LECTURES ON LOCAL SYSTEMS IN ALGEBRAIC-ARITHMETIC GEOMETRY

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Abstract. The topological fundamental group of a smooth complex algebraic variety is poorly understood. One way to approach it is to consider its complex linear representations modulo conjugation, that is complex local systems. One fundamental problem is to recognize those coming from geometry, and more generally subloci of the moduli space of local systems with special arithmetic properties. This is the object of deep conjectures. We'll study some consequences of those, notably integrality and crystallinity properties.

1. General Introduction

The *topological fundamental group* $\pi_1^{\rm top}(X,x)$ of a finite CW-complex X based at a point x in topology, as defined by Poincaré, is a finitely presented group. In turn, any finitely presented group is the fundamental group of a finite CW complex. The finite generation enables one to define a 'moduli' (parameter) space $M_B(X,r)$ of all its semi-simple complex linear representations $\rho : \pi_1^{\rm top}(X,x) \to GL_r(\mathbb{C})$ in a given rank r, modulo conjugation, or equivalently, of all its rank r semi-simple complex local systems $\mathbb{L}$. It is called the character variety, also the *Betti moduli space* of X in rank r, and is a scheme of finite type defined over the ring of integers $\mathbb{Z}$.

We are interested in the case when X consists of the complex points of an algebraic manifold (smooth algebraic variety of finite type over the complex numbers $\mathbb{C}$). We know extremely little, if not nothing on the restrictions it imposes on $\pi_1^{\rm top}(X,x)$. On the other hand, there are naturally defined local systems $\mathbb{L}$: those which in restriction to some Zariski dense open $U \hookrightarrow X$ are subquotients (equivalently, summands by Deligne's semi-simplicity theorem) of a local system on U which comes from the variation of the cohomology of the fibers of a smooth projective morphism $g : Y \to U$. Such $\mathbb{L}$ are called *geometric*. For example, $U = X$ and g *finite* (equivalently, $\mathbb{L}$ has finite $\mathrm{Im}(\rho)$, which is called monodromy), which by the Riemann existence theorem is equivalent to g being finite étale, and thus relates $\pi_1^{\rm top}(X,x)$ to its profinite completion $\pi_1(X_{\mathbb{C}},x)$, the étale fundamental group, defined by Grothendieck, itself related to the Galois group of the field of functions of $X_{\mathbb{C}}$.

So it is very natural to try to single out geometric or even finite complex points of $M_B(X,r)$. More generally, it is natural to try to define a notion of geometric subloci of higher dimension. It is clearly an inaccessible task, which is reminiscent of the Hodge and the Tate conjectures: how can we construct g out of $\mathbb{L}$? There are several *conjectures* relying on various aspects of $M_B(X,r)$.

Grothendieck's p-curvature conjecture: It relies on the *Riemann-Hilbert correspondence* which equates the complex points $\mathbb{L}$ of $M_B(X,r)$ with algebraic integrable connections (E,∇) on X (say X projective for simplicity to avoid boundary growth conditions): we consider (E,∇) mod p for all large primes p and request this characteristic $p>0$ connection to be generated by flat sections. This is the original formulation and should characterize finite local systems $\mathbb{L}$. More generally, to characterize geometric local systems $\mathbb{L}$, we request (E,∇) mod p for all large p to be filtered so that the associated graded is spanned by flat sections. Since the work by Katz [Katz72] which roughly (a bit less) shows that on a geometric $\mathbb{L}$ we can characterize its finiteness like this, the ones by Chudnosvky [Chu85], Bost [Bos01] and André [And04] which handle the solvable case, and some remarks like [EK18], there is essentially no big progress on this viewpoint.

Gieseker-de Jong conjecture: It relies simply on the *finite generation* of $\pi_1^{\mathrm{top}}(X,x)$ which implies the theorem of Malčev-Grothendieck [Mal40], [Gro70] saying that $\pi_1(X_{\mathbb{C}},x)$ controls the size of $M_B(X,r)$: if $\pi_1(X_{\mathbb{C}},x)=\{1\}$ then $M_B(X,r)$ consists of one point, the trivial $\mathbb{L}$ of rank r (in fact there are no extensions as well). Gieseker's conjecture [Gie75] solved in [EM10] asserts an analog in characteristic $p>0$ for infinitesimal crystals, while de Jong's conjecture, which is still unsolved in its generality (see [ES18] for small steps) predicts an analog for isocrystals. It is also related to the Langlands program: if the ground field is $\bar{\mathbb{F}}_p$ and the isocrystal is endowed with a Frobenius structure, then the existence of ℓ-adic companions ([AE19], [Kel22], initially predicted by Deligne in Weil II [Del80]) proves the conjecture. It would be of interest to understand a generalization of de Jong's conjecture on prismatic crystals which encompasses his initial formulation.

Simpson's motivicity conjecture: Rigid Local Systems. Those are the 0-dimensional components of $M_B(X,r)$. Simpson [Sim92] predicted that they are all geometric. It relies on the corresponding theorem by Katz [Kat96] when X has dimension 1, in which case X has to be an open in $\mathbb{P}^1$ (so in the definition of $M_B(X,r)$ one has to fix conjugacy classes of quasi-unipotent monodromy at infinity). And it relies on the so-called Simpson's correspondence, which when X is projective equates real analytically $M_B(X,r)$ with the moduli space of semi-stable Higgs bundles with vanishing Chern classes. Those are endowed with a $\mathbb{C}^\times$-flow, thus rigid local systems, viewed on the Higgs side, are fixed by it, so by Simpson's theorem, underly a polarizable complex variation of Hodge structures. From there it is one short step to dream of geometricity.

Relative Fontaine-Mazur conjecture: Revisited by Petrov [Pet22], relying on p-adic geometry through the work by Scholze [Sch13] and Liu-Zhu [LZ17], it predicts that an irreducible $\mathbb{L}$, viewed ℓ-adically, comes from geometry if and only if it is defined over a form X_F of X where $F\subset\mathbb{C}$ is a subfield of finite type.

Higher dimensional special subloci: It is possible to define *arithmetic* subloci of $M_B(X,r)$ and ℓ-adic analogs. One predicts that they themselves come from geometry in a specific sense. This program is carried out in rank one in [EK20] over $\mathbb{C}$ and in [EK21] in characteristic $p>0$. Furthermore, over $\bar{\mathbb{F}}_p$ a special case of a general density conjecture of arithmetic local systems is proved in [EK22, EK22], while over $\mathbb{C}$, this density initially predicted in [EK23] can not be true for varieties defined over large fields ([LL22a], [LL22b]). That one may pose the *density conjecture* relies on de Jong's theorem [dJ01] on the structure of Mazur's deformation spaces, which are the complete local analogs of $M_B(X,r)$ over $\bar{\mathbb{F}}_p$, and on Drinfeld's use of it [Dri01] to prove Kashiwara's conjecture to the effect the direct image of (regular singular) D-modules by a proper morphism preserves semi-simplicity.

Short of being able to prove such so general conjectures, one can consider corollaries of them. May be the bigger progress in the last years has been reached for rigid local systems. Simpson had proven that if $\mathbb{L}$ is rigid, then an ℓ-adic completion of it for ℓ large is defined over a form X_F for F a field of finite type. Given Petrov's formulation of the relative Fontaine-Mazur conjecture, one can now say that Simpson's geometricity conjecture, which comes from complex geometry, is a special case of Fontaine-Mazur's one, which comes from geometry over a number field. If true, rigid local systems should be integral, that is stemming from a representation $\rho : \pi_1^{\rm top}(X,x) \to GL_r(\bar{\mathbb{Z}})$. This is called the *integrality conjecture*, formulated by Simpson himself in [Sim92]. It is proven in [EG18] under the extra condition that $\mathbb{L}$ as a complex point of $M_B(X,r)$ is smooth, that is has no multiplicity. It is a cohomological condition and for this reason one says that $\mathbb{L}$ is *cohomologically rigid*. The proof uses X modulo $p > 0$ for p large, and on it the ℓ-adic companions mentioned above (the existence of which, in the form used, has been proved by L. Lafforgue for curves as a consequence of the Langlands program [Laf02], and by Drinfeld in higher dimension [Dri12]). For example, Katz proves in [Kat96] that in dimension 1 rigid local systems are cohomologically rigid. Also, by Margulis superrigidity, on Shimura varieties of real rank ≥ 2, all local systems are semi-simple and cohomologically rigid.

Another consequence of Simpson's geometricity conjecture can be drawn: at a place of good reduction of residual characteristic $p > 0$ where X descends to $X_{W(\mathbb{F}_q)}$, and p is large, the induced p-adic local system on the underlying geometric p-adic variety $X_{\overline{{\rm Frac}(W(\mathbb{F}_q))}}$ descends to a *crystalline* p-adic local system on $X_{{\rm Frac}(W(\mathbb{F}_q))}$. We prove this fact in [EG20] for X smooth projective. We also prove this in [EG21] assuming X is a Shimura variety of real rank ≥ 2. This result *loc. cit.* is the building block of the recent proof of the André-Oort conjecture on such Shimura varieties [PST21]. One dream would be to be able to understand whether given a rigid local system in $M_B(X,r)$ for X projective, we can assign to it for p large a prismatic F-crystal in the sense of Bhatt-Scholze [BS21]. This would be the best possible generalization of [EG20].

However, as shown very recently in [dJEG22], it is *not* the case (unlike in dimension 1 and on Shimura varieties of real rank ≥ 2) that all rigid local systems are cohomolopgically rigid. So a new proof of Simpson's integrality conjecture has to be found.

2. Overview of the Content of the Lectures

• We should *define the general notions* used in the General Introduction: (projective) smooth complex algebraic varieties, model over rings of finite type over $\mathbb{Z}$, mod p reduction, topological fundamental groups, their profinite completion, their understanding as the étale fundamental group via Riemann existence theorem, Betti moduli spaces.

• On the *p-curvature conjecture* we won't really discuss as there is no substantially new input. Let us nonetheless mention one small point to help thinking philosophically of it: let $a = \mathrm{ex}(2\pi\sqrt{-1}b)$ be a complex number, so b is a complex number as well. How can we decide whether a is a root of unity, or equivalently whether $b \in \mathbb{Q}$ that is whether b is a rational number? There is one *analytic* answer: if we assume from the beginning that a is an algebraic integer, that is satisfies an equation $a^n + c_1 a^{n-1} + \ldots + c_n = 0$ with $c_i \in \mathbb{Z}$, then *Kronecker* says that a is a root of unity if and only if all the complex solutions of this equation have absolute value equal to one. A vast generalization of this theorem is at the core of Katz' proof mentioned above. Kronecker also gives an *algebraic* answer. Assume b is an algebraic number, that is satisfies an equation as above but with $c_i \in \mathbb{Q}$, so b lies in a number ring with finitely many places inverted. Then b lies in $\mathbb{Q}$, that is the prime field in characteristic 0, if and only if all its mod p reductions for p large enough lie in the prime field $\mathbb{F}_p$. Grothendieck's p-curvature conjecture is precisely a vast generalization of this.

• We shall survey *Malčev-Grothendieck* proof (it is an exercise) and the proof of the *Gieseker conjecture* for crystals in the infinitesimal site in characteristic $p > 0$. For this we shall introduce the moduli of vector bundles, also in unequal characteristic [Lan04], and those crystals which are nothing but D-modules [Gie75]. We should not lose too much time with those notions and with the proof, we can regard this as an invitation to shape a generalization of de Jong's conjecture as mentioned in the General Introduction.

• I am not the most competent person to explain Sasha Petrov's idea, perhaps someone closer to p-adic Hodge theory can summarize the argument, or we could invite Sasha to give a lecture (on zoom). If this is not possible, I shall at least summarize his result, which clarifies greatly the conjectural situation.

• On *higher dimensional special subloci*, we shall explain what is the Hard Lefschetz theorem with semi-simple coefficients, in what generality it is true in characteristic 0 (Hodge, Simpson and T. Mochizuki), explain what are arithmetic local systems, and mention that by Deligne [Del80] and its generalization by Beilinson-Bernstein-Deligne-Gabber [BBDG82], Hard Lefschetz holds for them, and explain the general density conjecture and strategy of proof for Hard Lefschetz in general for semi-simple coefficients, which works only in rank one so far ([EK21]).

• On *rigid local systems* we shall first give examples, essentially due to Katz. We shall explain the theory of ℓ-adic *companions*, and mention their *geometricity* on curves over a finite field, as a result of L. Lafforgue's proof of the Langlands program ([Laf02]). This, together with Grothendieck's specialization theory of the (tame) fundamental group is at the core of the proof of the *integrality* of cohomologically rigid local systems in [EG18], so we can see where the philosophy of the proof comes from. To explain the notion of *crystallinity* is not very easy. Depending on how far we have gone in the program listed above, we shall explain the way we think of this property via the Faltings-Fontaine-Lafaille theory ([EG20]).

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