Packets in Grothendieck’s section conjecture

Hélène Esnault a, Phùng Hồ Hai a,b,*

a Universität Duisburg-Essen, Mathematik, 45117 Essen, Germany
b Institute of Mathematics, Hanoi, Viet Nam

Received 21 September 2007; accepted 20 December 2007
Available online 8 February 2008
Communicated by Michael J. Hopkins

Abstract
The goal of this article is to show a part of Grothendieck’s section conjecture using the identification of sections with neutral fiber functors as defined in [H. Esnault, P.H. Hai, The fundamental groupoid scheme and applications, preprint, 2006, 34 pp.].

© 2008 Elsevier Inc. All rights reserved.

Keywords: Arithmetic fundamental groups; Finite connections; Neutral fiber functors

1. Introduction

Let $U$ be an absolutely connected, smooth scheme of finite type defined over field $k$ of characteristic 0. Then a section of the Galois group $\text{Gal}(\bar{k}/k)$ into Grothendieck’s fundamental group $\pi_1(U, \bar{u})$ based at a geometric point $\bar{u} \to U$ is identified in [6] with a neutral fiber functor $\rho$ of the Tannaka category $\text{FC}(U)$ of finite connections. To a neutral fiber functor $\rho$, one defines a $k$-form $s_\rho : U_\rho \to U$ of Grothendieck’s universal covering $U_{\bar{u}} \to U$ based at $\bar{u}$, which identifies $\rho$ with the cohomological fiber functor $H^0(U_\rho, s_\rho^*(-))$.

Grothendieck’s section conjecture predicts a geometric description of sections $\text{Gal}(\bar{k}/k) \to \pi_1(U, \bar{u})$, under a geometric condition on $U$ and an arithmetic condition on $k$. Without any condition, we can already say that the Tannaka description above is of geometric nature, if we think of neutral fiber functors as rational points of a gerbe. On the other hand, Tamagawa [12]...
showed a part of the conjecture following a suggestion of Grothendieck. The aim of this article is twofold. We reprove the known part of Grothendieck’s section conjecture using our identification of sections with neutral fiber functors as defined in [6]. We then use our method to define the packets and to show the properties predicted by the conjecture.

A smooth absolutely connected curve over a field \( k \) is called hyperbolic if the degree of the sheaf of 1-differential forms with logarithmic poles at \( \infty \) is strictly positive. Grothendieck’s section conjecture [8] (see precise formulation in Conjecture 2.2 of this article) predicts first that under the assumptions that \( U \) be a hyperbolic curve over \( k \), a field of finite type over \( \mathbb{Q} \), then rational points of \( U \) inject into the set of sections. Call such sections geometric. Let \( X(U) \) be the pro-system of smooth compactifications of the pro-system \( U \). Grothendieck predicts further that a geometric section yields a unique Gal(\( \bar{k} / k \))-invariant point on \( X(U) \), lying above a point in \( U(k) \), and that a non-geometric section also yields a unique Gal(\( \bar{k} / k \))-invariant point on \( X(U) \), thus lying above a point in \( (X \setminus U)(k) \), where \( X \supset U \) is the smooth compactification of \( U \). Grothendieck mentions in [8, p. 8] that those properties should be proven using a Mordell-Weil type argument. Indeed, it has been essentially worked out in [12, Section 2]. We show these properties anew in Section 5, using our method. It becomes then a simple consequence of the definitions.

According to Grothendieck, non-geometric sections should be subdivided into packets \( P_x \), \( x \in (X \setminus U)(k) \) [8, p. 9]. Each packet \( P_x \) should have the cardinality of the continuum, and, for \( x \neq y \), a section in \( P_x \) should not be equivalent under conjugation with \( \pi_1(\bar{U}, \bar{u}) \) to a section in \( P_y \).

The Tannaka method we use, which relies on Nori’s ideas [10,11], on Deligne’s non-neutral Tannaka theory [4], and on fiber functors at \( \infty \) as defined in [3, Section 15] and [9a], allows us to define these packets and to show the properties wanted.

In Section 2, we review Grothendieck’s theory of the arithmetic fundamental group [7] as used later on and formulate precisely his section conjecture. In Section 3 we review our theory as developed in [6] and suitable for our purpose here, and reformulate in this language Grothendieck’s section conjecture. In Section 4, we provide a homological interpretation of the part concerning geometric sections. In Section 5, we show that a section yields at most one fixed point on \( X(U) \) and define the packets. In Section 6, we show that each packet has the cardinality of the continuum. In fact, at each stage, we show what are the necessary conditions for those properties to hold.

What remains to be understood is the more difficult part of Grothendieck’s section conjecture, namely the existence of the fixed point. Call it the existence conjecture. Nothing is known on it, except Königsmann’s spectacular result [9a] using logic and model theory, if \( k \) of finite type over \( \mathbb{Q} \) is replaced by a \( p \)-adic field. We make in Section 7 a list of reductions. In particular, thinking of a fixed point as a \( k \)-rational point of a \( k \)-form of \( X(U) \), it would be enough to find a \( K \)-rational point for \( K \) a function field appearing in the prosystem defining the \( k \)-form of \( U \) (Proposition 7.1). Furthermore, the existence conjecture suggests a weaker conjecture on extensions of fiber functors which, if possible to understand, would in general reduce the existence conjecture to an open in \( \mathbb{P}^1 \), and over a number field to \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \) (Proposition 7.9).

2. Grothendieck’s section conjecture

Let \( k \) be a characteristic zero field, and fix an algebraic closure \( \bar{k} \) of it together with the canonical inclusion \( i : k \to \bar{k} \). Denote \( \Gamma := \text{Gal}(\bar{k}/k) \). Let \( \epsilon : X \to \text{Spec } k \) be an absolutely connected scheme over \( k \) and fix a closed geometric point \( \bar{x} : \text{Spec } \bar{k} \to X \) of \( X \). Let \( s_{\bar{x}} : X_{\bar{x}} \to X \)
be the fundamental pro-étale covering of $X$, which is equipped with a $\tilde{k}$-point $\tilde{x}$ with $s\tilde{x}(\tilde{x}) = \tilde{x}$. Thus the arithmetic fundamental group $\pi_1(X, \tilde{x})$ of $X$, is, as a set, the fiber $s\tilde{x}^{-1}(\tilde{x})$, in particular $\tilde{x}$ is identified with the unit element of $\pi_1(X, \tilde{x})$.

The morphism $\epsilon$ yields a homomorphism of fundamental groups

$$\epsilon_* : \pi_1(X, \tilde{x}) \rightarrow \pi_1(\text{Spec } k, i) = \text{Gal}(\tilde{k}/k) = \Gamma.$$ (2.1)

The assumption that $X$ is absolutely connected implies that $\epsilon_*$ is surjective. The kernel of $\epsilon_*$ is isomorphic to $\pi_1(\tilde{X}, \tilde{x})$, the geometric fundamental group of $\tilde{X} := \text{Spec } \tilde{k} \times_{\text{Spec } k} X$ with base point $\tilde{x}$. That is, we have the following exact sequence

$$1 \rightarrow \pi_1(\tilde{X}, \tilde{x}) \rightarrow \pi_1(X, \tilde{x}) \xrightarrow{\epsilon_*} \Gamma \rightarrow 1.$$ (2.2)

In particular we have the following commutative diagram

$$\begin{array}{ccc}
\text{Spec } \tilde{k} & \xrightarrow{\id \times \tilde{x}} & \text{Spec } \tilde{k} \times X = \tilde{X} \\
\downarrow t_{\tilde{x}, s\tilde{x}} & & \downarrow \\
\bar{X}_{\tilde{x}} & & \end{array}$$ (2.3)

Here we omit the subindex $\text{Spec } k$ for the fibre product of $k$-schemes over $\text{Spec } k$.

Assume that $\tilde{x}$ lies above a rational point $x : \text{Spec } k \rightarrow X$, that is $\tilde{x} = x \circ i$. This yields a homomorphism $x_* : \pi_1(\text{Spec } k, i) \rightarrow \pi_1(X, \tilde{x})$ which is a section of $\epsilon_*$ as a homomorphism of pro-finite groups.

In general, let $y$ be an arbitrary $k$-rational point of $X$. Then $\tilde{y} := y \circ i : \text{Spec } \tilde{k} \rightarrow X$ is a $\tilde{k}$-point of $X$. As above, $y_*$ is a section of $\epsilon_* : \pi_1(X, \tilde{y}) \rightarrow \Gamma$. Since $\pi_1(X, \tilde{x}) \cong \pi_1(X, \tilde{y})$ by an isomorphism which commutes with $\epsilon_*$, we can consider $y_*$ as a section of $\epsilon : \pi_1(X, \tilde{x}) \rightarrow \Gamma$, determined up to an inner conjugation by an element of $\pi_1(\tilde{X}, \tilde{x})$.

Grothendieck denotes the set of sections of $\epsilon_* : \pi_1(X, \tilde{x}) \rightarrow \Gamma$ (as pro-finite group homomorphisms) up to conjugation by elements of $\pi_1(\tilde{X}, \tilde{x}) = \text{Ker } \epsilon_*$ by

$$\text{Hom-ext}_\Gamma(\Gamma, \pi_1(X, \tilde{x}))$$

(“ext” stands for external; see [8, Eq. (6)]). Notice that such a section $\sigma : \Gamma \rightarrow \pi_1(X, \tilde{x})$ yields an action of $\Gamma$ on $\bar{X}_{\tilde{x}}$ making the morphism $t_{\tilde{x}} : \bar{X}_{\tilde{x}} \rightarrow \text{Spec } \tilde{k} \Gamma$-equivariant.

**Definition 2.1.** Let $U$ be a smooth absolutely connected curve defined over a field $k$ of characteristic 0. Let $X \supset U$ be its smooth compactification. Then $U$ is said to be **elliptic** if its log-dualizing sheaf $\omega_X(\log(X \setminus U))$ has degree 0, and is said to be **hyperbolic** if $\omega_X(\log(X \setminus U))$ has strictly positive degree. Equivalently, $U$ is elliptic if the Euler characteristic $\chi(U) := 2 - 2g(X) - \#(X \setminus U)(\tilde{k})$ is 0, and hyperbolic if it is strictly negative.

Let $k$ be a characteristic zero field, and $U$ be an absolutely connected curve over $k$. Let $\tilde{u}$ be a $\tilde{k}$-point of $U$. Then we have the exact sequence (2.2) with $(X, \tilde{x})$ replaced by $(U, \tilde{u})$. Let $X(U_{\tilde{u}})$ denote the compactification of $U_{\tilde{u}}$ above the inclusion $U \hookrightarrow X$, that is, the pro-system of the compactifications of the finite étale coverings of $U$. It projects onto $X$, is étale above $U$ and ramifies along $(X \setminus U)$. 
In a letter to Gerd Faltings dated June 27, 1983, Alexander Grothendieck proposed the following conjecture (see [8, p. 7–9]):

**Conjecture 2.2 (Grothendieck’s section conjecture).** Let $U$ be a hyperbolic curve defined over a field $k$ of finite type over $\mathbb{Q}$. Then

(SC1) The map

$$U(k) \to \text{Hom}_{\text{ext}}\left(\Gamma, \pi_1(U, \bar{u})\right), \quad y \mapsto y_*,$$

(2.4)

is injective. We call a section of the form $y_*$ a geometric section.

(SC2) For a geometric section $y_*$, the resulting action of $\Gamma$ on $X(U_{\bar{k}})$ has a unique fixed point which lies above $y$.

(SC3) For a non-geometric section, the resulting action of $\Gamma$ on $X(U_{\bar{k}})$ has also a unique fixed point, which lies at infinity $X(U_{\bar{k}}) \setminus U_{\bar{k}}$, and projects on a point in $(X \setminus U)(k)$. This implies in particular that the set of non-geometric sections is divided into disjoint “packets” $\mathcal{P}_x$, $x \in (X \setminus U)(k)$.

(SC4) Each packet $\mathcal{P}_x$ has the cardinality of the continuum.

Let us specialize to the case $X = U$. Then Grothendieck’s section conjecture is equivalent to the following.

(SC) The map

$$X(k) \to \text{Hom}_{\text{ext}}\left(\Gamma, \pi_1(X, \bar{x})\right)$$

(2.5)

is bijective. More precisely, let $\sigma$ be a section of $\epsilon_* : \pi_1(X, \bar{x}) \to \Gamma$. The scheme $X_{\bar{x}}$, considered as a $\text{Spec} \bar{k}$-scheme by means of the morphism $t_{\bar{x}}$, has a unique fixed point under the action of $\Gamma$ determined by means of $\sigma$. Let $y$ be the image of this point in $X$. It is a $k$-rational point of $X$ and $\sigma = y_*$. 

**Remark 2.3.** We notice that the $\Gamma$-fixed points of $X_{\bar{x}}$ are nothing but the $k$-rational points of the $k$-form of $X_{\bar{x}}$ determined by the action of $\Gamma$.

### 3. The fundamental groupoid scheme

As in the previous section, we fix a field $k$ of finite type over $\mathbb{Q}$ as well as an algebraic closure $\bar{k}$ and denote by $i : k \hookrightarrow \bar{k}$ the canonical inclusion. Let $X \hookrightarrow \text{Spec} \bar{k}$ be a smooth, absolutely connected scheme of finite type. We denote by $\text{FC}(X)$ the category of finite connections, defined in [6, Section 2]. This notion generalizes, in characteristic 0, Nori’s notion of finite bundles over proper absolutely connected schemes to the case of smooth absolutely connected varieties. As $X$ is absolutely connected, $\text{FC}(X)$ is a Tannaka category over $k$ (as a fibre functor one can always choose the tautological functor $\tau$ to $\text{QCoh}(X)$ which assigns to a connection its underlying algebraic bundle). Furthermore, $\text{FC}(X)$ satisfies base change for finite field extensions $K \supset k$. Let $\rho : \text{FC}(X) \to \text{QCoh}(S)$ be a fibre functor, where $S$ is a $k$-scheme. Then Tannaka duality yields a transitive $k$-pro-groupoid scheme $\Pi_{\rho} \xrightarrow{(t, s)} S \times_k S$ [4, Théorème 1.12]. In order to simplify notations, in the rest of the article, we will use the terminology “groupoid scheme” rather than
“pro-groupoid scheme.” Let $\Pi^A_\rho$ denote the diagonal part of $\Pi_\rho$, which is a flat group scheme over $S$. In [6, Theorem 1.1] we associate to $\rho$ a $\Pi^A_\rho$-principal bundle $X_\rho \to S \times_k X$. The original fibre functor $\rho$ is equivalent to the cohomological functor $\mathcal{H}^0_{DR}(X_\rho/S, s_\rho^*(\silent))$. This construction is functorial in $\rho$, $S$ and $X$. The following fibre functors are of special interest.

Let us denote by $\text{Fib}_k(X)$ the set of neutral fibre functors. Assume that $S = \text{Spec} \, k$. Thus $\rho \in \text{Fib}_k(X)$, hence $\Pi_\rho$ is a $k$-group scheme and $s_\rho : X_\rho \to X$ is a principal under $\Pi_\rho$. In particular, let $x \in X(k)$ and $\rho := \rho_x$, $\rho_x((V, \nabla)) = V|_x$ be the fibre functor at $x$. Then our construction agrees with Nori’s construction in case $X$ is proper.

In particular, let $\tilde{x}$ be a $\bar{k}$-rational point of $X$ and $\rho_{\tilde{x}}$ be the fibre functor at $\tilde{x}$. We call the $\bar{k}$-groupoid scheme $\Pi(X, \tilde{x}) \rightarrow \text{Spec} \bar{k} \times \text{Spec} \bar{k}$ the fundamental groupoid scheme of $X$ with base point at $\tilde{x}$ and denote it by $\Pi(X, \tilde{x})$. By functoriality we have

$$\Pi(X, \tilde{x}) \rightarrow X_{\rho_{\tilde{x}}} \quad E \rightarrow \quad X_{\tau} \rightarrow X \times X$$

In particular $\Pi(X, \tilde{x})$ is the fibre of $X_{\rho_{\tilde{x}}} \to \text{Spec} \bar{k} \times \text{Spec} \bar{k}$ a $\bar{k}$-scheme by means of the morphism $s : \Pi(X, \tilde{x}) \to \text{Spec} \bar{k}$, has the property that its $\bar{k}$-points build a profinite group which is identified with $\pi_1(X, \tilde{x})$.

We shall therefore use the notation $X_{\tilde{x}} := X_{\rho_{\tilde{x}}}$ and $\tilde{x} := x_{\rho_{\tilde{x}}}$.

Let $\bar{y}$ be another $\bar{k}$-point of $X$. Then $\Pi(X, \tilde{x})$ and $\Pi(X, \bar{y})$ are isomorphic as groupoid schemes acting upon $\text{Spec} \bar{k}$. Actually, any isomorphism is given by conjugating with an el-
element of $\text{Isom}_k^\otimes(\rho_{\bar{x}}, \rho_{\bar{y}})(\bar{k})$, which according to [5, Theorem 3.2] is isomorphic (as a set) to $\Pi(X, \bar{x})^A(\bar{k})$.

The following proposition was proved in [6, Theorems 1.2, 1.3], except that we did not underline in [6, Theorems 1.2, 1.3] that we keep the same $x$.

**Proposition 3.1.** There is a one-to-one correspondence between conjugation classes of sections of $\varepsilon_\ast: \pi_1(X, \bar{x}) \to \Gamma$ and equivalence classes of neutral fibre functors of $FC(X)$. A section $\sigma$ of $\varepsilon_\ast$ corresponds to a fibre functor $\rho$ if the $k$-form of $X_{\bar{x}}$ (see Remark 2.3) determined by $\sigma$ is isomorphic to $X_\rho$ over $X$. In particular a geometric section given by a point $x \in X(k)$ corresponds to the fibre functor $\rho_x$ at $x$.

**Proof.** According to the discussion above, $X_{\bar{x}} = X_{\rho_{\bar{x}}}$. Given a neutral fibre functor $\rho$ of $FC(X)$, then $\rho \otimes_k \bar{k} \cong \rho_{\bar{x}}$, hence $X_\rho$ is a form of $X_{\rho_{\bar{x}}}$. Conversely, each $k$-form of $X_{\rho_{\bar{x}}}$ determines a neutral fibre functor by taking cohomology. On the other hand, each section of $\varepsilon_\ast$ determines a $k$-form of $X_{\bar{x}}$ and conversely. □

By means of this proposition, Grothendieck’s section conjecture can be reformulated as follows.

**Conjecture 3.2** (Section conjecture reformulated). Let $U$ be a hyperbolic curve over a field $k$ of finite type over $\mathbb{Q}$ and $X$ be its compactification. For each neutral fibre functor $\rho$ of $FC(U)$ we denote by $X(U_\rho)$ the compactification of the $\Pi_\rho$-principal bundle $s_\rho: U_\rho \to U$ with respect to $X$. Then

1. **(SC1)** The map $\delta: U(k) \to \text{Fib}_k(X), x \mapsto \rho_x$ is injective.
2. **(SC2)** For each $u \in U(k)$, the pro-curve $X(U_{\rho_u})$ has a unique $k$-rational point which lies above $u$.
3. **(SC3)** Let $\rho \in \text{Fib}_k(X)$ which is not geometric (i.e. not isomorphic to $\rho_u$ for any $u \in U(k)$), then $X(U_\rho)$ has a unique $k$-rational point which lies above a point $x \in (X \setminus U)(k)$. Denote the “packet” of such fibre functor in $\text{Fib}_k(U)$ by $\mathcal{P}_x$. Thus,

$$\text{Fib}_k(U) = \left\{ \rho_u, u \in U(k) \right\} \sqcup \bigcup_{x \in (X \setminus U)(k)} \mathcal{P}_x.$$  \hspace{1cm} (3.4)

4. **(SC4)** Each packet $\mathcal{P}_x$ has the cardinality of the continuum.

Let us specialize to the case $X = U$. Then Grothendieck’s section conjecture reformulated is equivalent to the following.

**SC** The map

$$X(k) \to \text{Fib}_k(X), \quad x \mapsto \rho_x,$$

is bijective.
4. A cohomology-like interpretation of (SC1) and (SC2)

We continue to assume that $U$ is an absolutely connected curve over $k$, where $k$ is of finite type over $\mathbb{Q}$. Fix a neutral fibre functor $\rho$ for $\text{FC}(U)$ (in particular assume the existence of such a functor). Let $\eta$ be another neutral fibre functor for $\text{FC}(U)$. Then according to [5, Theorem 3.2], the functor $\text{Isom}_k^\otimes(\eta, \rho)$ is representable by a (profinite) scheme $E \to \text{Spec} k$, which is a principal bundle under both $\Pi_\rho$ (on the right) and $\Pi_\eta$ (on the left). Conversely, let $E \to \text{Spec} k$ be a $\Pi_\rho$-principal bundle over $\text{Spec} k$, in particular $\Pi_\rho$ acts on $E$ from the right, hence $k[E]$ is a representation of $\Pi_\rho$. Then we can define a fibre functor $\eta$ for $\text{FC}(U)$ as follows:

$$\eta(V) := (k[E] \otimes \rho(V))^{\Pi_\rho}, \quad \text{for all } V \in \text{FC}(U),$$

(4.1)

where $(-)^{\Pi_\rho}$ denotes the set of $\Pi_\rho$-invariants. Then $E$ becomes a left $\Pi_\eta$-principal bundle.

It is well known that $\Pi_\rho$-principal bundles over $\text{Spec} k$ are classified by the pro-system of pointed sets

$$H^1_{\text{ét}}(\text{Spec} k, \Pi_\rho) := \lim_{\Pi_\rho \to G, G \text{ finite}} H^1_{\text{ét}}(k, G),$$

(4.2)

where the distinguished element corresponds to a principal bundle $E$ with $E(k) \neq \emptyset$, thus $E \cong \Pi_\rho$ via the choice of the $k$-rational point. Consequently $\rho$ defines a bijective map $\delta_\rho : \text{Fib}_k(U) \to H^1_{\text{ét}}(\text{Spec} k, \Pi_\rho)$.

Now let us give ourselves $u \in U(k)$ and take $\rho = \rho_u$. We obtain a bijective map

$$\delta_u : \text{Fib}_k(U) \to H^1_{\text{ét}}(\text{Spec} k, \Pi(U, u)), \quad v \mapsto E_v,$$

(4.3)

where $E_v$ is a short hand for the $k$-scheme representing the functor $\text{Isom}_k^\otimes(\rho_v, \rho_u)$.

Observe that this map can be determined by the following diagram

$$\begin{array}{ccc}
E_v & \xrightarrow{i_v} & X_u \\
\downarrow & & \downarrow s_u \\
\text{Spec} k & \xrightarrow{v} & X.
\end{array}$$

(4.4)

The geometric part of Grothendieck section conjecture (i.e. the part concerning geometric sections) has the following cohomological interpretation. Consider the “sequence” of $k$-schemes

$$\Pi(U, u) \xrightarrow{i} U_u \xrightarrow{s_u} U,$$

(4.5)

where $i$ is the closed embedding $s_\rho^{-1}(u) = \Pi(U, u) \subset U_u$.

Let $i^0, s_0^u$ denote the corresponding maps on the sets of $k$-points. Define the “cohomology set” $H^1_{\text{ét}}(k, U_u)$ as the set of $k$-forms of $U_u \otimes \overline{k}$, and say that a $k$-form is trivial if it has a $k$-rational point. And define the map

$$i^1 : H^1_{\text{ét}}(k, \Pi(U, u)) \to H^1_{\text{ét}}(k, U_u)$$
that assigns to each \( \Pi(U,u) \)-principal bundle \( E \to \text{Spec} k \) the \( \Pi(U,u) \)-principal bundle \( \mathcal{U}_E \to E \) defined as follows: \( E \) determines a fibre functor \( \rho_E \) for \( \mathcal{F}_C(U) \), which in turn determines a \( \Pi(U,\rho_E) \)-principal bundle \( \mathcal{U}_{\rho_E} \to \mathcal{U} \), which is a form of \( \mathcal{U}_u \otimes \bar{k} \). Then we have:

**Lemma 4.1.** The sequence (4.5) induces the following “long exact” sequence of pointed sets

\[
e \to \Pi(U,u)(k) \xrightarrow{i_0} \mathcal{U}_u(k) \xrightarrow{s_0^U} U(k) \xrightarrow{\delta_u} H^1_{\text{ét}}(k,\Pi(U,u)) \xrightarrow{i_1} H^1_{\text{ét}}(k,\mathcal{U}_u).
\] (4.6)

**Proof.** The exactness at \( \Pi(U,u)(k) \) and at \( \mathcal{U}_u(k) \) is clear. Let \( v \in U(k) \) such that \( \delta_u(v) \) is trivial in \( H^1_{\text{ét}}(k,\Pi(U,u)) \), which means that \( E_v = \delta_u(v) \) has a \( k \)-rational point. According to diagram (4.4), \( \mathcal{U}_u \) has a \( k \)-rational point which lies above \( v \). Similarly, an element \( U_\rho \) of \( H^1_{\text{ét}}(k,\Pi(U,u)) \) is mapped to the trivial object of \( H^1_{\text{ét}}(k,X_u) \) if and only if it has a \( k \)-rational point. This mean \( \rho \) is geometric, i.e. \( U_{\rho} \) lies in the image of \( \delta_u \). \( \square \)

**Remarks 4.2.** (a) The cohomological notations such as \( H^1_{\text{ét}}(k,U_u) \) used in this section are just a support for thinking, they are not standard and also not properly defined as cohomologies.

(b) The claim (SC1) of Grothendieck section conjecture is equivalent to saying that \( \delta_u \) is injective, or equivalently, that \( s_0^U \) is trivial, i.e. maps \( U_u(k) \) to the single point \( u \in U(k) \), or equivalently that \( i_0 \) is a bijective map.

(c) The claim (SC2) is equivalent to saying that \( U_u(k) \) consists of a unique point above \( u \).

(d) If \( U = X \) (i.e. \( U \) is projective) then claim (SC3) is equivalent to saying that \( \delta_u \) is surjective.

5. The uniqueness of the fixed point and the packets

In this section, using our method we provide new proofs of SC1 and SC2 in Conjecture 3.2, which were shown by Tamagawa in [12, Section 2]. This leads in a natural way to the definition of the packets.

**Theorem 5.1.** Let \( U \) be smooth geometrically connected curve which is elliptic or hyperbolic (see Definition 2.1) and defined over a field \( k \) of finite type over \( \mathbb{Q} \). Let \( u \in U(k) \) be a \( k \)-rational point. Then the map

\[
\delta_u : U(k) \to H^1_{\text{ét}}(k,\Pi(U,u))
\] (5.1)

is injective.

**Proof.** If \( U \) is not projective, let \( U \subset V \) be a larger open which is still defined over \( k \). It yields the restriction functor \( \mathcal{F}_C(V) \xrightarrow{\text{rest}} \mathcal{F}_C(U) \), which defines \( \mathcal{F}_C(V) \) as a full subcategory of \( \mathcal{F}_C(U) \). So as long as \( V \) still fulfills the assumption elliptic or hyperbolic, the theorem for \( U \) follows from the theorem for \( V \). Consequently, if the genus of the smooth compactification \( X \) of \( U \) is \( \geq 1 \), we can assume \( U \) to be \( X \). If the genus of \( X \) is 0, the cardinality of \( (X \setminus U)(k) \) is at least 2. Let \( K \supset k \) be a finite field extension over which at least two of those points are rational. Then \( U \otimes K \subset \mathbb{G}_m \) and the statement for \( \mathbb{G}_m \) over \( K \) implies the statement for \( U \otimes K \). On the other hand, by base change, \( \Pi(U,u) \otimes K = \Pi(U \otimes K,u \otimes K) \), and the composite map \( U(k) \to H^1_{\text{ét}}(k,\Pi(U,u)) \to H^1_{\text{ét}}(k,\Pi(U,u) \otimes K) \) factors through \( U(k) \subset U(K) \). This reduces the statement to \( \mathbb{G}_m \) in this case.
Thus let $U$ be either a projective curve of genus $\geq 1$ or $\mathbb{G}_m$. Let $J(U)$ be the Jacobian of $U$. So it is an abelian variety defined over $k$ in genus $\geq 1$ or the semi-abelian variety $\mathbb{G}_m$ in genus 0. We fix $u \in U(k)$ and denote by $j : U \to J(U)$ the cycle map sending $u$ to the unit $e$ of $J(U)$. To simplify notations, in the genus 0 case, we assume $u = 1$ so $j$ is then the identity. We denote by $\text{FC}^{\text{ab}}(U)$ the full subcategory of $\text{FC}(U)$ consisting of those connections $V$ which split into a direct sum of rank 1 connections after base change by a finite extension of $k$. For a (neutral) fibre functor $\rho$, let $\Pi^{\text{ab}}_{\rho}$ denote the corresponding Tannaka $k$-group scheme. It is the maximal commutative quotient $k$-group scheme of $\Pi_{\rho}$.

The morphism $j : U \to J(U)$ defines by pull-back the tensor functor $j^* : \text{FC}(J(U)) \to \text{FC}(U)$, which factors through the subcategory $\text{FC}^{\text{ab}}(U)$. Therefore the resulting homomorphism $\tilde{j}_*: \Pi(U,u) \to \Pi(J(U),e)$ factors through the homomorphism

$$\tilde{j}_*: \Pi^{\text{ab}}(U,u) := \Pi^{\text{ab}}_{\rho u} \to \Pi(J(U),e). \quad (5.2)$$

**Lemma 5.2.** The homomorphism $\tilde{j}_*$ is an isomorphism.

**Proof.** Since $\text{FC}(U)$ is compatible with base change, so is $\text{FC}^{\text{ab}}(U)$. Since $k$ has characteristic 0, $\tilde{j}_*$ is an isomorphism if and only if it is an isomorphism after $\otimes \bar{k}$. But over $\bar{k}$, $j$ induces an isomorphism $j^*: \text{Pic}^0(J(U)) \to \text{Pic}^0(U)$ in genus $\geq 1$, while for $\mathbb{G}_m$, $j$ itself is the identity map. Thus in particular, $j_*$ induces an isomorphism on $k$-valued torsion points. Therefore $\tilde{j}_*$ yields an equivalence between $\text{FC}(J(U))$ and $\text{FC}^{\text{ab}}(U)$. \(\square\)

**Proposition 5.3.** Let $A/k$ be a semi-abelian variety defined over a field $k$ of finite type over $\mathbb{Q}$. Then the map

$$\delta_e : A(k) \to \text{Fib}_k(A) \quad (5.3)$$

is injective.

**Proof.** For a natural number $m \in \mathbb{N}$, let us denote by $m \cdot A(k)$ the set of $k$-rational points of $A$ which are divisible by $m$ as $k$-rational points. Since $k$ is of finite type over $\mathbb{Q}$, there are no $k$-rational points which are infinitely divisible, thus

$$\bigcap_{m \in \mathbb{N}} m \cdot A(k) = e. \quad (5.4)$$

Let $s_e : A_e \to A$ be as usual the universal covering of $A$ associated to the fibre functor at $e$. By Lemma 4.1 the injectivity of $\delta_e$ is equivalent to the triviality of $s^0_e$, that is, all $k$-points of $A_e$ lie above $e$. Assume the contrary, that is $A_e$ has a $k$-point $b$ lying above $a \in A(k)$, $a \neq e$. According to (5.4), there exists $m > 0$ such that there is no $a' \in A(k)$ with $m \cdot a' = a$. Observe that the covering $[m] : A \to A$ is a principal bundle under the $k$-group scheme $A[m] := \text{Ker}[m]$,
and \( e \in A(k) \) is a \( k \)-point lying above \( m \cdot e \in A(k) \) with respect to \([m]\). Therefore the universal covering \( s_e \) factors as

\[
\begin{array}{ccc}
A_e & \xrightarrow{p} & A \\
\downarrow{s_e} & & \downarrow{[m]} \\
A. & & \\
\end{array}
\]  
\( (5.5) \)

Let \( a' := p(b) \in A(k) \). Then \( m \cdot a' = a \), a contradiction. Thus \( s_e^0 \) is the trivial map, hence \( \delta_e \) is injective. \( \square \)

Proposition 5.3 finishes the proof of the theorem. \( \square \)

Remarks 5.4. Theorem 5.1 shows SC1 in Conjecture 3.2 under the weaker assumption on the geometry of \( U \): it is enough for \( U \) to be elliptic. The ellipticity assumption is sharp. If \( U = \mathbb{A}^1 \), then certainly \( \text{FC}(\mathbb{A}^1) \) is the trivial category, thus all neutral fiber functors are equivalent to \( M \mapsto H^0(\mathbb{A}^1, M) \), and \( \delta_u \) cannot be injective. It is also to be noted that for SC1, one only needs the maximal abelian subcategory of \( \text{FC}(U) \).

Theorem 5.5. Let \( X \) be a smooth projective curve over a field \( k \) of finite type over \( \mathbb{Q} \), and let \( \emptyset \neq U \subset X \) be Zariski open over \( k \). Let \( \rho \) be a neutral fibre functor for \( \text{FC}(U) \), defining the principal \( \Pi_T \)-bundle \( s_\rho: U_\rho \to U \) and the normalization \( X(U_\rho) \) of \( X \) in \( k(U_\rho) \), defining \( \tilde{\delta}_\rho: X(U_\rho) \to X \) (see Section 3). If the genus of \( X \) is \( \geq 1 \) (so \( U \) is elliptic or hyperbolic, see Definition 2.1), or if the genus of \( X \) is 0 and \( U \) is hyperbolic, then

\[
\#X(U_\rho)(k) \leq 1. \quad (5.6)
\]

Proof. We first consider the case \( g(X) \geq 1 \). Assume on the contrary that there exist \( \alpha, \beta \in X(U_\rho)(k), \alpha \neq \beta \). Let \( a, b \) be respectively their images in \( X(k) \). We use the notation \( \rho_X \) for the restriction of \( \rho \) to \( \text{FC}(X) \). Since \( \text{FC}(X) \) is a full subcategory of \( \text{FC}(U) \), we have the following commutative diagram

\[
\begin{array}{ccc}
U_\rho & \xrightarrow{\rho_X} & X(U_\rho) \\
\downarrow{s_\rho} & & \downarrow{\tilde{s}_\rho} \\
U & \xrightarrow{\rho_X} & X. \\
\end{array}
\]  
\( (5.7) \)

Denote the images of \( \alpha \) and \( \beta \) in \( X_\rho(k) \) by \( \tilde{\alpha} \) and \( \tilde{\beta} \). Their images in \( X(k) \) are then \( a \) and \( b \). Thus we have

\[
\rho_X \cong \rho_a \cong \rho_b \quad (5.8)
\]

where \( \rho_a \) and \( \rho_b \) denote the geometric fibre functors at \( a \) and \( b \) for \( \text{FC}(X) \). Indeed, as recalled in Section 3,
We conclude by Theorem 5.1 that \( a = b \).

There exists a principal bundle \( q : V \to U \) in the pro-system defining \( U_\rho \) such that the images \( \alpha', \beta' \) of \( \alpha \) and \( \beta \) in the compactification \( Y \) of \( V \) are distinct. Thus \( \rho \) induces in a canonical way a fibre functor \( \rho_V \) for FC\((V)\). Indeed, \( q_*N \) is an object of FC\((U)\) whenever \( N \) is an object of FC\((V)\), and since \( q^*q_*N \to N \), the pull-back \( N' \) of \( N \) to \( U_\rho \) is trivialized, and \( \rho_V \) is defined by \( N \mapsto H^0(U_\rho, N') \). It also shows that \( V_{\rho_V} = U_\rho \). Consequently \( X(U_\rho) \) is the compactification of \( U_\rho \) with respect to \( V \mapsto Y \). On the other hand, it is obvious that the pair \( (V, Y) \) satisfies the condition of the theorem. Thus we can repeat the above argument to conclude that \( \alpha' = \beta ' \). This is a contradiction.

To treat the case \( g(X) = 0 \) we first show the following lemma.

**Lemma 5.6.** Let \( k \) be an arbitrary field of characteristic 0 and \( U \subset X \) be defined over \( k \) such that \( g(X) = 0 \) and \( \chi(U) < 0 \). Let \( \rho \) be an arbitrary neutral fibre functor for FC\((U)\). Then in the pro-system defining \( U_\rho \), there exists a principal bundle \( V \to U \) such that the compactification \( Y \) of \( V \) has genus \( \geq 1 \).

**Proof.** As \( U_\rho \otimes \bar{k} \to U \otimes \bar{k} \) is isomorphic to Grothendieck’s universal covering, and the category of finite bundles is compatible with base change, we just have to know that \( \mathbb{P}^1 \) has coverings of genus \( \geq 1 \) ramified in 3 points, which is a well-known fact. \( \Box \)

We now finish the proof of Theorem 5.5. Let \( \rho \in \text{Fib}_k(U) \). According to Lemma 5.6, there exists a principal bundle \( V \to U \) in the pro-system defining \( U_\rho \) such that the genus of its compactification \( Y \) is at least 1. Further we can choose \( V \) such that the images \( \alpha', \beta' \) of \( \alpha \) and \( \beta \) in \( Y \) of \( V \) are distinct. Thus we can repeat the above argument to conclude that \( \alpha' = \beta ' \). This is a contradiction, that is, there exists at most one \( k \)-point in \( X(U_\rho) \). This finishes the proof. \( \Box \)

Theorem 5.5 suggests the following definition.

**Definition 5.7.** Let \( U \) be a hyperbolic curve and \( X \supset U, k \) be as in Theorem 5.5, where we allow \( k \) to be any field of characteristic 0, and let \( x \in (X \setminus U)(k) \). We define the packet \( \mathcal{P}_x \) as the set of those fibre functors \( \rho \in \text{Fib}_k(U) \) which have the property that the pro-scheme \( X(U_\rho) \) has a \( k \)-point, and this point lies above \( x \).

**Properties 5.8.** With the notations as in Definition 5.7, (5.9) shows that if \( \rho \in \mathcal{P}_x \), then \( \rho|_{\text{FC}(U \cup \{x\})} = \rho_x \). Vice-versa, if \( \rho \in \text{Fib}_k(U) \) such that \( \rho|_{\text{FC}(U \cup \{x\})} = \rho_x \) for some \( x \in (X \setminus U)(k) \), and \( k \) is of finite type over \( \mathbb{Q} \), then to conclude that \( \rho \in \mathcal{P}_x \) would be a consequence of SC3 in Conjecture 3.2.

6. The tangential fibre functor

Let \( U \) be a hyperbolic curve (see Definition 2.1) and defined over a field \( k \) of characteristic 0. We keep the notation as in Section 5. Let \( x \in (X \setminus U)(k) \) be a rational point at \( \infty \), with maximal
ideal sheaf \( m_x \). We denote by \( T_x = \text{Spec}(\bigoplus_{n=0}^{\infty} m_x^n/m_x^{n+1}) \) the fiber of the tangent bundle at \( X \) in \( x \), and by \( T^0_x = T_x \setminus \{x\} = \text{Spec}(\bigoplus_{n=0}^{\infty} m_x^n/m_x^{n+1}) \) the complement of the zero section. We denote by \( K_x \) the local field at \( x \) by \( R_x \) its valuation ring, so after the choice of a local parameter \( t \) at \( x \), one has \( K_x \cong k((t)) \), \( R_x \cong k[[t]] \). Denote

\[
S_x := \text{Spec } R_x, \quad S^0_x := S_x \setminus \{x\} = \text{Spec } K_x.
\]

Then there exists an exact tensor functor depending only on the choice of \( t \) in \( m_x^2 - \{0\} \)

\[
\text{FC}(S^0_x) \to \text{FC}(T^0_x)
\]

constructed by Deligne [3, Section 15] and by Katz [9b], which is an equivalence of categories, and which inverts the natural restriction functor \( \text{FC}(T^0_x) \to \text{FC}(S^0_x) \). So composing (6.2) with the restriction functor \( r_x : \text{FC}(U) \to \text{FC}(S^0_x) \) yields the tensor functor

\[
DK_x : \text{FC}(U) \to \text{FC}(T^0_x).
\]

This allows one to construct fibre functors for \( \text{FC}(U) \) by composing \( DK_x \) with fibre functors of \( \text{FC}(T^0_x) \). Since \( \text{FC}(S_x) \) is trivial, any fiber functor \( \rho \in \text{Fib}_k(U) \) obtained in this way has the property

\[
\rho|_{\text{FC}(X)} = \rho_x.
\]

Fibre functors of \( \text{FC}(T^0_x) \) can be explicitly described. The choice of a local parameter \( t \) at \( x \) identifies \( T^0_x \) with \( \mathbb{G}_m = \text{Spec } k[t^{\pm 1}] \). Then \( \text{FC}(T^0_x) \) is spanned (as an abelian category) by the connections

\[
L_a := (O \cdot e_a, \nabla), \quad \nabla(e_a) = \frac{a}{t} e_a, \quad a \in \mathbb{Q},
\]

where \( O := O_{T^0_x} \). The tensor product for \( L_a \) reads

\[
L_a \otimes L_b = L_{a+b}
\]

and there is an isomorphism

\[
\varphi_a : L_a \to L_{a+1}; \quad \varphi_a(e_a) = \frac{a+1}{t} e_{a+1}.
\]

Thus \( \text{FC}(T^0_x) \) is indeed generated (as a tensor category) by \( L_a, \ a \in \mathbb{Q}/\mathbb{Z} \). The Tannaka group of \( \text{FC}(T^0_x) \) with respect to the fibre functor at \( 1 \in \mathbb{G}_m \) identified with \( T^0_x \) via \( t \) is

\[
\Pi(T^0_x, 1) \cong \varprojlim_n \mu_n; \quad \mu_m \xrightarrow{(\cdot)^m} \mu_n.
\]

Consequently (see (4.2))

\[
\text{Fib}_k(T^0_x) = H^1_{\text{et}}(k, \varprojlim_n \mu_n) = (\text{Kummer theory}) \varprojlim_n \frac{k^\times}{(k^\times)^n}.
\]
Notice that $k^{\times}$ is naturally embedded in $\lim_{n \to \infty} \frac{k^{\times}}{(k^{\times})^n}$ and its image is precisely the set of geometric fibre functors of $\text{FC}(T_x^0)$. In particular we have proved the following

**Corollary 6.1.** The set $\text{Fib}_k(T_x^0)$ has the cardinality of the continuum.

The universal covering $(T_x^0)_\rho$ associated to a fibre functor $\rho$ of $\text{FC}(T_x^0)$ can be described as follows. Consider $\rho$ as an element of $\lim_{n \to \infty} \frac{k^{\times}}{(k^{\times})^n}$ and let $\rho_n \in k^{\times}$ be a representant of the image of $\rho$ in $\frac{k^{\times}}{(k^{\times})^n}$. Thus we have

$$\rho_n \equiv \rho_{m-n} \mod (k^{\times})^n. \quad (6.10)$$

Then

$$(T_x^0)_\rho = \text{Spec} \left( \frac{k[t_1^{\pm 1}, n \in \mathbb{N}]}{t_{1/m-n}^m = \frac{\rho_{m-n}}{\rho_n} t_{1/n}} \right). \quad (6.11)$$

The projection $(T_x^0)_\rho \to T_x^0$ is defined by $t \mapsto t_1$.

**Remark 6.2.** This remark echoes Remarks 5.4. The normalization $T_x((T_x^0)_\rho)$ of $T_x$ in $k((T_x^0)_\rho)$ is

$$T_x((T_x^0)_\rho) = \text{Spec} \left( \frac{k[t_1^{\pm 1}, n \in \mathbb{N}]}{t_{1/m-n}^m = \frac{\rho_{m-n}}{\rho_n} t_{1/n}} \right). \quad (6.12)$$

It has a unique $k$-point, which lies above $0 \in T_x$, if $\rho$ is not geometric. Otherwise, if $\rho = \rho_a$, $a \in T_x^0(k)$, then it has the rational point above $0$ and another one above $a$. We see in this way that $U \cong_k \mathbb{G}_m$ does not satisfy Theorem 5.5.

We denote by $T_x$ the set of fibre functors of $\text{FC}(U)$ obtained by composing $DK_x$ with a fibre functor $\rho$ of $\text{Fib}_k(T_x^0)$, and use the notation

$$\tau_{x,\rho} := \rho \circ DK_x. \quad (6.13)$$

Let us choose a $k$-rational point of $T_x^0$, which we denote by $1$, as it is the same as choosing $t \in m_x/m_x^2 \setminus \{0\}$. Tannaka duality for

$$\begin{array}{ccc}
\text{FC}(U) & \xrightarrow{DK_x} & \text{FC}(T_x^0) \\
\tau_{x,1} & \downarrow & \rho_1 \\
\text{Vect}_k & \end{array} \quad (6.14)$$

where $\rho_1$ denotes the fibre functor at $1 \in T_x^0$, yields a group scheme homomorphism

$$DK_x^*: \Pi(T_x^0, 1) \to \Pi(U, \tau_{x,\rho_1}). \quad (6.15)$$
Lemma 6.3. The homomorphism $DK^*_x$ in (6.15) is injective. Consequently each object of $\mathcal{FC}(T^0_x)$ is isomorphic to a direct summand of the image under $DK_x$ of an object of $\mathcal{FC}(U)$.

Proof. The homomorphism $DK^*_x$ respects base change. So we may assume that $k$ is of finite type over $\mathbb{Q}$, thus in particular, it is embeddable in $\mathbb{C}$. Then (6.15) is compatible with the homomorphism of topological fundamental groups

$$\pi_1(\mathbb{C}^\times, 1) \to \pi_1(U(\mathbb{C}), t)$$

(6.16)

after the choice of an embedding $k \to \mathbb{C}$. Here $\pi_1(U(\mathbb{C}), t)$ is the topological fundamental group based at the tangent vector $t$ at $x$ in the sense of Deligne (see [3, Section 15]). Ellipticity or hyperbolicity of $U$ implies that (6.16) is injective. Since by uniformization theory, the topological fundamental groups of $U(\mathbb{C})$ and of $\mathbb{C}^\times$ lie inside the complex points of an algebraic group, thus are residually finite, $DK^*_x$ is injective as well.

The last claim follows from [5, Theorem 2.11] and the fact that $\mathcal{FC}(U)$ is semisimple as an abelian category.

We mention the following result of [6, Theorem 5.7].

Proposition 6.4. Let $\tau := \tau_{x,\rho} \in T_x$. Then $X(U_\tau)$ has a $k$-point lying above $x$. Consequently $T_x \subset P_x$.

Fix $\rho \in \text{Fib}_k(U)$. Then each finite full abelian tensor subcategory $S \subset \mathcal{FC}(U)$ defines a finite quotient $\Pi_\rho \to G := \Pi_{\rho,S}$, and a principal bundle $U_G := U_{\rho,S} \overset{p}{\to} U$ under group $G$. Let $X(U_G)$ be the compactification of $U_G$, which then projects on $X$.

We set

$$S_{G,x} := S_x \times_X X(U_G) \quad \text{and} \quad S^0_{G,x} := S_x^0 \times_X U_G$$

where $S_x, S^0_x$ are defined in (6.1).

Proposition 6.5. The (fixed point free) action of $G$ on $U_G$ extends to a (not necessarily fixed point free) action

$$G \times X(U_G) \xrightarrow{\mu} X(U_G) \xleftarrow{\text{rest}} G \times S_{G,x} \xrightarrow{\mu} S_{G,x} \xleftarrow{\text{rest}} G \times S^0_{G,x} \xrightarrow{\mu} S^0_{G,x}$$

(6.17)

Proof. Denote for simplicity $V := U_G$ and $Y := X(U_G)$. We first start with the action on $Y$. The composition of the two maps $G \times V \xrightarrow{\mu} V \to Y$ factors (uniquely) through $G \times Y$ as $G \times Y$ is a smooth curve and $Y$ is projective (it is a removable singularity). The same argument yields that the two maps $G \times G \times V \xrightarrow{\mu,\mu} G \times V \to G \times Y$ defined by $\mu(gh, v)$ and $\mu(g, \mu(h, v))$ factor uniquely through $G \times G \times Y \xrightarrow{\mu,\mu} Y$. Since $\mu(gh, v) = \mu(g, \mu(h, v))$ on $V$, equality holds on $Y$ as...
well. This shows the existence of the extension of \( \mu \) to \( \bar{\mu} \) on the left triangle. On the other hand, since \( p_1 = p \circ \mu \) on \( V \), one has \( p_1 = \bar{p} \circ \bar{\mu} \) on \( Y \). This shows the left triangle. It also shows that the fiber \( X(U_G)_x = \bar{p}^{-1}(x) \) is left invariant by \( \bar{\mu} \). This finishes the proof.

Let \( \rho \in P_x \) and \( \alpha \in X(U_\rho) \) be the unique \( k \)-point, which lies above \( x \). Recall from Properties 5.8 that \( \rho \mid_{FC(U \cup \{x\})} = \rho_x \). On the other hand, \( FC(U \cup \{x\}) \) is a full abelian tensor subcategory of \( FC(U) \), which is closed under taking subquotients. Indeed, let \( (E, \nabla) \) be an object in \( FC(U) \), and let \( (E', \nabla') \) be its Deligne’s canonical extension to \( U \cup \{x\} \) [2]. Then \( (E, \nabla) \) lies in the image of \( FC(U \cup \{x\}) \) if and only if the residue of \( \nabla' \) at \( x \) is equal to 0, a condition which is compatible with exact sequences, and is stable by taking tensor products, duals and subquotients. Thus we have a surjective homomorphism \( \pi(U, \rho) \rightarrow \pi(U \cup \{x\}, \rho_x) \), the kernel of which will be denoted by \( K \).

Proposition 6.6. Let \( X \) be a (not necessarily projective) curve and \( U \subset X \) be such that \( X \setminus U \) consists of a unique \( k \)-point \( x \). Let \( \rho \in P_x \) and \( \alpha \in X(U_\rho) \). Then the map \( \bar{s}_\rho : X(U_\rho) \rightarrow X \) factors as

\[
\begin{array}{cccccc}
X(U_\rho) & \rightarrow & \alpha \\
\downarrow \bar{q}_\rho & & \downarrow \\
X_{\rho_x} & \rightarrow & x_{\rho_x} \\
\downarrow s_{\rho_x} & & \downarrow \\
X & \rightarrow & x
\end{array}
\]

with the property

\[
\bar{q}_\rho^{-1}(x_{\rho_x}) = \alpha
\]

i.e. \( \bar{q}_\rho \) fully ramifies at \( \alpha \).

Proof. Fixing a finite full abelian tensor subcategory \( S \) of \( FC(U) \) as in the discussion preceding Proposition 6.5 defines a finite quotient group scheme \( G \) of \( \pi(U, \rho) \). Recall that \( K := Ker(\pi(U, \rho) \rightarrow \pi(X, \rho_x)) \). Denote by \( A \) the image of \( K \) in \( G \), and consider the exact sequence

\[
1 \rightarrow A \rightarrow G \rightarrow H \rightarrow 1.
\]
The group $H$ is isomorphic to the Tannaka group of the restriction of $\rho$ to the intersection of $S$ with $FC(X)$. Thus $X_H$ is the compactification $X(U_H)$ of $U_H$ and we have the following commutative diagram

\[
\begin{array}{ccc}
U_G & \longrightarrow & X(U_G) \\
\downarrow q & & \downarrow \tilde{q} \\
U_H & \longrightarrow & X_H = X(U_H) \\
\downarrow & & \downarrow \\
U & \longrightarrow & X \\
\end{array}
\quad (6.21)
\]

where $q : U_G \to U_H$ is an $A$-principal bundle and $x_G, x_H$ are images of the point $\alpha \in X(U_\rho)(k)$ in $X(U_G)$ and $X(U_H)$, respectively.

Proposition 6.5 applied to the principal bundle $U_G \xrightarrow{\tilde{q}} U_H$ yields an action

$$A \times \tilde{q}^{-1}(x_H) \to \tilde{q}^{-1}(x_H)$$

of the $k$-algebraic group scheme $A$ on the $k$-scheme $\tilde{q}^{-1}(x_H)$.

One has from (6.21) that $x_G \in \tilde{q}^{-1}(x_H)$ and that $H$ acts freely on $X_H$. Consequently $B := \text{Stab}(x_G) \subset G$ projects trivially onto $H$, thus $B \subset A$. This yields the exact sequence

$$1 \to A/B \to G/B \to H \to 1$$

(6.23)

together with a principal bundle

$$A/B \times \tilde{q}^{-1}(x_H) \to x_H.$$

(6.24)

Thus one has a factorization

\[
\begin{array}{ccc}
U_G & \longrightarrow & X(U_G) \\
\downarrow B & & \downarrow q'' \\
U_{G/B} & \longrightarrow & X(U_{G/B}) \\
\downarrow A/B & & \downarrow q' \\
U_H & \longrightarrow & X_H \\
\end{array}
\quad (6.25)
\]

Denoting by $e$ the index of ramification of $x_G \otimes \tilde{k}$, one has that $|B(\tilde{k})| = e$ while $|(A/B)(\tilde{k})| = |\tilde{q}^{-1}(x_H)(k)|$. We conclude that

$$\left|q''^{-1}(x_H)(\tilde{k})\right| = \left|\tilde{q}^{-1}(x_H)(\tilde{k})\right| = \left|(A/B)(\tilde{k})\right|$$

(6.26)
and $\bar{q}'$ is étale. Thus $A/B$ factors as a quotient of $\pi(X, \rho_x)$. By definition (6.20), one concludes $A/B = 1$. This shows

$$\bar{q}^{-1}(x_H) = \{x_G\}. \quad (6.27)$$

Passing to the pro-system we deduce the claims of the proposition. □

**Corollary 6.7.** With the assumption of Proposition 6.6, for any $\rho \in \mathcal{P}_x$ there is a factorization

$$\begin{array}{ccc}
\text{FC}(U) & \xrightarrow{r_x} & \text{FC}(S_x^0) \\
\rho \downarrow & & \exists \eta \\
\text{Vec}_k. & & \\
\end{array} \quad (6.28)
$$

**Proof.** We adopt the notation of Proposition 6.6. By Hensel’s lemma, one has a unique lifting

$$X_{\rho_x} \xrightarrow{i} S_x \xrightarrow{s_{\rho_x}} X \quad (6.29)$$

with $x_{\rho_x} \in i(S_x)$. We define

$$\sigma : \mathcal{S}_x^0 := S_x^0 \times_{i(S_x)} U_\rho \to S_x^0,$$

$$\begin{array}{ccc}
\mathcal{S}_x^0 & \xrightarrow{\sigma} & U_\rho \\
\downarrow & & \downarrow \bar{q}_\rho \\
S_x^0 & \xrightarrow{i} & X_{\rho_x} \xrightarrow{s_{\rho_x}} X. \\
\end{array} \quad (6.30)
$$

According to Lemma 6.3, every object $N$ in $\text{FC}(S_x^0)$ is a summand of the restriction to $S_x^0$ of an object $M$ of $\text{FC}(U)$. Since $s_{\rho}^*(M)$ is trivializable, so is $\sigma^*(N)$. On the other hand, (6.19) implies

$$H^0(\mathcal{S}_x^0) = k. \quad (6.31)$$

Consequently, $\rho$ extends to $\text{FC}(S_x^0)$ by setting

$$\rho(N) = H^0(\sigma^*(N)). \quad (6.32)$$

**Theorem 6.8.** Let $U$ be a hyperbolic curve (see Definition 2.1) defined over a field $k$ of characteristic 0 and $X$ be its compactification. Then $T_x = \mathcal{P}_x$ for any $x \in (X \setminus U)(k)$. 
Proof. Let \( V := U \cup \{ x \} \). Denote by \( V(U_\rho) \) the compactification of \( U_\rho \) with respect to the inclusion \( U \to V \). Thus we have the following commutative diagram

\[
\begin{array}{ccc}
U_\rho & \xrightarrow{\iota} & V(U_\rho) \\
\downarrow & & \downarrow \\
X(U_\rho) & \xrightarrow{\phi} & X
\end{array}
\]

where the front square is cartesian. Consequently, any \( k \)-point \( \alpha \) of \( X(U_\rho) \) that lies above \( x \) should lie in \( V(U_\rho) \). Now Corollary 6.7 applies and shows that \( \rho \) is in fact a tangential functor at \( x \).

By means of the equivalence in (6.2), it remains to show the uniqueness of the functor \( \eta \) in diagram (6.28) for any given functor \( \rho \in \mathcal{P}_x \). Thus, let \( \rho \) and \( \eta \) be as in (6.28) and let \( \varphi \) be another fibre functor in \( \text{Fib}_k(S_0^0) \), such that \( \rho \cong \varphi \circ r_x \). Via the equivalence \( \text{FC}(S_0^0) \cong \text{FC}(T_0^0) \), we can consider \( \varphi \) as a functor in \( \text{Fib}_k(T_0^0) \). Now, as in the proof of Proposition 6.4, i.e. the proof of [6, Theorem 5.7], there exists a map

\[
f : (S_0^0)_\varphi \to U_\rho
\]

that determines the \( k \)-point of \( X(U_\rho) \), see diagram (5.11) of [6, Theorem 5.7]. The uniqueness of the \( k \)-point of \( X(U_\rho) \) shows that \( f \) is compatible with the morphisms \( i \) of (6.29) and \( \tilde{q}_\rho \) of (6.18) in the sense that we have the commutative diagram

\[
\begin{array}{ccc}
(S_0^0)_\varphi & \xrightarrow{f} & U_\rho \\
\downarrow s_\varphi & & \downarrow \tilde{q}_\rho \\
S_0^0 & \xrightarrow{i} & X_{\rho_x}.
\end{array}
\]

It follows from the universal property of \( S_0^0 \), i.e. (6.30), that there exists a map

\[
\theta : (S_0^0)_\varphi \to S_0^0
\]

which is compatible with the maps to \( S_0^0 \) and \( U_\rho \). That is, there exists a natural tensor transformation \( \varphi \to \eta \), which is then automatically a natural isomorphism (cf. [5, Proposition 1.13]). Thus \( \eta \) and \( \varphi \) are isomorphic. \( \square \)
Corollary 6.9. Let \( \eta \) and \( \eta' \) be non-isomorphic fibre functors of \( FC(T^0) \). Then \( \eta \circ DK_x \) and \( \eta' \circ DK_x \) are not isomorphic. Consequently, if \( k \) is of finite type over \( \mathbb{Q} \) then \( P_x \) has the cardinality of the continuum.

7. Remarks towards the existence

In this section, we gather a few facts towards SC3 in Conjecture 3.2.

Proposition 7.1. Let \( U \) be a geometrically connected curve defined over a characteristic 0 field \( k \). Let \( \rho \in \text{Fib}_k(U) \) and \( X(U_{\rho}) \) be as in Theorem 5.5. Then, if \( U \) is hyperbolic (see Definition 2.1), then \( X(U_{\rho})(k) = X(U_{\rho})(L) \) for \( L = k(V) \) where \( V \to U \) is any member of the pro-system \( s_{\rho} : U_{\rho} \to U \).

Proof. Let \( a : \text{Spec} \, L \to X(U_{\rho}) \). Then for \( W \to U \) in the pro-system defining \( U_{\rho} \), with \( W \neq V \), the hyperbolicity condition implies that the genus of the compactification of \( W \) is strictly larger than the genus of the compactification of \( V \). Therefore the induced point \( \text{Spec} \, L \to X(W) \) has to be a closed point. Let \( \kappa(a) \) be its residue field, thus \( k \subset \kappa(a) \) is finite, and \( \kappa(a) \subset L \). By the connectivity of \( V \), this is only possible if \( \kappa(a) = k \). This finishes the proof. \( \square \)

The following remark is contained in [12, Corollary 2.10], and seen here quite directly with our method.

Remark 7.2. Let \( U \) be a hyperbolic curve defined over a field \( k \) of finite type over \( \mathbb{Q} \). Let \( \rho \) be a neutral fibre functor of \( FC(U) \). Then \( X(U_{\rho}) \) has a \( k \)-point if and only if for any principal bundle \( V \) in the pro-system defining \( U_{\rho} \), \( X(V)(k) \neq \emptyset \), where \( X(V) \) is the compactification of \( V \).

Proof. The “only if” part is obvious since the image of the \( k \)-point in \( X(U_{\rho}) \) on each \( X(V) \) is a \( k \)-point of \( X(V) \). We prove the “if” part.

Assume that \( X \) has genus \( \geq 2 \). Then, as a consequence of Faltings’ theorem asserting that a genus \( \geq 2 \) curve over a number field has at most finitely many rational points, there is a Hausdorff topology on \( k \) that induces a topology on \( X(k) \) making it a compact space. For each \( S \subset FC(U) \) finite full subcategory we denote by \( U_S \) the principal bundle constructed from \( S \) and \( \rho \). We call a point \( a \in X(k) \) \( S \)-good if \( a \) lies in the image of \( X(U_S)(k) \), which, by assumption, is not empty. By assumption on the genus of \( X \), the genus of \( X(U_S) \) is also \( \geq 2 \). Thus for each \( S \), the set of \( S \)-good points is not empty, and is compact, being the image of a compact set. For \( S, S' \) finite subcategories, the set of points which are both \( S \)-good and \( S' \)-good is not empty neither: just consider \( S \cup S' \). Thus the set of \( \rho \)-good points in \( X(k) \), i.e. those which are \( S \)-good for any \( S \), is not empty. This can be repeated for any principal bundle \( V \) of \( U \) with the fibre functor induced from \( \rho \) (see proof of Theorem 5.5). Now we can conclude the existence of the \( k \)-point in the universal covering \( X_{\rho} \) as points in the limit of the pro-system of \( \rho \)-good points in each covering \( Y \) of \( X \).

If \( X \) has genus \( \leq 1 \), then we can still find in the pro-system defining \( U_{\rho} \) a principal bundle \( V \) with compactification \( Y \) of genus at least 2. We can then replace \( (U, X) \) by \( (V, Y) \) without lost of generality. \( \square \)
Corollary 7.3. Claim SC3 of Conjecture 3.2 holds for all hyperbolic curves $U$ over $k$ if and only if it does for the subfamily of all $U$ with $X(k) = \emptyset$, that is in this case $\text{Fib}_k(U) = \emptyset$.

Proof. Let $U$ be given. Assume $\rho$ given and $X(U_\rho)(k) = \emptyset$. Then, according to Remark 7.2 there is a $V \to U$ in the pro-system, with compactification $Y \to X$, such that $Y(k) = \emptyset$, as $Y$ is in the pro-system of $X(U_\rho)$. So $V$ violates the conjecture. □

Proposition 7.4. The assumptions are as in SC3 of Conjecture 3.2. Then $|X(U_\rho)(k)| = 1$ if $|X_L((U_L)_\rho)(L)| = 1$ for some $L \supset k$ finite. Consequently if the section conjecture holds for $U_L$ then it holds for $U$.

Proof. Assume $|X_L((U_L)_\rho)(L)| = 1$. One has a fiber square

$$
(U_L)_\rho \to U_\rho \to U.
$$

Since the normalization of $X$ in $k(U_\rho) \otimes k L$ is the normalization of $X_L$ in $k(U_\rho) \otimes k L$, (7.1) induces the fiber square

$$
X_L((U_L)_\rho) \to X(U_\rho) \to X.
$$

Assume $X_L((U_L)_\rho)(L) = \{\alpha\}$. Then $X((U_\rho)(L) = \{\alpha\}$. If there was no $a \in X(U_\rho)(k)$ below $\alpha$, then $X(U_\rho)L(L)$ would contain the conjugates of $\alpha$. This contradicts unicity. Then the assumption $X_L((U_L)_\rho)(L) = \{\alpha\}$ implies that $X(U_\rho)(k) = \{a\}$. Vice-versa, if $X(U_\rho)(k) = \{a\}$, then certainly $X(U_\rho)L(L)$ consists of one point $\alpha \otimes L$. □

Observe now the following consequence of the section conjecture.

Lemma 7.5. Assume SC3 of Conjecture 3.2 holds for a hyperbolic curve $U$ and let $V \subset U$ be an open defined over $k$. Then any neutral fibre functor of $\text{FC}(U)$ extends to a neutral fibre functor of $\text{FC}(V)$.

Proof. Let $\rho \in \text{Fib}_k(U)$. Thus either (i): $\rho = \rho_u$, $u \in U(k)$ or (ii): $\rho = \tau_{x,\eta}$, $x \in (X \setminus U)(k)$. In the first case, if $u \in V(k)$ then for the extension just take $\rho_u$, if $u \in (U \setminus V)(k)$ then for the extension just take any tangential fibre functor of $\text{FC}(V)$ at $u$. In the second case for the extension just take $\tau_{x,\eta}$ as tangential fibre functor for $\text{FC}(V)$. □

This lemma suggests to pose the following conjecture which is an immediate consequence of the section conjecture.
**Conjecture 7.6.** Let $U$ be a hyperbolic curve and $V \subset U$ defined over a field $k$ of finite type over $\mathbb{Q}$. Then any $\rho \in \text{Fib}_k(U)$ extends to a functor in $\text{Fib}_k(V)$.

The reason for us to make this conjecture is the possibility of using it to reduce the section conjecture for any curve to the case of $\mathbb{P}^1$ minus a 0-dimensional subscheme defined over $k$. First we need

**Lemma 7.7.** Let $U$ be a hyperbolic curve defined over a field $k$ of finite type over $\mathbb{Q}$ and $V \xrightarrow{p} U$ be a principal bundle under a finite $k$-group scheme $G$. Then $U_L$ satisfies SC3 of Conjecture 3.2 for any finite extension $L$ of $k$ if and only if $V_L$ does for those $L$.

**Proof.** Let $X, Y$ be the compactifications of $U, V$. Assume $U$ satisfies SC3. Let $\sigma$ be a functor in $\text{Fib}_k(V)$. Then $\rho := \sigma \circ p^*$ is a fibre functor in $\text{Fib}_k(U)$. Thus $V$ appears in the pro-system defining $U_\rho$ and hence $U_\rho \cong V_\sigma$. This implies $X(U_\rho) = Y(V_\sigma)$. Since $U$ satisfies the section conjecture, $X(U_\rho)$ has a unique $k$-point, thus $V$ also satisfies the section conjecture.

Vice-versa, assume $V_L$ satisfies SC3 for any finite extension $L$ of $k$. Let $\rho \in \text{Fib}_k(U)$. Let $S$ denote the full subcategory of $\text{Fib}_k(U)$ generated by $p_* \mathcal{O}_V$, which is finite in the sense of [6, Definition 2.5]. Then the functor $\eta := H^0_{\text{DR}}(V, p^*(-))$ is a neutral fibre functor for $S$. On the other hand, the functor $\rho$ restricted to $S$ yields a principal bundle $q : U_S \to U$. By the finiteness of $S$, there exists a finite extension $L$ of $k$ such that $\eta \otimes L$ and $\rho|_S \otimes L$ are isomorphic. Thus $(U_S)_L$ is isomorphic to $V_L$ over $U_L$. Consequently $V_L$ appears in the pro-system defining $(U_L)_{\rho_L} = U_\rho \times \text{Spec } L$. This means the projection $s_{\rho_L} : (U_L)_{\rho_L} \to U_L$ factors through $s_\sigma : (U_L)_{\rho_L} \to V_L$, which is the universal covering associated to the functor $\sigma := H^0_{\text{DR}}((U_L)_{\rho_L}, s_\sigma^*(-)) \in \text{Fib}_L(V_L)$. By assumption on $V_L$, $X_L((U_L)_{\rho_L}) = Y_L((V_L)_{\sigma})$ has an $L$-point. Now the proof of Proposition 7.4 applies and yields a $k$-point of $X(U_\rho)$. □

**Corollary 7.8.** Let $V \xrightarrow{p} U$ be an étale covering of hyperbolic curves over $k$. Then $U_L$ satisfies SC3 of Conjecture 3.2 for any finite extension $L \supset k$ if and only if $V_L$ does for those $L$.

**Proof.** Let $W \xrightarrow{s} V$ be an étale covering such that $s : W \xrightarrow{s} V \xrightarrow{p} U$ is a Galois covering. Let $k' := H^0_{\text{DR}}(W, \mathcal{O}_W)$. It is a finite field extension of $k$. After base change $k \to k'$, $W$ is the union of $k'$-curves $W_i, i = 1, \ldots, d$, $d = |\text{Gal}(k'/k)|$. Choose one of them, say $W_1$. Thus $W_1 \to U_{k'}$ and $W_1 \to V_{k'}$ are Galois coverings. According to Lemma 7.7, for any $L \supset k'$, $U_L$ satisfies SC3 if and only if $W_1L = W_1 \times_{k'} \text{Spec } L$ does, if and only if $V_L$ does. According to Proposition 7.4, if $U_L$ satisfies SC3 then so does $U$, the same holds of course for $V$. The claim of the corollary now follows. □

**Proposition 7.9.** Assume that Conjecture 7.6 holds true. Then SC3 of Conjecture 3.2 holds true for any finite field extension of $k$ if it holds true for hyperbolic $U \subset \mathbb{P}^1$ over any finite field extensions of $k$. In particular the section conjecture holds for any number field if it holds for $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ over any number field.

**Proof.** Let $U$ be a hyperbolic curve and denote by $X$ its compactification. By assumption on the validity of Conjecture 7.6, Lemma 7.5 shows that removing a $k$-subscheme of dimension 0 makes the problem harder. Let $f$ be a non-trivial element of $k(X)$. It defines a ramified covering of $f : U \to \mathbb{P}^1$. Replacing $U$ by a smaller Zariski open, we can assume that $f^{-1}(U) = U$, so
f is an étale covering of \( W = f(U) \subset \mathbb{P}^1 \). Now Corollary 7.8 tells us that the section conjecture holds true for \( U_L, L \supset k \), if it holds true for \( W_L, L \supset k \).

Let now \( k \) be a number field. Then Belyı’s theorem [1] asserts that we can choose a map \( f : X \to \mathbb{P}^1 \) in such a way that \( V := f^{-1}(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \subset U \). Thus Corollary 7.8 again implies that \( U \) satisfies the section conjecture if \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \) does. \( \square \)

Acknowledgments

Our first attempt was to try to contradict the existence conjecture using our method of neutral fiber functor [6]. Pierre Deligne showed us our mistake in the computation of the fiber functor in the example we sent him. Thinking of his answer helped us to develop the formalism of this article. We thank him for his interest and the clarity of his answer. We profoundly thank the referee for a very careful reading, and for pointing out in an earlier version of our article some inaccuracies. We also thank the Deutsche Forschungsgemeinschaft (German Research Council) for its constant support.

References