ON NORI'S FUNDAMENTAL GROUP SCHEME

HÉLÈNE ESNAULT, PHÙNG HÔ HAI, AND XIAOTAO SUN

ABSTRACT. The aim of this note is to give a structure theorem on Nori's fundamental group scheme of a proper connected variety defined over a perfect field and endowed with a rational point.

1. Introduction

For a proper connected reduced scheme X defined over a perfect field k Nori introduced in [8] and [9] the notion of *essentially finite bundles*. He shows that they form a k-linear abelian rigid tensor category, denoted subsequently by $\mathcal{C}^N(X)$. A k-rational point x of X endows $\mathcal{C}^N(X)$ with a fiber functor $V \mapsto V|_x$ with values in the category of finite dimensional vector spaces over k. This makes $\mathcal{C}^N(X)$ a Tannaka category, thus by Tannaka duality ([1, 12]), the fiber functor establishes an equivalence between $\mathcal{C}^N(X)$ and the representation category $\mathrm{Rep}(\pi^N(X,x))$ of an affine group scheme $\pi^N(X,x)$, which turns out to be a pro-finite group scheme (see Section 2 for an account of Nori's construction). The purpose of this note is to study the structure of this Tannaka group scheme.

To this aim, we define two full tensor subcategories $\mathcal{C}^{\acute{e}t}(X)$ and $\mathcal{C}^F(X)$. The objects of the first one are *étale finite* bundles, that is bundles for which the corresponding representation of $\pi^N(X,x)$ factors through a finite étale group scheme, and the objects of the second one are F-*finite* bundles, that is bundles for which the corresponding representation of $\pi^N(X,x)$ factors through a finite local group scheme. As Tannaka subcategories they are the representation categories of Tannaka group schemes $\pi^{\acute{e}t}(X,x)$ and $\pi^F(X,x)$.

In fact $\pi^{\acute{e}t}(X,x)$ relates closely to the more familiar fundamental group $\pi_1(X,\bar{x})$ defined by Grothendieck ([4, Exposé V]), where $\bar{x}$ is a geometric point above x, which is a pro-finite group. One has

$$\pi^{\acute{e}t}(X,x)(\bar{k}) \cong \pi_1(X \times_k \bar{k}, \bar{x}) \tag{1.1}$$

(see Remarks 2.10 for a detailed discussion). Thus the étale piece of Nori's group scheme takes into account only the geometric fundamental group and ignores somehow arithmetics. On the other hand, $\pi^F(X,x)$ reflects the purely inseparable covers of X. That k is perfect guarantees that inseparable covers come only from geometry, and not from the ground field.

Date: Mai 9, 2006.

Partially supported by the DFG Leibniz Preis, the DFG Heisenberg program and the grant NFSC 10025103.

The inclusion of $\mathcal{C}^{ét}(X)$ (resp. $\mathcal{C}^F(X)$) in $\mathcal{C}^N(X)$ as a full tensor subcategory induces a surjective homomorphism of groups schemes $r^{ét} : \pi^N(X,x) \to \pi^{ét}(X)$ (resp. $r^F : \pi^N(X) \to \pi^F(X)$). Our first remark is that the natural homomorphism

$$(r^{ét}, r^F) : \pi^N(X,x) \to \pi^{ét}(X,x) \times \pi^F(X,x) \tag{1.2}$$

is surjective but generally not injective. We give an example which is based on Raynaud's work [11] on coverings of curves producing a new ordinary part in the Jacobian (see Corollary 3.7). In particular, it is given as a rank 1 bundle in Pic of this covering, and thus does not come from a rank 1 bundle on X. The referee observes here that the morphism induced from (1.2) on the maximal abelian quotients of $\mathcal{C}^N(X), \mathcal{C}^F(X), \mathcal{C}^{ét}(X)$ is however an isomorphism. This provides one reason for the determination of the representation category of the kernel of (1.2): our work gives some information on the non-abelian part of Nori's category.

The central theorem of our note is the determination by its objects and morphisms of a k-linear abelian rigid tensor category $\mathcal{E}$, which is equivalent to the representation category of $\mathrm{Ker}(r^{ét}, r^F)$ (see Definition 4.3 for the construction and Proposition 4.4 and Theorem 4.5 to see that it computes what one wishes). This is the most delicate part of the construction. If S is a finite subcategory of $\mathcal{C}^N(X)$ with an étale finite Tannaka group scheme $\pi(X,S,x)$, then the total space X_S of the $\pi(X,S,x)$-principal bundle $\pi_S : X_S \to X$ which trivializes all the objects of S has the same property as X. It is proper, reduced and connected. However, if S is finite but $\pi(X,S,x)$ is not étale, then Nori shows that X_S is still proper connected, but may not be reduced. We give a concrete example in Remark 2.3, 2), which is due to P. Deligne.

In order to describe $\mathcal{E}$, we need in some sense an extension of Nori's theory to those non-reduced covers. We define on each such X_S a full subcategory $\mathcal{F}(X_S)$ of the category of coherent sheaves, the objects of which have the property that their push down on X lies in $\mathcal{C}^N(X)$ (see Definition 2.4). We show that indeed those coherent sheaves have to be vector bundles (Proposition 2.7), so in a sense, even if the scheme X_S might be bad, objects which push down to Nori's bundles on X are still good. In particular, $\mathcal{C}^N(X_S) = \mathcal{F}(X_S)$ if $\pi(X,S,x)$ is étale (Theorem 2.9), so the definition generalizes slightly Nori's one. For given finite subcategories S and T of $\mathcal{C}^N(X)$, with $\pi(X,S,x)$ étale and $\pi(X,T,x)$ local, we introduce in Definition 4.1 a full subcategory $\mathcal{E}(X_{S\cup T}) \subset \mathcal{F}(X_{S\cup T})$ consisting of those bundles V, the push down of which on X_S is F-finite. Now the objects of $\mathcal{E}$ are pairs $(X_{S\cup T}, V)$ for V an object in $\mathcal{E}(X_{S\cup T})$. Morphisms are subtle as they do take into account the whole inductive system of such $T' \subset \mathcal{C}^F(X)$. We can formulate our main theorem (see Theorem 4.5 for a precise formulation).

Main Theorem: The functor $\mathcal{C}^N(X) \to \mathcal{E}$ which assigns $(X_S, \pi_S^*(V))$ to V, where S is the maximal étale subcategory the subcategory $\langle V \rangle$ spanned by

$V \in \mathrm{Obj}(\mathcal{C}^N(X))$, identifies the representation category of $\mathrm{Ker}(r^{\acute{e}t}, r^F)$ with $\mathcal{E}$.

We now describe our method of proof. We proceed in two steps. As mentioned above,the homomorphism $r^{\acute{e}t} : \pi^N(X,x) \to \pi^{\acute{e}t}(X,s)$ is surjective. We denote its kernel by $L(X,x)$ and determine its representation category in section 3. The computation is based on two results. The first one of geometric nature asserts that sections of an F-finite bundle can be computed on any principal bundle $X_S \to X$ with finite étale group scheme (see Proposition 3.2). The second one is the key to the categorial work and comes from [3, Theorem 5.8]. (For the reader's convenience, we give a short account of the categorial statement in Appendix A). It gives a criterion for the exactness of a sequence of affine group schemes

$$1 \to L \to G \to A \to 1$$

in terms of their representation categories. Roughly speaking, assuming the exactness at L and A then the exactness at G holds if and only if the following conditions hold: (i) a representation of G becomes trivial when restricted to L if and only if it comes from a reprsentation of A; (ii) for a representation V of G considered as representation of L, its subspace of L-invariants is invariant under G; (iii) each representation of L is embedable into the restriction to L of a representation of G.

We show that the category $\mathrm{Rep}(L(X,x))$ of (finitely dimensional) representations of $L(X,x)$ is equivalent to the category $\mathcal{D}$, whose objects are pairs (X_S, V) where $X_S \to X$ is a principal bundle under an étale finite group scheme and V is a F-finite bundle on X_S. Morphisms are defined naturally via Proposition 3.2.

By definition of $L(X,x)$, the kernel of $(r^{\acute{e}t}, r^F)$ is the kernel of restriction $r^F|_L : L(X,x) \to \pi^F(X,x)$ of r^F to $L(X,x)$. The second step consists in showing that the category $\mathcal{E}$ constructed in Section 4 is equivalent to the representation category of the kernel of $r^F|_L$. The proof is based on the strengthening of Proposition 3.2, namely Proposition 3.6 and Proposition 4.6 as well as the criterion mentioned above.

Beyond the technicalities of the proof, let us remark that any finite k-group scheme G has two natural quotients: its maximal étale quotient $G^{\acute{e}t}$ and its maximal local quotient G^F. The kernel $G^0 := \mathrm{Ker}(G \to G^{\acute{e}t})$ is the 1-component of G, in particular is local. If G is abelian, the morphism $G^0 \to G^F$ is an isomorphism, and then G is the product of $G^{\acute{e}t}$ with G^F. In general, $G^0 \to G^F$ is surjective. The article here deals in some sense with the prosystem of the kernels of $G^0 \to G^F$.

Acknowledgements: Pierre Deligne sent us his enlightening example which we reproduced in Remark 2.3, 2). It allowed us to correct the main definition of our category $\mathcal{E}$ (Section 5)) which was wrongly stated in the first version of this article. We profoundly thank him for his interest, his encouragement and his help. We also warmly thank Michel Raynaud for answering all our questions on his work on theta characteristics on curves and on their Jacobians. Finally we thank the referee for careful reading, accurate remarks and helpful suggestions.

2. Nori's category

Throughout this work we shall fix a proper reduced scheme X over a perfect field k, which is *connected* in the sense that $H^0(X, \mathcal{O}_X) = k$. We assume that $X(k) \neq \emptyset$ and fix a rational point $x \in X(k)$.

In [8], [9], Nori defines a category of "essentially finite vector bundles" which we recall now. A vector bundle on X is called *semi-stable of degree* 0 if it is semi-stable of degree 0 while restricted to each proper curve in X. This is a full subcategory of the category $\mathrm{Qcoh}(X)$ of quasi-coherent sheaves on X, is abelian [8, Lemma 3.6] and will be denoted by $\mathcal{S}(X)$. A vector bundle V on X is called *finite* if there are polynomials $f \neq g$ whose coefficients are non-negative integers such that $f(V)$ and $g(V)$ are isomorphic. Nori proves that finite bundles are semi-stable of degree 0 [8, Corollary 3.5] and that the full abelian subcategory of $\mathcal{S}(X)$, consisting of those bundles which are subquotients in $\mathcal{S}(X)$ of a direct sum of finite bundles, is a k-linear abelian rigid tensor category. We shall denote this category by $\mathcal{C}^N(X)$ and call its objects *Nori finite bundles*.

The fiber functor at x (where by assumption $\kappa(x) = k$)

$$|_x : \mathcal{C}^N(X) \to \mathrm{Vect}_k,\ V \mapsto V|_x := V \otimes_{\mathcal{O}_X} \kappa(x) \tag{2.1}$$

with values in the category of finite dimensional k-linear vector spaces, implies that $\mathcal{C}^N(X)$ is a Tannaka category. We denote by $\pi^N(X, x)$ the corresponding Tannaka group scheme over k. Tannaka duality ([1, Theorem 2.11]) yields an equivalence of categories

$$\mathcal{C}^N(X) \xrightarrow{|_x \ \cong} \mathrm{Rep}(\pi^N(X, x)). \tag{2.2}$$

We denote by η the inverse functor

$$\eta : \mathrm{Rep}(\pi^N(X, x)) \to \mathcal{C}^N(X). \tag{2.3}$$

Recall that for an affine group scheme G over k, a k-morphism $j : P \to X$ is said to be *a principal G-bundle on X* if

(i) j is a faithfully flat affine morphism
(ii) $\phi : P \times G \to P$ defines an action of G on P such that $j \circ \phi = j \circ p_1$
(iii) $(p_1, \phi) : P \times G \to P \times_X P$ is an isomorphism.

Given such a principal G-bundle P one associates to it an exact tensor functor

$$\eta_P : \mathrm{Rep}(G) \to \mathrm{Qcoh}(X) \tag{2.4}$$

as follows. For each representation V of G, one has the diagonal action of G on the trivial bundle $\mathcal{O}_P \otimes_k V$. Using Grothendieck flat descent [4, Exposé VIII], one obtains a vector bundle $\eta_P(V)$ on X by taking the G invariants of $\mathcal{O}_P \otimes_k V$, denoted by $P \times^G V$. Conversely, consider the regular representation of G in $k[G]$ given by $(gf)(h) = f(hg),\ g, h \in G, h \in k[G]$. Then a functor $\eta : \mathrm{Rep}(G) \to$

$\mathrm{Qcoh}(X)$ yields a principal G-bundle on X, which is the spectrum of the $\mathcal{O}_X$-algebra $\eta(k[G])$. These two constructions are inverse to each other.

Consider the regular representation of $\pi^N(X,x)$ in $k[\pi^N(X,x)]$. Then the discussion above applied to functor η in (2.3) yields a (universal) principal $\pi^N(X,x)$-bundle $\widetilde{\pi} : \widetilde{X} \to X$ together with the identity $\widetilde{\pi}^{-1}(x) = \pi^N(X,x)$. The unit element of $\pi^N(X,x)$ yields a distinguished rational point of $\widetilde{X}$ lying above x.

The projection $\widetilde{\pi} : \widetilde{X} \to X$ is in fact pro-finite in the following sense. Let S be an abelian tensor full subcategory of $\mathcal{C}^N(X)$ generated by finitely many objects $S_i, i = 1, \dots r$. That is, objects of S are subquotients of direct sums of tensor products of copies of S_i and $S_i^\vee$ (the dual bundle to S_i). In the sequel we shall simply call S a *finitely generated tensor subcategory* of $\mathcal{C}^N(X)$. S is a Tannaka category by means of the fibre functor at x and denote its Tannaka group by $\pi(X,S,x)$. The discussion above applied to the forgetful functor $\eta_S : S \to \mathrm{Qcoh}(X)$ yields a $\pi(X,S,x)$-principal bundle $\pi_S : X_S \to X$ and a rational point x_S lying above x. Then $\pi(X,S,x)$ is a finite group scheme, which is a quotient of $\pi^N(X,x)$ and

$$\pi^N(X,x) = \varprojlim_S \pi(X,S,x), \widetilde{X} = \varprojlim_S X_S,\ \widetilde{\pi} = \varprojlim_S \pi_S, \quad \widetilde{x} = \varprojlim_S x_S, \tag{2.5}$$

where S runs in the pro-system of finitely generated full abelian tensor subcategories of $\mathcal{C}^N(X)$. Moreover the scheme X_S is connected:

$$H^0(X_S, \mathcal{O}_{X_S}) = k \tag{2.6}$$

Furthermore, π_S is universal in the following sense:

$$V \in \mathrm{Obj}(S) \Longleftrightarrow \pi_S^*(V) \text{ trivializable.} \tag{2.7}$$

Indeed, if $\pi_S^* V$ is trivializable on X_S, then the injective map $V \hookrightarrow \pi_{S*}\pi_S^* V \cong \pi_{S*}\mathcal{O}_{X_S}^{\oplus d}$, where d is the rank of V, shows that $V \in \mathrm{Obj}(S)$. Conversely, for $V \in S$ the construction in (2.4) shows that $\pi^* V$ is trivializable on X_S.

Finally we notice that $\pi^N(X,x)$ respects base change for algebraic extensions of k, that is

$$\pi^N(X \times_k K, x \times_k K) \cong \pi^N(X,x) \times_k K \tag{2.8}$$

for any algebraic extension $K \supset k$, in particular for $K = \bar{k}$. We refer to [9, Chapters I,II] for the exposition above.

For a finite bundle V, denote by $\langle V \rangle$ the tensor subcategory generated by V.

Definition 2.1. An *étale finite* bundle is a Nori finite bundle for which $\pi(X, \langle V \rangle, x)$ is étale (equivalently is smooth). If k has characteristic $p > 0$, an F-*finite* bundle is a Nori finite bundle for which $\pi(X, \langle V \rangle, x)$ is local. We denote by $\mathcal{C}^{ét}(X)$, resp. $\mathcal{C}^F(X)$, the full tensor subcategory of $\mathcal{C}^N(X)$ of étale, resp. F-, finite bundles.

The categories $\mathcal{C}^{ét}(X)$ and $\mathcal{C}^F(X)$ are both abelian tensor full subcategories, thus via the fiber functor at x they yield Tannaka k-group schemes $\pi^{ét}(X,x)$ and

$\pi^F(X,x)$, respectively. Furthermore, one has

$$\mathcal{C}^{\acute{e}t}(X) \cap \mathcal{C}^F(X) = \{\text{trivial objects}\} \tag{2.9}$$

where {trivial objects} means the full subcategory of $\mathcal{C}^N(X)$ consisting of trivializable bundles.

Following the method of [9, II, Proposition 5] we obtain the following lemma.

Lemma 2.2. *The group schemes $\pi^{\acute{e}t}(X,x)$ and $\pi^F(X,x)$ respect base change for algebraic extensions of k, that is 2.8 holds with N replaced by ét and F.*

Remark 2.3. 1) It is shown in [7, Section 2] that a Nori finite bundle V is F-finite if and only if there is a natural number $N > 0$, such that $(F_{\text{abs}}^N)^*(V)$ is trivial, where F_{abs} is the absolute Frobenius.

2) If $S \subset \mathcal{C}^N(X)$ is a finite subcategory, with $\pi(X,S,x)$ étale, then X_S is still proper, reduced and connected. However, if $\pi(X,S,x)$ is finite but not étale, X_S is still proper and connected [9, Chapter II, Proposition 3], but not necessarily reduced. Indeed there are principal bundles $Y \to X$ under a finite local group scheme, such that the total space Y is not reduced. We reproduce here an example due to P. Deligne. Let k be algebraically closed of characteristic $p > 0$, and let $X \subset \mathbb{P}^2$ be the union of a smooth conic X' and a tangent line X''. Thus $X' \cap X''$ is isomorphic to Spec $k[\epsilon]/(\epsilon^2)$ as a k-scheme. One constructs $\pi : Y \to X$ by gluing the trivial μ_p-torsors $X' \times_k \mu_p$ to $X'' \times_k \mu_p$ along a non-constant section of Spec $k[\epsilon]/(\epsilon^2) \times_k \mu_p \to$ Spec $k[\epsilon]/(\epsilon^2)$. For example, one may take the non-constant section Spec $k[\epsilon]/(\epsilon^2) \to$ Spec $k[\epsilon]/(\epsilon^2) \times_k \mu_p$ defined by $k[\xi,\epsilon]/(\xi^p - 1, \epsilon^2) \to k[\epsilon]/(\epsilon^2), \xi \mapsto 1 + \epsilon$. Then Y is projective, non-reduced, and yet fulfills the condition $H^0(Y, \mathcal{O}_Y) = k$.

If X_S is not reduced, there is no good notion of semi-stable vector bundles on X_S. However, for later use in this article, we introduce a category $\mathcal{F}(X_S)$ on the principal $\pi(X,S,x)$-bundle $\pi_S : X_S \to X$ where $S \subset \mathcal{C}^N(X)$ is a finitely generated full tensor subcategory. $\mathcal{F}(X_S)$ will play on X_S the rôle $\mathcal{C}^N(X)$ plays on X.

Definition 2.4. Let $S \subset \mathcal{C}^N(X)$ be a finitely generated abelian tensor full subcategory. Define $\mathcal{F}(X_S) \subset \text{Qcoh}(X_S)$ to be the full subcategory of $\text{Qcoh}(X_S)$, the objects of which are quasi-coherent sheaves V on X_S such that $(\pi_S)_*V \in \mathcal{C}^N(X)$.

Notice that $\mathcal{F}(X_S)$ is an abelian category. In fact, for a morphism $f : V \to W$ in $\mathcal{F}(X_S)$, by the exactness of $(\pi_S)_*$, we have $(\pi_S)_*\text{Ker}f = \text{Ker}((\pi_S)_*f) \in \mathcal{C}^N(X)$, as $\mathcal{C}^N(X)$ is full in $\text{Qcoh}(X)$, and the same holds for $\text{im}f$. We will show in Proposition 2.7 below that $\mathcal{F}(X_S)$ is k-linear abelian rigid tensor category and its objects are vector bundles on X_S, and when $\pi(X,S,x)$ is reduced $\mathcal{F}(X_S)$ coincides with $\mathcal{C}^N(X_S)$.

Let $S \subset S' \subset \mathcal{C}^N(X)$ be finitely generated abelian tensor full subcategories. Then one has the following commutative diagram

$$\begin{tikzcd} X_{S'} \arrow[rr, "\pi_{S',S}"] \arrow[dr, "\pi_{S'}"'] & & X_S \arrow[dl, "\pi_S"] \\ & X & \end{tikzcd} \tag{2.10}$$

Further one has a surjective (hence faithfully flat) homomorphism

$$G_{S'} := \pi(X, S', x) \to \pi(X, S, x) =: G_S \tag{2.11}$$

Lemma 2.5. *The morphism $\pi_{S',S} : X_{S'} \to X_S$ is a principal bundle under a group $G_{S',S}$ which is the kernel of the homomorphism* (2.11)*.*

Proof. It is well-known that the morphism $G_{S'} \to G_S$ is a principal bundle under the group $G_{S',S}$. In fact, the map $G_{S'} \times_{G_S} G_{S'} \to G_{S'} \times G_{S',S}$ is given by $(g, h) \mapsto (g, g^{-1}h)$ and its inverse is given by $(g, k) \mapsto (g, gk)$, where $g, h \in G_{S'}$, $k \in G_{S',S}$. Now apply the fibre functor $\eta_{S'}$ to the corresponding function algebras $k[G_{S'}]$ and $k[G_S]$ we obtain the required isomorphism

$$X_{S'} \times_{X_S} X_{S'} \cong X_{S'} \times G_{S',S}$$

(recall that $\eta_{S'}(k[G_{S'}])$ is the $\mathcal{O}_X$-algebra that determines $X_{S'}$, similarly $\eta_{S'}(k[G_S]) = \eta_S[k(G_S)]$ is the one that determines X_S). $\square$

The principal bundle $\pi_{S',S} : X_{S'} \to X_S$ yields a tensor functor

$$\eta_{S',S} : \mathrm{Rep}(G_{S',S}) \to \mathrm{Qcoh}(X_S), \quad \eta_{S',S}(V) := X_{S'} \times^{G_{S',S}} V. \tag{2.12}$$

Lemma 2.6. *The functor $\eta_{S',S}$ in* (2.12) *is fully faithful and exact. Consequently $G_{S',S}$ is isomorphic to the Tannaka group of the category* $\mathrm{im}(\eta_{S',S})$*.*

Proof. It is enough to check that $\eta := \eta_{S',S}$ is full, i.e., any morphism $\eta(V) \to \eta(W)$ in $\mathrm{Qcoh}(X_S)$ is induced by a morphism $V \to W$ in $\mathrm{Rep}(G_{S',S})$. This is equivalent to showing $H^0(X_S, \eta(V)) \cong V^{G_{S',S}}$ for any $V \in \mathrm{Rep}(G_{S',S})$. Recall that $\eta(V) := X_{S'} \times^{G_{S',S}} V$. Thus

$$H^0(X_S, \eta(V)) \cong H^0(X_{S'}, \mathcal{O}_{X_{S'}} \otimes_k V)^{G_{S',S}} \cong V^{G_{S',S}} \tag{2.13}$$

since $H^0(X_{S'}, \mathcal{O}_{X_{S'}}) = k$. $\square$

Proposition 2.7. *The category $\mathcal{F}(X_S)$ defined in Definition 2.4 is a Tannaka category, whose objects are vector bundles.*

Proof. We first show that for any $V \in \mathcal{F}(X_S)$, there are $W_1, W_2 \in \mathcal{C}^N(X)$ and a morphism $f : \pi_S^* W_1 \to \pi_S^* W_2$ in $\mathrm{Qcoh}(X_S)$ such that $V = \mathrm{coker}(f)$. One takes $W_2 := (\pi_S)_* V$ which by definition lies in $\mathcal{C}^N(X)$, and defines V_1 to be the kernel of the surjection $\pi_S^* W_2 \twoheadrightarrow V$. Then $W_1 := (\pi_S)_* V_1 \in \mathcal{C}^N(X)$ and $f : \pi_S^* W_1 \twoheadrightarrow V_1 \hookrightarrow \pi_S^* W_2$ satisfying $\mathrm{coker}(f) = V$.

Let $S' \subset \mathcal{C}^N(X)$ be the full tensor subcategory generated by W_1, W_2 and S. Then the pullbacks of $\pi_S^* W_1$ and $\pi_S^* W_2$ to $X_{S'}$ (under $\pi_{S',S} : X_{S'} \to X_S$) become trivial, thus $\pi_S^* W_1$ and $\pi_S^* W_2$ are in the image of

$$\eta_{S',S} : \mathrm{Rep(G_{S',S})} \to \mathrm{Qcoh(X_S)}. \tag{2.14}$$

By Lemma 2.6, the functor $\eta_{S',S}$ is fully faithful, thus $V = \mathrm{coker}(f)$ is also in the image of $\eta_{S',S}$. In particular, V is a vector bundle.

It is now easy to check that $\mathcal{F}(X_S)$ is a k-linear abelian rigid tensor category. Since X_S has a rational point x_S, $\mathcal{F}(X_S)$ is a Tannaka category. □

Next we show $\mathcal{F}(X_S) = \mathcal{C}^N(X_S)$ when $\pi(X, S, x)$ is reduced. To this aim, recall that a bundle V on X is said to be *strongly semi-stable of degree* 0 if for any non-singular projective curve C and any morphism $f : C \to X$, the pullback f^*V is semi-stable of degree 0 on C. It is known that strongly semi-stable bundles of degree 0 on X form a k-linear Tannaka full subcategory of $\mathrm{Qcoh}(X)$ [10, Theorem 3.23]. On the other hand as Nori finiteness is preserved under pull-back by any $f : C \to X$, we see that $\mathcal{C}^N(X)$ is a full subcategory of the category of strongly semi-stable bundles of degree 0.

Lemma 2.8. *If $\pi(X, S, x)$ is a smooth finite group scheme, and if $V \in \mathcal{C}^N(X_S)$, then $\pi_S^*(\pi_{S_*} V) \in \mathcal{C}^N(X_S)$ and $W := \pi_{S_*} V$ is strongly semi-stable of degree* 0.

Proof. Let $G := \pi(X, S, x)$, $\pi := \pi_S$, $Y := X_S$ and $y := x_S$. Since strong semi-stability and Nori finiteness are compatible with base change by algebraic field extensions of k, we can assume that $k = \bar{k}$. Consider

$$\begin{tikzcd} Y \times_k G \arrow[r, "\cong"] \arrow[rr, bend left, "\mu"] \arrow[dr, "p_1"'] & Y \times_X Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\pi"] \\ & Y \arrow[r, "\pi"'] & X \end{tikzcd} \tag{2.15}$$

where $\mu : Y \times_k G \to Y$ is the action of G, p_1 is the projection to the first factor and $Y \times_k G \cong Y \times_X Y$ is induced by (p_1, μ). Then

$$\pi^*\pi_* V \cong p_{1*}\mu^* V = \bigoplus_{g \in G(k)} V_g \tag{2.16}$$

where V_g is the translation of V by g. Thus $\pi^*\pi_* V \in \mathcal{C}^N(Y)$.

To show that $W := \pi_* V$ is strongly semi-stable, we consider the fiber square

$$\begin{tikzcd} Y_C \arrow[r, "g"] \arrow[d, "\pi_C"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\pi"] \\ C \arrow[r, "f"'] & X \end{tikzcd}$$

So in particular, π_C is still a principal bundle under G. Since π is finite, one has

$$f^*W = f^*(\pi_* V) = \pi_{C*}(g^*V).$$

Denote $V_C := g^*V$. Since $V_C \in \mathcal{C}^N(Y_C)$, the discussion above shows that $\pi_C^*(f^*W) = \pi_C^*(\pi_{C*}V_C) \in \mathcal{C}^N(Y_C)$. In particular, $\pi_C^*(f^*W)$ is semi-stable of degree 0, which implies that f^*W is semi-stable of degree 0. Indeed, for any subbundle $U \subset f^*W$, the bundle π_C^*U is a subbundle of $\pi_C^*(f^*W)$, hence has negative degree, consequently the degree of U is also negative. □

Theorem 2.9. *Assume that $\pi(X, S, x)$ is a smooth finite group scheme. Then $\mathcal{F}(X_S) = \mathcal{C}^N(X_S)$ and there is an exact sequence of group schemes*

$$1 \to \pi^N(X_S, x_S) \to \pi^N(X, x) \to \pi(X, S, x) \to 1 \tag{2.17}$$

Proof. By Proposition 2.7, $\mathcal{F}(X_S) \subset \mathcal{C}^N(X_S)$. We prove the inverse inclusion. Thus let $V \in \mathcal{C}^N(X_S)$. By Lemma 2.8, $W := \pi_{S_*}V$ semi-stable of degree 0 and $\pi_S^*W \in \mathcal{C}^N(X_S)$. Let $\langle W\rangle \cup S$ be the full tensor subcategory generated by W and objects of S in the Tannaka category of strongly semi-stable bundles of degree 0 and denote by G its Tannaka group with respect to the fiber functor at x. To show $W \in \mathcal{C}^N(X)$, it suffices to show that G is a finite group scheme (see construction in (2.4)).

The full subcategory of $\langle W\rangle \cup S$ whose objects become trivial when pulled-back to X_S is precisely S (see (2.7)). The functor $\pi_S^* : \langle W\rangle \cup S \to \langle \pi_S^*W\rangle$ yields a sequence of homomorphisms of group schemes

$$1 \to \pi(X_S, \langle \pi_S^*W\rangle, x_S) \to G \to \pi(X, S, x) \to 1 \tag{2.18}$$

which we claim to be exact.

The surjectivity of $G \to \pi(X, S, x)$ and the injectivity of $\pi(X_S, \langle \pi_S^*W\rangle, x_S) \to G$ follow from the definition and A.1, (i), (ii). We show the exactness at G, using Theorem A.1, (iii). Condition (a) in A.1, (iii), follows from (2.7).

We check condition (c). Let $M \in \langle \pi_S^*W\rangle$. By definition, M is a subquotient of π_S^*N, $N \in \langle W\rangle \cup S$. Thus $\pi_{S_*}M$ is a subquotient of $\pi_{S_*}\pi_S^*N = N \otimes \pi_{S_*}\pi_S^*\mathcal{O}_{X_S} \in \langle W\rangle \cup S$. Hence $\pi_{S_*}M$ lies in $\langle W\rangle \cup S$. Now we have the required surjective map $\pi_S^*(\pi_{S_*}M) \to M$.

As for (b) we use projection formula

$$H^0(X_S, \pi_S^*N) = H^0(X, \pi_{S_*}\pi_S^*N) = \mathrm{Hom}_{\mathcal{O}_X}(\pi_{S_*}\mathcal{O}_{X_S}^\vee, N) = \bigoplus_{i=1}^r k \cdot \phi_i \tag{2.19}$$

where $\phi_i : (\pi_{S_*}\mathcal{O}_{X_S})^\vee \to N$. Let $N_0 = \sum_i \mathrm{im}(\phi_i) \subset N$. Then N_0 is in S and any morphism $\phi : (\pi_{S_*}\mathcal{O}_{X_S})^\vee \to N$ has image in N_0. By comparing the ranks, we see that $\pi_S^*N_0$ is the maximal trivial subbundle in π_S^*N.

Thus the sequence in (2.18) is exact, hence G is finite. The exactness of (2.17) follows from the exactness of (2.18) by taking the projective limit on S. □

Remarks 2.10. The group scheme $\pi^{ét}(X,x)$ can be considered as the k-linearization of Grothendieck's fundamental group ([4, Exposé V]), which we recall now. Grothendieck considers the category of finite étale coverings of X with morphisms being X-morphisms. A geometric point $\bar{x} \in X(K)$ $(K=\bar{K})$ defines a fiber functor from this category to the category of finite sets: $\mathcal{G}r : (Y \xrightarrow{\pi} X) \mapsto \pi^{-1}(\bar{x})$. The fundamental group $\pi_1(X,\bar{x})$ of the connected scheme X with base point $\bar{x}$ is defined to be the automorphism group of the fiber functor. This is a pro-finite group, hence has a natural topology in which subgroups of finite index are open and form a basis of topology at the unit element. The main theorem claims an equivalence between the category of finite étale coverings and the category of finite sets with continuous action of $\mathcal{G}r$ (finite sets are endowed with discrete topology). Further there exists a pro-finitie étale covering $\hat{\pi} : \widehat{X} \to X$ which is universal in the sense that

$$\mathrm{Mor}_X(\widehat{X}, Y) \cong \mathcal{G}r(Y) \tag{2.20}$$

for any finite covering $Y \to X$ ([4, Theorem V.4.1]). One recovers the group $\pi_1(X,\bar{x})$ as the fiber $\hat{\pi}^{-1}(\bar{x})$. Notice that it suffices to check (2.20) for Galois coverings $Y \to X$.

Assume that k is moreover algebraically closed. Then the fundamental group $\pi_1(X,x)$ with base point at x is called the geometric fundamental group. Upon the algebraically closed field k, a reduced finite group scheme is uniquely determined by its k-points, which is a finite group. Therefore for any $S \subset \mathcal{C}^{ét}(X)$, $\pi_S : X_S \to X$ is a Galois covering of X under the group $\pi(X,S,x)(k)$. Conversely, any Galois covering $Y \xrightarrow{\pi} X$ under a finite group H can be considered as a principal bundle under the constant (finite) group scheme defined by H. It is easy to check that the covering $\pi^{ét} : X_{\mathcal{C}^{ét}(X)} \to X$ given in (2.5) satisfies the universal property (2.20). We conclude that the group of k-points of $\pi^{ét}(X,x)$ is isomorphic to $\pi_1(X,x)$.

If k is perfect but not algebraically closed, take $X = \mathrm{Spec}(k)$ with the rational point point $x = X \in X(k)$. Then $\mathcal{C}^{ét}(X)$ is equivalent via the fibre functor to Vect_k, and consequently $\pi^{ét}(X,x) = \{1\}$. On the other hand, Grothendieck's fundamental group is then $\mathrm{Gal}(\bar{k}/k)$, which is highly nontrivial. However, according to Lemma 2.2 we have an isomorphism of $\bar{k}$-group schemes

$$\pi^{ét}(X \times_k \bar{k}, x \times_k \bar{k}) \xrightarrow{\cong} \pi^{ét}(X,x) \times_k \bar{k}.$$

So we conclude in general

$$\pi_1(X \times_k \bar{k}, \bar{x}) = \pi^{ét}(X,x)(\bar{k}).$$

The aim of our article is to understand the relationship between the groups $\pi^N(X,x)$, $\pi^{ét}(X,x)$ and $\pi^F(X,x)$. We first notice that Theorem A.1, (i), applied to the full subcategories $\mathcal{C}^{ét}(X) \to \mathcal{C}^N(X)$, resp. $\mathcal{C}^F(X) \to \mathcal{C}^N(X)$, shows that the restriction homomorphisms $\pi^N(X,x) \xrightarrow{r^{\acute{e}t}} \pi^{\acute{e}t}(X,x)$, resp. $\pi^N(X,x) \xrightarrow{r^F} \pi^F(X,x)$ are faithfully flat.

Notation 2.11. We set $L(X,x) = \mathrm{Ker}(\pi^N(X,x) \to \pi^{\acute{e}t}(X,x))$.

3. The representation category of the difference between Nori's fundamental group scheme and it's étale quotient

We continue to fix X/k and $x \in X(k)$ as in section 2. The purpose of this section is to determine the representation category of the kernel L of the map $\pi^N(X,x) \to \pi^{\acute{e}t}(X,x)$. To this aim, we first observe the following.

Lemma 3.1. *The group scheme $\pi^{\acute{e}t}(X,x)$ is the largest quotient pro-finite group scheme of $\pi^N(X,x)$ which is reduced.*

Proof. For $S \subset \mathcal{C}^N(X)$ an abelian tensor full subcategory generated by finitely many objects we set

$$S^{\acute{e}t} := S \cap \mathcal{C}^{\acute{e}t}(X), \tag{3.1}$$

i.e. the full subcategory consisting of objects in $\mathcal{C}^N(X)$, isomorphic both to an object in S and an object in $\mathcal{C}^{\acute{e}t}(X)$. Thus $\pi(X, S^{\acute{e}t}, x)$ is a reduced quotient of $\pi(X,S,x)$. We claim that this is the largest quotient of $\pi(X,S,x)$.

Tannaka duality shows that any quotient map $\pi(X,S,x) \twoheadrightarrow H$ of group schemes over k yields a fully faithful functor from $\mathrm{Rep}(H)$ to $\mathcal{C}^N(X)$ with image, say (H), lying in S, and consequently yielding an H-principal bundle $\pi_{(H)} : X_{(H)} \to X$ which is proper, connected, with a rational point $x_{(H)}$ mapping to x, so that $V \in (H)$ if and only if $\pi^*_{(H)}(V)$ is trivial. If H is reduced then (H) consists only of étale finite bundles, thus $(H) \subset S^{\acute{e}t}$. Hence $\pi(X,S,x) \to H$ factors through $\pi(X,S,x) \to \pi(X,S^{\acute{e}t},x) \to H$. This shows that $\pi(X,S^{\acute{e}t},x)$ is the maximal reduced quotient of $\pi(X,S,x)$. Now the claim of Lemma follows by passing to the limit. □

In the rest of this section, S will denote a finitely generated tensor subcategory of $\mathcal{C}^{\acute{e}t}(X)$. Thus $\pi_S : X_S \to X$ is étale and X_S is reduced.

Proposition 3.2. *Let X be a proper reduced connected scheme defined over a perfect field k. Let V be an F-finite bundle on X. Then*

$$\pi_S^* : H^0(X,V) \to H^0(X_S, \pi_S^* V)$$

is an isomorphism.

Proof. To simplify the notations, we set $Y := X_S$, $\pi := \pi_S$. Let V_0 be the maximal trivial subobject of V. Since π is étale, the bundles associated to $\pi_*\mathcal{O}_Y$, and therefore to $(\pi_*\mathcal{O}_Y)^\vee$, are étale finite. The image under a morphism of

$(\pi_*\mathcal{O}_Y)^\vee$ to V is therefore at the same time étale- and F-finite, hence (see (2.9)) lies in the maximal trivial subobject V_0 of V. By projection formula we have

$$\begin{aligned} H^0(Y,\pi^*V) = H^0(X,(\pi_*\mathcal{O}_Y)\otimes V) &\cong \mathrm{Hom}_X((\pi_*\mathcal{O}_Y)^\vee, V) \\ \subset \mathrm{Hom}_X((\pi_*\mathcal{O}_Y)^\vee, V_0) &\cong H^0(Y,\pi^*V_0) = H^0(X,V_0). \end{aligned} \tag{3.2}$$

as $H^0(Y,\mathcal{O}_Y)=k$ by (2.6). Hence

$$H^0(X,V_0)\subset H^0(X,V)\subset H^0(Y,\pi^*V)\subset H^0(Y,\pi^*V_0) = H^0(X,V_0), \tag{3.3}$$

so one has everywhere equality. $\square$

Denote the directed system of finitely generated tensor subcategories of $\mathcal{C}^{\acute{e}t}(X)$ with respect to the inclusion

$$\mathcal{S}^{\acute{e}t} = \left\{ S\subset \mathcal{C}^{\acute{e}t}(X),\ \text{finitely generated}\right\}. \tag{3.4}$$

Let $\widetilde{X}^{\acute{e}t}\to X$ denote the universal pro-étale covering associated to $\pi^{\acute{e}t}(X,x)$, defined similarly as in (2.5). Thus

$$\widetilde{X}^{\acute{e}t} = \varprojlim_{S\in\mathcal{S}^{\acute{e}t}} X_S, \quad \widetilde{\pi}_S:\widetilde{X}\twoheadrightarrow X_S. \tag{3.5}$$

By means of Proposition 3.2 we have the following isomorphism for any $V\in \mathcal{C}^F(X_S)$,

$$H^0(X_S,V)\cong H^0(\widetilde{X},\widetilde{\pi}_S^*V). \tag{3.6}$$

Definition 3.3. The category $\mathcal{D}$ has for objects pairs (X_S,V) where $S\in\mathcal{S}^{\acute{e}t}$, $V\in\mathcal{C}^F(X_S)$, and for morphisms

$$\mathrm{Hom}((X_{S_1},V),(X_{S_2},W)) := \mathrm{Hom}_{\widetilde{X}}(\widetilde{\pi}_{S_1}^*V,\widetilde{\pi}_{S_2}^*W).$$

For any two abelian tensor full subcategories $S_1,S_2\in\mathcal{S}^{\acute{e}t}$, denote $S_1\cup S_2$ the abelian tensor full subcategory generated by objects of S_1 and S_2. One has $S_1\cup S_2\in\mathcal{S}^{\acute{e}t}$. We also extend this notation for several subcategories.

Proposition 3.4. *The category $\mathcal{D}$ is an abelian, rigid k-linear tensor category, with the tensor structure defined by*

$$(X_{S_1},V)\otimes(X_{S_2},W) = (X_{S_1\cup S_2}, \pi^*_{S_1\cup S_2,S_1}(V)\otimes \pi^*_{S_1\cup S_2,S_2}(W)) \tag{3.7}$$

The functor

$$\omega:\mathcal{D}\to \mathrm{Vect}_k,\ (X_S,V)\mapsto V|_{x_S} \tag{3.8}$$

makes $\mathcal{D}$ a Tannaka category.

Proof. We define the kernel, image and cokernel of a homomorphism $f:(X_{S_1},V)\to (X_{S_2},W)$ in $\mathcal{D}$ as follows. By means of (3.6), one has an isomorphism

$$\mathrm{Hom}_{\widetilde{X}^{\acute{e}t}}(\widetilde{\pi}_{S_1}^*V,\widetilde{\pi}_{S_2}^*W)\cong \mathrm{Hom}_{X_S}(\pi^*_{S_1\cup S_2,S_1}V,\pi^*_{S_1\cup S_2,S_2}W), \tag{3.9}$$

under which f corresponds to f_S. Then the kernel, image and cokernel of f are defined to be the kernel, image and cokernel of f_S respectively. It is clear that $\mathcal{D}$ is an abelian category.

The unit object is $(X, \mathcal{O}_X)$, the endomorphism ring of the unit object is thus k. The dual object is given by $(X_S, V)^\vee = (X_S, V^\vee)$. □

We observe that $(X_S, \mathcal{O}_{X_S})$ is isomorphic to $(X, \mathcal{O}_X)$ in $\mathcal{D}$. More generally, for $S_1 \in \mathcal{S}^{ét}$,

$$(X_{S_1}, V) \text{ is isomorphic to } (X_{S_1\cup S_2}, \pi^*_{S_1\cup S_2, S_1} V) \text{ in } \mathcal{D} \text{ for all } S_2 \in \mathcal{S}^{ét}. \tag{3.10}$$

For $V \in \mathcal{C}^N(X)$, the category $\langle V\rangle^{ét}$ is defined as in (3.1). According to Lemma 2.5, for $S' = \langle V\rangle$ and $S = \langle V\rangle^{ét}$, $X_{\langle V\rangle} \to X_{\langle V\rangle^{ét}}$ is a principal bundle under the group $H = \mathrm{Ker}(G_{\langle V\rangle} \to G_{\langle V\rangle^{ét}})$, which, according to the proof of Lemma 3.1, is a local group. Therefore $\pi^*_{\langle V\rangle^{ét}}(V)$ is an F-finite bundle on $X_{\langle V\rangle^{ét}}$. Define the functor

$$q : \mathcal{C}^N(X) \to \mathcal{D}, \quad V \mapsto (X_{\langle V\rangle^{ét}}, \pi^*_{\langle V\rangle^{ét}}(V)). \tag{3.11}$$

Then q is an exact tensor functor which is compatible with the fiber functors ω and $|_x$. Thus q yields a homomorphism of group schemes

$$q^* : G(\mathcal{D}) \to \pi^N(X, x). \tag{3.12}$$

Denote by $G(\mathcal{D})$ the Tannaka group scheme over k with respect to ω. The functor q has the property that $V \in \mathcal{C}^N(X)$ is étale finite if and only if $q(V)$ is trivial in $\mathcal{D}$. Therefore the composition

$$G(\mathcal{D}) \xrightarrow{q^*} \pi^N(X, x) \to \pi^{ét}(X, x) \tag{3.13}$$

is the trivial homomorphism. That is q^* factors though a homomorphism (denoted by the same letter) to L (see Notation 2.11).

Theorem 3.5. *The representation category of the kernel $L(X, x)$ of the homomorphism $r^{ét} : \pi^N(X, x) \to \pi^{ét}(X, x)$ is equivalent to $\mathcal{D}$ by means of the functor q.*

Proof. We show that the sequence of k-group schemes (3.13) is exact. We shall use the criterion given in Theorem A.1, (iii). Condition (a) there is satisfied by (2.8).

Let (X_S, V) be an object in $\mathcal{D}$. Then, by Theorem 2.9, $W := (\pi_S)_* V$ is an object in $\mathcal{C}^N(X)$. Moreover one has a surjection $q(W) \twoheadrightarrow (X_S, V)$ in $\mathcal{D}$. Thus every object of $\mathcal{D}$ is a quotient of the image by q of an object in $\mathcal{C}^N(X)$. Condition (c) of A.1, (iii), is satisfied.

It remains to check condition (b) of A.1, (iii). For $V \in \mathcal{C}^N(X)$ set $S = \langle V\rangle^{ét}$ then $q(V) = (X_S, \pi_S^* V)$. Applying projection formula we obtain

$$H^0(X_S, \pi_S^*(V)) = \mathrm{Hom}_{\mathcal{O}_X}((\pi_{S*}\mathcal{O}_{X_S})^\vee, V) = \bigoplus_{i=1}^r k \cdot \varphi_i, \tag{3.14}$$

where $\varphi_i : (\pi_{S*}\mathcal{O}_{X_S})^\vee \to V$. Let $V_{ét} \subset V$ be the image of

$$\oplus_1^r \varphi_i : \quad \bigoplus_1^r (\pi_{S*}\mathcal{O}_{X_S})^\vee \to V. \tag{3.15}$$

As $(\pi_{S*}\mathcal{O}_{X_S})^\vee$ is étale finite, $V_{ét}$ is étale finite and lies in S, hence $\pi_S^* V_{ét}$ is a trivial bundle by (2.7). Thus $q(V_{ét})$ is a trivial subobject of $q(V)$. We show that it is the largest one. By the definition of $V_{ét}$, one has

$$H^0(X, (\pi_{S*}\mathcal{O}_{X_S}) \otimes V) = H^0(X, (\pi_{S*}\mathcal{O}_{X_S}) \otimes V_{ét}). \tag{3.16}$$

Applying the projection formula again, one has

$$H^0(X, (\pi_{S*}\mathcal{O}_{X_S}) \otimes V_{ét}) = H^0(X_S, \pi_S^*(V_{ét})). \tag{3.17}$$

That is, $q(V_{ét})$ is the maximal trivial subobject of $q(V)$. □

Our next aim is to study the kernel of $r^{ét} \times r^F$. For this we shall need a strengthening of Proposition 3.2.

Proposition 3.6. *Let $S \subset \mathcal{C}^{ét}(X)$ be a finitely generated tensor subcategory. Then the homomorphism of group schemes $\pi^F(X_S, x_S) \to \pi^F(X, x)$ induced by π_S^* is surjective.*

Proof. To simplify notations, we set $Y := X_S$, $\pi := \pi_S$ in the proof. According to Lemma 2.2 if suffices to consider the case where k is algebraically closed. According to Theorem A.1, (i), one has to show that for any $V \in \mathcal{C}^F(X)$ and any inclusion $\varphi : W \hookrightarrow \pi^* V$ in $\mathcal{C}^F(X_S)$, there exists an inclusion $\iota : V_0 \hookrightarrow V$ in $\mathcal{C}^F(X)$, such that $\varphi = \pi^*\iota : \pi^* V_0 \to \pi^* V$. We first assume that W is simple. The bundle V has a decomposition series $V_0 \subset V_1 \subset \ldots \subset V_N = V$ with V_i/V_{i-1} simple. Then there exists an index i such that the image of $\varphi(W)$ in $\pi^*(V_i/V_{i-1})$ is not zero. Thus we may assume that V itself is simple. It suffices now to show that φ is an isomorphism.

Using the adjointness between π_* and π^* we have

$$\begin{aligned} \mathrm{Hom}_{X_S}(W, \pi^*V) \cong \mathrm{Hom}_{X_S}(\pi^*V^\vee, W^\vee) \cong \mathrm{Hom}_X(V^\vee, \pi_*(W^\vee)) \cong \\ \mathrm{Hom}((\pi_*(W^\vee))^\vee, V). \end{aligned} \tag{3.18}$$

Thus φ corresponds to a non-zero morphism $\psi : \pi_*(W^\vee)^\vee \to V$. Since V is simple, ψ is surjective and hence so is $\pi^*\psi : \pi^*\pi_*(W^\vee)^\vee \to \pi^* V$. On the other hand, as in Lemma 2.8, we have

$$\pi^*\pi_* W = \bigoplus_{g \in G(k)} W_g. \tag{3.19}$$

Since W is simple, so are W_g, $g \in G(k)$. This shows that $\pi^* V$, being a quotient of a direct sum of simple objects, is semi-simple. According to Proposition 3.2, $\pi^* V$ has to be simple. Therefore $W = \pi^* V$.

The general case follows by induction on the length of the decomposition of W. □

Corollary 3.7. *The natural map* $\pi^N(X,x) \xrightarrow{r^{ét}\times r^F} \pi^{ét}(X,x)\times\pi^F(X,x)$ *is surjective and in general not an isomorphism.*

Proof. In fact, the surjectivity of $r^{ét}\times r^F$ holds for any pro-finite group, as it holds for finite groups. In our case this can also be seen from the proof of Proposition 3.6.

The claim of Corollary is equivalent to showing that the induced homomorphism $r^F|_L : L(X,x) \to \pi^F(X,x)$ is surjective and not necessarily an isomorphism. This homomorphism is Tannaka dual to the restriction $q|_{\mathcal{C}^F(X)} : \mathcal{C}^F(X) \to \mathcal{D}$, for functor q defined in (3.11), which is the identity functor $q|_{\mathcal{C}^F(X)}(V) = (X,V)$. Now the proof of Proposition 3.6 and the injectivity criterion A.1, (ii), prove the corollary.

It remains to exhibit an example when $r^F|_L$ is not an isomorphism. According to the discussion above, this amounts to finding an F-finite bundle on X_S which does not come from X. By [11], Théorème 4.3.1, if X is a smooth projective curve of genus $g \geq 2$ over an algebraic closed field k of characteristic $p > 0$, then for $\ell \neq p$ prime with $\ell + 1 \geq (p-1)g$, there is a cyclic covering $\pi : Y \to X$ of degree ℓ (thus étale), such that $\mathrm{Pic}^0(Y)/\mathrm{Pic}^0(X)$ is ordinary. Since π is Galois cyclic of order ℓ, it is defined as $\mathrm{Spec}_X(\oplus_0^{\ell-1}L^i)$ for some L étale finite of rank 1 over X and of order ℓ, thus $\pi = \pi_{\langle L\rangle}$ and $Y = X_{\langle L\rangle}$. We conclude that there are p-power torsion rank 1 bundles on $X_{\langle L\rangle}$ which do not come from X. □

4. The representation category of the difference between Nori's fundamental group and the product of its étale and local quotients

The aim of this section is to describe the representation category of the kernel of the homomorphism $r^{ét}\times r^F$. Recall that X is a reduced proper scheme over a perfect field of characteristic $p > 0$ with a rational point $x \in X(k)$ and is connected in the sense that $H^0(X,\mathcal{O}_X) = k$.

In order to determine the representation category $\mathcal{E}$ of the kernel of $r^{ét}\times r^F$ we shall need an auxiliary category $\mathcal{E}(X_{S\cup T})$, where S is a finitely generated tensor full subcategory of $\mathcal{C}^{ét}(X)$ and T is a finitely generated tensor full subcategory of $\mathcal{C}^F(X)$.

Definition 4.1. For S a finitely generated tensor subcategory of $\mathcal{C}^{ét}(X)$ and T a finitely generated tensor full subcategory of $\mathcal{C}^F(X)$, one defines $\mathcal{E}(X_{S\cup T}) \subset \mathcal{F}(X_{S\cup T})$ (for the definition of $\mathcal{F}(X_{S\cup T})$, see Definition 2.4) to be the full subcategory whose objects V have the property that $(\pi_{S\cup T,S})_*V \in \mathcal{C}^F(X_S)$.

Denote the directed system of finitely generated tensor subcategories of $\mathcal{C}^F(X)$ with respect to inclusion

$$\mathcal{T}^\ell := \left\{T \subset \mathcal{C}^F(X),\ \text{finitely generated}\right\}. \tag{4.1}$$

Lemma 4.2. *Let* $S \subset S' \in \mathcal{S}^{ét}$, $T \subset T' \in \mathcal{T}^\ell$ *and* $V \in \mathcal{E}(X_{S\cup T})$. *Then:*

1) *The following commutative diagram is cartesian*

$$\begin{tikzcd} X_{S'\cup T} \arrow[r, "\pi_{S'\cup T,S'}"] \arrow[d, "\pi_{S'\cup T,S\cup T}"'] & X_{S'} \arrow[d, "\pi_{S',S}"] \\ X_{S\cup T} \arrow[r, "\pi_{S\cup T,S}"'] & X_S \end{tikzcd}$$

2) $\mathcal{E}(X_{S\cup T})$ *is a k-linear abelian, rigid tensor category.*
3) $\pi^*_{S\cup T',S\cup T}V \in \mathcal{E}(X_{S\cup T'})$.
4) $\pi^*_{S'\cup T,S\cup T}V \in \mathcal{E}(X_{S'\cup T})$.
5) *The canonical homomorphism*

$$H^0(X_{S\cup T}, V) \to H^0(X_{S'\cup T}, \pi^*_{S'\cup T,S\cup T}V) \tag{4.2}$$

is an isomorphism.

Proof. 1) We first show the following commutative diagram

$$\begin{tikzcd} X_{S\cup T} \arrow[r, "\pi_{S\cup T,S}"] \arrow[d, "\pi_{S\cup T,T}"'] & X_S \arrow[d, "\pi_S"] \\ X_T \arrow[r, "\pi_T"'] & X \end{tikzcd} \tag{4.3}$$

is cartesian. Indeed, $\pi_S : X_S \to X$ is a principal bundle under $\pi(X, S, x)$, and similarly for $\pi_T, \pi_{S\cup T}$. So the assertion is equivalent to showing that the natural homomorphism

$$\pi(X, S \cup T, x) \to \pi(X, S, x) \times \pi(X, T, x) \tag{4.4}$$

induced by the embeddings $S \subset (S \cup T), T \subset (S \cup T)$ of categories is an isomorphism. Since $\pi(X, S, x)$ (resp. $\pi(X, T, x)$) is a reduced (resp. local) quotient of $\pi(X, S \cup T, x)$, (4.4) is surjective. On the other hand, by definition, every object in $S \cup T$ is a subquotient of tensors of objects in S and objects in T, thus by A.1,(ii), (4.4) is injective. Therefore we have

$$X_{S'} \times_{X_S} X_{S\cup T} = X_{S'} \times_{X_S} (X_S \times_X X_T) = X_{S'\cup T}. \tag{4.5}$$

This shows 1).

2) Note that $\pi_{S\cup T,S} : X_{S\cup T} \to X_S$ is a principal $\pi(X, T, x)$-bundle since (4.3) is cartesian, thus $(\pi_{S\cup T,S})_*\mathcal{O}_{X_{S\cup T}}$ is F-finite on X_S. The pullback by $\pi_{S\cup T,S}$ of any F-finite bundle on X_S lies in $\mathcal{E}(X_{S\cup T})$. Then it is easy to write any $V \in \mathcal{E}(X_{S\cup T})$ as a cokernel of a morphism $\pi^*_{S\cup T,S}W_1 \to \pi^*_{S\cup T,S}W_2$, where W_1, $W_2 \in \mathcal{C}^F(X_S)$. Thus 2) follows.

3) and 4) are easy by chasing diagrams and using projection formula, as we did already many times.

To show 5), one uses Proposition 3.2 and projection formula

$$H^0(X_{S'\cup T}, \pi^*_{S'\cup T, S\cup T} V) = H^0(X_{S'}, (\pi_{S'\cup T, S'})_* \pi^*_{S'\cup T, S\cup T} V) \tag{4.6}$$

$$= H^0(X_{S'}, \pi^*_{S',S}(\pi_{S\cup T,S})_* V) \overset{\text{(Prop. 3.2)}}{=} H^0(X_S, (\pi_{S\cup T,S})_* V) = H^0(X_{S\cup T}, V).$$

□

Fix S in $\mathcal{S}^{ét}$ and consider the principal bundle $X_{S\cup\mathcal{C}^F(X)}$ associated to $S\cup \mathcal{C}^F(X)$, that is

$$X_{S\cup\mathcal{C}^F(X)} = \varprojlim_{T\in\mathcal{T}^\ell} X_{S\cup T}, \quad \widetilde{\pi}_{S,T} : X_{S\cup\mathcal{C}^F(X)} \twoheadrightarrow X_{S\cup T}. \tag{4.7}$$

The cartesian diagram in (4.3) implies that $X_{S\cup\mathcal{C}^F(X)}$ is the product of X_S with $\widetilde{X}^F = X_{\mathcal{C}^F(X)}$ over X. For each $V \in \mathcal{E}(X_{S\cup T})$, set

$$H^0_{\mathcal{T}^\ell}(X_{S\cup T}, V) := H^0(X_{S\cup\mathcal{C}^F(X)}, \widetilde{\pi}^*_{S,T} V). \tag{4.8}$$

Recall that (2.6) implies that $H^0(X_{S\cup\mathcal{C}^F(X)}, \mathcal{O}) = k$. Consequently, $H^0(X_{S\cup\mathcal{C}^F(X)}, \widetilde{\pi}^*_{S,T} V)$ is a finite dimensional k-vector space and one has

$$H^0(X_{S\cup\mathcal{C}^F(X)}, \widetilde{\pi}^*_{S,T} V) = \varinjlim_{T\subset T'\in\mathcal{T}^\ell} H^0(X_{S\cup T'}, \pi^*_{S\cup T', S\cup T} V). \tag{4.9}$$

So in fact,

(4.10)
$$H^0(X_{S\cup\mathcal{C}^F(X)}, \widetilde{\pi}^*_{S,T} V) = H^0(X_{S\cup T'}, \pi^*_{S\cup T', S\cup T} V) \text{ for some } T' \supset T, T' \in \mathcal{T}^\ell.$$

Denote $\mathcal{U} := \mathcal{C}^{ét}(X) \cup \mathcal{C}^F(X)$ and $X_{\mathcal{U}}$ the associated principal bundle. Thus

$$X_{\mathcal{U}} = \varprojlim_{\substack{S\in\mathcal{S}^{ét}\\ T\in\mathcal{T}^\ell}} X_{S\cup T}, \quad \widetilde{\pi}_{S\cup T} : X_{\mathcal{U}} \twoheadrightarrow X_{S\cup T}. \tag{4.11}$$

Then for any bundle $V \in \mathcal{E}(X_{S\cup T})$ we have, by means of (4.2),

$$H^0(X_{\mathcal{U}}, \widetilde{\pi}^*_{S\cup T} V) \cong H^0_{\mathcal{T}^\ell}(X_{S\cup T}, V). \tag{4.12}$$

Definition 4.3. The category $\mathcal{E}$ has for objects pairs $(X_{S\cup T}, V)$, where $S \in \mathcal{S}^{ét}$, $T \in \mathcal{T}^\ell$ and $V \in \mathcal{E}(X_{S\cup T})$, and for morphisms

$$\mathrm{Hom}_{\mathcal{E}}((X_{S_1\cup T_1}, V), (X_{S_2\cup T_2}, W)) := \mathrm{Hom}_{X_{\mathcal{U}}}(\widetilde{\pi}^*_{S_1\cup T_1} V, \widetilde{\pi}^*_{S_2\cup T_2} W). \tag{4.13}$$

Proposition 4.4. *The category $\mathcal{E}$ in Definition 4.3 is a Tannaka cateogry over k, with tensor product defined by ($S := S_1 \cup S_1$, $T := T_1 \cup T_2$)*

$$(X_{S_1\cup T_1}, V) \otimes (X_{S_2\cup T_2}, W) := (X_{S\cup T}, \pi^*_{S\cup T, S_1\cup T_1}(V) \otimes \pi^*_{S\cup T, S_2\cup T_2}(W))$$

the unit object is $(X, \mathcal{O}_X)$, and a fiber functor

$$\mathcal{E} \to \mathrm{Vect}_k, \quad (X_{S\cup T}, V) \mapsto V|_{x_{S\cup T}}. \tag{4.14}$$

Proof. We first show that $\mathcal{E}$ is an abelian category. The kernel, image and cokernel of a morphism $f : ((X_{S_1\cup T_1}, V), (X_{S_2\cup T_2}, W))$ are defined as follows. Set $S := S_1 \cup S_2$, $T := T_1 \cup T_2$. By means of (4.12), f corresponds to an element f_S of $H^0_{\mathcal{T}^\ell}(X_{S\cup T}, \pi^*_{S\cup T,S_1\cup T_1}V^\vee \otimes \pi^*_{S\cup T,S_2\cup T_2}W)$, which by means of (4.10) is represented by an element $f_{S\cup T'}$ in $\mathrm{Hom}_{X_{S\cup T'}}(\pi^*_{S\cup T',S_1\cup T_1}V, \pi^*_{S\cup T',S_2\cup T_2}W)$. Now we define the kernel, image and cokernel of f to be the kernel, image and cokernel of $f_{S\cup T'}$ respectively. It is clear that $\mathcal{E}$ is thus an abelian category. With respect to the tensor product in $\mathcal{E}$, the dual of an object is defined by $(X_{S\cup T}, V)^\vee = (X_{S\cup T}, V^\vee)$. □

One notices that the kernel (image, cokernel) of f does depend on the choice of T', in particular, for an object $(X_{S\cup T}, V)$ in $\mathcal{E}$, and for any $T' \in \mathcal{T}^\ell$, $T' \supset T$

$$(X_{S\cup T}, V) \text{ is isomorphic with } (X_{S\cup T'}, \pi^*_{S\cup T',S\cup T}(V)) \text{ in } \mathcal{E}. \tag{4.15}$$

Let $G(\mathcal{E}, x)$ be the Tannaka group scheme of $\mathcal{E}$ with respect to the given fiber functor. Consider the tautological functor $p : \mathcal{D} \to \mathcal{E}, (X_S, V) \mapsto (X_S, V)$ which is clearly compatible with the fiber functors of $\mathcal{D}$ and $\mathcal{E}$. It yields a homomorphism $p^* : G(\mathcal{E}, x) \to L(X, x)$ which is clearly injective.

Theorem 4.5. *The k-group scheme homomorphism $q^* : G(\mathcal{E}, x) \to L(X, x)$ is the kernel of the homomorphism $L(X, x) \to \pi^F(X, x)$ and consequently is the kernel of $r^{ét} \times r^F : \pi^N(X, x) \to \pi^{ét}(X, x) \times \pi^F(X, x)$. In other words the representation category of* $\mathrm{Ker}(r^{ét} \times r^F)$ *is equivalent to $\mathcal{E}$.*

Proof. We use Theorem A.1, (iii), to show that the sequence

$$G(\mathcal{E}) \to L(X, x) \to \pi^F(X, x) \tag{4.16}$$

is exact.

If $(X_{S\cup T}, V)$ is an object in $\mathcal{E}$, then

$$\pi^*_{S\cup T,S}(\pi_{S\cup T,S})_*V \twoheadrightarrow V \tag{4.17}$$

and since $(X_S, (\pi_{S\cup T,S})_*V)$ is an object of $\mathcal{D}$, every object of $\mathcal{E}$ is the quotient of an object coming from $\mathcal{D}$ via q. Thus condition (c) in A.1, (iii), is fulfilled.

The maximal trivial subobject of (X_S, V) in $\mathcal{E}$ is an object $(X_{S\cup T}, V_0)$ for some $T \in \mathcal{T}^\ell$ and V_0 is the maximal trivial subobject of $\pi^*_{S\cup T,S}(V)$ in $\mathcal{E}(X_{S\cup T})$. Proposition 4.6 below shows that there exists an F-finite bundle $W \in T$ on X and an inclusion $j : \pi^*_S W \hookrightarrow V$, such that

$$\pi^*_{S\cup T,S}(j) : \pi^*_{S\cup T}W \cong V_0.$$

Thus condition (b) in A.1, (iii), is fulfilled.

Finally recall that the homomorphism $L(X, x) \to \pi^F(X, x)$ is induced from the functor $\mathcal{C}^F(X, x) \to \mathcal{D}$, $V \mapsto (X, V)$. On the other hand the above discussion also shows that (X_S, V) is trivial if and only if the inclusion $j : \pi^*_S W \to V$ is an isomorphism, that is (X_S, V) is isomorphic to (X, W) in $\mathcal{D}$ (see 3.10). Thus condition (a) is also fulfilled. □

It just remains to prove

Proposition 4.6. *Let $V \in \mathcal{C}^F(X_S)$, and $V_0 \subset \pi^*_{S\cup T,S}(V)$ be the maximal trivial subobject of $\pi^*_{S\cup T,S}(V)$ in $\mathcal{E}(X_{S\cup T})$. Then there exists an F-finite bundle $W \in T$ on X equipped with an inclusion $j : \pi^*_S(W) \hookrightarrow V$ such that*

$$\pi^*_{S\cup T,S}(j) : \pi^*_{S\cup T}(W) = V_0 \subset \pi^*_{S\cup T,S}(V).$$

Proof. Using the cartesian diagram (4.3), we have

$$\begin{aligned} H^0(X_{S\cup T}, \pi^*_{S\cup T,S}V) &\cong H^0(X_T, \pi_{S\cup T,T*}\pi^*_{S\cup T,S}V) \\ &\overset{(4.3)}{\cong} H^0(X_T, \pi^*_T\pi_{S*}V) \cong H^0(X, \pi_{T*}\mathcal{O}_{X_T} \otimes \pi_{S*}V) \\ &\cong \mathrm{Hom}_X((\pi_{T*}\mathcal{O}_{X_T})^\vee, \pi_{S*}V) = \oplus_1^r k \cdot \varphi_i \end{aligned} \tag{4.18}$$

for some morphisms $\varphi_i : (\pi_{T*}\mathcal{O}_{X_T})^\vee \to \pi_{S*}V$. Let $W \subset \pi_{S*}V$ be the image of

$$\oplus_1^r\varphi_i : \oplus_1^r(\pi_{T*}\mathcal{O}_{X_T})^\vee \to \pi_{S*}V. \tag{4.19}$$

Then W is trivializable by π_T and in particular $W \in T$ is F-finite. We show that the map $j : \pi^*_S W \to V$, induced from the inclusion $i : W \hookrightarrow \pi_{S*}V$, is injective.

Indeed, let $V' = \mathrm{im} j \subset V$ thus V' is a quotient of $\pi^*_S W$ and according to Proposition 3.6 and Theorem A.1, (i), we conclude that there is a quotient $q : W \to W'$ of W such that the quotient map $\pi^*W \to V'$ is the pull back $\pi^*_S q : \pi^*_S W \to \pi^*_S W'$. Now the inclusion $\pi^*_S W' = V' \subset V$ corresponds to a morphism $i' : W' \to V$ which is compatible with i in the sense that $i = i' \circ q$. By assumption q is surjective and i is injective, hence i' is an isomorphism, consequently j is injective.

On the other hand, one has from (4.19)

$$\mathrm{Hom}_X((\pi_{T*}\mathcal{O}_{X_T})^\vee, \pi_{S*}V) = \mathrm{Hom}_X((\pi_{T*}\mathcal{O}_{X_T})^\vee, W) \tag{4.20}$$

Thus

$$H^0(X_{S\cup T}, \pi^*_{S\cup T,S}V) = H^0(X_T, \pi^*_T W) = H^0(X_{S\cup T}, \pi^*_{S\cup T}W) \tag{4.21}$$

which means that $\pi^*_{S\cup T}(W) = V_0 \subset \pi^*_{S\cup T,S}(V)$. □

Remark 4.7. Using Proposition 2.7, replacing X by X_S and X_S by $X_{S\cup T}$, one sees that the objects of $\mathcal{E}(X_{S\cup T})$ are precisely those bundles which come from a representation of a local fundamental group over k.

Appendix A. Exact sequences of Tannaka group schemes

In this appendix, we summarize the material on Tannaka categories which we used throughout the article. The statements and their proofs are taken from [3], but for the reader's convenience, we gather the information in a compact form here.

Let $L \xrightarrow{q} G \xrightarrow{p} A$ be a sequence of homomorphism of affine group scheme over a field k. It induces a sequence of functors

$$\mathrm{Rep}(A) \xrightarrow{p^*} \mathrm{Rep}(G) \xrightarrow{q^*} \mathrm{Rep}(L) \tag{A.1}$$

where Rep denotes the category of finite dimensional representations over k.

Theorem A.1. *With the above settings we have*

(i) *The map $p : G \to A$ is faithfully flat (and in particular surjective) if and only if $p^*\mathrm{Rep}(A)$ is a full subcategory of $\mathrm{Rep}(G)$, closed under taking subquotients.*

(ii) *The map $q : L \to G$ is a closed immersion if and only if any object of $\mathrm{Rep}(L)$ is a subquotient of an object of the form $q^*(V)$ for some $V \in \mathrm{Rep}(G)$.*

(iii) *Assume that q is a closed immersion and that p is faithfully flat. Then the sequence $L \xrightarrow{q} G \xrightarrow{p} A$ is exact if and only if the following conditions are fulfilled:*
   (a) *For an object $V \in \mathrm{Rep}(G)$, $q^*(V)$ in $\mathrm{Rep}(L)$ is trivial if and only if $V \cong p^*U$ for some $U \in \mathrm{Rep}(A)$.*
   (b) *Let W_0 be the maximal trivial subobject of $q^*(V)$ in $\mathrm{Rep}(L)$. Then there exists $V_0 \subset V$ in $\mathrm{Rep}(G)$, such that $q^*(V_0) \cong W_0$.*
   (c) *Any W in $\mathrm{Rep}(L)$ is embeddable in (hence, by taking duals, a quotient of) $q^*(V)$ for some $V \in \mathrm{Rep}(G)$.*

Proof. The statements (i) and (ii) are due to Saavedra [12]. We refer also to [1, Proposition 2.21] for a nice proof. We show (iii).

Assume that $q : L \to G$ is the kernel of $p : G \to A$. Then (a), (b) follow from the well-know properties of normal subgroups (cf. [13, Chapter 13]). It remains to show (c).

Let $\mathrm{Ind} : \mathrm{Rep}(L) \to \mathrm{Rep}(G)$ be the induced representation functor, it is the right adjoint functor to the restriction functor $\mathrm{Res} : \mathrm{Rep}(G) \to \mathrm{Rep}(L)$ that is, one has a functorial isomorphism

$$\mathrm{Hom}_G(V, \mathrm{Ind}(W)) \xrightarrow{\cong} \mathrm{Hom}_L(\mathrm{Res}(V), W). \tag{A.2}$$

It is easy to check

$$\mathrm{Ind}(W) \cong (k[G] \otimes_k W)^L \tag{A.3}$$

where L acts on $k[G]$ on the right. It is well-known that $k[G]$ is faithfully flat over it subalgebra $k[A]$ ([13, Chapter 13]) and there is the following isomorphism

$$k[G] \otimes_{k[A]} k[G] \cong k[L] \otimes_k k[G] \tag{A.4}$$

which precisely means that $G \to A$ is a principal bundle under L. Consequently

$$k[G] \otimes_{k[A]} \mathrm{Ind}(W) \cong k[A] \otimes_k V \tag{A.5}$$

Thus the functor $\mathrm{Ind} : \mathrm{Rep}(L) \to \mathrm{Rep}(G)$ is exact.

Setting $V = \mathrm{Ind}(W)$ in (A.2), one obtains a canonical map $u_W : \mathrm{Ind}(W) \to W$ in $\mathrm{Rep}(L)$ which gives back the isomorphism in (A.2) as follows:

$$\mathrm{Hom}_G(V, \mathrm{Ind}(W)) \ni h \mapsto u_W \circ h \in \mathrm{Hom}_L(\mathrm{Res}(V), W).$$

The map u_W is non-zero whenever W is non-zero. Indeed, since Ind is faithfully exact, $\mathrm{Ind}(W)$ is non-zero whenever W is non-zero. Thus, if $u_W = 0$, then both sides of (A.2) were zero for any V. On the other hand, for $V = \mathrm{Ind}(W)$, the right hand side contains the identity map. This show that u_W can't vanish.

We want to show that u_W is always surjective. Let $U = \mathrm{Im}(u_W)$ and $T = W/U$. We have the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ind}(U) \arrow[r] \arrow[d, "u_U"'] & \mathrm{Ind}(W) \arrow[r] \arrow[d, "u_T"'] & \mathrm{Ind}(T) \arrow[r] \arrow[d, "u_W"'] & 0 \\ 0 \arrow[r] & U \arrow[r] & W \arrow[r] & T \arrow[r] & 0 \end{tikzcd} \tag{A.6}$$

By assumption, the composition $\mathrm{Ind}(W) \twoheadrightarrow \mathrm{Ind}(T) \to T$ is 0, therefore $\mathrm{Ind}(T) \to T$ is a zero map, implying $T = 0$.

Since $\mathrm{Ind}(W)$ is the union of its finite dimensional subrepresentations, we can find a finite dimensional G-subrepresentation $W_0(W)$ of $\mathrm{Ind}(W)$ which still maps surjectively on W. In order to obtain the statement on the embedding of W, we dualize $W_0(W^\vee) \twoheadrightarrow W^\vee$.

Conversely, assume that (a), (b), (c) are satisfied. Then it follows from (a) that for $U \in \mathrm{Rep}(A)$, $q^*p^*(U) \in \mathrm{Rep}(L)$ is trivial. Hence $pq : L \to A$ is the trivial homomorphism. Recall that by assumption, q is injective, p is surjective. Let $\bar{q} : \bar{L} \to G$ be the kernel of p. Then we have commutative diagram

$$\begin{tikzcd} L \arrow[rr, "i"] \arrow[dr, "q"'] & & \bar{L} \arrow[dl, "\bar{q}"] \\ & G & \end{tikzcd} \iff \begin{tikzcd} \mathrm{Rep}(L) & & \mathrm{Rep}(\bar{L}) \arrow[ll, "i^*"'] \\ & \mathrm{Rep}(G) \arrow[ul, "q^*"] \arrow[ur, "\bar{q}^*"'] & \end{tikzcd} \tag{A.7}$$

with injective homomorphisms. It remains to show that i is surjective, which amounts to saying that the category $i^*\mathrm{Rep}(\bar{L})$ in $\mathrm{Rep}(L)$ is full and closed under taking subquotients.

We first show the fullness. Let $\bar{W}_0, \bar{W}_1$ be objects in $\mathrm{Rep}(\bar{L})$ and $\varphi : W_0 := i^*(\bar{W}_0) \to i^*(\bar{W}_1) =: W_1$ be a morphism in $\mathrm{Rep}(L)$. Since $\mathrm{Rep}(\bar{L})$ also satisfies (c), there exists V_0, V_1 in $\mathrm{Rep}(G)$ with a surjective morphism $\pi : \bar{q}^*(V_0) \to \bar{W}_0$, and an injective morphism $\iota : \bar{W}_1 \to \bar{q}^*(V_1)$. These yield a morphism

$$\psi := i^*(\iota)\varphi i^*(\pi) : q^*(V_0) \to q^*(V_1) \tag{A.8}$$

The morphism ψ induces and element in $H^0(L, q^*(V_0^\vee \otimes V_1))$. Now, by (b) and by the fact that $\mathrm{Rep}(\bar{L})$ also satisfies (b) we conclude that $\psi = i^*(\bar{\psi})$, for some $\bar{\psi} : \bar{q}^*(V_0) \to \bar{q}^*(V_1)$. Since ι is injective and π is surjective, we conclude that

$\varphi = \bar{\varphi}$, for some $\bar{\varphi} : \bar{W}_0 \to \bar{W}_1$ in $\mathrm{Rep}(\bar{L})$. Thus the category $i^*\mathrm{Rep}(\bar{L})$ is full in $\mathrm{Rep}(L)$.

On the other hand, for any $W \in \mathrm{Rep}(L)$, by (c) there exist V_0, V_1 in $\mathrm{Rep}(G)$ and $\varphi : q^*(V_0) \to q^*(V_1)$ such that $W \cong \mathrm{im}\varphi$. By the fullness of $i^*\mathrm{Rep}(\bar{L})$ in $\mathrm{Rep}(L)$, $\varphi = i^*\bar{\varphi}$, hence $W \cong i^*(\mathrm{im}\bar{\varphi})$. Thus we have proved that any object in $\mathrm{Rep}(L)$ is isomorphic to the image under i^* of an object in $\mathrm{Rep}(\bar{L})$. Together with the discussion above this implies that $L \cong \bar{L}$. □

References

[1] Deligne, P., J. Milne: Tannakian Categories, Lectures Notes in Mathematics **900**, 101–228, Springer-Verlag (1982).

[2] Deligne, P.: Catégories tannakiennes, The Grothendieck Festschrift, Vol. II, 111–195, Progr. Math. **87**, Birkhäuser (1990).

[3] Esnault, H., Phùng Hô Hai: The Gauß-Manin connection and Tannaka duality, International Mathematics Research Notices, **93878** (2006), 1-35.

[4] Grothendieck, A. : Revêtements étales et groupe fondamental, SGA 1, Lect. Notes in Mathematics **224** (1970), Springer Verlag.

[5] Katz, N.: On the calculation of some differential Galois groups, Invent. math. **87** (1987), 13-61.

[6] Langer, H., Stuhler, U.: Vektorbündel auf Kurven und Darstellungen der algebraischen Fundamentalgruppe, Math. Z. **156** (1977), 73-83.

[7] Mehta, V. B., Subramanian, S.: On the fundamental group scheme, Invent. math. **148** (2002), 143-150.

[8] Nori, M.: On the representation of the fundamental group, Compositio math. **33** (1976), 29-41.

[9] Nori, M.: The fundamental group scheme, Proc. Indian Acad. Sci. **91** (1982), 73-122.

[10] Ramanan, S., Ramanathan, A.: Some remarks on the instability flag, Tohoku Math. Journal **36** (1984), 269-291.

[11] Raynaud M.: Sections de fibrés vectoriels sur une courbe, Bull. Soc. Math. France **110** (1982), 103-125.

[12] N. Saavedra-Rivano: Catégories Tannakiennes, Lectures Notes in Mathematics **265**, Springer-Verlag (1972).

[13] Waterhouse, W.C.: Introduction to affine group schemes, Graduate Texts in Mathematics **66**, Springer-Verlag (1979).

Universität Duisburg-Essen, Mathematik, 45117 Essen, Germany
E-mail address: `esnault@uni-due.de`

Universität Duisburg-Essen, Mathematik, 45117 Essen, Germany and Institute of Mathematics, Hanoi, Vietnam
E-mail address: `hai.phung@uni-duisburg-essen.de`

Institute of Mathematics AMSS, Chinese Academy of Sciences, Beijing 100080, P. R. of China
E-mail address: `xsun@math.ac.cn`