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Packets in Grothendieck's section conjecture ☆

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Abstract

The goal of this article is to show a part of Grothendieck's section conjecture using the identification of sections with neutral fiber functors as defined in [H. Esnault, P.H. Hai, The fundamental groupoid scheme and applications, preprint, 2006, 34 pp.].
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1. Introduction

Let U be an absolutely connected, smooth scheme of finite type defined over field k of characteristic 0. Then a section of the Galois group $\mathrm{Gal}(\bar{k}/k)$ into Grothendieck's fundamental group $\pi_1(U, \bar{u})$ based at a geometric point $\bar{u} \to U$ is identified in [6] with a neutral fiber functor ρ of the Tannaka category $\mathsf{FC}(U)$ of finite connections. To a neutral fiber functor ρ, one defines a k-form $s_\rho : U_\rho \to U$ of Grothendieck's universal covering $U_{\bar{u}} \to U$ based at $\bar{u}$, which identifies ρ with the cohomological fiber functor $H^0(U_\rho, s_\rho^*(-))$.

Grothendieck's section conjecture predicts a geometric description of sections $\mathrm{Gal}(\bar{k}/k) \to \pi_1(U, \bar{u})$, under a geometric condition on U and an arithmetic condition on k. Without any condition, we can already say that the Tannaka description above is of geometric nature, if we think of neutral fiber functors as rational points of a gerbe. On the other hand, Tamagawa [12]

showed a part of the conjecture following a suggestion of Grothendieck. The aim of this article is twofold. We reprove the known part of Grothendieck's section conjecture using our identification of sections with neutral fiber functors as defined in [6]. We then use our method to define the packets and to show the properties predicted by the conjecture.

A smooth absolutely connected curve over a field k is called hyperbolic if the degree of the sheaf of 1-differential forms with logarithmic poles at ∞ is strictly positive. Grothendieck's section conjecture [8] (see precise formulation in Conjecture 2.2 of this article) predicts first that under the assumptions that U be a hyperbolic curve over k, a field of finite type over $\mathbb{Q}$, then rational points of U inject into the set of sections. Call such sections geometric. Let $X(U_{\bar{u}})$ be the pro-system of smooth compactifications of the pro-system $U_{\bar{u}}$. Grothendieck predicts further that a geometric section yields a unique $\mathrm{Gal}(\bar{k}/k)$-invariant point on $X(U_{\bar{u}})$, lying above a point in $U(k)$, and that a non-geometric section also yields a unique $\mathrm{Gal}(\bar{k}/k)$-invariant point on $X(U_{\bar{u}})$, thus lying above a point in $(X \setminus U)(k)$, where $X \supset U$ is the smooth compactification of U. Grothendieck mentions in [8, p. 8] that those properties should be proven using a Mordell-Weil type argument. Indeed, it has been essentially worked out in [12, Section 2]. We show these properties anew in Section 5, using our method. It becomes then a simple consequence of the definitions.

According to Grothendieck, non-geometric sections should be subdivided into packets $\mathcal{P}_x$, $x \in (X \setminus U)(k)$ [8, p. 9]. Each packet $\mathcal{P}_x$ should have the cardinality of the continuum, and, for $x \neq y$, a section in $\mathcal{P}_x$ should not be equivalent under conjugation with $\pi_1(\bar{U}, \bar{u})$ to a section in $\mathcal{P}_y$.

The Tannaka method we use, which relies on Nori's ideas [10,11], on Deligne's non-neutral Tannaka theory [4], and on fiber functors at ∞ as defined in [3, Section 15] and [9a], allows us to define these packets and to show the properties wanted.

In Section 2, we review Grothendieck's theory of the arithmetic fundamental group [7] as used later on and formulate precisely his section conjecture. In Section 3 we review our theory as developed in [6] and suitable for our purpose here, and reformulate in this language Grothendieck's section conjecture. In Section 4, we provide a homological interpretation of the part concerning geometric sections. In Section 5, we show that a section yields at most one fixed point on $X(U_{\bar{u}})$ and define the packets. In Section 6, we show that each packet has the cardinality of the continuum. In fact, at each stage, we show what are the necessary conditions for those properties to hold.

What remains to be understood is the more difficult part of Grothendieck's section conjecture, namely the existence of the fixed point. Call it the existence conjecture. Nothing is known on it, except Königsmann's spectacular result [9a] using logic and model theory, if k of finite type over $\mathbb{Q}$ is replaced by a p-adic field. We make in Section 7 a list of reductions. In particular, thinking of a fixed point as a k-rational point of a k-form of $X(U_{\bar{u}})$, it would be enough to find a K-rational point for K a function field appearing in the prosystem defining the k-form of $U_{\bar{u}}$ (Proposition 7.1). Furthermore, the existence conjecture suggests a weaker conjecture on extensions of fiber functors which, if possible to understand, would in general reduce the existence conjecture to an open in $\mathbb{P}^1$, and over a number field to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ (Proposition 7.9).

2. Grothendieck's section conjecture

Let k be a characteristic zero field, and fix an algebraic closure $\bar{k}$ of it together with the canonical inclusion $i : k \to \bar{k}$. Denote $\Gamma := \mathrm{Gal}(\bar{k}/k)$. Let $\epsilon : X \to \mathrm{Spec}\, k$ be an absolutely connected scheme over k and fix a closed geometric point $\bar{x} : \mathrm{Spec}\, \bar{k} \to X$ of X. Let $s_{\bar{x}} : X_{\bar{x}} \to X$

be the fundamental pro-étale covering of X, which is equipped with a $\bar{k}$-point $\tilde{x}$ with $s_{\tilde{x}}(\tilde{x}) = \bar{x}$. Thus the arithmetic fundamental group $\pi_1(X, \bar{x})$ of X, is, as a set, the fiber $s_{\tilde{x}}^{-1}(\bar{x})$, in particular $\tilde{x}$ is identified with the unit element of $\pi_1(X, \bar{x})$.

The morphism ϵ yields a homomorphism of fundamental groups

$$\epsilon_* : \pi_1(X, \bar{x}) \to \pi_1(\operatorname{Spec} k, i) = \operatorname{Gal}(\bar{k}/k) = \Gamma. \tag{2.1}$$

The assumption that X is absolutely connected implies that ϵ_* is surjective. The kernel of ϵ_* is isomorphic to $\pi_1(\bar{X}, \bar{x})$, the geometric fundamental group of $\bar{X} := \operatorname{Spec} \bar{k} \times_{\operatorname{Spec} k} X$ with base point $\bar{x}$. That is, we have the following exact sequence

$$1 \to \pi_1(\bar{X}, \bar{x}) \to \pi_1(X, \bar{x}) \xrightarrow{\epsilon_*} \Gamma \to 1. \tag{2.2}$$

In particular we have the following commutative diagram

$$\begin{array}{ccc}
& & X_{\bar{x}} \\
& \nearrow^{\tilde{x}} & \downarrow{\scriptstyle t_{\bar{x}}, s_{\bar{x}}} \\
\operatorname{Spec} \bar{k} & \xrightarrow{\operatorname{id} \times \bar{x}} & \operatorname{Spec} \bar{k} \times X = \bar{X}.
\end{array} \tag{2.3}$$

Here we omit the subindex $\operatorname{Spec} k$ for the fibre product of k-schemes over $\operatorname{Spec} k$.

Assume that $\bar{x}$ lies above a rational point $x : \operatorname{Spec} k \to X$, that is $\bar{x} = x \circ i$. This yields a homomorphism $x_* : \pi_1(\operatorname{Spec} k, i) \to \pi_1(X, \bar{x})$ which is a section of ϵ_* as a homomorphism of pro-finite groups.

In general, let y be an arbitrary k-rational point of X. Then $\bar{y} := y \circ i : \operatorname{Spec} \bar{k} \to X$ is a $\bar{k}$-point of X. As above, y_* is a section of $\epsilon_* : \pi_1(X, \bar{y}) \to \Gamma$. Since $\pi_1(X, \bar{x}) \cong \pi_1(X, \bar{y})$ by an isomorphism which commutes with ϵ_*, we can consider y_* as a section of $\epsilon : \pi_1(X, \bar{x}) \to \Gamma$, determined up to an inner conjugation by an element of $\pi_1(\bar{X}, \bar{x})$.

Grothendieck denotes the set of sections of $\epsilon_* : \pi_1(X, \bar{x}) \to \Gamma$ (as pro-finite group homomorphisms) up to conjugation by elements of $\pi_1(\bar{X}, \bar{x}) = \operatorname{Ker} \epsilon_*$ by

$$\operatorname{Hom-ext}_\Gamma\big(\Gamma, \pi_1(X, \bar{x})\big)$$

("ext" stands for external; see [8, Eq. (6)]). Notice that such a section $\sigma : \Gamma \to \pi_1(X, \bar{x})$ yields an action of Γ on $X_{\bar{x}}$ making the morphism $t_{\bar{x}} : X_{\bar{x}} \to \operatorname{Spec} \bar{k}$ Γ-equivariant.

Definition 2.1. Let U be a smooth absolutely connected curve defined over a field k of characteristic 0. Let $X \supset U$ be its smooth compactification. Then U is said to be *elliptic* if its log-dualizing sheaf $\omega_X(\log(X \setminus U))$ has degree 0, and is said to be *hyperbolic* if $\omega_X(\log(X \setminus U))$ has strictly positive degree. Equivalently, U is elliptic if the Euler characteristic $\chi(U) := 2 - 2g(X) - \#(X \setminus U)(\bar{k})$ is 0, and hyperbolic if it is strictly negative.

Let k be a characteristic zero field, and U be an absolutely connected curve over k. Let $\bar{u}$ be a $\bar{k}$-point of U. Then we have the exact sequence (2.2) with $(X, \bar{x})$ replaced by $(U, \bar{u})$. Let $X(U_{\bar{u}})$ denote the compactification of $U_{\bar{u}}$ above the inclusion $U \hookrightarrow X$, that is, the pro-system of the compactifications of the finite étale coverings of U. It projects onto X, is étale above U and ramifies along $(X \setminus U)$.

In a letter to Gerd Faltings dated June 27, 1983, Alexander Grothendieck proposed the following conjecture (see [8, p. 7–9]):

Conjecture 2.2 *(Grothendieck's section conjecture). Let U be a hyperbolic curve defined over a field k of finite type over $\mathbb{Q}$. Then*

(SC1) *The map*

$$U(k) \to \text{Hom-ext}_\Gamma\big(\Gamma, \pi_1(U, \bar{u})\big), \quad y \mapsto y_*, \qquad (2.4)$$

is injective. We call a section of the form y_* a geometric section.

(SC2) *For a geometric section y_*, the resulting action of Γ on $X(U_{\bar{u}})$ has a unique fixed point which lies above y.*

(SC3) *For a non-geometric section, the resulting action of Γ on $X(U_{\bar{u}})$ has also a unique fixed point, which lies at infinity $X(U_{\bar{u}}) \setminus U_{\bar{u}}$, and projects on a point in $(X \setminus U)(k)$. This implies in particular that the set of non-geometric sections is divided into disjoint "packets" $\mathcal{P}_x$, $x \in (X \setminus U)(k)$.*

(SC4) *Each packet $\mathcal{P}_x$ has the cardinality of the continuum.*

Let us specialize to the case $X = U$. Then Grothendieck's section conjecture is equivalent to the following.

(SC) The map

$$X(k) \to \text{Hom-ext}_\Gamma\big(\Gamma, \pi_1(X, \bar{x})\big) \qquad (2.5)$$

is bijective. More precisely, let σ be a section of $\epsilon_* : \pi_1(X, \bar{x}) \to \Gamma$. The scheme $X_{\bar{x}}$, considered as a $\text{Spec}\,\bar{k}$-scheme by means of the morphism $t_{\bar{x}}$, has a unique fixed point under the action of Γ determined by means of σ. Let y be the image of this point in X. It is a k-rational point of X and $\sigma = y_*$.

Remark 2.3. We notice that the Γ-fixed points of $X_{\bar{x}}$ are nothing but the k-rational points of the k-form of $X_{\bar{x}}$ determined by the action of Γ.

3. The fundamental groupoid scheme

As in the previous section, we fix a field k of finite type over $\mathbb{Q}$ as well as an algebraic closure $\bar{k}$ and denote by $i : k \to \bar{k}$ the canonical inclusion. Let $X \xrightarrow{\epsilon} \text{Spec}\,k$ be a smooth, absolutely connected scheme of finite type. We denote by $\mathsf{FC}(X)$ the category of finite connections, defined in [6, Section 2]. This notion generalizes, in characteristic 0, Nori's notion of finite bundles over proper absolutely connected schemes to the case of smooth absolutely connected varieties. As X is absolutely connected, $\mathsf{FC}(X)$ is a Tannaka category over k (as a fibre functor one can always choose the tautological functor τ to $\mathsf{QCoh}(X)$ which assigns to a connection its underlying algebraic bundle). Furthermore, $\mathsf{FC}(X)$ satisfies base change for finite field extensions $K \supset k$. Let $\rho : \mathsf{FC}(X) \to \mathsf{QCoh}(S)$ be a fibre functor, where S is a k-scheme. Then Tannaka duality yields a transitive k-pro-groupoid scheme $\Pi_\rho \xrightarrow{(t,s)} S \times_k S$ [4, Théorème 1.12]. In order to simplify notations, in the rest of the article, we will use the terminology "groupoid scheme" rather than

"pro-groupoid scheme." Let Π_ρ^Δ denote the diagonal part of Π_ρ, which is a flat group scheme over S. In [6, Theorem 1.1] we associate to ρ a Π_ρ^Δ-principal bundle $X_\rho \xrightarrow{(t_\rho, s_\rho)} S \times_k X$. The original fibre functor ρ is equivalent to the cohomological functor $H_{DR}^0(X_\rho/S, s_\rho^*(-))$. This construction is *functorial* in ρ, S and X. The following fibre functors are of special interest.

Let us denote by $\mathsf{Fib}_k(X)$ the set of neutral fiber functors.

Assume that $S = \operatorname{Spec} k$. Thus $\rho \in \mathsf{Fib}_k(X)$, hence Π_ρ is a k-group scheme and $s_\rho : X_\rho \to X$ is a principal under Π_ρ. In particular, let $x \in X(k)$ and $\rho := \rho_x$, $\rho_x((V, \nabla)) = V|_x$ be the fibre functor at x. Then our construction agrees with Nori's construction in case X is proper. In this case, by definition $\Pi_{\rho_x} = s_\rho^{-1}(x)$ as a k-scheme. The unit of Π_{ρ_x} yields a k-rational point x_{ρ_x} of X_{ρ_x} lying above x. This was a key point in Nori's theory. We use the following simplified notations:

$$X_x := X_{\rho_x}, \quad \Pi(X, x) := \Pi_{\rho_x}, \quad s_x := s_{\rho_x}, \quad \hat{x} := x_{\rho_x}. \tag{3.1}$$

For the tautological fibre functor τ, we have $X_\tau = \Pi_\tau \xrightarrow{(t,s)} X \times X$. We call it the total fundamental groupoid scheme of X.

Let $\bar{x}$ be a $\bar{k}$-rational point of X and $\rho_{\bar{x}}$ be the fibre functor at $\bar{x}$. We call the k-groupoid scheme $\Pi_{\rho_{\bar{x}}} \xrightarrow{t,s} \operatorname{Spec} \bar{k} \times \operatorname{Spec} \bar{k}$ the fundamental groupoid scheme of X with base point at $\bar{x}$ and denote it by $\Pi(X, \bar{x})$. By functoriality we have

$$
\begin{array}{ccc}
X_{\rho_{\bar{x}}} \longrightarrow X_\tau & \qquad & \Pi(X, \bar{x}) \longrightarrow X_\tau \\
\scriptstyle t_{\bar{x}}, s_{\bar{x}} \downarrow \quad \square \quad \downarrow \scriptstyle t,s & & \scriptstyle t,s \downarrow \quad \square \quad \downarrow \scriptstyle t,s \\
\operatorname{Spec} \bar{k} \times X \xrightarrow[\bar{x} \times \operatorname{id}]{} X \times X & & \operatorname{Spec} \bar{k} \times \operatorname{Spec} \bar{k} \xrightarrow[\bar{x} \times \bar{x}]{} X \times X.
\end{array}
\tag{3.2}
$$

In particular $\Pi(X, \bar{x})$ is the fibre of $X_{\rho_{\bar{x}}} \xrightarrow{s_{\bar{x}}} X$ at $\bar{x}$. Thus we have the following commutative diagram

$$
\begin{array}{ccc}
& \Pi(X, \bar{x}) \xrightarrow{\quad i \quad} X_{\rho_{\bar{x}}} & \\
\scriptstyle e \nearrow & \scriptstyle t,s \downarrow \quad \square \quad \downarrow \scriptstyle t_{\bar{x}}, s_{\bar{x}} & \\
\operatorname{Spec} \bar{k} \xrightarrow[\Delta]{} \operatorname{Spec} \bar{k} \times \operatorname{Spec} \bar{k} \xrightarrow[\operatorname{id} \times \bar{x}]{} \operatorname{Spec} \bar{k} \times X &
\end{array}
\tag{3.3}
$$

(with $x_{\rho_{\bar{x}}}$ labeling the curved arrow)

where e denotes the unit of $\Pi(X, \bar{x})$. The composition $x_{\rho_{\bar{x}}} = i \circ e$ is a $\bar{k}$-point of $X_{\rho_{\bar{x}}}$ lying above $\bar{x}$. It is shown in [6, Theorem 1.2] that $(X_{\rho_{\bar{x}}}, x_{\rho_{\bar{x}}})$ is the universal pro-étale covering of X with respect to the fibre functor $F_{\bar{x}}$. Furthermore, $\Pi(X, \bar{x})$ considered as a $\bar{k}$-scheme by means of the morphism $s : \Pi(X, \bar{x}) \to \operatorname{Spec} \bar{k}$, has the property that its $\bar{k}$-points build a profinite group which is identified with $\pi_1(X, \bar{x})$.

We shall therefore use the notation $X_{\bar{x}} := X_{\rho_{\bar{x}}}$ and $\tilde{x} := x_{\rho_{\bar{x}}}$.

Let $\bar{y}$ be another $\bar{k}$-point of X. Then $\Pi(X, \bar{x})$ and $\Pi(X, \bar{y})$ are isomorphic as groupoid schemes acting upon $\operatorname{Spec} \bar{k}$. Actually, any isomorphism is given by conjugating with an el-

ement of $\mathrm{Isom}_{\bar{k}}^{\otimes}(\rho_{\bar{x}}, \rho_{\bar{y}})(\bar{k})$, which according to [5, Theorem 3.2] is isomorphic (as a set) to $\Pi(X, \bar{x})^{\Delta}(\bar{k})$.

The following proposition was proved in [6, Theorems 1.2, 1.3], except that we did not underline in [6, Theorems 1.2, 1.3] that we keep the same x.

Proposition 3.1. *There is a one-to-one correspondence between conjugation classes of sections of $\epsilon_* : \pi_1(X, \bar{x}) \to \Gamma$ and equivalence classes of neutral fibre functors of $\mathsf{FC}(X)$. A section σ of ϵ_* corresponds to a fibre functor ρ if the k-form of $X_{\bar{x}}$ (see Remark 2.3) determined by σ is isomorphic to X_ρ over X. In particular a geometric section given by a point $x \in X(k)$ corresponds to the fibre functor ρ_x at x.*

Proof. According to the discussion above, $X_{\bar{x}} = X_{\rho_{\bar{x}}}$. Given a neutral fibre functor ρ of $\mathsf{FC}(X)$, then $\rho \otimes_k \bar{k} \cong \rho_{\bar{x}}$, hence X_ρ is a form of $X_{\rho_{\bar{x}}}$. Conversely, each k-form of $X_{\rho_{\bar{x}}}$ determines a neutral fibre functor by taking cohomology. On the other hand, each section of ϵ_* determines a k-form of $X_{\bar{x}}$ and conversely. $\quad\square$

By means of this proposition, Grothendieck's section conjecture can be reformulated as follows.

Conjecture 3.2 (*Section conjecture reformulated*). *Let U be a hyperbolic curve over a field k of finite type over $\mathbb{Q}$ and X be its compactification. For each neutral fibre functor ρ of $\mathsf{FC}(U)$ we denote by $X(U_\rho)$ the compactification of the Π_ρ-principal bundle $s_\rho : U_\rho \to U$ with respect to X. Then*

(SC1) *The map $\delta : U(k) \to \mathsf{Fib}_k(X)$, $x \mapsto \rho_x$ is injective.*
(SC2) *For each $u \in U(k)$, the pro-curve $X(U_{\rho_u})$ has a unique k-rational point which lies above u.*
(SC3) *Let $\rho \in \mathsf{Fib}_k(X)$ which is not geometric (i.e. not isomorphic to ρ_u for any $u \in U(k)$), then $X(U_\rho)$ has a unique k-rational point which lies above a point $x \in (X \setminus U)(k)$. Denote the "packet" of such fibre functor in $\mathsf{Fib}_k(U)$ by $\mathcal{P}_x$. Thus*

$$\mathsf{Fib}_k(U) = \{\rho_u, u \in U(k)\} \sqcup \bigsqcup_{x \in (X \setminus U)(k)} \mathcal{P}_x. \tag{3.4}$$

(SC4) *Each packet $\mathcal{P}_x$ has the cardinality of the continuum.*

Let us specialize to the case $X = U$. Then Grothendieck's section conjecture reformulated is equivalent to the following.

(SC) The map

$$X(k) \to \mathsf{Fib}_k(X), \quad x \mapsto \rho_x, \tag{3.5}$$

is bijective.

4. A cohomology-like interpretation of (SC1) and (SC2)

We continue to assume that U is an absolutely connected curve over k, k is of finite type over $\mathbb{Q}$. Fix a neutral fibre functor ρ for $\mathsf{FC}(U)$ (in particular assume the existence of such a functor). Let η be another neutral fibre functor for $\mathsf{FC}(U)$. Then according to [5, Theorem 3.2], the functor $\mathrm{Isom}_k^\otimes(\eta, \rho)$ is representable by a (profinite) scheme $E \to \mathrm{Spec}\, k$, which is a principal bundle under both Π_ρ (on the right) and Π_η (on the left). Conversely, let $E \to \mathrm{Spec}\, k$ be a Π_ρ-principal bundle over $\mathrm{Spec}\, k$, in particular Π_ρ acts on E from the right, hence $k[E]$ is a representation of Π_ρ. Then we can define a fibre functor η for $\mathsf{FC}(U)$ as follows:

$$\eta(V) := \left(k[E] \otimes \rho(V)\right)^{\Pi_\rho}, \quad \text{for all } V \in \mathsf{FC}(U), \tag{4.1}$$

where $(-)^{\Pi_\rho}$ denotes the set of Π_ρ-invariants. Then E becomes a left Π_η-principal bundle.

It is well known that Π_ρ-principal bundles over $\mathrm{Spec}\, k$ are classified by the pro-system of pointed sets

$$H^1_{\text{ét}}(\mathrm{Spec}\, k, \Pi_\rho) := \varprojlim_{\Pi_\rho \twoheadrightarrow G, G \text{ finite}} H^1_{\text{ét}}(k, G), \tag{4.2}$$

where the distinguished element corresponds to a principal bundle E with $E(k) \neq \emptyset$, thus $E \cong \Pi_\rho$ via the choice of the k-rational point. Consequently ρ defines a bijective map $\delta_\rho : \mathsf{Fib}_k(U) \to H^1_{\text{ét}}(\mathrm{Spec}\, k, \Pi_\rho)$.

Now let us give ourselves $u \in U(k)$ and take $\rho = \rho_u$. We obtain a bijective map

$$\delta_u : \mathsf{Fib}_k(U) \to H^1_{\text{ét}}\left(\mathrm{Spec}\, k, \Pi(U, u)\right), \quad v \mapsto E_v, \tag{4.3}$$

where E_v is a short hand for the k-scheme representing the functor $\mathrm{Isom}^\otimes(\rho_v, \rho_u)$.

Observe that this map can be determined by the following diagram

$$
\begin{array}{ccc}
E_v & \xrightarrow{\;i_v\;} & X_u \\
\downarrow & \square & \downarrow{\scriptstyle s_u} \\
\mathrm{Spec}\, k & \xrightarrow{\;\;v\;\;} & X.
\end{array}
\tag{4.4}
$$

The geometric part of Grothendieck section conjecture (i.e. the part concerning geometric sections) has the following cohomological interpretation. Consider the "sequence" of k-schemes

$$\Pi(U, u) \xrightarrow{\;i\;} U_u \xrightarrow{\;s_u\;} U, \tag{4.5}$$

where i is the closed embedding $s_\rho^{-1}(u) = \Pi(U, u) \subset U_u$.

Let i^0, s_u^0 denote the corresponding maps on the sets of k-points. Define the "cohomology set" $H^1_{\text{ét}}(k, U_u)$ as the set of k-forms of $U_u \otimes \bar{k}$, and say that a k-form is trivial if it has a k-rational point. And define the map

$$i^1 : H^1_{\text{ét}}\left(k, \Pi(U, u)\right) \to H^1_{\text{ét}}(k, U_u)$$

that assigns to each $\Pi(U, u)$-principal bundle $E \to \operatorname{Spec} k$ the $\Pi(U, u)$-principal bundle $U_E \to E$ defined as follows: E determines a fibre functor ρ_E for $\mathsf{FC}(U)$, which in turn determines a $\Pi(U, \rho_E)$-principal bundle $U_{\rho_E} \to U$, which is a form of $U_u \otimes \bar{k}$. Then we have:

Lemma 4.1. *The sequence* (4.5) *induces the following "long exact" sequence of pointed sets*

$$e \to \Pi(U, u)(k) \xrightarrow{i^0} U_u(k) \xrightarrow{s_u^0} U(k) \xrightarrow{\delta_u} H^1_{\text{ét}}\big(k, \Pi(U, u)\big) \xrightarrow{i^1} H^1_{\text{ét}}(k, U_u). \qquad (4.6)$$

Proof. The exactness at $\Pi(U, u)(k)$ and at $U_u(k)$ is clear. Let $v \in U(k)$ such that $\delta_u(v)$ is trivial in $H^1_{\text{ét}}(k, \Pi(U, u))$, which means that $E_v = \delta_u(v)$ has a k-rational point. According to diagram (4.4), U_u has a k-rational point which lies above v. Similarly, an element U_ρ of $H^1_{\text{ét}}(k, \Pi(U, u))$ is mapped to the trivial object of $H^1_{\text{ét}}(k, X_u)$ if and only if it has a k-rational point. This mean ρ is geometric, i.e. U_ρ lies in the image of δ_u. $\quad\square$

Remarks 4.2. (a) The cohomological notations such as $H^1_{\text{ét}}(k, U_u)$ used in this section are just a support for thinking, they are not standard and also not properly defined as cohomologies.

(b) The claim (SC1) of Grothendieck section conjecture is equivalent to saying that δ_u is injective, or equivalently, that s_u^0 is trivial, i.e. maps $U_u(k)$ to the single point $u \in U(k)$, or equivalently that i^0 is a bijective map.

(c) The claim (SC2) is equivalent to saying that $U_u(k)$ consists of a unique point above u.

(d) If $U = X$ (i.e. U is projective) then claim (SC3) is equivalent to saying that δ_u is surjective.

5. The uniqueness of the fixed point and the packets

In this section, using our method we provide new proofs of SC1 and SC2 in Conjecture 3.2, which were shown by Tamagawa in [12, Section 2]. This leads in a natural way to the definition of the packets.

Theorem 5.1. *Let U be smooth geometrically connected curve which is elliptic or hyperbolic (see Definition 2.1) and defined over a field k of finite type over $\mathbb{Q}$. Let $u \in U(k)$ be a k-rational point. Then the map*

$$\delta_u : U(k) \to H^1_{\text{ét}}\big(k, \Pi(U, u)\big) \qquad (5.1)$$

is injective.

Proof. If U is not projective, let $U \subset V$ be a larger open which is still defined over k. It yields the restriction functor $\mathsf{FC}(V) \xrightarrow{\text{rest}} \mathsf{FC}(U)$, which defines $\mathsf{FC}(V)$ as a full subcategory of $\mathsf{FC}(U)$. So as long as V still fulfills the assumption elliptic or hyperbolic, the theorem for U follows from the theorem for V. Consequently, if the genus of the smooth compactification X of U is $\geqslant 1$, we can assume U to be X. If the genus of X is 0, the cardinality of $(X \setminus U)(\bar{k})$ is at least 2. Let $K \supset k$ be a finite field extension over which at least two of those points are rational. Then $U \otimes K \subset \mathbb{G}_m$ and the statement for $\mathbb{G}_m$ over K implies the statement for $U \otimes K$. On the other hand, by base change, $\Pi(U, u) \otimes K = \Pi(U \otimes K, u \otimes K)$, and the composite map $U(k) \to H^1_{\text{ét}}(k, \Pi(U, u)) \to H^1_{\text{ét}}(k, \Pi(U, u) \otimes K)$ factors through $U(k) \subset U(K)$. This reduces the statement to $\mathbb{G}_m$ in this case.

Thus let U be either a projective curve of genus $\geqslant 1$ or $\mathbb{G}_m$. Let $J(U)$ be the Jacobian of U. So it is an abelian variety defined over k in genus $\geqslant 1$ or the semi-abelian variety $\mathbb{G}_m$ in genus 0. We fix $u \in U(k)$ and denote by $j : U \to J(U)$ the cycle map sending u to the unit e of $J(U)$. To simplify notations, in the genus 0 case, we assume $u = 1$ so j is then the identity. We denote by $\mathsf{FC}^{\mathrm{ab}}(U)$ the full subcategory of $\mathsf{FC}(U)$ consisting of those connections V which split into a direct sum of rank 1 connections after base change by a finite extension of k. For a (neutral) fibre functor ρ, let Π_ρ^{ab} denote the corresponding Tannaka k-group scheme. It is the maximal commutative quotient k-group scheme of Π_ρ.

The morphism $j : U \to J(U)$ defines by pull-back the tensor functor $j^* : \mathsf{FC}(J(U)) \to \mathsf{FC}(U)$, which factors through the subcategory $\mathsf{FC}^{\mathrm{ab}}(U)$. Therefore the resulting homomorphism $j_* : \Pi(U, u) \to \Pi(J(U), e)$ factors through the homomorphism

$$\bar{j}_* : \Pi^{\mathrm{ab}}(U, u) := \Pi_{\rho_u}^{\mathrm{ab}} \to \Pi\big(J(U), e\big). \tag{5.2}$$

Lemma 5.2. *The homomorphism $\bar{j}_*$ is an isomorphism.*

Proof. Since $\mathsf{FC}(U)$ is compatible with base change, so is $\mathsf{FC}^{\mathrm{ab}}(U)$. Since k has characteristic 0, $\bar{j}_*$ is an isomorphism if and only if it is an isomorphism after $\otimes \bar{k}$. But over $\bar{k}$, j induces an isomorphism $j^* : \mathrm{Pic}^0(J(U)) \to \mathrm{Pic}^0(U)$ in genus $\geqslant 1$, while for $\mathbb{G}_m$, j itself is the identity map. Thus in particular, j_* induces an isomorphism on $\bar{k}$-valued torsion points. Therefore $\bar{j}^*$ yields an equivalence between $\mathsf{FC}(J(U))$ and $\mathsf{FC}^{\mathrm{ab}}(U)$. $\square$

Proposition 5.3. *Let A/k be a semi-abelian variety defined over a field k of finite type over $\mathbb{Q}$. Then the map*

$$\delta_e : A(k) \to \mathsf{Fib}_k(A) \tag{5.3}$$

is injective.

Proof. For a natural number $m \in \mathbb{N}$, let us denote by $m \cdot A(k)$ the set of k-rational points of A which are divisible by m as k-rational points. Since k is of finite type over $\mathbb{Q}$, there are no k-rational points which are infinitely divisible, thus

$$\bigcap_{m \in \mathbb{N}} m \cdot A(k) = e. \tag{5.4}$$

Let $s_e : A_e \to A$ be as usual the universal covering of A associated to the fibre functor at e. By Lemma 4.1 the injectivity of δ_e is equivalent to the triviality of s_e^0, that is, all k-points of A_e lie above e. Assume the contrary, that is A_e has a k-point b lying above $a \in A(k)$, $a \neq e$. According to (5.4), there exists $m > 0$ such that there is no $a' \in A(k)$ with $m \cdot a' = a$. Observe that the covering $[m] : A \to A$ is a principal bundle under the k-group scheme $A[m] := \mathrm{Ker}[m]$,

and $e \in A(k)$ is a k-point lying above $m \cdot e \in A(k)$ with respect to $[m]$. Therefore the universal covering s_e factors as

$$
\begin{array}{ccc}
A_e & \xrightarrow{\ p\ } & A \\
& \searrow_{s_e} & \downarrow{[m]} \\
& & A.
\end{array}
$$

(5.5)

Let $a' := p(b) \in A(k)$. Then $m \cdot a' = a$, a contradiction. Thus s_e^0 is the trivial map, hence δ_e is injective. $\square$

Proposition 5.3 finishes the proof of the theorem. $\square$

Remarks 5.4. Theorem 5.1 shows SC1 in Conjecture 3.2 under the weaker assumption on the geometry of U: it is enough for U to be elliptic. The ellipticity assumption is sharp. If $U = \mathbb{A}^1$, then certainly $\mathsf{FC}(\mathbb{A}^1)$ is the trivial category, thus all neutral fiber functors are equivalent to $M \mapsto H^0(\mathbb{A}^1, M)$, and δ_u cannot be injective. It is also to be noted that for SC1, one only needs the maximal abelian subcategory of $\mathsf{FC}(U)$.

Theorem 5.5. *Let X be a smooth projective curve over a field k of finite type over $\mathbb{Q}$, and let $\emptyset \neq U \subset X$ be Zariski open over k. Let ρ be a neutral fibre functor for $\mathsf{FC}(U)$, defining the principal Π_ρ-bundle $s_\rho : U_\rho \to U$ and the normalization $X(U_\rho)$ of X in $k(U_\rho)$, defining $\bar{s}_\rho : X(U_\rho) \to X$ (see Section 3). If the genus of X is $\geqslant 1$ (so U is elliptic or hyperbolic, see Definition 2.1), or if the genus of X is 0 and U is hyperbolic, then*

$$
\#X(U_\rho)(k) \leqslant 1.
$$

(5.6)

Proof. We first consider the case $g(X) \geqslant 1$. Assume on the contrary that there exist $\alpha, \beta \in X(U_\rho)(k)$, $\alpha \neq \beta$. Let a, b be respectively their images in $X(k)$. We use the notation ρ_X for the restriction of ρ to $\mathsf{FC}(X)$. Since $\mathsf{FC}(X)$ is a full subcategory of $\mathsf{FC}(U)$, we have the following commutative diagram

$$
\begin{array}{ccc}
U_\rho & \hookrightarrow X(U_\rho) \longrightarrow & X_{\rho_X} \\
s_\rho \downarrow & \bar{s}_\rho \downarrow & \swarrow_{s_{\rho_X}} \\
U & \hookrightarrow X. &
\end{array}
$$

(5.7)

Denote the images of α and β in $X_\rho(k)$ by $\tilde{a}$ and $\tilde{b}$. Their images in $X(k)$ are then a and b. Thus we have

$$
\rho_X \cong \rho_a \cong \rho_b
$$

(5.8)

where ρ_a and ρ_b denote the geometric fibre functors at a and b for $\mathsf{FC}(X)$. Indeed, as recalled in Section 3,

$$\rho_X \text{ is equivalent to } \mathsf{FC}(X) \to \mathsf{Vec}_k, \quad (V, \nabla) \mapsto H^0\big(X_{\rho_X}, s_{\rho_X}^*(V, \nabla)\big),$$

$$\text{thus to } (V, \nabla) \mapsto V|_{\tilde{\alpha}} = V|_a,$$

$$\text{thus to } (V, \nabla) \mapsto V|_a = V|_b. \tag{5.9}$$

We conclude by Theorem 5.1 that $a = b$.

There exists a principal bundle $q : V \to U$ in the pro-system defining U_ρ such that the images a', b' of α and β in the compactification Y of V are distinct. Thus ρ induces in a canonical way a fibre functor ρ_V for $\mathsf{FC}(V)$. Indeed, $q_* N$ is an object of $\mathsf{FC}(U)$ whenever N is an object of $\mathsf{FC}(V)$, and since $q^* q_* N \twoheadrightarrow N$, the pull-back N' of N to U_ρ is trivialized, and ρ_V is defined by $N \mapsto H^0(U_\rho, N')$. It also shows that $V_{\rho_V} = U_\rho$. Consequently $X(U_\rho)$ is the compactification of U_ρ with respect to $V \hookrightarrow Y$. On the other hand it is obvious that the pair (V, Y) satisfies the condition of the theorem. Thus we can repeat the above argument to conclude that $a' = b'$. This is a contradiction.

To treat the case $g(X) = 0$ we first show the following lemma.

Lemma 5.6. *Let k be an arbitrary field of characteristic 0 and $U \subset X$ be defined over k such that $g(X) = 0$ and $\chi(U) < 0$. Let ρ be an arbitrary neutral fibre functor for $\mathsf{FC}(U)$. Then in the pro-system defining U_ρ, there exists a principal bundle $V \to U$ such that the compactification Y of V has genus $\geqslant 1$.*

Proof. As $U_\rho \otimes \bar{k} \to U \otimes \bar{k}$ is isomorphic to Grothendieck's universal covering, and the category of finite bundles is compatible with base change, we just have to know that $\mathbb{P}^1$ has coverings of genus $\geqslant 1$ ramified in 3 points, which is a well-known fact. $\quad\square$

We now finish the proof of Theorem 5.5. Let $\rho \in \mathsf{Fib}_k(U)$. According to Lemma 5.6, there exists a principal bundle $V \to U$ in the pro-system defining U_ρ such that the genus of its compactification Y is at least 1. Further we can choose V such that the images a', b' of α and β in Y of V are distinct. Thus we can repeat the above argument to conclude that $a' = b'$. This is a contradiction, that is, there exists at most one k-point in $X(U_\rho)$. This finishes the proof. $\quad\square$

Theorem 5.5 suggests the following definition.

Definition 5.7. Let U be a hyperbolic curve and $X \supset U, k$ be as in Theorem 5.5, where we allow k to be any field of characteristic 0, and let $x \in (X \setminus U)(k)$. We define the packet $\mathcal{P}_x$ as the set of those fibre functors $\rho \in \mathsf{Fib}_k(U)$ which have the property that the pro-scheme $X(U_\rho)$ has a k-point, and this point lies above x.

Properties 5.8. *With the notations as in Definition 5.7, (5.9) shows that if $\rho \in \mathcal{P}_x$, then $\rho|_{\mathsf{FC}(U \cup \{x\})} = \rho_x$. Vice-versa, if $\rho \in \mathsf{Fib}_k(U)$ such that $\rho|_{\mathsf{FC}(U \cup \{x\})} = \rho_x$ for some $x \in (X \setminus U)(k)$, and k is of finite type over $\mathbb{Q}$, then to conclude that $\rho \in \mathcal{P}_x$ would be a consequence of SC3 in Conjecture* 3.2.

6. The tangential fibre functor

Let U be a hyperbolic curve (see Definition 2.1) and defined over a field k of characteristic 0. We keep the notation as in Section 5. Let $x \in (X \setminus U)(k)$ be a rational point at ∞, with maximal

ideal sheaf $\mathfrak{m}_x$. We denote by $T_x = \mathrm{Spec}(\bigoplus_0^\infty \mathfrak{m}_x^n / \mathfrak{m}_x^{n+1})$ the fiber of the tangent bundle at X in x, and by $T_x^0 = T_x \setminus \{x\} = \mathrm{Spec}(\bigoplus_{-\infty}^\infty \mathfrak{m}_x^n / \mathfrak{m}_x^{n+1})$ the complement of the zero section. We denote by K_x the local field at x, by R_x its valuation ring, so after the choice of a local parameter t at x, one has $K_x \cong k((t))$, $R_x \cong k[[t]]$. Denote

$$S_x := \mathrm{Spec}\, R_x, \qquad S_x^0 := S_x \setminus \{x\} = \mathrm{Spec}\, K_x. \tag{6.1}$$

Then there exists an exact tensor functor depending only on the choice of t in $\mathfrak{m}/\mathfrak{m}_x^2 - \{0\}$

$$\mathsf{FC}\big(S_x^0\big) \to \mathsf{FC}\big(T_x^0\big) \tag{6.2}$$

constructed by Deligne [3, Section 15] and by Katz [9b], which is an equivalence of categories, and which inverts the natural restriction functor $\mathsf{FC}(T_x^0) \to \mathsf{FC}(S_x^0)$. So composing (6.2) with the restriction functor $r_x : \mathsf{FC}(U) \to \mathsf{FC}(S_x^0)$ yields the tensor functor

$$DK_x : \mathsf{FC}(U) \to \mathsf{FC}\big(T_x^0\big). \tag{6.3}$$

This allows one to construct fibre functors for $\mathsf{FC}(U)$ by composing DK_x with fibre functors of $\mathsf{FC}(T_x^0)$. Since $\mathsf{FC}(S_x)$ is trivial, any fiber functor $\rho \in \mathsf{Fib}_k(U)$ obtained in this way has the property

$$\rho|_{\mathsf{FC}(X)} = \rho_x. \tag{6.4}$$

Fibre functors of $\mathsf{FC}(T_x^0)$ can be explicitly described. The choice of a local parameter t at x identifies T_x^0 with $\mathbb{G}_m = \mathrm{Spec}\, k[t^{\pm 1}]$. Then $\mathsf{FC}(T_x^0)$ is spanned (as an abelian category) by the connections

$$L_a := (\mathcal{O} \cdot e_a, \nabla), \qquad \nabla(e_a) = \frac{a\, dt}{t} e_a, \quad a \in \mathbb{Q}, \tag{6.5}$$

where $\mathcal{O} := \mathcal{O}_{T_x^0}$. The tensor product for L_a reads

$$L_a \otimes L_b = L_{a+b} \tag{6.6}$$

and there is an isomorphism

$$\varphi_a : L_a \to L_{a+1}; \qquad \varphi_a(e_a) = \frac{a+1}{t} e_{a+1}. \tag{6.7}$$

Thus $\mathsf{FC}(T_x^0)$ is indeed generated (as a tensor category) by L_a, $a \in \mathbb{Q}/\mathbb{Z}$. The Tannaka group of $\mathsf{FC}(T_x^0)$ with respect to the fibre functor at $1 \in \mathbb{G}_m$ identified with T_x^0 via t is

$$\Pi\big(T_x^0, 1\big) \cong \varprojlim_n \mu_n; \qquad \mu_{m \cdot n} \xrightarrow{(-)^m} \mu_n. \tag{6.8}$$

Consequently (see (4.2))

$$\mathsf{Fib}_k\big(T_x^0\big) = H_{\mathrm{\acute{e}t}}^1\Big(k, \varprojlim_n \mu_n\Big) = \text{(Kummer theory)} \varprojlim_n \frac{k^\times}{(k^\times)^n}. \tag{6.9}$$

Notice that $k^\times$ is naturally embedded in $\varprojlim_n \frac{k^\times}{(k^\times)^n}$ and its image is precisely the set of geometric fibre functors of $\mathsf{FC}(T_x^0)$. In particular we have proved the following

Corollary 6.1. *The set* $\mathsf{Fib}_k(T_x^0)$ *has the cardinality of the continuum.*

The universal covering $(T_x^0)_\rho$ associated to a fibre functor ρ of $\mathsf{FC}(T_x^0)$ can be described as follows. Consider ρ as an element of $\varprojlim_n \frac{k^\times}{(k^\times)^n}$ and let $\rho_n \in k^\times$ be a representant of the image of ρ in $\frac{k^\times}{(k^\times)^n}$. Thus we have

$$\rho_n \equiv \rho_{m \cdot n} \mod (k^\times)^n. \tag{6.10}$$

Then

$$(T_x^0)_\rho = \mathrm{Spec}\left(\frac{k[t_{1/n}^{\pm 1}, n \in \mathbb{N}]}{(t_{1/m \cdot n})^m = \frac{\rho_{m \cdot n}}{\rho_n} t_{1/n}} \right). \tag{6.11}$$

The projection $(T_x^0)_\rho \to T_x^0$ is defined by $t \mapsto t_1$.

Remark 6.2. This remark echoes Remarks 5.4. The normalization $T_x((T_x^0)_\rho)$ of T_x in $k((T_x^0)_\rho)$ is

$$T_x\big((T_x^0)_\rho\big) = \mathrm{Spec}\left(\frac{k[t_{1/n}, n \in \mathbb{N}]}{(t_{1/m \cdot n})^m = \frac{\rho_{m \cdot n}}{\rho_n} t_{1/n}} \right). \tag{6.12}$$

It has a unique k-point, which lies above $0 \in T_x$, if ρ is not geometric. Otherwise, if $\rho = \rho_a$, $a \in T_x^0(k)$, then it has the rational point above 0 and another one above a. We see in this way that $U \cong_k \mathbb{G}_m$ does not satisfy Theorem 5.5.

We denote by $\mathcal{T}_x$ the set of fibre functors of $\mathsf{FC}(U)$ obtained by composing DK_x with a fibre functor ρ of $\mathsf{Fib}_k(T_x^0)$, and use the notation

$$\tau_{x,\rho} := \rho \circ DK_x. \tag{6.13}$$

Let us choose a k-rational point of T_x^0, which we denote by 1, as it is the same as choosing $t \in \mathfrak{m}_x/\mathfrak{m}_x^2 \setminus \{0\}$. Tannaka duality for

$$
\begin{array}{ccc}
\mathsf{FC}(U) & \xrightarrow{\ DK_x\ } & \mathsf{FC}(T_x^0) \\
& \searrow{\scriptstyle \tau_{x,\rho_1}} & \big\downarrow{\scriptstyle \rho_1} \\
& & \mathsf{Vect}_k
\end{array}
\tag{6.14}
$$

where ρ_1 denotes the fibre functor at $1 \in T_x^0$, yields a group scheme homomorphism

$$DK_x^* : \Pi\big(T_x^0, 1\big) \to \Pi(U, \tau_{x,\rho_1}). \tag{6.15}$$

Lemma 6.3. *The homomorphism DK_x^* in (6.15) is injective. Consequently each object of $\mathsf{FC}(T_x^0)$ is isomorphic to a direct summand of the image under DK_x of an object of $\mathsf{FC}(U)$.*

Proof. The homomorphism DK_x^* respects base change. So we may assume that k is of finite type over $\mathbb{Q}$, thus in particular, it is embeddable in $\mathbb{C}$. Then (6.15) is compatible with the homomorphism of topological fundamental groups

$$\pi_1(\mathbb{C}^\times, 1) \to \pi_1(U(\mathbb{C}), t) \qquad (6.16)$$

after the choice of an embedding $k \to \mathbb{C}$. Here $\pi_1(U(\mathbb{C}), t)$ is the topological fundamental group based at the tangent vector t at x in the sense of Deligne (see [3, Section 15]). Ellipticity or hyperbolicity of U implies that (6.16) is injective. Since by uniformization theory, the topological fundamental groups of $U(\mathbb{C})$ and of $\mathbb{C}^\times$ lie inside the complex points of an algebraic group, thus are residually finite, DK_x^* is injective as well.

The last claim follows from [5, Theorem 2.11] and the fact that $\mathsf{FC}(U)$ is semisimple as an abelian category. $\quad\square$

We mention the following result of [6, Theorem 5.7].

Proposition 6.4. *Let $\tau := \tau_{x,\rho} \in \mathcal{T}_x$. Then $X(U_\tau)$ has a k-point lying above x. Consequently $\mathcal{T}_x \subset \mathcal{P}_x$.*

Fix $\rho \in \mathsf{Fib}_k(U)$. Then each finite full abelian tensor subcategory $S \subset \mathsf{FC}(U)$ defines a finite quotient $\Pi_\rho \twoheadrightarrow G := \Pi_{\rho,S}$, and a principal bundle $U_G := U_{\rho,S} \xrightarrow{p} U$ under group G. Let $X(U_G)$ be the compactification of U_G, which then projects on X.

We set

$$S_{G,x} := S_x \times_X X(U_G) \quad \text{and} \quad S_{G,x}^0 := S_x^0 \times_U U_G$$

where S_x, S_x^0 are defined in (6.1).

Proposition 6.5. *The (fixed point free) action of G on U_G extends to a (not necessarily fixed point free) action*

$$
\begin{array}{ccccccccc}
G \times X(U_G) & \xrightarrow{\bar\mu} & X(U_G) & \xleftarrow{\text{rest}} & G \times S_{G,x} & \xrightarrow{\bar\mu} & S_{G,x} & \xleftarrow{\text{rest}} & G \times S_{G,x}^0 & \xrightarrow{\mu} & S_{G,x}^0 \\
{\scriptstyle p_1}\downarrow\!\!\searrow & & \downarrow{\scriptstyle \bar p} & & {\scriptstyle p_1}\downarrow\!\!\searrow & & \downarrow{\scriptstyle \bar p} & & {\scriptstyle p_1}\downarrow\!\!\searrow & & \downarrow{\scriptstyle p} \\
& & X & & & & S_x & & & & S_x^0 .
\end{array}
$$

$$(6.17)$$

Proof. Denote for simplicity $V := U_G$ and $Y := X(U_G)$. We first start with the action on Y. The composition of the two maps $G \times V \xrightarrow{\mu} V \to Y$ factors (uniquely) through $G \times Y$ as $G \times Y$ is a smooth curve and Y is projective (it is a removable singularity). The same argument yields that the two maps $G \times G \times V \rightrightarrows G \times V \to G \times Y$ defined by $\mu(gh, v)$ and $\mu(g, \mu(h, v))$ factor uniquely through $G \times G \times Y \rightrightarrows Y$. Since $\mu(gh, v) = \mu(g, \mu(h, v))$ on V, equality holds on Y as

well. This shows the existence of the extension of μ to $\bar{\mu}$ on the left triangle. On the other hand, since $p_1 = p \circ \mu$ on V, one has $p_1 = \bar{p} \circ \bar{\mu}$ on Y. This shows the left triangle. It also shows that the fiber $X(U_G)_x = \bar{p}^{-1}(x)$ is left invariant by $\bar{\mu}$. This finishes the proof. $\quad\square$

Let $\rho \in \mathcal{P}_x$ and $\alpha \in X(U_\rho)$ be the unique k-point, which lies above x. Recall from Properties 5.8 that $\rho|_{\mathsf{FC}(U \cup \{x\})} = \rho_x$. On the other hand, $\mathsf{FC}(U \cup \{x\})$ is a full abelian tensor subcategory of $\mathsf{FC}(U)$, which is closed under taking subquotients. Indeed, let (E, ∇) be an object in $\mathsf{FC}(U)$, and let (E', ∇') be its Deligne's canonical extension to $U \cup \{x\}$ [2]. Then (E, ∇) lies in the image of $\mathsf{FC}(U \cup \{x\})$ if and only if the residue of ∇' at x is equal to 0, a condition which is compatible with exact sequences, and is stable by taking tensor products, duals and subquotients. Thus we have a surjective homomorphism $\pi(U, \rho) \to \pi(U \cup \{x\}, \rho_x)$, the kernel of which will be denoted by K.

Proposition 6.6. *Let X be a (not necessarily projective) curve and $U \subset X$ be such that $X \setminus U$ consists of a unique k-point x. Let $\rho \in \mathcal{P}_x$ and $\alpha \in X(U_\rho)$. Then the map $\bar{s}_\rho : X(U_\rho) \to X$ factors as*

$$
\begin{array}{ccc}
X(U_\rho) & & \alpha \\
\Big\downarrow{\scriptstyle \bar{q}_\rho} & & \Big\downarrow \\
X_{\rho_x} & & x_{\rho_x} \\
\Big\downarrow{\scriptstyle s_{\rho_x}} & & \Big\downarrow \\
X & & x
\end{array}
\qquad (6.18)
$$

with the property

$$
\bar{q}_\rho^{-1}(x_{\rho_x}) = \alpha \qquad (6.19)
$$

i.e. $\bar{q}_\rho$ fully ramifies at α.

Proof. Fixing a finite full abelian tensor subcategory S of $\mathsf{FC}(U)$ as in the discussion preceding Proposition 6.5 defines a finite quotient group scheme G of $\pi(U, \rho)$. Recall that $K := \mathrm{Ker}(\pi(U, \rho) \to \pi(X, \rho_x))$. Denote by A the image of K in G, and consider the exact sequence

$$
1 \to A \to G \to H \to 1. \qquad (6.20)
$$

The group H is isomorphic to the Tannaka group of the restriction of ρ to the intersection of S with $\mathsf{FC}(X)$. Thus X_H is the compactification $X(U_H)$ of U_H and we have the following commutative diagram

$$
\begin{array}{ccccc}
U_G & \longrightarrow & X(U_G) & & x_G \\
{\scriptstyle A}\big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle \bar{q}} & & \big\downarrow \\
U_H & \longrightarrow & X_H = X(U_H) & & x_H \\
{\scriptstyle H}\big\downarrow & & \big\downarrow & & \big\downarrow \\
U & \longrightarrow & X & & x
\end{array}
\qquad (6.21)
$$

where $q : U_G \to U_H$ is an A-principal bundle and x_G, x_H are images of the point $\alpha \in X(U_\rho)(k)$ in $X(U_G)$ and $X(U_H)$, respectively.

Proposition 6.5 applied to the principal bundle $U_G \xrightarrow{q} U_H$ yields an action

$$
A \times \bar{q}^{-1}(x_H) \to \bar{q}^{-1}(x_H) \qquad (6.22)
$$

of the k-algebraic group scheme A on the k-scheme $\bar{q}^{-1}(x_H)$.

One has from (6.21) that $x_G \in \bar{q}^{-1}(x_H)$ and that H acts freely on X_H. Consequently $B := \mathrm{Stab}(x_G) \subset G$ projects trivially onto H, thus $B \subset A$. This yields the exact sequence

$$
1 \to A/B \to G/B \to H \to 1 \qquad (6.23)
$$

together with a principal bundle

$$
A/B \times \bar{q}^{-1}(x_H) \to x_H. \qquad (6.24)
$$

Thus one has a factorization

$$
\begin{array}{ccccc}
U_G & \longrightarrow & X(U_G) & & x_G \\
{\scriptstyle B}\big\downarrow{\scriptstyle q''} & & \big\downarrow{\scriptstyle \bar{q}''} & & \big\downarrow \\
U_{G/B} & \longrightarrow & X(U_{G/B}) & & x_{G/B} \\
{\scriptstyle A/B}\big\downarrow{\scriptstyle q'} & & \big\downarrow{\scriptstyle \bar{q}'} & & \big\downarrow \\
U_H & \longrightarrow & X_H & & x_H.
\end{array}
\qquad (6.25)
$$

Denoting by e the index of ramification of $x_G \otimes \bar{k}$, one has that $|B(\bar{k})| = e$ while $|(A/B)(\bar{k})| = |\bar{q}^{-1}(x_H)(\bar{k})|$. We conclude that

$$
\big|(\bar{q}')^{-1}(x_H)(\bar{k})\big| = \big|\bar{q}^{-1}(x_H)(\bar{k})\big| = \big|(A/B)(\bar{k})\big| \qquad (6.26)
$$

and $\bar{q}'$ is étale. Thus A/B factors as a quotient of $\pi(X, \rho_x)$. By definition (6.20), one concludes $A/B = 1$. This shows

$$\bar{q}^{-1}(x_H) = \{x_G\}. \tag{6.27}$$

Passing to the pro-system we deduce the claims of the proposition. $\square$

Corollary 6.7. *With the assumption of Proposition* 6.6, *for any* $\rho \in \mathcal{P}_x$ *there is a factorization*

$$\begin{array}{ccc}
\mathsf{FC}(U) & \xrightarrow{\ r_x\ } & \mathsf{FC}(S_x^0) \\
{\scriptstyle\rho}\Big\downarrow & {\scriptstyle\exists\eta}\swarrow & \\
\mathsf{Vec}_k. & &
\end{array} \tag{6.28}$$

Proof. We adopt the notation of Proposition 6.6. By Hensel's lemma, one has a unique lifting

$$\begin{array}{ccc}
 & & X_{\rho_x} \\
 & {\scriptstyle i}\nearrow & \Big\downarrow {\scriptstyle s_{\rho_x}} \\
S_x & \longrightarrow & X
\end{array} \tag{6.29}$$

with $x_{\rho_x} \in i(S_x)$. We define

$$\sigma : \mathfrak{S}_x^0 := S_x^0 \times_{i(S_x)} U_\rho \to S_x^0,$$

$$\begin{array}{ccc}
\mathfrak{S}_x^0 & \longrightarrow & U_\rho \\
{\scriptstyle\sigma}\Big\downarrow & \square & \Big\downarrow {\scriptstyle\bar{q}_\rho} \quad {\scriptstyle s_\rho}\searrow \\
S_x^0 & \xrightarrow[\ i\ ]{} & X_{\rho_x} \xrightarrow[\ s_{\rho_x}\ ]{} X.
\end{array} \tag{6.30}$$

According to Lemma 6.3, every object N in $\mathsf{FC}(S_x^0)$ is a summand of the restriction to S_x^0 of an object M of $\mathsf{FC}(U)$. Since $s_\rho^*(M)$ is trivializable, so is $\sigma^*(N)$. On the other hand, (6.19) implies
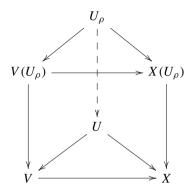
$$H^0(\mathfrak{S}_x^0) = k. \tag{6.31}$$

Consequently, ρ extends to $\mathsf{FC}(S_x^0)$ by setting

$$\rho(N) = H^0(\sigma^*(N)). \qquad \square \tag{6.32}$$

Theorem 6.8. *Let* U *be a hyperbolic curve (see Definition* 2.1*) defined over a field* k *of characteristic* 0 *and* X *be its compactification. Then* $\mathcal{T}_x = \mathcal{P}_x$ *for any* $x \in (X \setminus U)(k)$.

Proof. Let $V := U \cup \{x\}$. Denote by $V(U_\rho)$ the compactification of U_ρ with respect to the inclusion $U \to V$. Thus we have the following commutative diagram

$$
\begin{array}{ccc}
 & U_\rho & \\
 \swarrow & \vdots & \searrow \\
 V(U_\rho) & \longrightarrow & X(U_\rho) \\
 & \vdots & \\
 & U & \\
 \swarrow & & \searrow \\
 V & \longrightarrow & X
\end{array}
$$

where the front square is cartesian. Consequently, any k-point α of $X(U_\rho)$ that lies above x should lie in $V(U_\rho)$. Now Corollary 6.7 applies and shows that ρ is in fact a tangential functor at x.

By means of the equivalence in (6.2), it remains to show the uniqueness of the functor η in diagram (6.28) for any given functor $\rho \in \mathcal{P}_x$. Thus, let ρ and η be as in (6.28) and let φ be another fibre functor in $\mathsf{Fib}_k(S_x^0)$, such that $\rho \cong \varphi \circ r_x$. Via the equivalence $\mathsf{FC}(S_x^0) \equiv \mathsf{FC}(T_x^0)$, we can consider φ as a functor in $\mathsf{Fib}_k(T_x^0)$. Now, as in the proof of Proposition 6.4, i.e. the proof of [6, Theorem 5.7], there exists a map

$$
f : \left(S_x^0\right)_\varphi \to U_\rho \tag{6.33}
$$

that determines the k-point of $X(U_\rho)$, see diagram (5.11) of [6, Theorem 5.7]. The uniqueness of the k-point of $X(U_\rho)$ shows that f is compatible with the morphisms i of (6.29) and $\bar{q}_\rho$ of (6.18) in the sense that we have the commutative diagram

$$
\begin{array}{ccc}
(S_x^0)_\varphi & \xrightarrow{\ f\ } & U_\rho \\
{\scriptstyle s_\varphi}\downarrow & & \downarrow{\scriptstyle \bar{q}_\rho} \\
S_x^0 & \xrightarrow{\ i\ } & X_{\rho_x}.
\end{array}
\tag{6.34}
$$

It follows from the universal property of $\mathfrak{S}_x^0$, i.e. (6.30), that there exists a map

$$
\theta : \left(S_x^0\right)_\varphi \to \mathfrak{S}_x^0 \tag{6.35}
$$

which is compatible with the maps to S_x^0 and U_ρ. That is, there exists a natural tensor transformation $\varphi \to \eta$, which is then automatically a natural isomorphism (cf. [5, Proposition 1.13]). Thus η and φ are isomorphic. $\square$

Corollary 6.9. *Let η and η' be non-isomorphic fibre functors of $\mathsf{FC}(T_x^0)$. Then $\eta \circ DK_x$ and $\eta' \circ DK_x$ are not isomorphic. Consequently, if k is of finite type over $\mathbb{Q}$ then $\mathcal{P}_x$ has the cardinality of the continuum.*

7. Remarks towards the existence

In this section, we gather a few facts towards SC3 in Conjecture 3.2.

Proposition 7.1. *Let U be a geometrically connected curve defined over a characteristic 0 field k. Let $\rho \in \mathsf{Fib}_k(U)$ and $X(U_\rho)$ be as in Theorem 5.5. Then, if U is hyperbolic (see Definition 2.1), then $X(U_\rho)(k) = X(U_\rho)(L)$ for $L = k(V)$ where $V \to U$ is any member of the pro-system $s_\rho : U_\rho \to U$.*

Proof. Let $a : \operatorname{Spec} L \to X(U_\rho)$. Then for $W \to U$ in the pro-system defining U_ρ, with $W \neq V$, the hyperbolicity condition implies that the genus of the compactification of W is strictly larger than the genus of the compactification of V. Therefore the induced point $\operatorname{Spec} L \to X(W)$ has to be a closed point. Let $\kappa(a)$ be its residue field, thus $k \subset \kappa(a)$ is finite, and $\kappa(a) \subset L$. By the connectivity of V, this is only possible if $\kappa(a) = k$. This finishes the proof. $\square$

The following remark is contained in [12, Corollary 2.10], and seen here quite directly with our method.

Remark 7.2. Let U be a hyperbolic curve defined over a field k of finite type over $\mathbb{Q}$. Let ρ be a neutral fibre functor of $\mathsf{FC}(U)$. Then $X(U_\rho)$ has a k-point if and only if for any principal bundle V in the pro-system defining U_ρ, $X(V)(k) \neq \emptyset$, where $X(V)$ is the compactification of V.

Proof. The "only if" part is obvious since the image of the k-point in $X(U_\rho)$ on each $X(V)$ is a k-point of $X(V)$. We prove the "if" part.

Assume that X has genus $\geqslant 2$. Then, as a consequence of Faltings' theorem asserting that a genus $\geqslant 2$ curve over a number field has at most finitely many rational points, there is a Hausdorff topology on k that induces a topology on $X(k)$ making it a compact space. For each $S \subset \mathsf{FC}(U)$ finite full subcategory we denote by U_S the principal bundle constructed from S and ρ. We call a point $a \in X(k)$ S-good if a lies in the image of $X(U_S)(k)$, which, by assumption, is not empty. By assumption on the genus of X, the genus of $X(U_S)$ is also $\geqslant 2$. Thus for each S, the set of S-good points is not empty, and is compact, being the image of a compact set. For S, S' finite subcategories, the set of points which are both S-good and S'-good is not empty neither: just consider $S \cup S'$. Thus the set of ρ-good points in $X(k)$, i.e. those which are S-good for any S, is not empty. This can be repeated for any principal bundle V of U with the fibre functor induced from ρ (see proof of Theorem 5.5). Now we can conclude the existence of the k-point in the universal covering X_ρ as points in the limit of the pro-system of ρ-good points in each covering Y of X.

If X has genus $\leqslant 1$, then we can still find in the pro-system defining U_ρ a principal bundle V with compactification Y of genus at least 2. We can then replace (U, X) by (V, Y) without lost of generality. $\square$

Corollary 7.3. *Claim* SC3 *of Conjecture* 3.2 *holds for all hyperbolic curves U over k if and only if it does for the subfamily of all U with X(k) = ∅, that is in this case* $\mathsf{Fib}_k(U) = ∅$.

Proof. Let U be given. Assume $ρ$ given and $X(U_ρ)(k) = ∅$. Then, according to Remark 7.2 there is a $V → U$ in the pro-system, with compactification $Y → X$, such that $Y(k) = ∅$, as Y is in the pro-system of $X(U_ρ)$. So V violates the conjecture. □

Proposition 7.4. *The assumptions are as in* SC3 *of Conjecture* 3.2. *Then* $|X(U_ρ)(k)| = 1$ *if* $|X_L((U_L)_{ρ_L})(L)| = 1$ *for some* $L ⊃ k$ *finite. Consequently if the section conjecture holds for* U_L *then it holds for* U.

Proof. Assume $|X_L((U_L)_{ρ_L})(L)| = 1$. One has a fiber square

$$
\begin{array}{ccc}
(U_L)_{ρ_L} & \longrightarrow & U_ρ \\
\downarrow & \square & \downarrow \\
U_L & \longrightarrow & U.
\end{array}
\qquad (7.1)
$$

Since the normalization of X in $k(U_ρ) ⊗_k L$ is the normalization of X_L in $k(U_ρ) ⊗_k L$, (7.1) induces the fiber square

$$
\begin{array}{ccc}
X_L((U_L)_{ρ_L}) & \longrightarrow & X(U_ρ) \\
\downarrow & \square & \downarrow \\
X_L & \longrightarrow & X.
\end{array}
\qquad (7.2)
$$

Assume $X_L((U_L)_{ρ_L})(L) = \{α\}$. Then $X((U)_ρ)(L) = \{α\}$. If there was no $a ∈ X(U_ρ)(k)$ below $α$, then $X(U_ρ)_L(L)$ would contain the conjugates of $α$. This contradicts unicity. Then the assumption $X_L((U_L)_{ρ_L})(L) = \{α\}$ implies that $X(U_ρ)(k) = \{a\}$. Vice-versa, if $X(U_ρ)(k) = \{a\}$, then certainly $X(U_ρ)_L(L)$ consists of one point $α ⊗ L$. □

Observe now the following consequence of the section conjecture.

Lemma 7.5. *Assume* SC3 *of Conjecture* 3.2 *holds for a hyperbolic curve U and let* $V ⊂ U$ *be an open defined over k. Then any neutral fibre functor of* FC(U) *extends to a neutral fibre functor of* FC(V).

Proof. Let $ρ ∈ \mathsf{Fib}_k(U)$. Thus either (i): $ρ = ρ_u$, $u ∈ U(k)$ or (ii): $ρ = τ_{x,η}$, $x ∈ (X ∖ U)(k)$. In the first case, if $u ∈ V(k)$ then for the extension just take $ρ_u$, if $u ∈ (U ∖ V)(k)$ then for the extension just take any tangential fibre functor of FC(V) at u. In the second case for the extension just take $τ_{x,η}$ as tangential fibre functor for FC(V). □

This lemma suggests to pose the following conjecture which is an immediate consequence of the section conjecture.

Conjecture 7.6. *Let U be a hyperbolic curve and $V \subset U$ defined over a field k of finite type over $\mathbb{Q}$. Then any $\rho \in \mathsf{Fib}_k(U)$ extends to a functor in $\mathsf{Fib}_k(V)$.*

The reason for us to make this conjecture is the possibility of using it to reduce the section conjecture for any curve to the case of $\mathbb{P}^1$ minus a 0-dimensional subscheme defined over k. First we need

Lemma 7.7. *Let U be a hyperbolic curve defined over a field k of finite type over $\mathbb{Q}$ and $V \xrightarrow{p} U$ be a principal bundle under a finite k-group scheme G. Then U_L satisfies* SC3 *of Conjecture* 3.2 *for any finite extension L of k if and only if V_L does for those L.*

Proof. Let X, Y be the compactifications of U, V. Assume U satisfies SC3. Let σ be a functor in $\mathsf{Fib}_k(V)$. Then $\rho := \sigma \circ p^*$ is a fibre functor in $\mathsf{Fib}_k(U)$. Thus V appears in the pro-system defining U_ρ and hence $U_\rho \cong V_\sigma$. This implies $X(U_\rho) = Y(V_\sigma)$. Since U satisfies the section conjecture, $X(U_\rho)$ has a unique k-point, thus V also satisfies the section conjecture.

Vice-versa, assume V_L satisfies SC3 for any finite extension L of k. Let $\rho \in \mathsf{Fib}_k(U)$. Let S denote the full subcategory of $\mathsf{FC}(U)$ generated by $p_* \mathcal{O}_V$, which is finite in the sense of [6, Definition 2.5]. Then the functor $\eta := H^0_{DR}(V, p^*(-))$ is a neutral fibre functor for S. On the other hand, the functor ρ restricted to S yields a principal bundle $q : U_S \to U$. By the finiteness of S, there exists a finite extension L of k such that $\eta \otimes L$ and $\rho|_S \otimes L$ are isomorphic. Thus $(U_S)_L$ is isomorphic to V_L over U_L. Consequently V_L appears in the pro-system defining $(U_L)_{\rho_L} = U_\rho \times \operatorname{Spec} L$. This means the projection $s_{\rho_L} : (U_L)_{\rho_L} \to U_L$ factors through $s_\sigma : (U_L)_{\rho_L} \to V_L$, which is the universal covering associated to the functor $\sigma := H^0_{DR}((U_L)_{\rho_L}, s_\sigma^*(-)) \in \mathsf{Fib}_L(V_L)$. By assumption on V_L, $X_L((U_L)_{\rho_L}) = Y_L((V_L)_\sigma)$ has an L-point. Now the proof of Proposition 7.4 applies and yields a k-point of $X(U_\rho)$. $\quad\square$

Corollary 7.8. *Let $V \xrightarrow{p} U$ be an étale covering of hyperbolic curves over k. Then U_L satisfies* SC3 *of Conjecture* 3.2 *for any finite extension $L \supset k$ if and only if V_L does for those L.*

Proof. Let $W \xrightarrow{r} V$ be an étale covering such that $s : W \xrightarrow{r} V \xrightarrow{p} U$ is a Galois covering. Let $k' := H^0_{DR}(W, \mathcal{O}_W)$. It is a finite field extension of k. After base change $k \to k'$, W is the union of k'-curves $W_i, i = 1, \ldots, d, d = |\operatorname{Gal}(k'/k)|$. Choose one of them, say W_1. Thus $W_1 \to U_{k'}$ and $W_1 \to V_{k'}$ are Galois coverings. According to Lemma 7.7, for any $L \supset k'$, U_L satisfies SC3 if and only if $W_{1,L} = W_1 \times_{k'} \operatorname{Spec} L$ does, if and only if V_L does. According to Proposition 7.4, if U_L satisfies SC3 then so does U, the same holds of course for V. The claim of the corollary now follows. $\quad\square$

Proposition 7.9. *Assume that Conjecture* 7.6 *holds true. Then* SC3 *of Conjecture* 3.2 *holds true for any finite field extension of k if it holds true for hyperbolic $U \subset \mathbb{P}^1$ over any finite field extensions of k. In particular the section conjecture holds for any number field if it holds for $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ over any number field.*

Proof. Let U be a hyperbolic curve and denote by X its compactification. By assumption on the validity of Conjecture 7.6, Lemma 7.5 shows that removing a k-subscheme of dimension 0 makes the problem harder. Let f be a non-trivial element of $k(X)$. It defines a ramified covering of $f : U \to \mathbb{P}^1$. Replacing U by a smaller Zariski open, we can assume that $f^{-1}f(U) = U$, so

f is an étale covering of $W = f(U) \subset \mathbb{P}^1$. Now Corollary 7.8 tells us that the section conjecture holds true for U_L, $L \supset k$, if it holds true for W_L, $L \supset k$.

Let now k be a number field. Then Belyĭ's theorem [1] asserts that we can choose a map $f : X \to \mathbb{P}^1$ in such a way that $V := f^{-1}(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \subset U$. Thus Corollary 7.8 again implies that U satisfies the section conjecture if $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ does.   □

Acknowledgments

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