Simply connected projective manifolds in characteristic $p > 0$ have no nontrivial stratified bundles

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Abstract We show that simply connected projective manifolds in characteristic $p > 0$ have no nontrivial stratified bundles. This gives a positive answer to a conjecture by D. Gieseker (Ann. Sc. Norm. Super. Pisa, 4 Sér. 2(1):1–31, 1975). The proof uses Hrushovski's theorem on periodic points.

1 Introduction

Let X be a smooth complex variety. The category of bundles with integrable connections on X is the full subcategory of the category of coherent $\mathcal{D}_X$-modules which are $\mathcal{O}_X$-coherent as well. It is a $\mathbb{C}$-linear abelian rigid category. If X is projective or if we restrict to connections which are regular singular at infinity, then it is equivalent by the Riemann-Hilbert correspondence to the $\mathbb{C}$-linear abelian rigid category of local systems of complex vector spaces [2]. Upon neutralizing those categories by the choice of a point $x \in X(\mathbb{C})$, the Riemann-Hilbert correspondence translates via the Tannaka formalism into an isomorphism between the proalgebraic completion of the topological fundamental group $\pi_1^{\text{top}}(X, x)$ and the Tannaka pro-algebraic group $\pi_1^{\text{strat}}(X, x)$ of flat bundles. Malcev [14] and Grothendieck [7] showed

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that if the étale fundamental group $\pi_1^{\text{ét}}(X, x)$ is trivial, that is if X does not have any nontrivial connected finite étale covering, then $\pi_1^{\text{strat}}(X, x)$ is trivial as well, thus there are no nontrivial flat bundles. The proof has nothing to do with flat bundles, but with the fact that $\pi_1^{\text{top}}(X, x)$ is an abstract group of finite type, and that, as a consequence of the Riemann existence theorem, $\pi_1^{\text{ét}}(X, x)$ is its profinite completion. The theorem says that if the profinite completion of a group of finite type is trivial, so is its proalgebraic completion.

Let X now be a smooth variety defined over a perfect field k of characteristic $p > 0$. The full subcategory of the category of coherent $\mathcal{D}_X$-modules which are $\mathcal{O}_X$-coherent is again a k-linear rigid tensor category. Katz shows [5, Theorem 1.3] that it is equivalent to the category of objects $E = (E_n, \sigma_n)_{n \in \mathbb{N}}$ where E_n is a vector bundle, $\sigma_n : F^* E_{n+1} \simeq E_n$ is a $\mathcal{O}_X$-linear isomorphism, and where the morphisms respect all the structures. Let us call it the category $\mathsf{Strat}(X)$ of stratified bundles. One neutralizes the category via the choice of a rational point $i_x : x \to X(k)$, defining the functor $\omega_x : \mathsf{Strat}(X) \to \mathsf{Vec}_k$, $\omega_x(E) = i_x^* E_0$. This defines the pro-algebraic group $\pi_1^{\text{strat}}(X, x) = \mathrm{Aut}^{\otimes}(\omega_x)$.

For a rational point $x \in X(k)$, we denote by $\bar{x}$ a geometric point above it. The purpose of this article is to show, when X is projective, the analog in characteristic $p > 0$ of Malcev-Grothendieck theorem (see Theorem 3.15):

Theorem 1.1 *Let X be a smooth connected projective variety defined over a perfect field k of characteristic $p > 0$. Let $\bar{x} \in X$ be a geometric point. If $\pi_1^{\text{ét}}(X \otimes_k \bar{k}, \bar{x}) = \{1\}$, there are no nontrivial stratified bundles.*

This gives a (complete) positive answer to Gieseker's conjecture [5, p. 8]. If X has a rational point $x \in X(k)$, one can rephrase by saying that under the assumptions of the theorem, $\pi_1^{\text{strat}}(X, x) = \{1\}$. The theorem implies that if $\pi_1^{\text{ét}}(X \otimes_k \bar{k}, \bar{x})$ is a finite group, then irreducible stratified bundles come from representations of $\pi_1^{\text{ét}}(X \otimes_k \bar{k}, \bar{x})$ (see Corollary 3.16).

Dos Santos [3] studied the k-pro-group $\pi_1^{\text{strat}}(X, x)$ when $k = \bar{k}$. He showed that all quotients in $GL(\omega_x(E))$ are smooth algebraic groups ([3, Corollary 12], see also [15]). In fact, the proof is written only for the finite part, but it applies more generally. This is an important fact pleading in favor of the conjecture.

We now describe the philosophy of the proof of Theorem 1.1. Since there is no known group of finite type which controls $\pi_1^{\text{strat}}(X, x)$ and $\pi_1^{\text{ét}}(X, x)$, it is impossible to adapt Grothendieck's proof. Instead, one can first think of the full subcategory spanned by rank one objects. Over $\bar{k}$, the maximal abelian quotient $\pi_1^{\text{ab}}(X, x)$ of $\pi_1^{\text{ét}}(X, x)$ nearly controls the Picard variety: if $\pi_1^{\text{ab}}(X, x) = \{1\}$, then, for any prime $\ell \neq p$, the ℓ-adic Tate module $\varprojlim_n \mathrm{Pic}^0(X)(\bar{k})[\ell^n]$ is trivial, thus $\mathrm{Pic}^{\tau}(X)$ is finite. So a rank one stratified

bundle $L = (L_n, \sigma_n)_{n \in \mathbb{N}}$ on $X \otimes \bar{k}$ must have the property that for a strictly increasing sequence n_i, $i \geq 0$, the line bundles L_{n_i} are all isomorphic, thus all L_{n_i} are fixed by some power of the Frobenius, so define a Kummer covering of X, which then has to be trivial by our assumption. This implies that all the L_n, $n \geq 0$ are trivial. Since on X proper, a stratified bundle $E = (E_n, \sigma_n)_{n \in \mathbb{N}}$ is uniquely recognized by the bundles E_n [5, Proposition 1.7], this shows the statement.

Thus the rank one case does not rely directly on the Tannaka property of the category. It rather uses the representability of the Picard functor together with the fact that those bundles with are fixed by a power of the Frobenius define étale coverings on one hand, and are dense in $\mathrm{Pic}^\tau(X \otimes \bar{k})$ on the other. We try to follow the same idea in the higher rank case.

We now describe the main steps of the proof over $k = \bar{k}$. In order to be able to use moduli of vector bundles, we first reduce the problem to the case where all the underlying bundles E_n of the stratified bundles $E = (E_n, \sigma_n)_{n \in \mathbb{N}}$ are stable with vanishing Chern classes. Here we use μ- (or slope) stability with respect to a fix polarization, which is defined by Mumford by the growth of the degree of subsheaves. Brenner-Kaid [1, Lemma 2.2] show that if E is a stratified bundle, then E_n is semistable of slope 0 for n large. Using Langer's boundedness [12, Theorem 4.2], the authors apply the same argument as for the classical one in rank 1 sketched above, to conclude that if all the E_n are defined over the same finite field $\mathbb{F}_q$ and $\pi_1^{\text{ét}}(X_{\bar{\mathbb{F}}_q}) = 0$, then all stratified bundles are trivial [1, Lemma 2.4]. We show in general that the stratified bundle $E(n_0) = (E_{n_0+m}, \sigma_{n_0+m})_{m \in \mathbb{N}}$, for n_0 large enough, is always a successive extension of stratified bundles $(U_m, \tau_m)_{m \in \mathbb{N}}$ such that all the U_m are μ-stable. This, together with dos Santos' theorem [3, (9)], describing with the projective system $\varprojlim_n H^1(X, \mathcal{O}_X)$ over the Frobenius, the full subcategory of $\mathsf{Strat}(X)$ spanned by successive extensions of the trivial object by itself, is enough to perform the reduction (see Proposition 2.4).

The moduli scheme M of χ-stable torsionfree sheaves of Hilbert polynomial $p_{\mathcal{O}_X}$ and of rank $r > 0$ was constructed by Gieseker in [6, Theorem 0.3] in dimension 2 over $k = \bar{k}$, and by Langer in [13, Theorem 4.1] in general. Here χ-stability is taken with respect to a fix polarization, and is defined by Gieseker [6, Sect. 0] by the growth of the Hilbert polynomial of subsheaves. Examples of χ-stable torsionfree sheaves with Hilbert polynomial $p_{\mathcal{O}_X}$ are torsionfree sheaves with $\mu = 0$ [9, Lemma 1.2.13].

While on $\mathrm{Pic}(X)$, the *Verschiebung* morphism, which, to a line bundle L, assigns its Frobenius pullback $V(L) := F^*(L) \simeq L^p$, is well defined, on M it is not. As is well known [4, Theorem 1], even if E is stable, the bundle $F^*(E)$ need not be stable. We define the sublocus M^s of μ-stable points $[E]$ for which there is a stratified bundle $(E_n, \sigma_n)_{n \in \mathbb{N}}$ with $E_0 \cong E$. The bundles E_n are necessarily μ-stable, thus define points $[E_n] \in M^s$. In particular, $[E] = V([E_1])$ and M^s lies in the image of the sublocus of M^s on

which V is well defined. If we assume that there are nontrivial stratified bundles with $[E_n] \in M$, this sublocus is not empty. We define $N \subset M$ to be the Zariski closure of M^s. We show that the Verschiebung is a rational dominant morphism $V : N \dashrightarrow N$ (see Lemma 3.8). On the other hand, there is a smooth affine variety S, defined over $\mathbb{F}_p$, such that X has a smooth model $X_S \to S$, the Frobenius $F_k : \operatorname{Spec} k \to \operatorname{Spec} k$ descends to the absolute Frobenius $F_S : S \to S$, and the absolute Frobenius $F : X \to X$ over F_k descends to the absolute Frobenius $F_{X_S} : X_S \to X_S$ over F_S. By the representability theorem [13, Theorem 4.1], M has a model M_S with the property that for all morphisms $T \to S$ of $\mathbb{F}_p$-varieties, $M_S \times_S T = M_T$, where M_T is the moduli of stable vector bundles of the same Hilbert polynomial and the same rank $r > 1$ on $X_T \to T$, where $X_T = X_S \times_S T$. We take T smooth so that $N \subset M$ has a model $N_T \subset M_T$, and its connected components are also defined over T. We further require that V has a model $V_T : N_T \dashrightarrow N_T$. There is a dense open subvariety $T^0 \subset T$ such that for all closed points $t \in T^0$, the reduction $V_t : N_t \dashrightarrow N_t$ is still defined as a rational map, is dominant, and some power fixes the irreducible components of N_t. We show that V_t is the Verschiebung from M_t restricted to N_t (see Lemma 3.10).

Replacing N_t by an irreducible component Y say, we denote by $\Gamma \subset Y \times_t Y$ the Zariski closure of the graph, where it is defined, of a power of V_t respecting Y. One can apply Hrushovski's fundamental theorem [8, Corollary 1.2] to find a dense subset of closed points of the shape $(u, \Phi_{q(t)}^m(u)) \in \Gamma$, where $\Phi_{q(t)}$ is the geometric Frobenius of N_t raising coordinates to the $q(t)$-th power, and where $\mathbb{F}_{q(t)}$ is the residue field of t. From this and the representability theorem *loc. cit.*, one deduces that if N is not empty, $N_t \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ contains closed points which are fixed under the Verschiebung (see Theorem 3.14). Since by Grothendieck's specialisation theorem [17, Théorème 3.8], $\pi_1^{\text{ét}}(X_t \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q) = \{1\}$, we conclude that N_t is empty, so thus is N.

The reduction to the stable case is written in Sect. 2, the proof is performed in Sect. 3. In fact in Sect. 3, we do a bit more. We show that torsion points (see Definition 3.12) are dense in good models of stratified schemes (see Definition 3.4 and Theorem 3.14). In Sect. 4, we make a few remarks and raise some questions. In particular, if X is quasi-projective, in view to Grothendieck's theorem in characteristic 0, one would expect a relation between the fundamental group $\pi_1^{\text{ét}}(X \otimes_k \bar{k}, \bar{x})$, and tame stratified bundles. Without resolution of singularities and without theory of canonical extension like Deligne's one [2] in characteristic 0, our method of proof can't be extended to this case.

2 Reduction to the case where the bundles are stable

Let X be a smooth connected projective variety defined over a perfect field k of characteristic $p > 0$.

By Katz' equivalence of category between coherent $\mathcal{D}_X$-modules which are $\mathcal{O}_X$-coherent and stratified bundles [5, Theorem 1.3], any stratified bundle $E = (E_n, \sigma_n)_{n \in \mathbb{N}}$, $\sigma_n : F^* E_{n+1} \simeq E_n$ of rank $r > 0$ is isomorphic as a stratified bundle to one for which for all $n \geq 0$ and all $a \geq 0$, $(F^n)^{-1} E_{n+a} \subset E_a$ is a subsheaf of abelian groups, which is a $(F^n)^{-1} \mathcal{O}_X$-locally free module of rank r. From now on, we will always take a representative of a stratified bundle which has this property and we will drop the isomorphisms σ_n from the notation.

Let $\mathcal{O}_X(1)$ be an ample line bundle. For any bundle E of rank $r > 0$, one defines the *Hilbert polynomial of E relative to $\mathcal{O}_X(1)$* by $p_E(m) = \frac{1}{r}\chi(X, E(m)) \in \mathbb{Q}[m]$ for $m >> 0$ large enough so that it is a polynomial, and where $E(m) = E \otimes_{\mathcal{O}_X} \mathcal{O}_X(m)$. Recall [6, Sect. 0] that E is said to be χ-stable (χ-semistable) if for all subsheaves $U \subset E$ one has $p_U < p_E$ ($p_U \leq p_E$). Here the order is defined by the values of the polynomials for n large, and the rank of the (necessarily torsionfree) subsheaf U is its generic rank. Recall that E is said to be μ-stable (μ-semistable) if for all subsheaves $U \subset E$ on has $\mu(U) < \mu(E)$ ($\mu(U) \leq \mu(E)$).

We use the notation $CH^i(X)$ for the Chow group of codimension $i \geq 0$ cycles and denote by $\cdot$ the cup-product $CH^i(X) \times CH^j(X) \to CH^{i+j}(X)$. If X is connected and has dimension d, we denote by $\deg : CH^d(X) \to \mathbb{Z}$ the degree homomorphism.

With this notation, the *slope $\mu(E)$* of a torsionfree sheaf E is defined by the formula $\mu(E) = \frac{1}{\text{rank}(E)}\deg(c_1(E) \cdot \mathcal{O}_X(1)^{d-1})$, where $d = \dim_k X$.

Lemma 2.1 *Let X be a smooth connected projective variety of dimension $d \geq 1$ defined over a perfect field k of characteristic $p > 0$. Let $E = (E_n)_{n \in \mathbb{N}}$ be a stratified bundle. For any class $\xi \in CH^i(X)$, for all $n \geq 0$, and all $0 \leq i \leq d - 1$, one has $\deg(\xi \cdot \gamma_{d-i}(E_n)) = 0$, where γ_{d-i} is any homogeneous polynomial of degree $d - i$ with rational coefficients in the Chern classes.*

Proof One has $\deg(\xi \cdot c_{d-i}(E_n)) = \deg(\xi \cdot c_{d-i}((F^a)^* E_{n+a}))$ for all $a \geq 0$. On the other hand, for any bundle E, one has $\gamma_{d-i}(F^* E) = p^{d-i}\gamma_{d-i}(E)$. Thus $\deg(\xi \cdot c_{d-i}(E_n)) \in \frac{1}{D}\mathbb{Z}$ is infinitely p-divisible, where D is the bounded denominator $\in \mathbb{N} \setminus \{0\}$. Thus this is 0. $\square$

Corollary 2.2 *Let X be a smooth connected projective variety of dimension $d \geq 1$ defined over a perfect field k of characteristic $p > 0$. Let $E = (E_n)_{n \in \mathbb{N}}$ be a stratified bundle. Then $p_{E_n} = p_{\mathcal{O}_X}$ and $\mu(E_n) = 0$ for any $n \geq 0$.*

Proposition 2.3 *Let X be a smooth connected projective variety defined over a perfect field k of characteristic $p > 0$. For any stratified bundle $E = (E_n)_{n \in \mathbb{N}}$, there is a $n_0 \in \mathbb{N}$ such that the stratified bundle $E(n_0) = (E_n)_{n \geq n_0, n \in \mathbb{N}}$ is a successive extension of stratified bundles $U = (U_n)_{n \in \mathbb{N}}$*

with the property that all U_n for $n \in \mathbb{N}$ are μ-stable of slope 0. In particular, all U_n for $n \in \mathbb{N}$ are χ-stable bundles of Hilbert polynomial $p_{\mathcal{O}_X}$.

Proof Since μ-stability implies χ-stability [9, Lemma 1.2.13], it is enough to prove the proposition with μ-stability. We first show that for n_0 large enough, $E_n, n \geq n_0$ is μ-semistable (see [1, Lemma 2.2]). Let U_n be the a nontrivial subsheaf of E_n. Assume $\mu(U_n) > 0 = \mu(E_n)$ (Corollary 2.2). Since $(F^n)^*(U_n) \subset E_0$, $\mu((F^n)^*(U_n)) = p^n \mu(U_n)$ is bounded by $\mu_{\max}(E_0)$, the slope of the maximal destabilizing subsheaf of E_0. One concludes that there is a $n_0 \geq 0$ such that for all $n \geq n_0$, one has $\mu_n(U_n) \leq 0$. Thus E_n is μ-semistable for $n \geq n_0$.

To show the proposition, we may now assume that $n_0 = 0$, that is all E_n are μ-semistable of slope 0. Let $U_n \subset E_n$ be the socle of E_n, that is the maximal nontrivial subsheaf which is μ-polystable of slope 0. Then $(F^n)^*(U_n) \subset E_0$ has still slope 0, thus has to lie in U_0. This yields a decreasing sequence $\cdots \subset (F^{n+1})^*(U_{n+1}) \subset (F^n)^*(U_n) \subset \cdots \subset E_0$, which has to be stationary for n large. Thus there is a $n_1 \geq 0$ such that for all $n \geq n_1$, $F^*U_{n+1} = U_n$. So $(U_{n_1}, U_{n_1+1}, \ldots) \subset E(n_1)$ is a substratified sheaf. Thus it is a substratified bundle, as $\mathcal{O}_X$-coherent $\mathcal{D}_X$-coherent modules are locally free [3, Lemma 6].

So we may assume $n_1 = 0$. Write $U_n = \bigoplus_{b=1}^{a(n)} S_n^b$ where S_n^b is μ-stable of slope 0. Then one has an exact sequence $0 \to F^*S_{n+1}^b \cap S_n^c \to F^*S_{n+1}^b \oplus S_n^c \to F^*S_{n+1}^b + S_n^c \to 0$ where the sum on the right is taken in U_n. Thus $\mu(F^*S_{n+1}^b + S_n^c) \leq 0$ and $\mu(F^*S_{n+1}^b \cap S_n^c) \leq 0$, and we conclude that $F^*S_{n+1}^b \cap S_n^c$ is either S_n^c or else is equal to 0. We conclude that $U = (U_n)_{n \in \mathbb{N}} \subset E = (E_n)_{n \in \mathbb{N}}$ is a direct sum $U = \bigoplus_{b=1}^a U^b$ where $U^b = (U_n^b)_{n \in \mathbb{N}}$ is a stratified bundle which has the property that U_n^b is μ-stable of slope 0.

We now finish the proof using that the category of stratified bundles is abelian: we replace E by E/U, which has a strictly lower rank, and we redo the argument. $\qquad\square$

Proposition 2.4 *Let X be a smooth projective variety defined over a perfect field k of characteristic $p > 0$. Let $x \to X$ be a geometric point. If $\pi_1^{\text{ét}}(X \otimes_k \bar{k}, x)$ has no quotient isomorphic to $\mathbb{Z}/p$, then stratified bundles which are successive extensions of the trivial stratified bundle by itself are trivial.*

Proof We assume $k = \bar{k}$. We have to show that a stratified bundle $E = (E_n)_{n \in \mathbb{N}}$ which is a successive extension of $\mathbb{I} := (\mathcal{O}_X, \mathcal{O}_X, \ldots)$ by itself is trivial. By dos Santos' theorem [3, (9)], the isomorphism class of E lies in the projective system $\varprojlim_n H^1(X, \mathcal{O}_X)$ where the transition maps are the pullback maps $F^* : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ via the absolute

Frobenius. On the other hand, since k is perfect, one has the decomposition $H^1(X, \mathcal{O}_X) = H^1(X, \mathcal{O}_X)_{ss} \oplus H^1(X, \mathcal{O}_X)_{nilp}$ where $H^1(X, \mathcal{O}_X)_{ss} = H^1(X, \mathbb{Z}/p\mathbb{Z}) \otimes_{\mathbb{F}_p} k$ and $H^1(X, \mathcal{O}_X)_{nilp} \subset H^1(X, \mathcal{O}_X)$ is defined as the k-subvectorspace of classes on which F^* is nilpotent. Since $H^1(X, \mathbb{Z}/p\mathbb{Z}) = \mathrm{Hom}(\pi_1^{\text{ét}}(X), \mathbb{Z}/p\mathbb{Z})$, this group is 0 by the assumption. Since $H^1(X, \mathcal{O}_X)$ is a finite dimensional vectorspace, and F is semilinear, there is a $N \in \mathbb{N} \setminus \{0\}$ such that F^{N*} annihilates $H^1(X, \mathcal{O}_X)_{nilp}$. Thus we conclude $\varprojlim_n H^1(X, \mathcal{O}_X)$ $= \varprojlim_n H^1(X, \mathcal{O}_X)_{nilp} = 0$. Thus the stratified bundle E is trivial. This finishes the proof over $k = \bar{k}$. In general, this shows that if E is a stratified bundle on X which is a successive extension of the trivial stratified bundle by itself, then $E \otimes_k \bar{k}$ is a trivial stratified bundle on $X \otimes_k \bar{k}$. Thus for any $n \geq 0$, $H^0(X, E_n) \otimes_k \mathcal{O}_X \to E_n$ is an isomorphism after tensoring with $\bar{k}$ over k, thus it is an isomorphism. Thus E_n is trivial for all $n \geq 0$, so E is trivial. This finishes the proof in general. $\qquad\square$

3 The proof of the main theorem and its corollary

Let X be a smooth projective connected variety defined over a perfect field k of characteristic $p > 0$. We fix an ample line bundle $\mathcal{O}_X(1)$, a rank $r > 1$ and consider the quasi-projective moduli scheme M of χ-*stable* torsionfree sheaves with Hilbert polynomial $p_{\mathcal{O}_X}$ and rank r, as defined by Gieseker [6, Theorem 0.2] in dimension 2 and Langer [13, Theorem 4.1] in any dimension.

Even if one can reduce the statement of Theorem 1.1 to the dimension 2 case by a Lefschetz type argument on stratified bundles, we will need the strength of Langer's theorem. Let us recall from *loc. cit.* that if S is a smooth absolutely connected affine variety over a finite field $\mathbb{F}_q$ such that $\mathbb{F}_q(S) \subset k$, and if $X_S \to S$ is a smooth projective model of X, then there is a quasiprojective scheme $M_S \to S$ which universally corepresents the functor of families of χ-stable torsionfree sheaves of rank r and Hilbert polynomial $p_{\mathcal{O}_X}$ on geometric fibers. This concept is due to Simpson [16, p. 60]. The scheme M_S has several properties. It represents the étale sheaf associated to the moduli functor. If $s \in S$ is any closed point, then $M_S \times_S s$ is M_s. If $u \in M_S$ is any closed point above the closed point $s \in S$, there is a χ-stable torsionfree sheaf E on $X_S \times_S s$ with moduli point u.

We will use the notation $[E] \in M$ to indicate the closed point in M which represents the stable bundle E. As is well known [4, Theorem 1], even if E is stable, the bundle $F^*(E)$ need not be stable. On the other hand, stability is an open condition. Thus there is an open subscheme $M^0 \subset M$ such that $F^*(E)$ is stable for all $[E] \in M^0$.

Definition 3.1 The morphism $\quad V : M^0 \to M \quad$ is called the *Verschiebung*.

Definition 3.2 We define the locus $M^s \subset M$ consisting of points $[E] \in M$ which are μ-stable of slope 0 such that there is a statified bundle $(E_n)_{n \in \mathbb{N}}$ with $E = E_0$. (The upper script s here stands for stratified.)

This is the key definition of the article due to the following remark.

Remark 3.3 All the bundles $E_n, n \geq 0$ in Definition 3.2 have the property that they are μ-stable of slope 0 by the computation of Proposition 2.3: $\mu((F^n)^*U_n) = p^n \mu(U_n)$, hence $\mu(U_n) < 0$. Thus a point $[E] \in M^s$ defines a sequence of points $[E_n]_{n \in \mathbb{N}, n \geq 0} \in M^s$ such that $(E_n)_{n \in \mathbb{N}}$ is a stratified bundle.

We now define various closed subschemes of M using M^s.

Definition 3.4 Let $E = (E_n)_{n \in \mathbb{N}}$ be a stratified bundle with $E_0 \in M^s$ (thus $E_n \in M^s$ for all $n \geq 0$).

(1) We define $A(E)$ to be the Zariski closure of the locus $\{E_n, n \geq 0\}$ in M.
(2) We define $N(E) = \bigcap_{n \geq 0} A(E(n))$ where $E(n) = (E_{n+m, m \geq 0})$.

Closed subschemes of M of type $N(E)$ as in 2) are called *subschemes of M spanned by stratifications*.

Remarks 3.5

(1) As M^s is defined by its k-points, any subscheme of M spanned by stratifications lies in M_{red}.
(2) Let $E = (E_n)_{n \in \mathbb{N}}$ be a stratified bundle with $E_n \in M^s$ for all $n \geq 0$. As $A(E(n+1)) \subset A(E(n))$ are closed subschemes of M_{red}, by the noetherian property there is a $n_0 \geq 0$ such that $A(E(n)) = A(E(n_0)) = N(E)$ for all $n \geq n_0$. Thus there is a hierarchy in the definition: subschemes of type $N(E)$ are of type $A(E)$, and to show that M^s is empty is equivalent to showing that subschemes of M spanned by stratifications are empty.

Definition 3.6 A closed subscheme $N \subset M_{\text{red}}$ is called *Verschiebung divisible* if $V|_N : N \dashrightarrow N$ is a rational map which is dominant on all the components of N. We denote by $N^1 \subset N$ the dense locus on which $V|_N$ is defined.

Lemma 3.7 *Let N be a Verschiebung divisible subscheme of M. For all natural numbers $a \in \mathbb{N} \setminus \{0\}$, the composite $V|_N^a := V|_N \circ \cdots \circ V|_N$ (a-times) is a dominant rational map $V|_N^a : N \dashrightarrow N$ and there is a natural number $a \in \mathbb{N} \setminus \{0\}$ such that $V|_N^a$ stabilizes all the irreducible components of N.*

Proof Since $V|_N$ is a rational dominant self-map, the iterate maps $V^a|_N$ are rational dominant self-maps as well. On the other hand, the image of an irreducible component by a rational map lies in an irreducible component. So by

a counting argument, a rational dominant self-map permutes the irreducible components. Thus an iterate of the map stabilizes the components. ☐

Lemma 3.8 *Subschemes of M which are spanned by stratifications are Verschiebung divisible.*

Proof If N is empty, there is nothing to prove. Else N is constructed as the Zariski closure of family of points $S = \{e_n, n \geq n_0\}$ in M which have the property that for all $n \geq n_0$, there is a point $e_{n+1} \in N$ with $V(e_{n+1}) = e_n$, and which in addition have the property that N is also the Zariski closure of the family of points $S[m] = \{e_n, n \geq m + n_0\}$ for any $m \geq 0$. Taking $m = 1$, one has $V(S) = S[1]$. Let $N^1 \subset N$ be the locus on which V is defined and $V(N^1) \subset N$. Then N^1 contains $S[1]$. Thus $V|_N : N^1 \to N$ is dominant on N and is dense in N. This finishes the proof. ☐

Recall that if Z is any scheme of finite type defined over k, and S is a smooth affine variety, which is defined over a finite field $\mathbb{F}_q$, is geometrically irreducible over $\mathbb{F}_q$, then a *model* Z_S/S is a S-flat scheme such that $Z_S \otimes_S k = Z$.

Let S_0 be a smooth affine variety, which is defined over a finite field $\mathbb{F}_{q_0}$, is geometrically irreducible over $\mathbb{F}_{q_0}$, such that

(i) X/k has a smooth projective model $X_{S_0} \to S_0$,
(ii) the absolute Frobenius $F = F_X : X \to X$ over the Frobenius $F_k : \operatorname{Spec} k \to \operatorname{Spec} k$ has a model $F_{X_{S_0}} : X_{S_0} \to X_{S_0}$ over the absolute Frobenius $F_{S_0} : S_0 \to S_0$.

By Langer's theorem [12, Theorem 4.1], there is then a quasiprojective model $M_{S_0} \to S_0$ of M/k which universally corepresents the functor of χ-stable torsionfree sheaves of rank r and Hilbert polynomial $p_{\mathcal{O}_X}$.

Definition 3.9 Let $S \to S_0$ by a morphism, such that S is a smooth affine variety, geometrically irreducible over a finite extension $\mathbb{F}_q$ of $\mathbb{F}_{q_0}$, with $\mathbb{F}_{q_0}(S_0) \subset \mathbb{F}_q(S) \subset k$. Let X_S, M_S the base changed varieties X_{S_0}, M_{S_0}. Let N be a Verschiebung divisible subscheme of M_{red}. Then a model N_S of N is called *a good model* if

(a) all irreducible components $N_i \subset N \subset M_{\mathrm{red}}$ of N, for $i = 1, \ldots, \rho$, have a model $N_{iS} \subset M_S$ over S,
(b) $V|_N : N^1 \to N$ has a model $V_S : N_S^1 \to N_S$, where $N_S = \bigcup_{i=1}^{\rho} N_{iS} \subset M_S$ and $N_S^1 \subset N_S$ is dense.

Note, for some closed point $s \to S$, $N_{iS} \times_S s$ might be reducible.

For a closed point $s \to S_0$, we denote by $M_{S_0} \times_{S_0} s$, $X_{S_0} \times_{S_0} s = X_s$, $F_{S_0} \times_{S_0} s$ the reductions to s. By the universal corepresentability, one

has $M_{S_0} \times_{S_0} s = M_s$, Langer's moduli scheme of χ-stable torsionfree sheaves
of rank r with Hilbert polynomial $p_{\mathcal{O}_X} = p_{\mathcal{O}_{X_s}}$. By definition, $F_{S_0} \times_{S_0} s$ is
the absolute Frobenius F_s. We denote by V_s the Verschiebung on M_s and
by $V_{S_0} \times_S s$ the reduction of V_S to s on the locus where it is defined. Let
$M_S^0 \subset M_S$ be a model of M^0, with $S \to S_0$, with S smooth affine over a finite
extension $\mathbb{F}_q$ of $\mathbb{F}_{q_0}$. Then by definition, $V_S \times_S s$ is defined on $M_S^0 \times_S s$.

Lemma 3.10 *One has $V_s = V_S \times_S s$ on $M_S^0 \times_S s$.*

Proof Let $t \to M_S^0 \times_S s$ be a closed point. Since $\mathbb{F}_q$ is perfect, $t \to s$ is étale.
There is a morphism of schemes $T \to M_S^0$, such that T is irreducible over
$\mathbb{F}_q$, the composite $T \to M_S^0 \to S$ is finite and étale onto its image which is
a neighbourhood containing s, $t \in M_S^0$ is the image of $t_0 \in T$, $T \to M_S^0$ is
a closed embedding in a Zariski open containing t_0. In particular, one has a
factorization $\mathbb{F}_q(S) \subset \mathbb{F}_q(T) \subset k$. By the universal corepresentability, there
is a torsionfree sheaf E_T over X_T, such that $E_T \times_T k$ is χ-stable of rank
r and with Hilbert polynomial $p_{\mathcal{O}_X}$. By definition of M_S^0, $[E_T \times_T k] \in M^0$,
thus $F^*(E_T \times_T k)$ is χ-stable of Hilbert polynomial $p_{\mathcal{O}_X}$ and rank r. The
sheaf $F^*(E_T \times_T k)$ has a model which we explain now. The pullback of
the absolute Frobenius $F_S : S \to S$ by $T \to S$ is $F_T : T \to T$, as $T \to S$ is
étale. Thus the pullback of the absolute Frobenius $F_{X_S} : X_S \to X_S$ over F_S
by $T \to S$ is the absolute Frobenius $F_{X_T} : X_T \to X_T$. This defines the model
$F_{X_T}^* E_T$ of $F^*(E_T \times_T k)$. We conclude that $F_{X_T}^*(E_T) \times_T t_0 = F_{X_{t_0}}^* E_{t_0}$. By
definition of t_0, $F_{X_{t_0}}^* E_{t_0} = F_{X_t}^* E_t$. By definition, $[F_{X_t}^* E_t] \in M_t$. We conclude
$[F_{X_t}^* E_t] = V_t([E_t])$. $\square$

Corollary 3.11 *Let N be a Verschiebung divisible subscheme of M_{red}, and let
N_S be a good model. Let $a \in \mathbb{N} \setminus \{0\}$ such that $V|_N^a$ stabilizes the irreducible
components of N as in Lemma 3.7. Then*

(i) *there is a nontrivial open subscheme $T \subset S$ such that for all closed points
$s \in T$, and for all $i \in \{1, \ldots, \rho\}$, $V_T|_{N_{iT}}^a \times_T s$ is a dominant rational self
map of $N_{iT} \times_T s$;*

(ii) *for any closed point $s \in T$, there is a dense open subscheme of $N_{iT} \times_T s$
on which $V_T|_{N_{iT}}^a \times_T s$ is well defined and is equal to $V_s^a = V_s \circ \cdots \circ V_s$
(a-times).*

Proof By definition, the restriction of V on N is a rational dominant self-
map. As V_S is a good model, $V_S : N_S^1 \to N_S$ is a well defined dominant
map, and $N_S^1 \subset N_S$ is dense. Thus there is a nontrivial open $S_a \subset S$ and a
dense subscheme $N_{S_a}^a \subset N_S^1$, such that the composite $V_{S_a}^a = V_{S_a} \circ \cdots \circ V_{S_a}$
(a-times) is well defined and dominant. So there is a nontrivial open $T \subset S_a$

such that for all closed points $t \in T$, the restriction $V_T^a \times_T t : N_T^a \times_T t \to N_T \times_T t = N_t$ is well defined and is dominant. By the choice of a, V_T^a respects the components V_{iT}. This shows (i). As for (ii), this is then a direct consequence of Lemma 3.10. This finishes the proof. $\qquad\square$

Definition 3.12 Let X be a smooth projective variety defined over a perfect field k of characteristic $p > 0$. Let M be the moduli of χ-stable sheaves of rank r and Hilbert polynomial $p_{\mathcal{O}_X}$.

(1) A closed point $[E] \in M$ is called a *torsion point* if there is a $m \in \mathbb{N} \setminus \{0\}$ such that $(F^m)^*E \cong E$.
(2) Let S be a smooth affine variety defined over $\mathbb{F}_q$ such that X has a smooth projective model X_S, and let M_S Langer's quasi-projective moduli S-scheme. Let $u \in M_S$ be a closed point, mapping to the closed point $s \in S$. Then u is said to be a *torsion point* if u is a torsion point viewed as a closed point in M_s.

Remark 3.13 If $u \in M_S$ is a torsion point mapping to $s \in S$, then the torsionfree sheaf E on X_s it corresponds to has a stratification $E = E_0$, $E_1 = (F^{N-1})^*E, \ldots, E_N = E, E_{N+1} = E_1, \ldots$. It follows by Katz' theorem [5, Theorem 1.3] that $E = (E_n)_{n \in \mathbb{N}}$ is a stratified bundle (that is, all the E_n are locally free). Furthermore, all E_n are stable of slope 0, so all E_n define modular points in M_s.

Theorem 3.14 *Let X be a smooth projective variety defined over a perfect field k of characteristic $p > 0$. Then M be the moduli of χ-stable torsionfree sheaves of rank r. Let $N \subset M_{\mathrm{red}}$ be a Verschiebung divisible subscheme, and let N_S be a good model of N. Then the torsion points of N_S are dense in N_S.*

Proof Our goal is to show that if $Z \subset N_S$ is any strict closed subscheme, then $N_S \setminus Z$ contains torsion points. So it is enough to show that $N_T \setminus Z$ contains torsion points, where T is defined in Corollary 3.11. Let $s_0 \in T$ be a closed point and let $q(s_0)$ be the cardinality of the residue field $\kappa(s_0)$. We want to show that there are torsion points in $N_T \times_T s_0 \setminus (Z \cap N_T \times_T s_0)$.

We apply Hrushovski's theorem [8, Corollary 1.2]: let $i \in \{1, \ldots, \rho\}$. This fixes the component N_{iT} we consider. We define κ to be a finite extension of $\kappa(s_0)$ of cardinality q, so that the irreducible components of $N_{is} := N_{iT} \otimes_T \kappa$ are defined over κ. Here $s = s_0 \otimes_{\kappa(s_0)} \kappa$. We pick one such irreducible component of maximal dimension, say Y'. Since V_s^a stabilizes N_{is}, the same argument as in Lemma 3.7 implies that V_s^{ab} stabilizes Y' for some $b \in \mathbb{N} \setminus \{0\}$. We choose an open dense affine subvariety $Y \subset Y'$. Then Y is irreducible over $\mathbb{F}_q$. Let $\Gamma \subset Y \times_s Y$ be the Zariski closure of the graph of $\Psi := V_s^{ab}$ where it is well defined. Then the first projection $\Gamma \to Y$ is birational (and therefore dominant) and the second projection $\Gamma \to Y$ is dominant as Ψ is

dominant. Let $\Phi_q : Y \otimes_\kappa \bar{\mathbb{F}}_q \to Y \otimes_\kappa \bar{\mathbb{F}}_q$ be the Frobenius raising coordinates to the q-th power. Hrushovski's theorem *loc. cit.* asserts that for any proper closed subvariety $W \subset Y$, there is a closed point $u \in Y \setminus W$, thus that for a suited $m \in \mathbb{N} \setminus \{0\}$ large enough, $(u, \Phi_q^m(u))$ is a closed point of Γ. Taking W to contain both the locus on which Ψ is not defined, and $Z \cap Y$, we obtain that $(u, v) \in \Gamma$ if and only if $v = \Psi(u)$, thus $\Phi_q^m(u) = \Psi(u)$, and $u \notin Z \cap Y$. On the other hand, one has $\Phi_q \circ V_s(u) = V_s \circ \Phi_q(u)$ as V_s is defined over κ. Thus from $\Phi_q^m(u) = V_s^{ab}(u)$, we deduce that $\Phi_q^{mc}(u) = V_s^{abc}(u)$ for all $c \in \mathbb{N} \setminus \{0\}$. As $u \in T$ is a closed point, there is a c such that $\Phi_q^{mc}(u) = u$. We deduce that for all proper closed subvarieties $W \subset Y$ containing the indeterminacy locus of Ψ and $Z \cap Y$, there is a closed point $u \in Y \setminus W$ and a natural number $d \in \mathbb{N} \setminus \{0\}$, such that $u = V_s^d(u)$. Thus u is a torsion point. This finishes the proof. $\qquad\square$

Theorem 3.15 *Let X be a smooth connected projective variety defined over a perfect field k of characteristic $p > 0$. Let $\bar{x} \to X$ be a geometric point. If $\pi_1^{\text{ét}}(X \otimes_k \bar{k}, \bar{x}) = \{1\}$, there are no nontrivial stratified bundles.*

Proof We consider rank $r \geq 2$ stratified bundles, as for rank 1, we gave the proof in the introduction. We want to show that M^s is empty. This is equivalent to saying that the Zariski closure of M^s in M is empty, and is also equivalent to saying that any subscheme N of M_{red} spanned by stratifications is empty (see Remarks 3.5(2)). Let N_S be a good model of N. By Lemma 3.8 together with Theorem 3.14, torsion points are dense in N_S. By Remark 3.13, a torsion point $u \in N_T$ mapping to $s \in S$ represents in particular a vector bundle E on X_s. By the theorem of Lange-Stuhler [11, Satz 1.4], there is a (non-commutative) geometrically connected étale finite covering $h : Z \to X_s$, such that $h^* E$ is trivial (beware that the other statement of *loc. cit*, asserting that for any vector bundle E, an étale finite covering h which trivializes $h^* E$ exists if and only if $(F^m)^* E \cong E$ for some $m \in \mathbb{N} \setminus \{0\}$ is not correct, although there are many ways to correct it). On the other hand, since $X_S \to S$ has good reduction and is proper, the specialization map $\pi_1^{\text{ét}}(X_{\bar{k}}) \to \pi_1^{\text{ét}}(X_s \times_s \text{Spec}\,\bar{\mathbb{F}}_q)$ is surjective [17, Exposé X, Théorème 3.8]. The assumption implies then that $\pi_1^{\text{ét}}(X_s \times_s \text{Spec}\,\bar{\mathbb{F}}_q) = \{1\}$, thus h is the identity, and $E \simeq \bigoplus_1^r \mathcal{O}_{X_s}$. But E has to be stable of rank $r \geq 2$. This is impossible. So we conclude that N_S is empty, thus N is empty. So there are no stratified bundles $E = (E_n)_{n \in N}$ with E_n μ-stable of rank r and slope zero, except $E = \mathbb{I}$. By Proposition 2.3, for any stratified bundle, $E(n_0)$ is a successive extension of $\mathbb{I}$ by itself for n_0 large enough. We apply Proposition 2.4 to finish the proof over $k = \bar{k}$. If k is perfect and not algebraically closed, as already noticed in the proof of Proposition 2.4, a stratified bundle E is trivial if and only if $E \otimes_k \bar{k}$ is trivial. This finishes the proof in general. $\qquad\square$

In the remaining part of this section, we illustrate the theorem with two examples. The second one is due to M. Raynaud.

Let X be a smooth projective variety defined over an algebraically closed field $k = \bar{k}$ of characteristic $p > 0$. We know that if $k \neq \bar{\mathbb{F}}_p$, and if $H^1(X, \mathcal{O}_X)_{ss}$ has k-dimension at least ≥ 2, one easily constructs an infinite family of extensions of $\mathbb{I}$ by itself in $\mathsf{Strat}(X)$ (see [1, Proposition 2.9]). Furthermore, there are stratified bundles which are not semistable, so they can't be trivialized after a finite étale covering [5]. Thus the assumption on the finiteness of $\pi_1^{\text{ét}}(X \otimes_k \bar{k}, \bar{x})$ in the next corollary is really necessary.

Corollary 3.16 *Let X be a smooth connected projective variety defined over an algebraically closed field k of characteristic $p > 0$. Let $x \in X(k)$ be a rational point. Assume $\pi_1^{\text{ét}}(X, x)$ is finite. Then*

(i) *the surjective homomorphism $\pi^{\text{strat}}(X, x) \to \pi_1^{\text{ét}}(X, x)$ induces an isomorphism on irreducible representations;*
(ii) *if $\pi_1^{\text{ét}}(X, x)$ has order prime to p, then the surjective homomorphism $\pi^{\text{strat}}(X, x) \to \pi_1^{\text{ét}}(X, x)$ is an isomorphism; in particular, every stratified bundle is a direct sum of irreducible ones.*

Proof Let $h : Y \to X$ be the universal cover based at x, so it is a Galois cover under $\pi_1^{\text{ét}}(X, x)$. Let $\mathbb{I}_Y$ be the trivial stratified bundle on Y, and $\mathbb{E} = h_*\mathbb{I}_Y$ be its direct image. It is an object of $\mathsf{Strat}(X)$ which comes from the regular representation $k[\pi_1^{\text{ét}}(X, x)]$ of $\pi_1^{\text{ét}}(X, x)$. If $E = (E_n)_{n \in \mathbb{N}}$ is a stratified object, then by Theorem 3.15, h^*E is trivial, thus $E \subset H^0(Y, h^*E_0) \otimes \mathbb{E}$. We first show (i). If E is irreducible as a stratified bundle, there is a projection of $H^0(Y, h^*E_0)$ to a k-line ℓ such that E is still injective in $\ell \otimes_k \mathbb{E}$. Thus $E \subset \ell \otimes_k \mathbb{E}$ is a subrepresentation of $\pi^{\text{strat}}(X, x)$. Since $\pi^{\text{strat}}(X, x) \to \pi_1^{\text{ét}}(X, x)$ is surjective, $E \subset \ell \otimes_k \mathbb{E}$ is a subrepresentation of $\pi_1^{\text{ét}}(X, x)$. This shows (i). As for (ii), h then has degree prime to p, thus $\mathbb{E}$ is a direct sum of irreducible representations of $\pi^{\text{ét}}(X, x)$, thus $E \subset H^0(Y, h^*E_0) \otimes_k \mathbb{E}$ as well, and is in particular a subrepresentation of $\pi^{\text{ét}}(X, x)$. $\qquad\square$

We now illustrate Theorem 3.14 in rank 2 over $k = \bar{\mathbb{F}}_p$. To this aim, recall that if E is a stratified bundle over X projective smooth over $k = \bar{k}$, and if $x \in X(k)$, the Tannaka k-group $\text{Aut}^{\otimes}(\langle E \rangle, x) \subset GL(E_0|_x)$ is also called the *monodromy group*. Recall further that if $[E]$ is a torsion point (see Definition 3.12) in M, then its monodromy group is in fact a finite quotient of $\pi^{\text{ét}}(X, x)$, as E is trivialized on the Lange-Stuhler finite étale covering h (*loc. cit.*), so the monodromy group is a quotient of the Galois group of h.

Proposition 3.17 (M. Raynaud) *Let X be a smooth projective variety defined over $k = \bar{\mathbb{F}}_p$. Let $E = (E_n)_{n \in \mathbb{N}}$ be a rank 2 stratified bundle with $E_n \in M$,*

the moduli of χ-stable rank 2 torsionfree sheaves, and with $\det(E) = \mathbb{I}$ (i.e. $\det(E_n) = \mathcal{O}_X$ for all $n \geq 0$). Let $N(E)$ be defined in Definition 3.4. *We assume $N(E)$ irreducible of dimension > 0. Then*

(i) *either there is a dense subset of torsion points $a_i \in N(E)$ with monodromy group $SL(2, k_i) \subset GL(E_0|_x)$ where k_i are finite subfields of k of increasing order;*

(ii) *or there is a dense subset of torsion points $a_i \in N(E)$ with monodromy group a dihedral group D_{n_i} of order $2n_i$ with increasing n_i.*

One can characterize geometrically the second case: it happens precisely when there is an étale degree 2 covering $h : Y \to X$ and a rank one stratified bundle L on Y such that $h_* L = E$, or, equivalently, such that $h^* E$ becomes reducible: $h^* E \cong L \oplus L^\sigma$, where $\langle \sigma \rangle$ is the Galois group of h and L^σ is the Galois translate of L.

Proof Assume there if a degree 2 étale covering $h : Y \to X$ and a rank 1 stratified bundle L on Y such that $L \subset h^* E$. Then $L^\sigma \subset h^* E$ and thus $h^* E \cong L \oplus L^\sigma$. Since $\det(E) = \mathbb{I}$, one has $L^\sigma \cong L^{-1}$. One has in particular $h_* L_n = E_n$ for all $n \geq 0$. All L_n lie in $\mathrm{Pic}^\tau(Y)$. We define $N(L) \subset \mathrm{Pic}^\tau(Y)$ as in Definition 3.4. Then the morphism $\mathrm{Pic}^\tau(Y) \to M, \mathcal{L} \mapsto h_* \mathcal{L}$ sends $N(L)$ to a closed subscheme of M_{red}, as $\mathrm{Pic}^\tau(Y)$ is proper, which is contained in $N(E)$ by construction, but which contains all the $E_n = h_* L_n$. Thus $h_* N(L) = N(E)$. The subscheme $N(L)$ is the closure of an infinite family of torsion points t_m. Since $h_* t_m$ lies in $N(E)$, and since the locus of M on which the determinant is $\mathcal{O}_X$ is closed, all points of $N(E)$ have determinant equal to $\mathcal{O}_X$. In particular, one has $\det(h_*(t_m)) = \mathcal{O}_X$. This implies that $t_m^\sigma \cong t_m^{-1}$ as $\det(h^* h_* t_m) \cong \det(t_m \oplus t_m^\sigma) = \mathcal{O}_Y$. Let $y \in Y(k)$ mapping to $x \in X(k)$. The exact sequence $1 \to \pi_1^{\text{ét}}(Y, y) \to \pi_1^{\text{ét}}(X, x) \to \mathbb{Z}/2 \to 0$ induces by pushout an exact sequence $1 \to H \to D \to \mathbb{Z}/2 \to 0$ for any finite quotient $\pi_1^{\text{ét}}(Y, y) \twoheadrightarrow H$. Taking for H the cyclic quotients $\mathbb{Z}/n(m)\mathbb{Z}$ corresponding to the t_m yields a stratified line bundle $\mathcal{L}_m$ on Y, such that the generator $\sigma \in \pi_1^{\text{ét}}(X, x)$ of the automorphism group of Y over X acts via $x \mapsto -x$ on $\mathbb{Z}/n(m)\mathbb{Z}$. Thus D is the dihedral group $D_{n(m)}$. Summarizing: if there is a $h : Y \to X$ such that $h^* E$ becomes strictly semistable, then points with monodromy group a dihedral group $D_{n(m)}$ of increasing order $n(m)$ are dense in $N(E)$. (Note we do not need the irreducibility of $N(E)$ for this point.)

Vice-versa, assume the points δ_m with dihedral monodromy $D_{n(m)}$ are dense in $N(E)$. Since $\pi^{\text{ét}}(X, x)$ has only finitely many $\mathbb{Z}/2\mathbb{Z}$ quotients, there is an infinite sequence of such points δ_m such that the induced quotient $\pi^{\text{ét}}(X, x) \to \mathbb{Z}/2\mathbb{Z}$ is fixed. Let $h : Y \to X$ be the corresponding covering and $y \in Y(k)$ mapping to x. The representation $\pi^{\text{ét}}(X, x) \twoheadrightarrow D_{n(m)}$ induces a representation $\pi^{\text{ét}}(Y, y) \twoheadrightarrow \mathbb{Z}/n(m)\mathbb{Z}$, defining a stratified rank 1 bundle $\mathcal{L}_m$ on Y, such that $\mathcal{L}_m^\sigma \cong \mathcal{L}_m^{-1}$. Then the points $h_* \mathcal{L}_m$ are dense in $N(E)$. Thus

the Zariski closure Z of the $\mathcal{L}_m$ in $\text{Pic}^\tau(Y)$ has the property that $h_* Z = N(E)$ by the argument we had before. Thus there is a $L \in Z$ such that $h_* L = E$. This finishes the characterization of the second case.

To show that one has either case (i) or (ii), one has to appeal to the classification of finite groups of $SL(2, \bar{\mathbb{F}}_p)$: one deduces from Dickson's Theorem [10, Hauptsatz 8.27] that there are only 2 infinite families of finite subgroups of $SL(2, \bar{\mathbb{F}}_p)$ such that the natural rank 2 representation is geometrically irreducible, the dihedral subgroups D_m and the finite subgroups $SL(2, \mathbb{F}_{q_i})$. Arguing again that fixing a finite group H, there are finitely many surjections $\pi^{\text{ét}}(X, x) \twoheadrightarrow H$, one shows the dichotomy. □

4 Some remarks and questions

4.1

Using the effective Lefschetz properties [18, Corollaire 3.4], the property that stratified bundles uniquely lift to the formal neighbourhood of a smooth divisor [5, Proposition 1.5], and the fact that coherent $\mathcal{D}_X$-modules which are $\mathcal{O}_X$-coherent are locally free [3, Lemma 6], one shows that if $Y \subset X$ is a smooth ample divisor on a smooth projective variety defined over a perfect field of characteristic $p > 0$, and if $y \in Y(k)$, then the homomorphism $\pi_1^{\text{strat}}(Y, y) \to \pi_1^{\text{strat}}(X, x)$ induced by the restriction of bundles is an isomorphism, if $\dim_k(Y) \geq 2$. But in order to reduce the proof of the main theorem of this article to surfaces, one would need the full strength of Langer's theorem on corepresentability, which is hard to extract from Gieseker's article. So we did not present the argument via this reduction.

4.2

Simpson constructed in [16] quasi-projective moduli schemes of semi-simple bundles with flat connection on a complex smooth projective variety. It is not unlikely that his ideas, combined with Langer's methods in characteristic $p > 0$ [12] and Proposition 2.3, yield the existence of quasi-projective moduli schemes of stratified bundles. Those moduli, aside of their own interest, could then be used directly to prove Theorem 1.1.

4.3

As mentioned in the introduction, Grothendieck's theorem over $\mathbb{C}$ has nothing to do with stability questions, and applies to the algebraic completion of the topological fundamental group of any smooth complex quasi-projective manifold. If we translate his theorem via the Riemann-Hilbert correspondence on X smooth complex but not necessarily proper, then he shows that

if the profinite completion topological fundamental group is trivial, there is no nontrivial regular singular bundles with flat connection. The analog in characteristic $p > 0$ over a quasi-projective smooth variety X would relate Grothendieck's étale fundamental group over $\bar{k}$ and tame stratified bundles. This would require a completely different proof. At any rate, we do not even have at disposal a theory of good lattices in $j_* E$, where E is stratified on X and $j : X \to \hat{X}$ is a normal compactification. Unfortunately, we can not say anything on this subject.

4.4

The proof of the main theorem may be seen as an application of Hrushovski's theorem. On one hand, it is very nice to see how his profound theorem works concretely for some natural question coming from algebraic geometry. On the other hand, to have another proof anchored in algebraic geometry would perhaps shed more light on the original problem.

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