mathbb{F}}_p$ for $i = 1, \ldots, N$. This finishes the proof.

□

Remark 9.10. We hope in the near future ([EG22]) to strengthen the results and shorten the proofs of the existing ones using more p-adic methods.

10. Lecture 10: Comments and Questions

Abstract. The aim of this last Lecture is to list a few questions encountered during the Lectures.

10.1. **With respect to the p-curvature conjecture (Lecture 2).** What would be a formulation of the p-curvature conjecture for the formal completion along a non-normal subvariety of a smooth variety? The question is motivated by a discussion with Johan de Jong on [LR96].

10.2. **With respect to the Malčev-Grothendieck's theorem and its shadows in characteristic $p > 0$ (Lecture 3).** As already mentioned in Section 3.3 we have two versions of Malčev-Grothendieck theorem over an algebraically closed field of characteristic $p > 0$, one in the infinitesimal site (Gieseker's conjecture, solved, see Theorem 3.4), one in the crystalline site (unsolved) due to de Jong, but we do not have at present a formulation in the prismatic site, one difficulty being the *iso*-notion.

10.3. **With respect to Lubotzky's Theorem 4.2.** Notation as in *loc. cit.*. Assume π is a profinite group. What can be said on $\dim_{\mathbb{F}_\ell} H^i(\pi,\rho)$ for any $i \in \mathbb{N}$?

It holds $\dim_{\mathbb{F}_\ell} H^0(\pi,\rho) \le r$ for any representation, whatever π is.

It holds $\dim_{\mathbb{F}_\ell} H^1(\pi,\rho) \le \delta \cdot r$ for any π topologically spanned by δ elements, as a 1-continuous cocycle is uniquely defined by its value on topological generators of π.

For $i = 2$ Lubotzky's theorem yields even a characterization of finite presentation.

What does the growth of the cohomology for $i \ge 3$ encode as a property?

Do we have special properties for $\dim_{\mathbb{F}_\ell} H^i(\pi,\rho)$ for any $i \ge 3$ if π is the tame fundamental group of X smooth quasi-projective over any algebraically closed field?

10.4. **With respect to Theorem 4.7.** Notation as in *loc. cit.*. It is wishful that Theorem 4.5 be true under the assumption that X is quasi-projective normal over an algebraically closed characteristic $p > 0$ field k.

If $j : X \hookrightarrow \bar{X}$ is a normal compactification of a normal X over k, is there a formula which enables one to bound $H^2(\pi_1^t(X), M)$ by some cohomological invariant of a constructible sheaf on $\bar{X}$ built out of the local system $\underline{M}$?

What about higher cohomology $H^i(\pi_1^t(X), M)$, $i \ge 3$, also for X normal (see Section 10.3)?

10.5. **With respect to Theorem 5.3 and Theorem 5.10.** Can we extend those theorems to formally smooth proper schemes, to smooth rigid spaces, is there a version for normal varieties, for non-normal varieties involving the pro-étale fundamental group developed in [Sch13] and in all generality in [BS15] etc. We can also raise similar questions concerning the finite presentation of the (tame) fundamental group. (The question for formal and rigid spaces was posed to us by Piotr Achinger and Ben Heurer).

10.6. **With respect to Theorem 6.2 and** [dJE22]**.** Can we replace in the formulation the Betti moduli in a given rank of a normal complex variety by the Mazur or Chenevier deformation space of a given $\bar{\mathbb{F}}_\ell$-residual representation over X over $\bar{\mathbb{F}}_p$ to obtain Zariski density of the local systems with quasi-unipotent monodromies at infinity?

Even if we did not discuss in those Lecture Notes the recent integrality property in [dJE22], we can formulate the analog question: does [dJE22, Theorem 1.1] hold with X defined over $\bar{\mathbb{F}}_p$? Assume that in a given rank r, there is an irreducible ℓ-adic local system. Can we conclude that for all $\ell' \neq p$ there is an irreducible ℓ'-adic local system of rank r, and that perhaps by $\ell' = p$, there is an irreducible isocrystal of rank r?

10.7. **With respect to Lectures 7, 8, 9.** we do not comment as we essentially lack a new idea concerning integrality.

References

[AE19] Abe, T., Esnault, H.: *A Lefschetz theorem for overconvergent isocrystals with Frobenius structure*, Annales de l'École Normale Supérieure, **52** (4) (2019), 1243–1264.

[Ach17] Achinger, P.: *Wild ramification and $K(\pi, 1)$ spaces*, Invent. math., **210** (2017) (2), 453–499.

[AZ17] Achinger, P., Zdanowicz, M.: *Some elementary examples of non-liftable varieties*, Proc. Amer. Math. Soc. **145** (2017), 4717–4729.

[And89] André, Y.: *G-Functions and Geometry*, Aspects of Mathematics, Springer Verlag.

[And04] André, Y.: *Sur la conjecture des p-courbures de Grothendieck-Katz et un problème de Dwork*, in Geometric aspects of Dwork theory, Vol. I, II, 55–112, Walter de Gruyter (2004).

[BBDG82] Beilinson, A., Bernstein, J. , Deligne, P.: *Faisceaux pervers*, In Analysis and Topology on Singular Spaces, I (Luminy,1981), Astérisque **100** (1982), 5–171, Soc. Math. France, Paris.

[BL17] Bhatt, B., Lurie, J.: *A Riemann-Hilbert correspondence in characteristic* $p > 0$, `https://www.math.ias.edu/~lurie/papers/ModpRH.pdf`.

[BS15] Bhatt, B., Scholze, P.: *The pro-étale topology for schemes*, Astérisque **369** (2015), 99–201.

[BS21] Bhatt, B., Scholze, P.: *Prismatic F-crystals and crystalline Galois representations*, `http://www.math.uni-bonn.de/people/scholze/PrismaticCrystals.pdf`.

[Bel80] Belyĭ, G.: *Galois extensions of a maximal cyclotomic field*, translated by Neal Koblitz, Math. USSR Izv. **14** (2 (1980), 247–256.

[Ber74] Berthelot, P.: *Cohomologie cristalline des schémas de caractéristique* $p > 0$, Lecture Notes in Math. **407**, Springer-Verlag, 1974.

[Bos01] Bost, J.-B.: *Algebraic leaves of algebraic foliations over number fields*, Publ. math. I. H. É. S. 93 (2001), 161–221.

[BKT13] Brunebarbe, J., Klingler, B., Totaro, B.: *Symmetric differentials and the fundamental group*, Duke Math. J. **162** (2013), 1–17.

[Chu85] Chudnovsky, D, Chudnovsky, G.: *Applications of Padé approximation to the Grothendieck conjecture on linear differential equations*, in Number Theory, Lecture Notes in Math. **1135** (1985), 52–100.

[D'Ad20] D'Addezio, M.: *Some remarks on the companions conjecture for normal varieties*, `https://arxiv.org/abs/2006.09954`.

[dJ00] de Jong, J.: *A result on formal linearity*, J. of Algebra **225** (2000), 936–942.

[dJ01] de Jong, J.: *A conjecture on the arithmetic fundamental group*, Israel J. of Math. **121** (2001), 61–84.

[dJEG22] de Jong, J., Esnault, H., Groechenig, M.: *Rigid non-cohomologically rigid local systems*, preprint 2022, 6 pages, `http://page.mi.fu-berlin.de/esnault/helene_publ.html`.

[dJE22] de Jong, J., Esnault, H.: *Integrality of the Betti moduli space*, preprint 2022, https://page.mi.fu-berlin.de/esnault/preprints/helene/146_dJ_esn.pdf.

[Del70] Deligne, P.: *Équations différentielles à points singuliers réguliers*, Lecture Notes on Math, **163**, Springer Verlag (1970).

[Del72] Deligne, P.:*Les constantes des équations fonctionnelles des fonctions L*, Proc. Antwerpen Conference, vol. 2, Lecture Notes in Mathematics, **349**, pp. 501–597, Springer Verlag, Berlin.

[Del73] Deligne, P.: *Comparaison avec la théorie transcendente*, in: SGA 7 II, Exp. No. XIV, pp. 116–164, Lecture Notes in Mathematics, **340**, (1973).

[Del80] Deligne, P.: *La conjecture de Weil II*, Publ. math. I.H.É.S. **52** (1980), 137–252.

[DI87] Deligne, P., Illusie, L.: it Relèvements modulo p^2 et décomposition du complexe de de Rham, Invent. math. **89** (1987), 2, 247–270.

[Del12] Deligne, P.: *Finitude de l'extension de $\mathbb{Q}$ engendrée par des traces de Frobenius, en caractéristique finie*, Mosc. Math. J. **12** (2012), no.3, 497–514.

[Dri01] Drinfeld, V.: *On a conjecture of Kashiwara*, Math. Res. Lett. **8** (2001), no. 5-6, 713–728.

[Dri12] Drinfeld, V.: *On a conjecture of Deligne*, Volume dedicated to the memory of I. M. Gelfand, Moscow Math. J. **12** (2012) no. 3.

[Esn17] Esnault, H.: *Survey on some aspects of Lefschetz theorems in algebraic geometry*, Revista Matemática Complutense, **30** (2) (2017), 217–232.

[EM10] Esnault, H., Mehta, V.: *Simply connected projective manifolds in characteristic $p > 0$ have no nontrivial stratified bundles*, Inventiones math. **181** (2010), 449-465.

[ES15] Esnault, H., Shiho, A.: *Existence of locally free lattices of crystals*, preprint 2015, 2 pages, https://page.mi.fu-berlin.de/esnault/preprints/helene/119b_esn_shi.pdf

[ES18] Esnault, H., Shiho, A.: *Convergent isocrystals on simply connected varieties*, Annales de l'Institut Fourier **68** (2018) no 5, 2019–2148.

[ES19] Esnault, H., Shiho, A.: *Chern classes of crystals*, Transactions of the AMS **371** 2 (2019), 1333–1358.

[EK18] Esnault, H., Kisin, M.: *D-modules and finite monodromy*, Selecta Mathematica, volume dedicated to Alexander Beilinson **24** (1) (2018), 217–232.

[EG18] Esnault, H., Groechenig, M.: *Cohomologically rigid connections and integrality*, Selecta Mathematica **24** (5) (2018), 4279—4292.

[EG20] Esnault, H., Groechenig, M.: *Rigid connections and F-isocrystals*, Acta Mathematica **225** (1) (2020), 103—58.

[EG21] Esnault, H., Groechenig, M.: *Frobenius structures and unipotent monodromy at infinity*, preprint 2021, 8 pages. http://page.mi.fu-berlin.de/esnault/preprints/helene/143_appPST.pdf.

[EG22] Esnault, H., Groechenig, M.: in progress.

[EK11] Esnault, H., Kerz, M.: *Notes on Deligne's letter to Drinfeld dated March 5, 2007*, https://page.mi.fu-berlin.de/esnault/preprints/helene/103-110617.pdf.

[EK20] Esnault, H., Kerz, M.: *Arithmetic subspaces of moduli spaces of rank one local systems*, Cambridge Journal of Mathematics **8** (2020) (3), 453–478.

[EK21] Esnault, H., Kerz, M.: *Étale cohomology of rank one ℓ-adic local systems in positive characteristic*, Selecta Mathematica **27** (2021) no 4, paper 58, 25 pages.

[EK22] Esnault, H., Kerz, M.: *Density of Arithmetic Representations of Function Fields*, 18 pages, Epiga **6** (2022).

[EK23] Esnault, H., Kerz, M.: *Local systems with quasi-unipotent monodromy at infinity are dense*, preprint 2021, 9 pages to appear in Israel Journal of Mathematics, http://page.mi.fu-berlin.de/esnault/preprints/helene/140_esn_kerz.pdf.

[EK16] Esnault, H., Kindler L.: *Lefschetz theorems for tamely ramified coverings*, Proc. of the AMS **144** (2016), 5071–5080.

[ESS22] Esnault, H., Shusterman, M., Srinivas, V.: *Finite presentation of the tame fundamental group*, Selecta Mathematica **28** (2022) 2, paper 37, 19 pages.

[ESS22b] Esnault, H., Srinivas, V., Stix, J.: *An obstruction to lifting to characteristic* 0, preprint 2021, 20 pages, to appear in Algebraic Geometry, https://page.mi.fu-berlin.de/esnault/preprints/helene/142_ess.pdf.

[EV86] Esnault, H., Viehweg, E.: *Logarithmic de Rham complexes and vanishing theorems*, Inventiones math. **86** (1986), 161–194.

[Fal83] Faltings, G.: *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. math. **73** (1983), no. 3, 349–366.

[Fal88] Faltings, G.: *Crystalline cohomology and p-adic Galois-representations*, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), 25—80, Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[FL82] Fontaine, J. M., Laffaille, G.: *Construction de représentations p-adiques,* Ann. Sci. Éc. Norm. Sup. **15** (1982), 547–608.

[Gai07] Gaitsgory, D.: *On de Jong's conjecture*, Israel J. Math. **157** (2007), 155–191.

[Gie75] Gieseker, D.: *Flat vector bundles and the fundamental group in non-zero characteristics*, Ann. Sc. Norm. Super. Pisa, **4** Sér. 2(1) (1975), 1–31.

[Gri70] Griffiths P.: *Periods of integrals on algebraic manifolds: summary of results and discussion of open problems*, Bull. Bull. Amer. Math. Soc. **76** (1970), 228–296.

[GM88] Goldman, W., Millson, J.: *The deformation theory of representations of fundamental groups of compact Kähler manifolds*, Publ. math. I.H.É.S. **67** (1988), 43–96.

[Gro16] Groechenig, M.: *Moduli of flat connections in positive characteristic*, Math. Res. Let. **23** (2016) 4, 989–1047.

[Gro70] Grothendieck, A.: *Représentations linéaires et compactifications profinies des groupes discrets*, Manuscr. Math. **2** (1970), 375–396.

[Gro57] Grothendieck, A.: *Sur quelques points d'algèbre homologique*, Tohoku Math. J. **9** 2 (1957), 119–221.

[Hru04] Hrushovski, E.: *The elementary theory of Frobenius automorphisms*, `http://de.arxiv.org/pdf/math/0406514v1`.

[Ill79] Illusie, L.: *Complexe de de Rham-Witt et cohomologie cristalline*, Ann. scient. Éc. Norm. Sup. **12** (1979), 501–661.

[Kas81] Kashiwara, M.: *Quasi-unipotent constructible sheaves*, J. Fac. Sci. Univ. Tokyo, Sect. IA Math. **28** (1981) 3, 757–773.

[Kat70] Katz, N.: *Nilpotent connections and the monodromy theorem: applications of a result of Turrittin*, Publ. math. I.H.É.S. **39** (1970), 175–232.

[Kat72] Katz, N.: *Algebraic Solutions of Differential Equations (p-curvature and the Hodge Filtration)*, Invent. math. **18** (1972), 1–118.

[Kat96] Katz, N.: *Rigid local systems*, Princeton University Press (1996).

[KM74] Katz, N., Messing, W.: *Some Consequences of the Riemann Hypothesis for Varieties over Finite Fields*, Invent. math. **23** (1974), 73–77.

[Kel22] Kedlaya, K.: *Étale and crystalline companions*, to appear in Épiga, `https://kskedlaya.org/papers/`.

[KS10] Kerz, M., Schmidt, A.: *On different notions of tameness in arithmetic geometry*, Math. Ann. **346** (2010), no. 3, 641–668.

[Kro57] Kronecker, L.: *Zwei Sätze über Gleichungen mit ganzzahligen Coefficienten*, J. reine u. angew. Mathematik **53** (1857), 153–175.

[Laf02] Lafforgue, L.: *Chtoucas de Drinfeld et correspondance de Langlands*, Invent. math. **147** (2002), no. 1, 1–241.

[Lam22] Lam, Y. H. J.: *Motivic local systems on curves and Maeda's conjecture,* `https://arxiv.org/pdf/2211.06120.pdf`.

[LL22a] Landesman, A., Litt, D.: *Geometric local systems on very general curves and isomonodromy*, `https://arxiv.org/pdf/2202.00039.pdf`.

[LL22b] Landesman, A., Litt, D.: *Canonical representations of surface groups*, `https://arxiv.org/abs/2205.15352`.

[Lan04] Langer, A.: *Semistable sheaves in positive characteristic*, Ann. Math. **159** (2004), 251–276.

[Lan14] Langer, A.: *Semistable modules over Lie algebroids in positive characteristic*, Doc. Math. **19** (2014), 509–540.

[LR96] Lasell, B., Ramachandran, M.: *Observations on harmonic maps and singular varieties*, Annales Sci. de l'É.N.S. **29** 4o série (1996), 135–148.

[LSZ19] Lan, G., Sheng, M., Zuo, K.: *Semistable Higgs bundles, periodic Higgs bundles and representations of algebraic fundamental groups*, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 10, 3053–3112.

[LSYZ19] Lan, G., Sheng, M., Yang, Y., Zuo, K.: *Uniformization of p-adic curves via Higgs-de Rham flows*, Journal für die reine und angewandte Mathematik, (Crelles Journal) **747** (2019), 63–08.

[LZ17] Liu, R., Zhu, X.: *Rigidity and a Riemann-Hilbert correspondence for p-adic local systems*, Invent. math. **207** (2017), no. 1, 291–343.

[Lub01] Lubotzky, A.: *Pro-finite Presentations*, Journal of Algebra **242** 2 (2001), 672–690.

[Mal40] Malčev, A.: *On isomorphic matrix representations of infinite groups*, Mat. Sb. N. S. **8**(50) (1940), 405–422.

[Moc06] Mochizuki, T.: *Kobayashi-Hitchin correspondence for tame harmonic bundles and an application*, Astérisque **309** (2006), viii+117.

[Moc07] Mochizuki, T.: *Asymptotic behaviour of tame harmonic bundles and an application to pure twistor D-modules. II.* Mem. Amer. Math. Soc. **185** (2007), no. 870, xii+565 pp.

[OV07] Ogus, A., Vologodksy, V.: *Nonabelian Hodge theory in characteristic* p, Publ. math. Inst. Hautes Études Sci. **106** (2007), 1–138.

[Pet22] Petrov, A.: *Geometrically irreducible p-adic local systems are de Rham up to a twist*, to appear in Duke, `https://arxiv.org/abs/2012.13372`.

[PST21] Pila, J., Shankar, A., Tsimerman, J.: *Canonical Heights on Shimura Varieties and the André-Oort conjecture*, `https://arxiv.org/pdf/2109.08788.pdf`.

[Ray78] Raynaud, M.: *Contre-exemple au "Vanishing Theorem" en caractéristique* $p > 0$, C. P. Ramanujam - A tribute, Studies in Mathematics **8**, Tata Institute of Fundamental Research Bombay, 273–278 (1978).

[Roq70] Roquette, P.: *Abschätzung der Automorphismenanzahl von Funktionenkörpern bei Primzahlcharakteristik*, Math. Z. **117** (1970), 157–163.

[Sch13] Scholze, P.: *p-adic Hodge theory for rigid-analytic varieties*, Forum Math. Pi **1** (2013), e1, 77 pages.

[Ser61] Serre, J.-P.: *Exemples de variétés projectives en caractéristique p non relevables en car-actéristique zéro*, Proc. Nat. Acad. Sci. USA **47** (1961), 108–109.

[Sim92] Simpson, C.: *Higgs bundles and local systems*, Publ. math. I.H.É.S. **75** (1992), 5–95.

[SYZ22] Sun, R., Yang, J., Zuo, K.: *Projective crystalline representations of étale fundamental groups and twisted periodic Higgs-de Rham flow*, J. Eur. math. Soc. **24** (2022) 6, 1991–2076.

[vDdB21] van Dobben de Bruyn, R.: *A variety that cannot be dominated by one that lifts*, Duke J. Math. **170** (2021), no. 7, 1251–1289.

[Var18] Varshavski, Y.: *Intersection of a correspondence with a graph of Frobenius*, J. Alg. Geom. **27** (2018), 1, 1–20.

[Xu19] Xu, D: *Relèvement de la transformée de Cartier d'Ogus-Vologodsky modulo* p^n, Mémoires de la SMF **163** (2019), 6–144.

[Zuo00] Zuo, K.: *On the negativity of kernels of Kodaira-Spencer maps on Hodge bundles and applications.* Asian J. Math. **4** (2000), 279–302.

[EGAIV$_4$] Éléments de Géométrie Algébrique *Étude locale des schémas et des morphismes de schémas*, Quatrième partie, Publ.math.I.H.É.S. **32** (1967), 5–361.

[SGA1] Séminaire de Géométrie Algébrique: Revêtements étales et groupe fondamental, Lecture Notes in Mathematics **224**, Springer Verlag (1971).

[SGA4] Séminaire de Géométrie Algébrique: Théorie des Topos et cohomologie étale des schémas, Tome 3, Lecture Notes in Mathematics **305**, Springer Verlag (1973).

[SGA7.2] Séminaire de Géométrie Algébrique: Groupes de monodromie en géométrie algébrique, Lecture Notes in Mathematics **340**, Springer Verlag (1973).

[SP] Stack Project, `https://stacks.math.columbia.edu`

Freie Universität Berlin, Arnimallee 3, 14195, Berlin, Germany
Email address: `esnault@math.fu-berlin.de`