WEAK DENSITY OF THE FUNDAMENTAL GROUP SCHEME

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Abstract. Let X be a non-singular projective variety over an algebraically closed field k of characteristic 0. If $\pi_1^{\rm et}(X) = 0$, then for any ample line bundle H on X, any semistable bundle E on X with all Chern classes 0 is *trivial*. Over $\mathbb{C}$, this a consequence of the fact that $\pi_1^{\rm top}(X_{\rm an})$ is a finitely generated group and hence any representation into a linear group is residually finite. We prove an analog of this theorem in characteristic $p > 0$. Semistable bundles with vanishing Chern classes are replaced by Nori semistable bundles, that is those which are semistable of degree 0 on any curve mapping to X. The étale fundamental group is replaced by Nori's fundamental group scheme $\pi^N(X)$ ([16]), which is the profinite completion of the tensor automorphism group scheme $\pi^S(X)$ of Nori semistable bundles ([11]) studied by Langer ([10] [11]).

1. Introduction

Let X be a non-singular projective variety over the field $k = \mathbb{C}$ of complex numbers. Let H be a very ample line bundle on X. Assume that E is a stable vector bundle on X, with respect to H, with all the Chern classes $0 = c_i(E) \in H^{2i}(X, \mathbb{Q}(i)), 1 \le i \le d =$ dimension X. Then it is classical ([15], [13]) that E carries an integrable connection with unitary underlying monodromy. If E is assumed to be merely semistable, then one shows that there is a Jordan-Hölder filtration $0 \subset E_0 \subset \ldots \subset E_n = E$ such that each E_i/E_{i-1} is a stable bundle with all Chern classes 0.

Assume now that one is working over k, an arbitrary algebraically closed field of characteristic 0. If X and H are as above, and E is a μ-semistable bundle on X, with all Chern classes in ℓ-adic cohomology $H^{2i}(X, \mathbb{Q}_\ell(i))$ trivial, then again it follows from the Lefschetz principle, that there is a filtration $0 \subset E_0 \subset \ldots \subset E_n = E$, where each E_i/E_{i-1} is a stable bundle with all Chern classes 0.

Note that in characteristic 0, any semistable E on a smooth projective variety X with all Chern classes 0 has the following property : If C is a smooth curve and $f : C \to X$ is any map, then f^*E is semistable of degree 0 on C. This is seen as follows. By the Lefschetz principle, we may assume that we are over $\mathbb{C}$. If E is stable on X, then E corresponds to an irreducible unitary representation σ :

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$\pi^{\mathrm{top}}(X) \to U(r)$, where $U(r)$ is the complex unitray group of $r \times r$ matrices, and $r = \mathrm{rank} E$. Then f^*E corresponds to the composite representation $\pi^{\mathrm{top}}(C) \to \pi^{\mathrm{top}}(X) \to U(r)$. Thus f^*E is a direct sum of stable bundles of degree 0 on C. If E is semistable on X, then $f^*(E)$ is filtered, and the associated graded is a sum of stable bundles, each of degree 0.

This motivates the following:

Definition 1.1. Let X be a projective variety over any algebraically closed field. Let E be a vector bundle on X. Then E is *Nori semistable* if for all smooth projective curves C, all morphisms $f : C \to X$, the bundle $f^*(E)$ is semistable on C, of degree 0.

This notion has been introduced by Nori [16, Definition p.81]. We have seen above that in characteristic 0, any semistable E on a smooth X with all Chern classes 0 is Nori semistable. It is indeed true for $k = \mathbb{C}$ and thus true for all $k = \bar{k}$ of characteristic 0 by the Lefschetz principle.

In characteristic $p > 0$, a bundle can be stable and not Nori semistable, as Frobenius pull backs of semistable bundles are no longer semistable, even for curves [5, Theorem 1]. This is one reason why one works with Nori semistable bundles.

Let $\mathsf{Ns}(X)$ be the category of Nori semistable bundles on a smooth projective connected variety X defined over an algebraically closed field k of characteristic $p > 0$. Nori shows [16, Lemma 3.6] that this is a k-linear, abelian, rigid tensor category. Fix a k-rational point x in X. Let $\omega_x : \mathsf{Ns}(X) \to \mathsf{Vec}_k$ be the tensor functor $W \to W|_x$. Then $(\mathsf{Ns}(X), \omega_x)$ is a Tannaka category, and one defines $\pi^S(X, x) := \mathrm{Aut}^{\otimes}(\mathsf{Ns}(X), \omega_x)$ to be the associated Tannaka group scheme.

This notation $\pi^S(X, x)$ was introduced in [1] on curves and [10] in any dimension. More precisely, in [11, 1.2], Langer mentions that $\mathsf{Ns}(X)$ is identical to the category of vector bundles E which are *numerically flat*, which means that both the tautological line bundle $\mathcal{O}(1)$ on $\mathbb{P}(E)$ and the one on $\mathbb{P}(E^\vee)$ are numerically effective. We recall this fact in Lemma 2.3. He also shows [10, Proposition 5.1] that $\mathsf{Ns}(X)$ is the category of strongly semistable reflexive sheaves with the property that $\deg\big(\mathrm{ch}_1(E) \cdot H^{d-1}\big) = \deg\big(\mathrm{ch}_2(E) \cdot H^{d-2}\big) = 0$ for a fixed polarization H. So in particular, it does not depend on the chosen polarization.

Nori [16, Definition p.82] constructed the category $\mathcal{C}^N(X)$ of essentially finite bundles as the full subcategory of $\mathsf{Ns}(X)$ spanned by *finite* bundles, where finiteness is in the sense of Weil: E is finite if there are two polynomials $f \neq g$, $f, g \in \mathbb{N}[T]$, such that $f(E) \cong g(E)$. Thus by definition,

$$\pi^S(X, x) \twoheadrightarrow \pi^N(X, x)$$

is the profinite completion homomorphism of the k-progroup scheme $\pi^S(X, x)$.

We prove:

Theorem 1.2. *Let X be a smooth projective connected variety over an algebraically closed field of characteristic $p > 0$. Let $x \in X(k)$. If $\pi^N(X, x) = \{1\}$, then $\pi^S(X, x) = \{1\}$ as well.*

In characteristic 0, one has $\pi^N(X, x)(k) = \pi^{\mathrm{et}}(X, x)$, and thus Theorem 1.2 is a generalization of the theorem discussed above. Note also that if $k = \bar{\mathbb{F}}_p$, then every E in $\mathsf{Ns}(X)$ is actually essentially finite, thus in this case $\pi^S(X, x) = \pi^N(X, x)$ and the theorem is trivial. Over $\mathbb{C}$, it is known that the finite bundles, that is the bundles which are trivializable after a finite étale covering of X, are not Zariski dense in the moduli space of μ-stable bundles of degree 0 on a curve, except in rank 1. Indeed, by the Narasimhan-Seshadri correspondence [15], μ-stable bundles of degree 0 are flat bundles corresponding to a unitary representation of $\pi^{\mathrm{top}}(X)$, thus finite stable bundles are flat bundles with finite irreducible monodromy. By Jordan's theorem [7], a finite irreducible subgroup G of $GL(r, \mathbb{C})$ has a cardinality bounded by a number depending only on r. But $\pi^{\mathrm{top}}(X)$ has finite isomorphism classes of finite quotients of bounded cardinality. Thus there are finitely many points of the moduli of μ-stable bundles of degree 0 corresponding to finite monodromy.

In positive characteristic, it is known that the étale trivializable bundles are dense in the moduli space if $\dim X = 1$ ([17], [2]). In section 2, we sketch that on X smooth projective in characteristic 0, then $\pi_1^{\mathrm{et}}(X) = 0$ implies that $\pi^S(X) = 0$. This justifies the title of this note.

The main point of our article is to prove Theorem 1.2 over an arbitrary field of characteristic $p > 0$. The proof is a variant of the proof of [4, Theorem 1.1]. Apart from the existence of quasi-projective moduli spaces due to Langer [9, Theorem 4.1], the main tool is Hrushovski's fundamental theorem [6, Corollary 1.2]. The difference with the proof of the main theorem in [4] relies in the choice of the sublocus of the moduli on which we ultimately wish to apply Hrushovski's theorem. In [4], we defined Verschiebung *divisible* subschemes of M (see [4, Definition 3.6]), while here the notion which works is reversed. We do not discuss this in the note, but the locus we defined is rather Verschiebung *multiplicative*, that is if one moduli point $[E]$ lies in it, then $[F^*E]$ lies in it as well. Another difference is that in [4], even over $\bar{\mathbb{F}}_p$, we had to appeal to Hrushovski's theorem. In the present situation, where we deal with bundles going up by Frobenius, over $\bar{\mathbb{F}}_p$, we can appeal to Lange-Stuhler theorem [8, Satz 1.4]. Only for an arbitrary algebraically closed field k, does one use Hrushovski's theorem.

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2. Proof of Theorem 1.2

Throughout this section, X is a smooth connected projective variety of dimension d over an algebraically closed field k of characteristic $p > 0$, $F : X \to X$ is the absolute Frobenius morphism of X, $x \in X(k)$ is a rational point, and H is a fixed ample line bundle on it. We prove Theorem 1.2 in a series of lemmas.

Lemma 2.1. *Let X be a smooth projective variety over an algebraic closed field k of characteristic 0. If $\pi^{\mathrm{et}}(X) = 0$, then $\pi^N(X, x) = 0$.*

Proof. By the Lefschetz principle, we may assume that $k = \mathbb{C}$. For a stable vector bundle E on X with all Chern classes 0 over $k = \mathbb{C}$, let $\rho : \pi^{\mathrm{top}}(X_{\mathrm{an}}) \to U(r)$ be the associated irreducible unitary representation. Then the image G of ρ is a finitely generated subgroup of $GL(r, \mathbb{C})$. By Malcev theorem [12], G is residually finite, that is G injects into its profinite completion $\widehat{G}$. If $G \neq \{1\}$, there is a homomorphism of G onto a finite group, thus a homomorphism from $\pi^{\mathrm{top}}(X)$ onto a finite group. If one now assumes that $\pi^{\mathrm{et}}(X) = 0$, this implies that $G = \{1\}$, thus E is trivial. If E is only semistable, then the above argument implies that the associated graded bundle $\oplus_i(E_i/E_{i-1})$ is trivial. On the other hand, $\pi^{\mathrm{et}}(X) = 0$ implies that $H^1_{\mathrm{et}}(X, \mathbb{Z}/n) = \mathrm{Hom}(\pi_1^{\mathrm{et}}(X), \mathbb{Z}/n) = 0$, thus ℓ-adic cohomology $H^1_{\mathrm{et}}(X, \mathbb{Z}_\ell)$ is trivial, thus, by the comparison theorem, Betti cohomology $H^1(X_{\mathrm{an}}, \mathbb{Z})$ is trivial, thus by Hodge theory, $H^1(X, \mathcal{O}_X) = 0$. So E, which is a successive extension of $\mathcal{O}_X$ by itself, is trivial as well.

□

Lemma 2.2. *If $\pi^N(X, x) = 0$, then $\pi^{\mathrm{et}}(X, x) = 0$.*

Proof. As well known (see e.g. [3, Remarks 2.10]), thinking of $\pi^{\mathrm{et}}(X, s)$ as a constant $k = \bar{k}$-progroup scheme, the k-homomorphism $\pi^N(X, x) \to \pi^{\mathrm{et}}(X, x)$ is surjective. In fact, this is the pro-smooth quotient of $\pi^N(X, x)$. □

Recall [10, 1.2] that a bundle E is said to be *numerically effective* if for any smooth projective curve $f : C \to X$, the minimal slope of f^*E is nonnegative. This is equivalent to saying that the tautological line bundle $\mathcal{O}(1)$ on $\mathbb{P}(E)$ is numerically effective.

Lemma 2.3. *(See [11, 1.2]) A vector bundle E on X is in $\mathsf{Ns}(X)$ if and only if both E and $E^\vee$ are numerically effective on X.*

Proof. Let E be in $\mathsf{Ns}(X)$, and $f : C \to X$ be a morphism of a smooth projective curve. By definition, f^*E is semistable of degree 0. The minimal slope of f^*E is the slope of some stable quotient bundle, thus by semistability, it has to be nonnegative. As $E^\vee$ is in $\mathsf{Ns}(X)$ as well, we conclude that both E and $E^\vee$ are numerically effective. Vice-versa, if E and $E^\vee$ are numerically effective, then both f^*E and $f^*E^\vee$ are numerically effective on C, thus E is in $\mathsf{Ns}(X)$. □

Lemma 2.4. *If $E \in \mathsf{Ns}(X)$, then $\deg\big(c_i(E) \cdot H^{d-i}\big) = 0$ for all $i \geq 1$ and all ample line bundles H.*

Proof. This is the argument of the proof of [10, Theorem 4.1,Proof]): since E is strongly semistable, by boundedness, there are finitely many such natural numbers $\deg\big(c_i(F^*E)\cdot H^{d-i}\big) = p^i \cdot \deg\big(c_i(E)\cdot H^{d-i}\big)$, one concludes that they are 0. $\square$

Let E be an essentially finite bundle. Let $\langle E\rangle$ be the full subcategory of $\mathcal{C}^N(X)$ spanned by E, and set $G(E,x) = \mathrm{Aut}^{\otimes}(\langle E\rangle, \omega_x)$. Then there is an exact sequence $1 \to G(E,x)^0 \to G(E,x) \to G(E,x)^{\mathrm{et}} \to 1$ where $G(E,x)^0$ is local and $G(E,x)^{\mathrm{et}}$ is étale and is a quotient of $\pi_1^{\mathrm{et}}(X,x)$. We denote by $\mathcal{C}^{\mathrm{loc}}(X) \subset \mathcal{C}^N(X)$ the full subcategory of essentially finite bundles E which have the property that $G(E,x)^0 = G(E,x)$. We denote by $\pi^{\mathrm{loc}}(X,x)$ the Tannaka group scheme $\varprojlim_{E,\ G(E,x)^0=G(E,x)} G(E,x)^0$. This group has been studied in [3] (where it is denoted by $\pi^F(X,x)$) and in [14]. It is a quotient k-group scheme of $\pi^N(X,x)$.

Lemma 2.5. *On has the following implications.*

1. *If* $\pi_1^{\mathrm{et}}(X,x) = \{1\}$, *and* $H^0(X,\Omega^1_X) = 0$, *then* $\pi_1^N(X,x) = \{1\}$.
2. *If* $\pi_1^N(X,x) = \{1\}$, *then* $\pi^{\mathrm{loc}}(X,x) = \{1\}$ *and* $H^1(X,\mathcal{O}_X) = 0$.

Proof. We first prove (1). Let E be an essentially finite bundle. Since $G(E,x)^{\mathrm{et}}$ is a quotient of $\pi_1^{\mathrm{et}}(X,x) = \{1\}$, one has $G(E,x)^0 = G(E,x)$. This implies that there is a $n \in \mathbb{N}\setminus\{0\}$ such that $(F^n)^*E$ is trivial ([14, p. 145]). Since $H^0(X,\Omega^1_X) = 0$, there is only one connection on $(F^n)^*E$, the trivial one. Thus the connection with flat sections $(F^{n-1})^*E$ is trivial, thus $(F^{n-1})^*E$ is trivial. Repeating the argument, we see that E is trivial. This finishes the proof of (1).
We prove (2). The absolute Frobenius map $F : X \to X$ induces a p-linear endomorphism F^* on $H^1(X,\mathcal{O}_X)$. This induces a decomposition

$$H^1(X,\mathcal{O}_X) = H^1(X,\mathcal{O}_X)_{\mathrm{ss}} \oplus H^1(X,\mathcal{O}_X)_{\mathrm{nilp}},$$

where F^* is bijective on the first factor and nilpotent on the second factor. Since

$$H^1(X,\mathcal{O}_X)_{\mathrm{ss}} = H^1(X_{\mathrm{et}},\mathbb{Z}/p)\otimes_{\mathbb{F}_p} k = \mathrm{Hom}(\pi^{\mathrm{et}}(X),\mathbb{Z}/p)\otimes_{\mathbb{F}_p} k,$$

one concludes $H^1(X,\mathcal{O}_X)_{\mathrm{ss}} = 0$. Let $0 \neq y \in H^1(X,\mathcal{O}_X)_{\mathrm{nilp}}$. There exists $t \in \mathbb{N}\setminus\{0\}$ such that $(F^{t-1})^*(y) \neq 0$, but $(F^t)^*(y) = 0$. Then $(F^{t-1})^*(y)$ defines an α_p-torsor, a contradiction. This proves $H^1(X,\mathcal{O}_X) = 0$. $\square$

Lemma 2.6. *If* $\pi^N(X,x) = \{1\}$, *the line bundles in* $\mathsf{Ns}(X)$ *are trivial.*

Proof. As both L and $L^\vee$ are numerically effective on X, L is numerically equivalent to 0. The group $\mathrm{Num}_0(X)/\mathrm{Alg}_0(X)$ is a finite group. As $H^1(X,\mathcal{O}_X) = 0$, the 1-component of the reduced Picard scheme of X is a point. Hence L has finite order. But any non-trivial line bundle of finite order on X defines a torsor over X under a non-trivial finite group-scheme, a contradiction as $\pi^N(X,x)$ is trivial. So L is trivial. $\square$

Lemma 2.7. *(See* [10, Theorem 4.1] *Let $E \in \mathsf{Ns}(X)$. Then the following properties hold true.*

(a) *E is semistable with respect to H.*
(b) *If $0 \subset E_0 \subset E_1 \subset \ldots \subset E_n = E$ is the Jordan-Hölder filtration (or the stable filtration) of E, then each subquotient E_i/E_{i-1} is locally free, with $\deg\big(c_j(E_i/E_{i-1}) \cdot H^{d-i}\big) = 0$, for all $j > 0$, all $i > 0$ and all ample line bundles H.*

Now we continue with the proof of Theorem 1.2. For E in $\mathsf{Ns}(X)$, we want to show that E is trivial. We may assume, by induction on the rank, that if $W \in \mathsf{Ns}(X)$, with rank $W < r = \mathrm{rank}(E)$, then W is trivial, using Lemma 2.6. Define a sequence of bundles on X by $E_0 := E$, and $E_n = F^*(E_{n-1})$, for $n \geq 1$. This is a sequence of bundles on X and $\deg\big(c_i(E_n) \cdot H^{d-i}\big) = 0$ for all $n \geq 0$ and all $i \geq 1$.

Lemma 2.8. *$E_i \not\simeq E_j$, for all i, j with $i \neq j$.*

Proof. Assume there are i, j such that $i < j$ and $E_i \simeq E_j$. Then $(F^t)^*(E_i) \simeq E_i$, where $t = j - i \neq 0$. If E_i is not trivial, by the theorem of Lange-Stuhler [8, Satz 1.4], E_i becomes trivial on a non-trivial étale finite covering of X. As $\pi^{\mathrm{et}}(X, x) = \{1\}$, one must have E_i trivial. But by definition, $E_i = (F^i)^*(E_0)$, thus $G(\langle E_0\rangle, x)$ is local, thus E_0 is trivial as $\pi^{\mathrm{loc}}(X, x)$ is trivial by Lemma 2.5, 2). □

Lemma 2.9. *In order to prove Theorem 1.2, we may assume that all the E_i are stable on X.*

Proof. We may assume the proposition is true for all W in $\mathsf{Ns}(X)$ with $\mathrm{rank}(W) < \mathrm{rank}(E)$. Suppose that $E_n = (F^n)^*(E_0)$ is strictly semistable. For $m \geq n$, let $s(m)$ be the number of stable components of the Jordan-Hölder components of E_m. It is clear that $s(m)$ is a non-decreasing function of m. As the set $\{s(m), m \geq n\}$ is bounded above by $r = \mathrm{rank}(E)$, $s(m)$ is *constant* for all $m \geq n_0$ for some $n_0 \geq 0$. So if $W_m \subset E_m$ is the socle of E_m, then $F^*W_m = W_{m+1}$ is the socle of E_{m+1} for any $m \geq n_0$. The sequence $W_n, n \geq n_0$ is a sequence of stable subbundles of E_n. By Lemma 2.7, one has $\deg\big(c_i(W_n) \cdot H^{d-i}\big) = 0\ \forall i \geq 1,\ \forall n \geq n_0$. So by Lemma 2.3, together with [10, Theorem 5.1], we conclude $W_n \in \mathsf{Ns}(X)\ \forall n \geq n_0$, hence $E_n/W_n \in \mathsf{Ns}(X)\ \forall n \geq n_0$ as well.

We apply the induction hypothesis to $\{W_n, n \geq n_0\}$ and $\{E_n/W_n, n \geq n_0\}$, to assert that $\{W_n, n \geq n_0\}$ and $\{E_n/W_n, n \geq n_0\}$ are trivial bundles on X. Then E_n are extensions of trivial bundles $\forall n \geq n_0$. Applying Lemma 2.5 (2), we conclude that E_n is trival. But $E_n \simeq (F^a)^*(E_{n-a}),\ 0 \leq a \leq n$. Since $\pi^N(X, x) = \{1\}$, E_{n-a} is trivial as well. This finishes the proof. □

We continue with the proof of Theorem 1.2. Let E be in $\mathsf{Ns}(X)$. Let M be the moduli space of μ-stable bundles of degree 0, which is open in the moduli of χ-stable torsionfree sheaves with Hilbert polynomial $p_E = p_{\mathcal{O}_X}$, as constructed by Langer in [9, Theorem 4.1]. It is a quasi-projective scheme, of finite type over k.

Define $T := \{E_0, E_1, \ldots\}$ as a sublocus of $M_{\rm red}$. Here we identify a μ-stable bundle W with $\mu(W) = 0, p_W = p_{\mathcal{O}_X}$ of rank r with its moduli point $|W| \in M$. Let N be the *Zariski closure* of T in $M_{\rm red}$. We give N the reduced structure. By Lemma 2.8, the dimension of N is at least 1.

Consider the decomposition of N into its irreducible components, $A_i, i \in I$, and $N_j, j \in J$. This labeling is chosen such that A_i is finite $\forall i \in I$, and $T \cap N_j$ is infinite $\forall j \in J$. Recall that $F : X \to X$ is the absolute Frobenius morphism of X. Let $V : M \dashrightarrow M$ be the *Verschiebung*, which is the rational map defined by $[W] \mapsto [F^*W]$. As F preserves T, V maps N into itself rationally. Moreover, since the image of this rational map contains E_n for all $n \geq 1$, V is dominant on the Zariski closure of $\{E_1, E_2, \ldots\}$. This implies that V maps each N_i into some other N_j, *dominantly*. We conclude that V induces a permutation of the set J. Hence some nonzero power of V preserves each N_j.

We can now argue precisely as [4, Section 3]. We choose a scheme S, smooth, of finite type, geometrically irreducible over $\mathbb{F}_q$, such that

(a) X has a model smooth projective $X_S \to S$,
(b) M has a flat model $M_S \to S$,
(c) all the irreducible components N_j of N have a flat model $N_{jS} \to S$,
(d) V has a model V_S on M_S.

Recall by Langer's theorem [9, Theorem 4.1], $M_S \to S$ universally corepresents the functor of families of stable bundles on the closed fibres of $X_Y \to Y$, thus in particular, for all closed points $s \in Y$, one has $M_S \times_S s = M_s$, where Y is any Noetherian scheme over S. Applying [4, Corollary 3.11], we obtain that the specialization $V_S \times_S s$ of V_S over closed points $s \in S$ is the Verschiebung of M_s. We argue as in the proof of [4, Theorem 3.14] to show that Hrushovski's theorem [6, Corollary 1.2] implies the existence of closed points u in each irreducible component N_{iS} of N_S, mapping to a closed points $s \in S$, such that V_s^m is defined on u and fulfills $F_{X_s}^m(u) = u$. Here $F_{X_s} : X_s \to X_s$ is the absolute Frobenius endomorphism of X_s. Recall briefly that by corepresentability, the restriction to s of V is V_s, that after a finite base change of $k = \mathbb{F}_{q^m} \supset k(s)$, all components of $N_s = N_S \otimes k$ are geometriclaly irreducible, that some power of V_s is k-linear, stabilizes them, and is dominant. This is the situation in which one can apply Hrushovsky's theorem to find $\bar{k}$-points of the shape $(u, \Phi_{\mathbb{F}_q}(u))$ on the graph of V_s on one such irreducible component, where $\Phi_{\mathbb{F}_q}$ is the $\mathbb{F}_q$-linear Frobenius. Those points correspond to Frobenius periodic bundles. Applying further as in *loc. cit.* Lange-Stuhler Theorem [8, Satz 1.4], we conclude that $N_j = \emptyset$ for all $j \in J$.

Thus T has only finitely many components A_i of dimension 0. This implies that T is finite, thus we find $n \in \mathbb{N}$ and $t \in \mathbb{N} \setminus \{0\}$ such that $(F^t)^*E_n \cong E_n$. Applying again Lange-Stuhler Theorem *loc. cit.*, this shows that E_n is trivial. This implies by definition that $E_m, m \geq n$ is trivial, and by Lemma 2.5 (2) that $E_m, 0 \leq m < n$ are trivial as well. This finishes the proof.

Remark 2.10. Over $k = \bar{\mathbb{F}}_p$, assume that X, E and M are defined over some $\mathbb{F}_q$, so $X = X_0 \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, E = E_0 \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, M = M_0 \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. Let $F : X_0 \to X_0$ be the Frobenius of X over $\mathbb{F}_q$, and assume that $(F^i)^*(E_0)$ are μ-stable as well for all $i \geq 1$. Then the moduli points of $(F^i)^*(E_0)$ are in $M_0(\mathbb{F}_q)$, thus there $F^a(E_0)$ is isomorphic to $F^b(E_0)$ for distinct $a, b \in \mathbb{N} \setminus \{0\}$. Thus by [8, Satz 1.4], this produces an étale covering of X, a contradiction.

3. Remarks

3.1. If one knew that the Verschiebung $V : M \dashrightarrow M$ introduced in the proof of Theorem 1.2 was dominant, then we could apply Hrushovski's theorem directly to it and the proof would be much more direct. In fact, this would prove that torsion points are dense in the sense of Theorem [4, Theorem 3.14]. We do not know this in general, but do know it in dimension 1 [17, Theorem 6].

3.2. In [16, Chapter II,Proposition 8], Nori shows that if X is projective smooth and geometrically irreducible, then $\pi_1^N(X, x)$ is a birational invariant among the smooth projective models of $k(X)$. Langer in [10, Lemma 8.3] shows that blow ups with smooth centers do not affect $\pi^S(X, s)$ and raises the question whether Nori's result extends to $\pi^S(X, x)$. Theorem 1.2 shows that this is true under the assumption that $\pi^N(X, x)$ is trivial.

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