ON THE ALGEBRAIC FUNDAMENTAL GROUP OF SMOOTH VARIETIES IN CHARACTERISTIC $p > 0$

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Abstract. We define an analog in characteristic $p > 0$ of the proalgebraic completion of the topological fundamental group of a complex manifold.

1. Introduction

Let X be a smooth algebraic variety defined over a field k endowed with a rational point $x \in X(k)$.

If k is the field of complex numbers $\mathbb{C}$, the proalgebraic completion $\pi^{\rm alg,rs}(X,x)$ of the topological fundamental group $\pi_1^{\rm top}(X,x)$ is defined as the prosystem $\varprojlim H$, where $H \subset GL(n,\mathbb{C})$ runs over the Zariski closures of the monodromy groups $\rho(\pi^{\rm top}(X,x))$ of complex linear representations $\rho : \pi_1^{\rm top}(X,x) \to GL(n,\mathbb{C})$. The profinite completion $\varprojlim H$, where H runs over the finite quotients of $\pi_1^{\rm top}(X,x)$, is, via the Riemann existence theorem, identified with Grothendieck's étale fundamental group $\pi_1^{\rm \acute{e}t}(X,x)$. Since any finite group is embeddable in $GL(n,\mathbb{C})$ for some n, this defines, thinking of $\pi_1^{\rm \acute{e}t}(X,x)$ as a complex (constant) proalgebraic group, a surjective homomorphism $\varphi_{\mathbb{C}}^{\rm rs} : \pi^{\rm alg,rs}(X,x) \to \pi_1^{\rm \acute{e}t}(X,x)$, and in fact $\pi_1^{\rm \acute{e}t}(X,x)$ is the profinite quotient of $\pi^{\rm alg,rs}(X,x)$. By the Riemann-Hilbert correspondence, $\pi^{\rm alg,rs}(X,x)$ is the Tannaka group-scheme of the category of $\mathcal{O}_X$-coherent regular singular $\mathcal{D}_X$-modules, which is a full subcategory of the category of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules. We denote by $\pi^{\rm alg}(X,x)$ the corresponding Tannaka group-scheme, and by $\varphi_{\mathbb{C}} : \pi^{\rm alg}(X,x) \twoheadrightarrow \pi^{\rm alg,rs}(X,x) \xrightarrow{\varphi_{\mathbb{C}}^{\rm rs}} \pi_1^{\rm \acute{e}t}(X,x)$ the composite morphism. It is surjective as well, and since any flat connection with finite monodromy is regular singular, $\pi_1^{\rm \acute{e}t}(X,x)$ is the profinite quotient of $\pi^{\rm alg}(X,x)$.

If k is a characteristic 0 field, $\pi^{\rm alg}(X,x)$ is defined as the Tannaka group-scheme of the k-linear tensor category of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules equipped with the fiber functor defined as the restriction of the module on x. The full subcategory of *finite objects*, that is objects with finite monodromy group-scheme, or said differently, objects which have the property that the full Tannaka subcategory which is spanned by it has a finite Tannaka group-scheme, defines a pro-finite k-group-scheme $\pi^{\rm \acute{e}t}(X,x)$. Since $\pi^{\rm \acute{e}t}(X,x)(\bar{k}) = \pi_1^{\rm \acute{e}t}(X,x)$ ([5, Remark 2.10]), and both $\pi^{\rm alg}(X,x)$ and $\pi^{\rm \acute{e}t}(X,x)$ satisfy base change for finite extensions $k \subset L$ ([6,

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Property 2.54)]), we see that the surjection $\varphi : \pi^{\mathrm{alg}}(X,x) \to \pi^{\text{ét}}(X,x)$ is a k-form of $\varphi_{\mathbb{C}}$ for any complex embedding $k \subset \mathbb{C}$. Moreover, by definition, φ induces the pro-finite quotient homomorphism.

If k is a characteristic $p > 0$ field, the category of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules is again a k-linear abelian tensor rigid category. It is part of Katz' theorem asserting that this category is equivalent to the category of stratified $\mathcal{O}_X$-coherent sheaves (see [9, Theorem 1.3], [3, Theorem 8], where it is shown over $k = \bar{k}$). If $k = \bar{k}$, its Tannaka group-scheme $\pi^{\mathrm{alg}}(X,x)$ is shown to be pro-smooth in [3, Corollary 12] (strictly speaking, it is shown there only for the profinite part, but dos Santos' proof applies more generally as mentioned in [4, Corollary 7]). The homomorphism φ is then defined by the full embedding of the subcategory of objects with finite monodromy group-scheme. So by definition, φ induces the pro-finite quotient homomorphism.

On the other hand, if X is a reduced connected scheme over a characteristic $p > 0$ field k, endowed with a rational point $x \in X(k)$, Nori [10, Chapter II] constructed a fundamental group-scheme $\pi^N(X,x)$ as the projective system of finite k-group-schemes G for which there is a G-torsor $h : Y \to X$ under G with trivialization at x. The pro-étale quotient of $\pi^N(X,x)$ is precisely $\pi^{\text{ét}}(X,x)$.

Summarizing, one has a diagram

$$\begin{array}{ccc} \pi^{\mathrm{alg}}(X,x) & \xrightarrow{\text{surj}} & \pi^{\text{ét}}(X,x) \\ & & \uparrow{\scriptstyle \text{surj}} \\ & & \pi^N(X,x) \end{array} \tag{1.1}$$

The aim of our article is to define a Tannaka category $\mathsf{Strat}(X,\infty)$ over a perfect field k, which contains the category of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules as a full subcategory, in such a way that its Tannaka group-scheme $\pi^{\mathrm{alg},\infty}(X,x)$, which thus surjects onto $\pi^{\mathrm{alg}}(X,x)$, also surjects onto $\pi^N(X,x)$. In other words, we complete (1.1) to

$$\begin{array}{ccc} \pi^{\mathrm{alg}}(X,x) & \xrightarrow{\text{surj}} & \pi^{\text{ét}}(X,x) \\ {\scriptstyle \text{surj}}\uparrow & & \uparrow{\scriptstyle \text{surj}} \\ \pi^{\mathrm{alg},\infty}(X,x) & \xrightarrow{\text{surj}} & \pi^N(X,x) \end{array} \tag{1.2}$$

As a byproduct, we obtain a purely tannakian geometric description of $\pi^N(X,x)$ (see Corollary 4.9). Recall that we assume that X is smooth. If in addition X is proper, Nori himself described his fundamental group-scheme $\pi^N(X,x)$ as the Tannaka group-scheme of the category of essentially finite bundles [10, Chapter I]. He extends in [10, Chapter III] his construction to non-proper curves by using parabolic bundles. Lacking desingularization in characteristic $p > 0$ makes it difficult to generalize his construction to the higher dimensional case. If k has characteristic 0, then, as already mentioned, $\pi^N(X,x) = \pi^{\text{ét}}(X,x)$ is the Tannaka group-scheme of the category of finite flat connections [6, Section 2], or, equivalently, of the category of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules with finite monodromy group-scheme.

Our construction (see Section 3, most particularly Definition 3.2) generalizes on a smooth variety defined over a perfect characteristic $p > 0$ field k the construction of the category of flat connections (*loc. cit*) in characteristic 0, and the construction of the stratified bundles (*loc. cit.*) in characteristic $p > 0$. We now explain the main idea.

For $i \in \mathbb{N}$, let us define inductively the relative Frobenius $F^{(i)} : X^{(i)} \to X^{(i+1)}$ over k in the usual manner. As k is assumed to be perfect, one defines $X^{(-1)} = X \otimes_{k, F_k^{-1}} k$ where $F_k : \operatorname{Spec} k \to \operatorname{Spec} k$ is the absolute Frobenius of k, together with the relative Frobenius $F^{(-1)} : X^{(-1)} \to X^{(0)}$. Then one iterates to define inductively $F^{(i)} : X^{(i)} \to X^{(i+1)}$ for $i \in \mathbb{Z}, i < 0$. For $a, b \in \mathbb{Z}, a < b$ we define $F^{(a,b)} : X^{(a)} \xrightarrow{F^{(a)} \circ \ldots \circ F^{(b-1)}} X^{(b)}$.

Recall that a stratified bundle is a sequence $(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N})$, where $E^{(i)}$ is a bundle on $X^{(i)}$, $\sigma^{(i)} : E^{(i)} \xrightarrow{\cong} F^{(i)*}E^{(i+1)}$ is a $\mathcal{O}_{X^{(i)}}$-isomorphism. For $t \in \mathbb{N}, t \neq 0$, we define an object of $\mathsf{Strat}(X,t)$ to be a sequence $(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N})$, where $E^{(i)}$ is a bundle on $X^{(i)}$, $\sigma^{(i)} : E^{(i)} \xrightarrow{\cong} F^{(i)*}E^{(i+1)}$ is a $\mathcal{O}_{X^{(i)}}$-isomorphism for all $i \geq 1$, but for $i = 0$, $\sigma_0 : F^{(-t,0)*}E^{(0)} \xrightarrow{\cong} F^{(-t,1)*}E^{(1)}$ is a $\mathcal{O}_{X^{(-t)}}$-isomorphism. The morphisms are the ones between the bundles which respect all the structures. We show (Theorem 3.4) that the obvious functor $\mathsf{Strat}(X,t) \subset \mathsf{Strat}(X,t+1)$, which assigns $(E_i, F^{(-t-1)*}\sigma_0, \sigma_i, i \geq 1)$ to $(E_i, \sigma_0, \sigma_i, i \geq 1)$, induces a full embedding of Tannaka categories, where the fiber functor is simply the restriction of $E^{(0)}$ to the rational point x. Then $\mathsf{Strat}(X,\infty)$ is defined as the inductive limit over $t \to \infty$ of the categories $\mathsf{Strat}(X,t)$ (Corollary 3.5). In order to show that the Tannaka group-scheme $\pi^{\text{alg},\infty}(X,x)$ of $\mathsf{Strat}(X,\infty)$ surjects onto $\pi^N(X,x)$, we use a slight modification of Nori's reconstruction theorem [10, Chapter I, Proposition 2.9] of a torsor $h : Y \to X$ under a finite group scheme G out of the induced functor $h^{\#} : \mathsf{Rep}_k(G) \to \mathsf{Coh}(X)$ which assigns to a finite dimensional k-linear representation V of G the vector bundle on X which is defined by flat descent for h on $\mathcal{O}_Y \times_k V$ (Theorem 2.4).

This allows us to define the group-scheme homomorphism $\pi^{\text{alg},\infty}(X,x) \to \pi^N(X,x)$ (Theorem 4.5). In order to show that this map induces the profinite quotient, we in particular use the categorial translation of injectivity and surjectivity of homomorphisms of Tannaka group-schemes ([2, Proposition 2.12]).

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2. Nori's fundamental group-scheme

Let k be a field of characteristic $p > 0$ and X be a k-scheme. Let $x \in X(k)$ be a rational point and $i_x : x \to X$ be the closed embedding.

Nori [10, Chapter II] defines the category $\mathsf{N}(X, x)$ of triples $(Y \xrightarrow{f} X, G, y)$ where

(a) G/k is a finite group scheme,
(b) $f : Y \to X$ is a G-torsor,
(c) y is a k-point of Y lying above x.

A morphism between two such triples $(Y_i \xrightarrow{f_i} X, G_i, y_i) \quad i = 1, 2$, is a pair $(\phi : G_1 \to G_2, \psi : Y_1 \to Y_2)$ such that ψ an X-morphism which is ϕ-equivariant and $\psi(y_1) = y_2$. Nori shows [10, Chapter II, Proposition 2] that if X is reduced and geometrically connected, then the projective limit $\varprojlim_{\mathsf{N}(X,x)} G$ exits. He defines

Definition 2.1. Let X be a reduced geometrically connected k-scheme, then it's Nori fundamental group-scheme is the profinite k-group-scheme

$$\pi^N(X, x) = \varprojlim_{\mathsf{N}(X,x)} G.$$

Since giving a rational point $y \in f^{-1}(x)$ is the same as giving a trivialization $f^{-1}(x) \cong_k G$, $\mathsf{N}(X, x)$ is equivalent to the category of triples $(h : Y \to X, G, f^{-1}(x) \cong_k G)$, where the morphisms between two such objects are defined by torsor morphisms which respect the trivialization. We will not need this slightly different phrasing.

Definition 2.2. Let G be a finite k-group-scheme, and let $h : Y \to X$ be a G-torsor. Then it induces a functor $h^{\#} : \mathsf{Rep}_k(G) \to \mathsf{Coh}(X)$ which assigns to a finite dimensional k-representation V the bundle on X which comes by flat descent from $\mathcal{O}_Y \otimes_k V$.

Properties 2.3.
1) The functor $h^{\#}$ defined in Definition 2.2 is exact, k-linear and compatible with the tensor structure. Thus it is a *fiber functor* in the sense of Deligne [1, 1.9]. Since $\mathsf{Rep}_k(G)$ is a Tannaka category, it follows [1, Corollaire 2.10] that $h^{\#}$ is faithful.
2) The functor $i_x^* : \mathsf{Coh}(X) \to \mathsf{Vec}_k$ defined as the restriction to the rational point, with values in the category of finite dimensional k-vector spaces, is a fiber functor on the subcategory of vector bundles. The composite functor $i_x^* \circ h^{\#} : \mathsf{Rep}_k(G) \to \mathsf{Vec}_k$ is a fiber functor.
3) Let $h_i : Y_i \to X$ be G_i-torsors where $i = 1, 2$. Let $\phi : G_1 \to G_2$ be a group homomorphism and $\psi : Y_1 \to Y_2$ be an equivariant map with respect to ϕ. We denote by ϕ^* the induced functor $\mathsf{Rep}_k(G_2) \to \mathsf{Rep}_k(G_1)$. Then

one has the equality $h_2^\# = h_1^\# \circ \phi^*$ of functors. Indeed, if V is a G_2-representation, $\psi^* : \mathcal{O}_{Y_2} \otimes_k V \to \psi_*(\mathcal{O}_{Y_1} \otimes_k \phi^*(V))$ induces a $\mathcal{O}_X$-linear map $h_2^\#(V) \to h_1^\#(V)$ between those two vector bundles, which, after composing with i_x^*, is the identity on V. So $h_2^\#(V) = h_1^\# \circ \phi^*(V)$.

4) Let $h : Y \to X$ be a G-torsor, let $b : X' \to X$ be a morphism, and let $x' \in X'(k)$ be a rational point with $b(x') = x$. Let $Y' = Y \times_X X' \to X'$ and $h' : Y' \to X'$ denote the projection. Then one has the equality $b^* \circ h^\# = h'^\#$ of functors. Indeed, denoting by $b' : Y' \to Y$ the induced morphism, if V is a G-representation, $(b')^* : \mathcal{O}_Y \otimes_k V \to (b')_* \mathcal{O}_{Y'} \otimes_k V$ induces $\mathcal{O}_{X'}$-linear map $b^* \circ h^\#(V) \to (h')^\#(V)$ between vector bundles, which is the identity on V after composing with $i_{x'}$. So $b^* \circ h^\# = (h')^\#$.

The following is a direct consequence of [10, Proposition 2.9].

Theorem 2.4. *Let G be a finite k-group-scheme and let $F : \mathsf{Rep}_k(G) \to \mathsf{Coh}(X)$ be a fiber functor such that $i_x^* \circ F$ is the forgetful functor $F_G : \mathsf{Rep}_k(G) \to \mathsf{Vec}_k$. Then there exists a unique object $(Y \xrightarrow{h} X, G, y)$ of $\mathsf{N}(X, x)$ such that $F = h^\#$ and $(h^{-1}(x), y) = (G, 1)$. For any other object $(Y' \xrightarrow{h'} X, G, y') \in \mathsf{N}(X, x)$ such that $F = h^\#$, there exists a unique isomorphism in $\mathsf{N}(X, x)$ between $(Y \xrightarrow{h} X, G, y)$ and $(Y' \xrightarrow{h'} X, G, y')$.*

Proof. By Nori's reconstruction theorem [10, Proposition 2.9], $F(k[G])$, where $k[G]$ is the regular representaton of G, is a finite $\mathcal{O}_X$-algebra. The G-torsor $h : Y \to X$ is defined to be $\mathrm{Spec}_X F(k[G])$. By Property 2.3 2), $i_x^* \circ F(k[G]) = F_G(k[G]) = k[G]$. Said differently, $h^{-1}(x) = \mathrm{Spec}_x k[G] = G$. Then y is the rational point of $h^{-1}(x)$ which is $1 \in G$. By the unicity in *loc. cit.*, h is uniquely defined. If $y' = g \in h^{-1}(x)(k)$ is another rational point, then multiplication $g : Y \to Y$ by g, together with the conjugation $G \to G, h \mapsto ghg^{-1}$ defines an isomorphism $(h : Y \to X, G, y) \to (h : Y \to X, G, y')$ in $\mathsf{N}(X, x)$. ☐

3. The category of generalized stratified bundles

The aim of this section is to define the category of *generalized stratified bundles.* We start with some notations.

Notations 3.1. Let k be a perfect field of characteristic $p > 0$, X be a *smooth* scheme over k which is geometrically irreducible.

For $i \in \mathbb{N}$, we define inductively the relative Frobenius $F^{(i)} : X^{(i)} \to X^{(i+1)}$ over k in the usual manner, by defining $X^{(0)} = X$, $X^{(i+1)}$ to be the fiber product of $X^{(i)} \otimes_{k, F_k} k$ over the absolute Frobenius $F_k : \mathrm{Spec}\, k \to \mathrm{Spec}\, k$ of k, and $F^{(i)}$ to be the factorization of the absolute Frobenius $F_{X^{(i)}} : X^{(i)} \to X^{(i)}$ morphism.

For $i \in \mathbb{Z}, i < 0$, we define inductively $F^{(i)} : X^{(i)} \to X^{(i+1)}$ over k as follows. First we set $X^{(-1)} = X \otimes_{F_k^{-1}} k$. Then we define $F^{(-1)} : X^{(-1)} \to X$ to be the relative Frobenius. Similarly, we define $X^{(-i-1)} = X^{(-i)} \otimes_{F_k^{-1}} k$ together with the relative Frobenius $F^{(-i-1)} : X^{(-i-1)} \to X^{(-i)}$ over k.

For $a, b \in \mathbb{Z}, a < b$ we define $F^{(a,b)} : X^{(a)} \xrightarrow{F^{(a)} \circ \ldots \circ F^{(b-1)}} X^{(b)}$.

Recall that a *stratified bundle* (see [9, Section 1]) is a sequence $(E^{(i)}, \sigma^{(i)}), i \in \mathbb{N}$, where $E^{(i)}$ is a $\mathcal{O}_X$-coherent sheaf on $X^{(i)}$, $\sigma^{(i)} : E^{(i)} \xrightarrow{\cong} F^{(i)*}E^{(i+1)}$ is a $\mathcal{O}_{X^{(i)}}$-isomorphism. One defines the *category* $\mathsf{Strat}(X)$ *of stratified bundles* by defining

$$\mathrm{Hom}((D^{(i)}, \tau^{(i)}), (E^{(i)}, \sigma^{(i)}))$$

to be set of sequences $f_i : D^{(i)} \to E^{(i)}$ of morphisms of $\mathcal{O}_{X^{(i)}}$-coherent sheaves, which commute with all the σ_i and τ_i. It is a fact (*loc. cit.*) that if $(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N})$ is a stratified sheaf, the $E^{(i)}$ are all locally free, and if $f = (f)_i, i \in \mathbb{N}$ is a morphism of stratified sheaves, then f_i are vector bundle maps (i.e. locally split), so the categroy is abelian, rigid, and monoidal. Moreover, the Hom-sets are finite dimensional k-vector spaces. As X is geometrically irreducible, the unit object $\mathbb{I} = (\mathcal{O}_X, \mathrm{Id}), i \in \mathbb{N}$ fulfills $\mathrm{End}(\mathbb{I}) = k$. If now X is endowed with a rational point $x \in X(k)$, then $\omega_x : \mathsf{Strat}(X) \to \mathsf{Vec}_k, (E^{(i)}, \sigma^{(i)}) \mapsto E_0|_x$ is a fiber functor in the sense of Deligne [1, 1.9], and thus yields the structure of a Tannaka category on $\mathsf{Strat}(X)$. A fundamental property due to dos Santos is that the corresponding Tannaka k-group-scheme $\mathrm{Aut}^{\otimes}(\omega_x)$ is pro-smooth ([3, Corollary 12], [4, Corollary 7]).

Definition 3.2. Let $t \geq 0$ be an integer. A *t-stratified bundle* is a sequence

$$(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}),$$

where $E^{(i)}$ is a $\mathcal{O}_X$-coherent sheaf on $X^{(i)}$,

$$\sigma^{(i)} : E^{(i)} \xrightarrow{\cong} F^{(i)*}E^{(i+1)}$$

is a $\mathcal{O}_{X^{(i)}}$-isomorphism for $i \geq 1$ and for $i = 0$,

$$\sigma^{(0)} : F^{(-t,0)*}E^{(0)} \xrightarrow{\cong} F^{(-t,1)*}E^{(1)}$$

is a $\mathcal{O}_{X^{(-t)}}$-isomorphism.

One defines the *category* $\mathsf{Strat}(X, t)$ *of t-stratified bundles* by defining

$$\mathrm{Hom}((D^{(i)}, \sigma^{(i)}), (E^{(i)}, \tau^{(i)}))$$

to be set of sequences $f_i : D^{(i)} \to E^{(i)}$ of morphisms of $\mathcal{O}_X$-coherent sheaves, which commute with all the σ_i and τ_i.

In particular, $\mathsf{Strat}(X, 0) = \mathsf{Strat}(X)$.

Example 3.3. We now give an example of a non-trivial 1-stratified bundle on $X = \mathbb{A}^1_k = \mathrm{Spec}\,(k([x])$. Thus $X^{(i)} = \mathrm{Spec}\,(k[x_i])$ where the relative Frobenius $X^{(i)} \to X^{(i+1)}$ is induced by $x_{i+1} \to x_i^p$. For simplicity let us assume $p = \mathrm{char}(k) = 2$. Let V be a 2-dimensional vector space over k with basis e_1, e_2. Define

$$E^{(i)} = \mathcal{O}_{X^{(i)}} \otimes_k V \quad \forall\ i\ \geq 0$$

and

$$\sigma^{(i)} : E^{(i)} \to F^{(i)*}E^{(i+1)}, \quad i \geq 1$$

to be the isomorphism induced by the identity on V. We define

$$\sigma^{(0)} : F^{(-1,0)*}E^{(0)} \to F^{(-1,1)*}E^{(1)}$$

to be the isomorphism defined by sending

$$e_1 \to e_1, \quad e_2 \to x_{-1}e_1 + e_2.$$

We claim that the -1-stratified bundle thus defined is not isomorphic to the trivial stratified bundle of rank 2. If indeed this were the case, then we would have a $k[x]$-module automorphism $\phi : k[x] \otimes_k V \to k[x] \otimes_k V$, such that

$$\phi \otimes_{k[x]} k[x_{-1}] = \sigma^{(0)}.$$

This is impossible since x_{-1} is not contained in $k[x]$. It can be shown (see (4.3)) that this -1-stratified bundle "arises" from the non-trivial α_p-torsor on $\mathbb{A}^1_k$ defined by the relative Frobenius of $\mathbb{A}^1_k$.

Theorem 3.4. *The notations are as in 3.1.*

1) *For every integer $t \geq 0$, $\mathsf{Strat}(X, t)$ is a k-linear, abelian, rigid, tensor category.*
2) *The functor*

$$(+) : \mathsf{Strat}(X, t) \subset \mathsf{Strat}(X, t+1)$$
$$(E_i, \sigma_0, \sigma_i, i \geq 1) \mapsto (E_i, F^{(-t-1)*}\sigma_0, \sigma_i, i \geq 1),$$

induces a full faithful embedding of k-linear, abelian, rigid, tensor categories.

3) *If $x \in X(k)$ is a rational point, the functor*

$$\omega_x : \mathsf{Strat}(X, t) \to \mathsf{Vec}_k$$
$$(E^{(i)}, \sigma^{(i)}) \mapsto E_0|_x$$

is a fiber functor, which makes $(\mathsf{Strat}(X, t), \omega_x)$ a Tannaka category.

Proof. We show 1). Since $\mathsf{Strat}(X, 0) = \mathsf{Strat}(X)$, we assume $t > 0$. If $(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N})$ is an object in $\mathsf{Strat}(X, t)$, then $(E^{(i)}_+ = E^{(i+1)}, \sigma^{(i)}_+ = \sigma^{(i+1)}, i \in \mathbb{N})$ is an object $\mathrm{Ver}(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}) \in \mathsf{Strat}(X^{(1)})$. Since $E^{(i)}$ is locally free, by the isomorphism $\sigma^{(0)}$, $F^{(-t,0)*}E^{(0)}$ is locally free. Since X is smooth, the relative Frobenius is flat, thus by flat descent, $E^{(0)}$ is locally free as well. So $\mathsf{Strat}(X)$ is rigid and monoidal. On the other hand,

$$\begin{aligned} (3.1)\quad & \mathrm{Hom}((D^{(i)}, \tau^{(i)}, i \in \mathbb{N}), (E^{(i)}, \sigma^{(i)}, i \in \mathbb{N})) \\ & \subset \mathrm{Hom}(\mathrm{Ver}(D^{(i)}, \tau^{(i)}, i \in \mathbb{N}), \mathrm{Ver}(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N})) \end{aligned}$$

and is obviously a k-vector space. So the Hom-sets are finite dimensional k-vector spaces. Moreover, any morphism $f = (f^{(i)}, i \in \mathbb{N})$ is such that $f^i, i \geq 1$

is a morphism of vector bundles. Thus by the ismorphisms $\tau^{(0)}, \sigma^0$, Ker, Im and Coker of $f^{(0)}$ are pulled back to vector bundles on $X^{(-t)}$ via $F^{(-t,0)}$, thus by flat descent again, there are vector bundles on X. We conclude that $\mathsf{Strat}(X,t)$ is an abelian category. This shows 1).

2) follows immediately from the factorization of (3.1) through (+).

We show 3): the point $x \in X(k)$ maps to $x^{(1)} \in X^{(1)}(k)$, and the map $x \to x^{(1)}$ is the identity on the residue fields $k(x) = k(x^{(1)}) = k$. If $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathsf{Strat}(X,t)$, then $0 \to \mathrm{Ver}(A) \to \mathrm{Ver}(B) \to \mathrm{Ver}(C) \to 0$ is an exact sequence in $\mathsf{Strat}(X^{(1)})$, thus $0 \to \omega_{x^{(1)}}(\mathrm{Ver}(A)) \to \omega_{x^{(1)}}(\mathrm{Ver}(B)) \to \omega_{x^{(1)}}(\mathrm{Ver}(C)) \to 0$ is an exact sequence in Vec_k. But

$$\omega_{x^{(1)}}(\mathrm{Ver}(A)) = \omega_x(A). \tag{3.2}$$

This shows that ω_x is exact. Furthermore, ω_x is obviously k-linear and compatible with the tensor structure. This finishes the proof.

□

Corollary 3.5. *Let the notations be as in Theorem 3.4. The category*

$$\mathsf{Strat}(X,\infty) = \varinjlim_{+,t\in\mathbb{N}} \mathsf{Strat}(X,t)$$

is a k-linear, abelian, rigid tensor category, on which, if X has a rational point $x \in X(k)$, the functor ω_x is a fiber functor.

Definition 3.6. The notations are as in Theorem 3.4.

1) We define $\pi^{\mathrm{alg}}(X,x)$ to be the Tannaka k-group scheme $\mathrm{Aut}^{\otimes}(\omega_x)$ of $(\mathsf{Strat}(X), \omega_x)$.
2) We define $\pi^{\mathrm{alg},\infty}(X,x)$ to be the Tannaka k-group scheme $\mathrm{Aut}^{\otimes}(\omega_x)$ of $(\mathsf{Strat}(X,\infty), \omega_x)$.

The functor $(+) : \mathsf{Strat}(X) \to \mathsf{Strat}(X,\infty)$ defines the homomorphism

$$(+)^* : \pi^{\mathrm{alg},\infty}(X,x) \to \pi^{\mathrm{alg}}(X,x). \tag{3.3}$$

Lemma 3.7. *The homomorphism $(+)^*$ in (3.3) is faithfully flat.*

Proof. We apply [2, Proposition 2.21]. As (+) is fully faithful, the lemma is equivalent to saying that if A is an object on $\mathsf{Strat}(X)$, and $B \subset (+)A$ is a subobject in $\mathrm{Strat}(X,\infty)$, then there is a subobject $B' \subset A$ in $\mathsf{Strat}(X)$ such that $B = (+)B'$. One has that $\mathrm{Ver}(B) \subset \mathrm{Ver}(A)$ is a subobject in $\mathsf{Strat}(X^{(1)})$, thus $F^{(0)*}B^{(1)} \subset A^{(0)}$ is a subvector bundle with the property that $F^{(-t,0)*} \circ F^{(0)*}B^{(1)} = F^{(-t,1)*}B^{(1)} = F^{(-t,0)*}B^{(0)}$. Thus $B' = (F^{(0)*}B^{(1)}, B^{(i)}, i \geq 1, F^{(0)*}, \sigma^{(i)}, i \geq 1) \subset A$ is a subobject of A such that $(+)B' = B$. This finishes the proof. □

4. Comparison of $\pi^{\mathrm{alg},\infty}(X,x)$ with $\pi_1^N(X,x)$

In order to achieve the comparison, we start with a construction.

Construction 4.1. The notations are as in 3.1, and $x \in X(k)$ is a rational point. Let $(h : Y \to X, G, y)$ be an object of $\mathsf{N}(X, x)$. Using this object, we construct a tensor functor

$$h^* : \mathsf{Rep}_k(G) \to \mathsf{Strat}(X, \infty)$$

together with a factorization of functors

$$\begin{tikzcd} \mathsf{Rep}_k(G) \arrow[r, "h^*"] \arrow[dr, "F_G"'] & \mathsf{Strat}(X, \infty) \arrow[d, "\omega_x"] \\ & \mathsf{Vec}_k \end{tikzcd} \tag{4.1}$$

Here $F_G : \mathsf{Rep}_k(G) \to \mathsf{Vec}_k$ is the forgetful functor.

Recall that if G is a finite k-group-scheme, there is an exact sequence of finite k-group schemes $1 \to G_0 \to G \to G_{\text{ét}} \to 1$, where G_0 is the 1-component of G and $G_{\text{ét}}$ is étale. Furthermore, as k is perfect, $G_{\text{red}} \subset G$ is a closed subgroup-scheme and the composite $G_{\text{red}} \xrightarrow{\iota} G \to G_{\text{ét}}$ is an isomorphism. Thus ι yields on G the structure of a semi-direct product of $G_{\text{ét}}$ by G_0. The construction of h^* will be such that the image of h^* is contained in $\mathsf{Strat}(X, t)$, where t is a natural number such that the image of the k-group-scheme homomorphism $G^{(-t)} \to G$ is equal to $G_{\text{ét}}$.

Let V be a finite dimensional k-representation of G. We set

$$E^{(0)} = h^{\#}(V). \tag{4.2}$$

For $i \in \mathbb{N} \setminus \{0\}$, the relative Frobenius is an isomorphism of the étale k-group-schemes

$$F^{(0,i)} : G_{\text{ét}} \xrightarrow{\cong} G_{\text{ét}}^{(i)}. \tag{4.3}$$

Thus $\iota(G) \circ F^{(0,i)-1} : G_{\text{ét}}^{(i)} \subset G$ is a closed embedding and composing with it defines a $G_{\text{ét}}^{(i)}$-action on V. Since $h : Y \to X$ is a G-torsor, for $i \geq 0$, $h^{(i)} : Y^{(i)} \to X^{(i)}$ is also a $G^{(i)}$-torsor. Let $h_{\text{ét}}^{(i)} : Y_{\text{ét}}^{(i)} \to X^{(i)}$ be the induced $G_{\text{ét}}^{(i)}$-torsor obtained by moding out by $G_0^{(i)}$. We define

$$E^{(i)} = (h_{\text{ét}}^{(i)})^{\#}(V). \tag{4.4}$$

One has

$$\sigma^{(i)} : E^{(i)} \xrightarrow{=} F^{(i)*} E^{(i+1)}, \ i \in \mathbb{N} \setminus \{0\}. \tag{4.5}$$

The object $h^*(V) \in \mathsf{Strat}(X, t)$ which we wish to construct will have the property

$$\mathrm{Ver}(h^*(V)) = (E^{(i)}, \sigma^{(i)}, i \geq 1). \tag{4.6}$$

It remains to define $\sigma^{(0)}$. By definition,

$$F^{(0)*} E^{(1)} = (h_{\text{ét}}^{(0)})^{\#}(V) = (h_{\text{ét}})^{\#}(V). \tag{4.7}$$

Let t be a natural number such that the image of $G^{(-t)} \to G$ is equal to $G_{\text{ét}}$. One has the following commutative diagram of k-varieties.

$$\begin{tikzcd}
Y^{(-t)} \arrow[rr, "F^{(-t,0)}"] \arrow[dd] \arrow[dr] & & Y \arrow[dd] \arrow[dr] & \\
& Y_{\text{ét}}^{(-t)} \arrow[ur, dashed, "\exists!\ \lambda"] \arrow[rr, "F^{(-t,0)}"] \arrow[dl, "h^{(-t)}"'] & & Y_{\text{ét}} \arrow[dl, "h"] \\
X^{(-t)} \arrow[rr, "F^{(-t,0)}"] & & X &
\end{tikzcd} \tag{4.8}$$

The morphism $F^{(-t,0)} : Y^{(-t)} \to Y$ is equivariant under $F^{(-t,0)} : G^{(-t)} \to G$. Likewise, the morphism $F^{(-t,0)} : Y_{\text{ét}}^{(-t)} \to Y_{\text{ét}}$ is equivariant under $F^{(-t,0)} : G_{\text{ét}}^{(-t)} \to G_{\text{ét}}$. The commutativity of the diagram implies that

$$\lambda^*(\mathcal{O}_Y \otimes_k V) = F^{(-t,0)*}(\mathcal{O}_{Y_{\text{ét}}} \otimes_k V) = F^{(-t,1)*}(\mathcal{O}_{Y_{\text{ét}}^{(1)}} \otimes_k V) \tag{4.9}$$

equivariantly for the action of $G_{\text{ét}}^{(-t)}$. Thus

$$(h_{\text{ét}}^{(-t)})^{\#}(V) = F^{(-t,0)*}E^{(0)} = F^{(-t,1)*}E^{(1)}. \tag{4.10}$$

We define $\sigma^{(0)} : F^{(-t,0)*}E^{(0)} = F^{(-t,1)*}E^{(1)}$ to be the equality of (4.10).

Thus, starting with $V \in \mathsf{Rep}_k(G)$, we have constructed an object $h^*(V) = (E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}) \in \mathsf{Strat}(X, t)$. Clearly, any $\phi \in \mathrm{Hom}_{\mathsf{Rep}_k(G)}(V, W)$ induces $h^*(\phi) \in \mathrm{Hom}_{\mathsf{Strat}(X,t)}(h^*(V), h^*(W))$. This defines the functor

$$h^* : \mathsf{Rep}_k(G) \to \mathsf{Strat}(X, \infty). \tag{4.11}$$

by composing with (+). Moreover, one has

$$h^*(V)_x = (\mathcal{O}_Y \otimes_k V)_y = V. \tag{4.12}$$

This shows the commutativity of (4.1).

Remark 4.2. In the above construction we use the fact that for a finite flat group scheme G over a perfect field k, the epimorphism $G \to G_{\text{ét}}$ admits a section (necessarily unique). In other words $G_{\text{ét}}$ can be canonically thought of as a subgroup scheme of G via the identification $G_{\text{red}} = G_{\text{ét}}$. When k is not a perfect field, G_{red} may not be a subgroup scheme, (for example, $G = \operatorname{Spec} k[t]/(t^{p^2} - at^p)$, $a \in k \setminus k^p$, see [8, Chapter III, Exercice (3.2)],) and the above construction of h^* does not make sense. This is the reason why we assume throughout k to be perfect. We thank Nguyên Duy Tân for this important remark.

Example 4.3. Let $p = \mathrm{char}(k) = 2$ for simplicity and let $G = \alpha_2 = \operatorname{Spec}(k[t]/t^2)$. Let $X = \mathbb{A}^1_k = \operatorname{Spec}(k[x])$. Let $P = \operatorname{Spec}(k[u])$, and $h : P \to X$ be the relative Frobenius defined by $x \to u^2$. Then h is a G-torsor. Thus by Construction (4.1), one has a functor

$$h^* : \mathsf{Rep}_k(G) \to \mathsf{Strat}(X, -1).$$

We compute now that $h^*(k[G])$ is nothing but the -1-stratified bundle defined in Example 3.3. Here $k[G] = k[v]/(v^2)$ is the regular representation of G. As in Example (3.3), let $X^{(i)} = k[x_i]$. Let $h^*(k[G]) = (E^{(i)}, \sigma^{(i)}, i \in \mathbb{N})$. As all schemes are affine, we confuse coherent sheaves with corresponding modules. Since $G_{\text{ét}}$ is trivial, by definition of h^* we see that

$$E^{(i)} = k[x_i] \otimes_k k[v]/(v^2) \quad \forall\; i \geq 1$$

with

$$\sigma^{(i)} : E^{(i)} \to F^{(i)*}E^{(i+1)} \quad i \geq 1$$

induced by the identity map on $k[v]/(v^2)$. Then $E^{(0)}$ is by definition the $k[x]$-module of invariants of $k[u] \otimes_k k[v]/(v^2)$, where the action of $G = \operatorname{Spec} k[t]/(t^2)$ is defined by

$$u \to u + t, \quad v \to v + t.$$

Since $(u + v)^2 = u^2 = x$, one has $E^{(0)} = k[x] \cdot 1 \oplus k[x] \cdot (u + v)$. On P we have an identification

$$h^*E^{(0)} = k[u] \otimes_k k[v]/(v^2)$$

defined by $\tau : 1 \mapsto 1 \otimes 1, u + v \mapsto u \otimes 1 + 1 \otimes v$. The map $\sigma^{(0)}$ is nothing but the pull back of τ via the isomorphism $X^{(-1)} \to P$ defined by

$$k[u] \to k[x_{-1}], \quad u \to x_{-1}.$$

We thus see that

$$\sigma^{(0)} : k[x_{-1}] \cdot 1 \oplus k[x_1] \cdot (u + v) \longrightarrow k[x_{-1}] \otimes k[v]/(v^2)$$

is defined by $1 \mapsto 1 \otimes, (u+v) \mapsto u \otimes 1 + 1 \otimes v$. It is then an elementary exercise to see that the stratified bundle $h^*(k[G])$ is isomorphic to the -1 stratified bundle defined in Example 3.3.

Lemma 4.4. *The functor h^* defined in* (4.11) *is k-linear, exact, compatible with the tensor structure and faithful.*

Proof. As already recalled in the Properties 2.3 1), faithfulness follows from the remaining properties. On the other hand, k-linearity, and compatibility with the tensor structures are straightforward. Exactness is proven as using Ver as in Theorem 3.4 3). Indeed, $\mathrm{Ver} \circ h^*$ with values in $\mathsf{Strat}(X^{(1)})$ is obviously exact, while a sequence in $\mathsf{Strat}(X, \infty)$ is exact if and only if it remains exact after applying Ver. □

If $(h_i : Y_i \to X, G_i, y_i)$ are objects in $\mathsf{N}(X, x)$ for $i = 1, 2$ and $(\psi : Y_1 \to Y_2, \phi : G_1 \to G_2, y_1 \to y_2)$ is a morphism in $\mathsf{N}(X, x)$, then Property 2.3 3) implies that $h_2^* = h_1^* \circ \phi^*$. On the other hand, the projective system of ϕ in $\mathsf{N}(X, x)$ induces an inductive system $\varinjlim_{\mathsf{N}(X,x),\phi^*} \mathsf{Rep}_k(G)$ which is a Tannaka category, with the forgetful functor F_G as the fiber functor. The Tannaka k-group-scheme

$\mathrm{Aut}^{\otimes}(F_G)$ is simply $\varprojlim_{\mathsf{N}(X,x),\phi} G$, which is precisely Nori's fundamental group-scheme $\pi^N(X,x)$. As in addition the construction is obviously functorial in h, we conclude:

Theorem 4.5. *Let the notations be as in Construction 4.1. The functor h^* defined in* (4.11) *for one object $(h : Y \to X, G, y)$ of $\mathsf{N}(X,x)$ induces a functor of Tannakian categories*

$$\mathfrak{h}^* : \Big(\varinjlim_{\mathsf{N}(X,x),\phi^*} \mathsf{Rep}_k(G), F_G\Big) \to \big(\mathsf{Strat}(X,\infty), \omega_x\big),$$

and the Tannaka-dual homomorphism of k-group-schemes

$$\mathfrak{h}^{*\vee} : \pi^{\mathrm{alg},\infty}(X,x) \to \pi^N(X,x)$$

which is functorial in X.

The aim of the rest of the section is to show that the homomorphim $\mathfrak{h}^{*\vee}$ is faithfully flat and induces the profinite quotient homomorphism.

Proposition 4.6. *Let $(Y \xrightarrow{h} X, G, y)$ be an object of $\mathsf{N}(X,x)$. The following conditions are equivalent.*

1) *The induced map $\pi^{\mathrm{alg},\infty}(X,x) \to G$ (see* (4.11)*) is an epimorphism.*
2) *The induced map $\pi^N(X,x) \to G$ is an epimorphism.*
3) *The functor h^* in* (4.11) *is fully faithful and its image is closed under taking subquotients in $\mathsf{Strat}(X,\infty)$.*

Proof. The equivalence (1) $\Leftrightarrow$ (3) follows from [2, Proposition 2.21]. Moreover, since by construction, the map $\pi^{\mathrm{alg},\infty}(X,x) \to G$ factors through $\pi^N(X,x)$, (1) $\Rightarrow$ (2) is obvious.

We show (2) $\Rightarrow$ (3). Let $\mathcal{C}$ denote the full subcategory of $\mathsf{Strat}(X,\infty)$ generated by subquotients in $\mathsf{Strat}(X,\infty)$ of objects which are in the image of $h^* : \mathsf{Rep}_k(G) \to \mathsf{Strat}(X,\infty)$. The property 3) is equivalent to saying that $h^* : \mathsf{Rep}_k(G) \to \mathcal{C}$ is an equivalenece of categories. By standard Tannaka formalism, $\mathcal{C}$ itself is a k-linear, abelian, rigid tensor subcategory of $\mathsf{Strat}(X,\infty)$, thus $(\mathcal{C}, \rho_x)$ is a Tannaka subcategory of $(\mathsf{Strat}(X,\infty), \omega_x)$, where $\rho_x = \omega_x|_{\mathcal{C}}$.

We show now that $h^* : \mathsf{Rep}_k(G) \to \mathcal{C}$ is an equivalence of categories. Let $H = \mathrm{Aut}(\rho_x)$ be the Tannaka k-group-scheme of $(\mathcal{C}, \rho_x)$. We claim that the induced homomorphism $H \to G$ is a closed immersion. This is equivalent ([2, Proposition 2.21]) to saying that every object of $\mathcal{C}$ is a subquotient in $\mathcal{C}$ of an object in $h^*(\mathsf{Rep}_k(G))$, which is true since by definition of $\mathcal{C}$, a subquotient in $\mathcal{C}$ of objects in $h^*(\mathsf{Rep}_k(G))$ is the same as a subquotient in $\mathsf{Strat}(X,\infty)$ of objects in $h^*(\mathsf{Rep}_k(G))$. We conclude in particular that H is a finite group scheme.

The fiber functor (in the sense of Deligne [1, 1.9], see Properties 2.3 1)) $\omega_X : \mathsf{Strat}(X,\infty) \to \mathsf{Coh}(X)$ defined by $(E_i, \sigma_i, i \in \mathbb{N}) \mapsto E_0$ restricts to the fiber functor $\rho_X : \mathcal{C} \to \mathsf{Coh}(X)$. One has a commutative diagram of functors

$$\begin{tikzcd} \mathsf{Rep}_k(G) \arrow[r, "h^*"] \arrow[dr, "h^\#"'] & \mathcal{C} \arrow[d, "\rho_X"] \\ & \mathsf{Coh}(X) \end{tikzcd} \tag{4.13}$$

and, upon applying i_x, (4.1) implies that $i_x \circ h^\# = F_G$. By applying Theorem 2.4, we obtain a morphism

$$(h_H : Y_H \to X, H, y_H) \to (h : Y \to X, G, y) \tag{4.14}$$

in $\mathsf{N}(X, x)$. This in turn induces a factorization of $\pi^N(X, x) \to G$ as

$$\begin{tikzcd} \pi^N(X,x) \arrow[r] \arrow[d] & G \\ H \arrow[ur] & \end{tikzcd} \tag{4.15}$$

But $\pi^N(X, x) \to G$ is assumed to be an epimorphism. Thus $H \to G$ must be an epimorphism. Since it is also a closed immersion, we conclude

$$H \xrightarrow{\cong} G. \tag{4.16}$$

In other words

$$h^* : \mathsf{Rep}_k(G) \xrightarrow{\cong} \mathcal{C}. \tag{4.17}$$

This finishes the proof. $\square$

Recall that k is perfect.

Lemma 4.7. *Let G be a finite k-group-scheme, let $h : Y \to X$ be a G-torsor. Then the following conditions are equivalent*

(i) *h admits a reduction (necessarily unique) of structure group to $G_{\mathrm{red}} = G_{\text{ét}} \subset G$.*

(ii) *For every natural number n, there is a G-torsor $h_n : Y_n \to X^{(n)}$ which pulls back via $X \xrightarrow{F^{(0,n)}} X^{(n)}$ to h.*

Proof. We show (i) $\Rightarrow$ (ii). Let $h_{\text{ét}} : Y_{\text{ét}} \to X$ be a $G_{\text{ét}}$-torsor which is a reduction of structure of h for the closed embedding $G_{\text{ét}} \subset G$. Thus $Y = Y_{\text{ét}} \times_{G_{\text{ét}}} G$. The isomomorphism (4.3) induces a cartesian diagram

$$\begin{tikzcd} Y_{\text{ét}} \arrow[r, "F^{(0,n)}"] \arrow[d, "h_{\text{ét}}"'] \arrow[dr, phantom, "\square"] & (Y_{\text{ét}})^{(n)} \arrow[d, "(h_{\text{ét}})^{(n)}"] \\ X \arrow[r, "F^{(0,n)}"'] & X^{(n)} \end{tikzcd} \tag{4.18}$$

We set $Y_n = (Y_{\text{ét}})^{(n)} \times_{G_{\text{ét}}} G,\ h_n = (h_{\text{ét}})^{(n)} \times_{G_{\text{ét}}} G$.

We show (ii) $\Rightarrow$ (i). For a large enough positive integer n, we consider the commutative diagram similar to (4.8):

(4.19)
$$\begin{tikzcd}
Y_n^{(-n)} \arrow[rr, "\gamma"] \arrow[dd] \arrow[dr] & & Y_n \arrow[dd, "h_n"] \\
& \left(Y_n^{(-n)}\right)_{\text{ét}} \arrow[ur, dashed, "\exists\,!"] \arrow[dl] & \\
X \arrow[rr] & & X^{(n)}
\end{tikzcd}$$

We explain the terms in the diagram: with Notations 3.1, one has $Y_n^{(-n)} = Y_n \otimes_{F_k^{-n}} k$, thus h_n induces $h_n \otimes_{F_k^{-n}} k : Y_n^{(-n)} \to \left(X^{(n)}\right)^{(-n)} = X$, which is a principal $G^{(-n)}$ bundle. The top horizontal map γ is equivariant with respect to $G^{(-n)} \xrightarrow{F^{(-n,0)}} G$. Since n is large, the image of $G^{(-n)} \to G$ is precisely $G_{\text{ét}} \subset G$. Therefore, γ factors uniquely through $\left(Y_n^{(-n)}\right)_{\text{ét}}$. Via the identification $G_{\text{ét}}^{(-n)} \xrightarrow{F^{(-n,0)}} G_{\text{ét}}$, the morphism $\left(Y_n^{(-n)}\right)_{\text{ét}} \to X^{(n)}$ is a $G_{\text{ét}}$-torsor. The above commutative diagram shows the existence of an equivariant map $\left(Y_n^{(-n)}\right)_{\text{ét}} \to Y_n \times_{X^{(n)}} X$. We conclude that the G-torsor $Y_n \times_{X^{(n)}} X \to X$ has a reduction of structure group to $G_{\text{ét}}$.

$\square$

Theorem 4.8. *Let the notations are as in 3.1 and let $x \in X(k)$ be a rational point. Then the homomorphism $\mathfrak{h}^{*\vee} : \pi^{\text{alg},\infty}(X,x) \to \pi^N(X,x)$ is the profinite quotient homomomorphism.*

Proof. We have already shown in Proposition 4.6 that the homomorphism $\mathfrak{h}^{*\vee}$ is surjective. In order to show that $\mathfrak{h}^{*\vee}$ is the profinite completion homomorphism, we need to show that any epimorphism

$$\phi : \pi^{\text{alg},\infty}(X,x) \to G,$$

where G is a k-finite group-scheme, factors through $\pi^N(X,x)$. This is equivalent to showing that given any finite Tannaka subcategory $\mathcal{T} \subset \mathsf{Strat}(X,\infty)$, i.e. with $G = \text{Aut}^{\otimes}(\mathcal{T},\rho_x)$ finite, where $\rho_x = \omega_x|_{\mathcal{T}}$, there exists an object $(h : Y \to X, G, y)$ in $\mathsf{N}(X,x)$ such that $\mathcal{T}$ is the image of the functor h^* constructed in (4.11). We do this in two steps.

Step(1): For each $n \geq 0$, we consider the fiber functor

(4.20)
$$\omega_{X^{(n)}} : \mathsf{Strat}(X,\infty) \to \mathsf{Coh}(X^{(n)}), \quad (E^{(i)},\sigma^{(i)}, i \in \mathbb{N}) \mapsto E^{(n)}.$$

It restricts to a fiber functor

$$P_n : \mathcal{T} \to \mathsf{Coh}(X^{(n)}).$$

Let $\delta : \mathsf{Rep}_k(G) \to \mathcal{T}$ be the equivalence of Tannaka categories defined by the inverse functor to the equivalence induced by ρ_x. Consider

$$P_n \circ \delta : \mathsf{Rep}_k(G) \to \mathsf{Coh}(X^{(n)}).$$

By Theorem 2.4, we obtain G-torsors $(h_n : Y_n \to X^{(n)})$ for every n, such that

$$h_n^{\#} = P_n \circ \delta. \tag{4.21}$$

Since the G-torsors thus obtained are unique upto isomorphism, the equality

$$P_n = F^{(n)*} \circ P_{n+1},\ \forall\ n \geq 1$$

implies that the torsor h_{n+1} pulls back to h_n. Thus by Lemma 4.7, each Y_n admits a reduction of structure group to $G_{\text{ét}} \subset G$ for all $n \geq 1$.

Step(2): Composing δ with the inclusion $\mathcal{T} \hookrightarrow \mathsf{Strat}(X, \infty)$ we obtain a functor from $\mathsf{Rep}_k(G) \to \mathsf{Strat}(X, \infty)$. We also have the functor $h_0^* : \mathsf{Rep}_k(G) \to \mathsf{Strat}(X, \infty)$ (see (4.11)) defined by the G-torsor $h_0 : Y_0 \to X$. In order to finish the proof we have to show that these two functors coincide. This is equivalent to saying that the following diagram of functors commutes.

$$\begin{tikzcd} \mathsf{Rep}_k(G) \arrow[r, "\delta"] \arrow[dr, "h_0^*"'] & \mathcal{T} \arrow[d, "\text{incl.}"] \\ & \mathsf{Strat}(X, \infty) \end{tikzcd} . \tag{4.22}$$

Let V be an object of $\mathsf{Rep}_k(G)$. We will show that there is an isomorphism between $i(V)$ and $h_0^*(V)$, which is functorial in V. This will finish the proof. Let $\delta(V) = (\delta(V)^{(n)}, \sigma^{(n)}, n \in \mathbb{N})$ and $h_0^*(V) = (E^{(n)}, \tau^{(n)}, n \in \mathbb{N})$.

We let $h_{n,\text{ét}} : Y_{n,\text{ét}} \to X^{(n)}$ be the $G_{\text{ét}}$-torsor induced by h_n for $n \geq 1$. Note that by construction 4.1 of the functor h_0^*, one has

$$E^{(n)} = h_{n,\text{ét}}^{\#}(V)\ \forall\ n \geq 1 \quad \text{and} \quad E^{(0)} = h_0^{\#}(V). \tag{4.23}$$

On the other hand, by definition of the functors P_n,

$$P_n(i(V)) = i(V)^{(n)}$$

Thus by (4.21), one has

$$i(V)^{(n)} = h_n^{\#}(V)\ \forall\ n \geq 0. \tag{4.24}$$

But as explained before, for every $n \geq 1$, $h_n : Y_n \to X^{(n)}$ admits a reduction of structure group to $G_{\text{ét}}$. Thus by Proposition 2.3(3),

$$h_n^{\#}(V) = h_{n,\text{ét}}^{\#}(V)\ \ \forall\ n \geq 1. \tag{4.25}$$

Thus we conclude

$$i(V) = h_0^*(V). \tag{4.26}$$

$\square$

If $\mathcal{T}$ is any k-linear, abelian, rigid tensor category, together with a neutral fiber functor $\omega : \mathcal{T} \to \mathsf{Vec}_k$, we denote by $\mathcal{T}^{\rm fin}$ the full subcategory spanned by objects E which have the property that the full tensor subcategory $\langle E \rangle \subset \mathcal{T}$ spanned by E and its dual $E^\vee$ has a finite Tannaka group scheme $\mathrm{Aut}^{\otimes}(\langle E\rangle, \omega|_{\langle E\rangle})$. So by construction, Theorem 4.8 has the following consequence:

Corollary 4.9. *With the notations as in Theorem 4.8, the full embedding*

$$\mathsf{Strat}(X,x)^{\rm fin} \subset \mathsf{Strat}(X,x)$$

induces via the fiber functor ω_x the quotient homomorphism

$$\pi^{\rm alg,\infty}(X,) \to \pi^N(X,x).$$

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