On the vanishing in cohomology of the restriction map to the generic point

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(joint work with Mark Kisin and Alexander Petrov, in progress)

1. Introduction

Let X be an irreducible smooth projective complex algebraic variety. Grothendieck's generalized Hodge conjecture (GGHC) [2] predicts that if $H \subset H^i(X, \mathbb{Q})$ is a Hodge substructure, with corner piece of the Hodge filtration lying in $H^{i-c,c}$, then there is a codimension $\geq c$ subscheme $Z \subset X$ such that the restriction homomorphism

$$H^i(X, \mathbb{Q}) \to H^i(X \setminus Z, \mathbb{Q})$$

in Betti cohomology dies.

While the notion of Hodge structures is analytic, more precisely harmonic, there is one case where GGHC can be expressed purely algebraically. This is when $H^{i,0}(X) = H^0(X, \Omega^i_X) = 0$ and $H = H^i(X, \mathbb{Q})$. In this case $c \geq 1$ and by the comparison isomomorphism between Betti and de Rham cohomology, the conjecture just predicts that there is a dense open $U \subset X$ such that the restriction homomorphism

$$H^i(X) \to H^i(U)$$

in de Rham cohomology dies.

As de Rham cohomology fulfils base change, this vanishing is equivalent to the one for de Rham cohomology of a field $K \subset \mathbb{C}$ of finite type over which X is defined. As usual, it enables one to consider the problem mod p, so over a finite field, for almost all or simply for many ps, or to restrict X over K to K_p a p-adic completion and try to think with modern p-adic methods.

Before doing this, let us emphasize that we know a positive answer to GGHC only in two cases: $i = 1$, then $H^0(X, \Omega^1) = 0$ implies $H^1(X, \mathcal{O}_X) = 0$ either by Hodge duality (analytic) of Hard Lefschetz (algebraic) which in turn implies that $H^1_{dR}(X) = 0$. And $i = 2$: the same argument yields then $H^2(X, \mathcal{O}_X) = 0$ which in turn by the exponential sequence and GAGA implies that $H^2(X, \mathbb{Q})$ is spanned by the Néron-Severi group, which is a finite dimensional $\mathbb{Q}$-vectorspace. So U can be taken to be the complement of the union of its generators.

2. mod p for many p's

Theorem 1. *Let S be a smooth affine scheme over $\mathbb{Z}$, let X_S/S be a smooth proper scheme over S. The following holds true.*

1) *If $H^0(X_S, \Omega^i_{X_S/S})$/torsion $= 0$ then there is a dense open $S^\circ \subset S$ such that for all closed points $s \in S^\circ$, for any dense affine open $U_s \subset X_s$, the restriction homomorphism $H^i(X_s) \to H^i(U_s)$ in de Rham cohomology dies.*

2) *If there is a dense set of closed points $s \in S$ such that for each such s, there is a dense affine open $V_s \subset X_s$ such that the restriction homomorphism $H^i(X_s) \to H^i(V_s)$ in de Rham cohomology dies, then $H^0(X_S, \Omega^i_{X_S/S})/\mathrm{torsion} = 0$.*

We may remark at this point that the formulation of Theorem 1 2) is mimicked from the one for the p-curvature conjecture [5], as coupled with GGHC it predicts:

> If there is a dense set of closed points $s \in S$ such that for each such s, there is a dense affine open $V_s \subset X_s$ such that the restriction homomorphism $H^i(X_s) \to H^i(V_s)$ in de Rham cohomology dies, then there is a dense open $U \subset X_S \times_S \mathrm{Spec}(\mathbb{C})$ such that the restriction homomorphism $H^i(X_S \times_S \mathrm{Spec}(\mathbb{C})) \to H^i(U)$ in de Rham cohomology dies.

Here $\mathrm{Spec}(\mathbb{C}) \to S$ is a generic point. So unlike for the p-curvature conjecture in general, and as the p-curvature conjecture in the particular case of a Gauß-Manin connection of a smooth proper family, the the prediction does not request all closed points of a dense $S^\circ \subset S$, only a dense subset.

Sketch of proof. The proof of 1) uses the Cartier isomomorphism, the proof of 2) uses in addition Deligne-Illusie's Hodge-to-de Rham degeneration [1]. □

3. Over $\mathbb{Z}_p$: p-derived complete de Rham cohomology

3.1. **In p-derived complete de Rham cohomology.** Let U be a smooth scheme over $\mathbb{Z}_p$. *p-derived complete de Rham cohomology* is defined as

$$H^i(\hat{U}) := R^i\mathrm{lim}_n(\Omega^\bullet_{U/\mathbb{Z}_p} \otimes^L_{\mathbb{Z}_p} \mathbb{Z}_p/p^n).$$

As $U/\mathbb{Z}_p$ is smooth, we can forget the ^{L}. Thus $H^i(\hat{U})$ is an extension of the classical limit $\mathrm{lim}_n H^i(U_n)$, where $U_n = U \bmod p^n$, with $R^1\mathrm{lim} H^{i-1}(U_n) \xrightarrow{\text{Bockstein } \cong} R^1\mathrm{lim}_n H^i(U)[p^n]$, yielding the exact sequence

$$0 \to R^1\mathrm{lim}_n H^i(U)[p^n] \to H^i(\hat{U}) \to \mathrm{lim}_n H^i(U_n) \to 0. \tag{1}$$

As a $\mathbb{Z}_p$-module, $H^i(\hat{U})$ is endowed with the p-adic topology, compatibly with (1). The classical limit is then the separated quotient of $H^i(\hat{U})$ and the kernel the p-adic closure of 0.

Theorem 2. *Let X be a smooth proper scheme over $\mathbb{Z}_p$. If $H^0(X_1, \Omega^i_{X_1}) = 0$ then for any dense affine open $U \subset X$ smooth over $\mathbb{Z}_p$, the restriction homomorphism*

$$H^i(\hat{X}) \to \mathrm{lim}_n H^i(U_n)$$

in the separated quotient of $H^i(\hat{U})$ dies.

Theorem 2 goes in the direction of GGHC. At the same time it says that *any dense affine* kills the restriction homomorphism in the separated quotient of $H^i(\hat{U})$, while in algebraic de Rham cohomology the predicted affine in GGHC should be "*small enough.*" For example for $X_\circ = \mathbb{P}^1 \times \mathbb{P}^1$ and $U_\circ = X \setminus \mathrm{Diagonal}$,

$H^2(\hat{X}_\circ) = NS \otimes_{\mathbb{Z}} \mathbb{Z}_p$, where the Néron-Severi group NS is equal to $\mathbb{Z}^2$, and one computes

Lemma 3.

$$\mathrm{Im}(H^2(X_\circ)) \subset R^1\mathrm{lim}_n H^i(U_\circ)[p^n] \subset H^2(\hat{U}_\circ)$$

is not torsion.

Given Theorem 2 one could ask the following.

Problem 4 (p-GGHS)**.** Let X be a smooth proper scheme over $\mathbb{Z}_p$. If $H^0(X_1, \Omega^i_{X_1}) = 0$, is it the case that there is a open dense $U \subset X$ smooth over $\mathbb{Z}_p$ with $U_1 \subset X_1$ dense such that the restriction homomorphism

$$H^i(\hat{X}) \to H^i(\hat{U})$$

dies?

Theorem 2 says that the p-adic closure of 0 is where the subtlety of p-GGHC is hidden. It raises immediately another problem: if we had a positive answer to p-GGHC, would this imply GGHC? This is to say: is it the case that the kernel of $H^i(U) \to H^i(\hat{U})$ is torsion? As this kernel is easily described to be the submodule of strongly p-divisible elements, i.e. of those $x \in H^i(U)$, such that there are $x_n \in H^i(U),\ n \in \mathbb{N}$ with $x = x_0,\ x_n = px_{n+1}$, the problem can be formulated as follows.

Problem 5. Let U be a good smooth affine scheme over $\mathbb{Z}_p$. Are the strongly p-divisible elements in $H^i(\hat{U})$ all torsion?

Here "good" means that U is the complement of the normal crossings smooth compactification relative to $\mathbb{Z}_p$. Indeed, M. D'Addezio computed that e.g. for $p \geq 3$, $H^1(U)$ possesses non-torsion strongly p-divisible elements, where U is the complement in $\mathbb{A}^1$ of a degree 2 integral point which ramifies mod p.

Sketch of proof of Theorem 2. The vanishing in $H^i(U_1)$ is the content of Theorem 1 1). We illustrate the proof by showing how to go from U_1 to U_2. We want to show

$$0 = \mathrm{Im}H^0(X_1, \mathcal{H}^i_{X_3}) \subset H^0(X_1, \mathcal{H}^i_{X_2})$$

where $\mathcal{H}^i_{X_m}$ is the de Rham cohomology Zariski sheaf. We write the exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^\bullet_{X_2} & \longrightarrow & \Omega^\bullet_{X_3} & \longrightarrow & \Omega^\bullet_{X_1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow {\scriptstyle =} & & \\ 0 & \longrightarrow & \Omega^\bullet_{X_1} & \longrightarrow & \Omega^\bullet_{X_2} & \longrightarrow & \Omega^\bullet_{X_1} & \longrightarrow & 0 \end{array}$$

inducing the exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{H}^i_{X_2}/\delta_{1,2}\mathcal{H}^{i-1}_{X_1} & \longrightarrow & \mathcal{H}^i_{X_3} & \longrightarrow & \mathcal{H}^i_{X_1} \\ & & \downarrow & & \downarrow & & \downarrow = \\ 0 & \longrightarrow & \mathcal{H}^i_{X_1}/\delta_{1,1}\mathcal{H}^{i-1}_{X_1} & \longrightarrow & \mathcal{H}^i_{X_2} & \longrightarrow & \mathcal{H}^i_{X_1} \end{array}$$

where $\delta_{m,n}$ is the Bockstein. Combined with the Cartier isomorphism $\mathcal{H}^i_{X_1} \cong \Omega^i_{X_1}$, $\delta_{1,1} : \mathcal{H}^{i-1}_{X_1} \to \mathcal{H}^i_{X_1}$ is easily computed to be the Kähler differential $d : \Omega^i_{X_1} \to \Omega^{i+1}_{X_1}$. Thus by our vanishing assumption, the composite

$$\mathcal{H}^i_{X_2}/\delta_{1,2}\mathcal{H}^{i-1}_{X_1} \to \mathcal{H}^i_{X_1}/\delta_{1,1}\mathcal{H}^{i-1}_{X_1} = \Omega^i_{X_1}/d\Omega^{i-1}_{X_1} \to d\Omega^i_{X_1}$$

dies after taking H^0. It follows

$$0 = H^0(X_1, \mathcal{H}^i_{X_1}) = \mathrm{Im} H^0(X_1, \mathcal{H}^i_{X_2}/\delta_{1,2}\mathcal{H}^{i-1}_{X_1}) \subset H^0(X_1, \mathcal{H}^i_{X_2}).$$

This finishes the proof.

The precise computation of $\mathcal{H}^i_{X_m}$ and of their Bockstein in [3] and [4] allow to generalize the argument for all pairs $(2n-1, n)$ in place of $(3, 2)$. □

4. Over $\mathbb{Z}_p[[u]]$: prismatic cohomology

Let us write $H^i_{\Delta}(\hat{U})$ for prismatic cohomology of a smooth affine U over $\mathbb{Z}_p$. With precisely the same assumptions as in Theorem 2, one should be able to replace the p-adic separated quotient of $H^i(\hat{U})$ with the $\mathfrak{m}$-adic separated quotient of $H^i_{\Delta}(\hat{U})$. We have gone as far as computing the vanishing of the restriction homomorphism $H^i_{\Delta}(\hat{X}) \to \lim_n H^i_{\Delta/(I^2,p^n)}(\hat{U})$ where I is the prismatic ideal. Peter Scholze explained to us that our proof of Theorem 2 generalizes well to prismatic cohomology. We haven't yet checked the details.

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