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The Gauss-Manin Connection and Tannaka Duality

Hélène Esnault and Phùng Hô Hai

1 Introduction

If $f : X \to \mathrm{Spec}(K)$ is a smooth, geometrically connected variety defined over a field of characteristic 0, $K \supset k$ is a field extension, and $x \in X(K)$ is a rational point, one considers three Tannaka categories $\mathcal{C}(X/K)$, $\mathcal{C}(X/k)$, $\mathcal{C}(K/k)$ of flat connections with compatible fiber functors. The objects of $\mathcal{C}(X/K)$ are bundles (i.e., locally free coherent modules of finite type) with relative flat connections $((\mathcal{V}, \nabla_{/K}), \nabla_{/K} : \mathcal{V} \to \Omega^1_{X/K} \otimes_{\mathcal{O}_X} \mathcal{V})$, the ones of $\mathcal{C}(X/k)$ are bundles with flat absolute connections $((\mathcal{V}, \nabla),\ \nabla : \mathcal{V} \to \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{V})$, and the ones of $\mathcal{C}(K/k)$ are K-vector spaces with flat connections $((V, \nabla),\ \nabla : V \to \Omega^1_{K/k} \otimes_K V)$. The morphisms are the flat morphisms and the fiber functor has values in the category of finite dimensional vector spaces Vec_K over K, defined by the restriction of $\mathcal{V}$ to x for $\mathcal{C}(X/K)$, $\mathcal{C}(X/k)$ and by V for $\mathcal{C}(K/k)$. Then $\mathcal{C}(X/K)$ is a neutral Tannaka category, and Tannaka duality [2, Theorem 2.11] yields the existence of a proalgebraic group scheme $G(X/K)$ over K, so that $\mathcal{C}(X/K)$ becomes equivalent to the representation category $\mathrm{Rep}_f(G(X/K))$ on finite dimensional K-vector spaces. The two other Tannaka categories $\mathcal{C}(X/k)$, $\mathcal{C}(K/k)$ are not necessarily neutral. We assume that they are defined over k, which is to say that k is the endomorphism ring $\mathrm{End}_{\mathcal{C}(K/k)}((K, d_{K/k}))$ of the unit object, which in this case is the same as the subfield of K of flat sections. Equivalently, this is saying that k is algebraically closed in K. Then Tannaka duality [3, théorème 1.12] yields the existence of groupoid schemes $G(X/k)$, $G(K/k)$ over k acting on $\mathrm{Spec}(K) \times_k \mathrm{Spec}(K)$, so that, in the groupoid sense, $\mathcal{C}(X/k)$ (resp., $\mathcal{C}(K/k)$) becomes equivalent to the representation category $\mathrm{Rep}_f(K :$

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$G(X/k)$) (resp., $\mathrm{Rep}_f(K : G(K/k))$) on finite dimensional K-vector spaces. (We refer to the appendix for the brief review of Deligne's theory of groupoid schemes.) Since the fiber functors are compatible, one has natural transformations $\mathcal{C}(X/k) \xrightarrow{\text{rest}} \mathcal{C}(X/K)$ mapping an absolute connection $(\mathcal{V}, \nabla)$ to the induced relative one $(\mathcal{V}, \nabla/K)$ and $\mathcal{C}(K/k) \xrightarrow{f^*} \mathcal{C}(X/k)$ mapping (V, ∇) to $f^*(V, \nabla)$. This yields the homomorphisms

$$G(X/K) \xrightarrow{\text{rest}} G(X/k)^{\Delta}, \qquad G(X/k) \xrightarrow{f^*} G(K/k), \tag{1.1}$$

where Δ defines the induced group scheme over K which is the restriction of $G(X/k)$ to the diagonal $\Delta = \mathrm{Spec}(K) \to \mathrm{Spec}(K) \times_k \mathrm{Spec}(K)$, viewed as a group scheme over K.

On the other hand, if V is an object in $\mathrm{Rep}_f(G(X/K))$, its cohomology group $H^i(G(X/K), V)$ is well defined [5, Section 1.4], and is represented by an i-extension. Via the Tannaka formalism, this i-extension yields an i-extension of connections in $\mathcal{C}(X/K)$ of the trivial connection $(\mathcal{O}_X, d_{/K})$ by $(\mathcal{V}, \nabla_{/K})$ corresponding to V. Via the connecting homomorphism $\delta : H^0_{DR}(X, (\mathcal{O}_X, d_{/K})) \to H^i_{DR}(X, (\mathcal{V}, \nabla_{/K}))$, where $_{DR}$ means de Rham cohomology relative to K, one defines a homomorphism

$$H^i\big(G(X/K), V\big) \longrightarrow H^i{}_{DR}\big(X, \big(\mathcal{V}, \nabla_{/K}\big)\big), \quad i\text{-extension} \longmapsto \delta(1). \tag{1.2}$$

We show the following (see Proposition 2.2).

Proposition 1.1. The homomorphism (1.2) is an isomorphism for $i = 0, 1$ and injective for $i = 2$, thus, in particular, is an isomorphism when X is an affine curve, and also is an isomorphism if X is a projective curve of genus $g \geq 1$. □

If V is an object of $\mathrm{Rep}_f(K : G(X/k))$, corresponding to the connection $(\mathcal{V}, \nabla)$ and its restriction $(\mathcal{V}, \nabla/K)$, then one has the Gauss-Manin connection defined on the finite dimensional K-vector space $H^i_{DR}(X, (\mathcal{V}, \nabla/K))$. Via (1.2), it corresponds to a groupoid action of $G(K/k)$ on $H^i(G(X/K), V)$ for $i = 0, 1$. We investigate the question of whether one can interpret the Gauss-Manin connection as one does in topology. For the category of local systems on complex manifolds, $G(X/K)$ corresponds to $\pi_1(f^{-1}f(x), x)$, $G(X/k)$ to $\pi_1(X, x)$, while $G(K/k)$ corresponds to $\pi_1(S, f(x))$. So via the standard exact sequence expressing the absolute fundamental group as an extension of the one on the base with the relative one, one defines an action of $\pi_1(S, f(x))$ on $H^i(\pi_1(f^{-1}f(x), x), \rho)$ for a representation ρ of $\pi_1(X, x)$.

In our setting, the map f^* of groupoids in (1.1) is surjective, and one defines its kernel,

$$0 \longrightarrow L^{\delta} \longrightarrow G(X/k) \xrightarrow{f^*} G(K/k) \longrightarrow 0, \tag{1.3}$$

as a discrete k-groupoid scheme, that is, the k-morphism $L^{\delta} \to \mathrm{Spec}(K) \times_k \mathrm{Spec}(K)$ factors through the diagonal $\Delta \to \mathrm{Spec}(K) \times_k \mathrm{Spec}(K)$. Its representation category is equivalent to the representation category of the underlying group scheme $L \subset G(X/k)^{\Delta}$ over K. We show the following (see Theorem 5.8).

Theorem 1.2. The representation category of L is equivalent to the full subcategory of $\mathcal{C}(X/K)$, the objects of which are both subobjects and quotients of objects in $\mathcal{C}(X/k)$. □

On the other hand, the homomorphism rest in (1.1) factors naturally through L. One defines the subgroup scheme

$$H := \mathrm{rest}\left(G(X/K)\right) \subset L, \tag{1.4}$$

over K of L and shows (Proposition 3.1) that its representation category is the full subcategory of $\mathcal{C}(X/K)$, the objects of which are subquotients of objects in $\mathcal{C}(X/k)$. This description allows us to define an obstruction, local at ∞ of X, for an object of $\mathcal{C}(X/K)$ to lie in $\mathrm{Rep}(H)$. We show that this obstruction does not necessarily vanish, thus, we have the following (see Proposition 3.2).

Proposition 1.3. The homomorphism $G(X/K) \to G(X/k)^{\Delta}$ of K-group schemes is not necessarily injective. □

More precisely we show the following (see Theorem 4.7).

Theorem 1.4. The kernel of $G(X/K) \to G(X/k)^{\Delta}$ has a nontrivial subgroup scheme which is defined in categorial terms, and which has the property that it has no nontrivial homomorphism into the additive group $\mathbb{G}_a$. □

On the other hand, Deligne shows that any relative subconnection of an absolute connection is also the quotient that acts as a relative connection of an absolute connection (Theorem 5.10). Thus the description of $\mathrm{Rep}(H)$ and $\mathrm{Rep}(L)$ as full subcategories of $\mathrm{Rep}(G(X/K))$ allows to conclude that $H = L$. The following is the main result of this article (see Theorem 5.11).

Theorem 1.5. The sequence

$$G(X/K) \longrightarrow G(X/k) \xrightarrow{f^*} G(K/k) \longrightarrow 0 \tag{1.5}$$

is exact, where exactness means that one sees the k-discrete groupoid scheme L^δ as a K-group scheme L, and then it is the image of $G(X/K)$. □

Thus there are at least two reasons why one cannot directly apply the standard argument describing a canonical action of $G(K/k)$ on $H^i(G(X/K), V)$, where V is a finite representation of $G(X/k)$. Firstly, (1.1) is a sequence of groupoid schemes rather than group schemes. This means that one has to redo for groupoids the classical theory available for groups. Secondly, $G(X/K) \to L$ is not injective, which is really a new phenomenon. The reason why, nevertheless, one has this $G(K/k)$-groupoid action on $H^i(G(X/K), V)$ comes from the following (see Corollary 4.3 and Theorem 5.12).

Corollary 1.6 and Theorem 1.7. The natural homomorphisms $H^i(L, V) \to H^i(H, V) \to H^i(G(X/K), V)$ defined by functoriality are all isomorphisms for $i = 0, 1$. □

As a corollary, one obtains a Tannaka theoretic formulation of the Gauss-Manin connection on $H^i_{DR}(X, (\mathcal{V}, \nabla/K))$ (Section 6).

Finally, one can develop the same theory using connections relative to a finite dimensional K-vector space D of $T_{K/k}$. More precisely, let us remark that a connection is the same as an $\mathcal{O}_X$-coherent module endowed with an action of the sheaf of differential operators. Let $D \subset T_{K/k}$ be a finite dimensional K-linear subspace of the tangent vectors of K/k, which is closed under brackets, such that $\mathrm{End}_D(K) = k$; and let $T_{X/k,D}$ be the inverse image of f^*D in $T_{X/k}$, the tangent sheaf of K/k, under the map $T_{X/k} \to f^*T_{K/k}$. Then $T_{X/k,D}$ is an extension of f^*D by $T_{X/K}$, which generates a subalgebra $\mathcal{D}_{X/k,D} \subset \mathcal{D}_{X/k}$. The methods developed in this paper could be used to treat the relations $\mathcal{C}(X/K)$, $\mathcal{C}(X/k, D)$, $\mathcal{C}(K/k, D)$ as well, where $\mathcal{C}(X/k, D)$ is the category of $\mathcal{O}_X$-coherent modules with an action of $\mathcal{D}_{X/k,D}$, and $\mathcal{C}(K/k, D)$ is the category of finite dimensional K-vector spaces with a $\mathcal{D}(K/k, D)$, the subalgebra of $\mathcal{D}(K/k)$ spanned by D. We do not write the details.

2 The neutral Tannaka category of flat connections

Let $f : X \to \mathrm{Spec}(K)$ be a smooth geometrically connected variety defined over a field of characteristic 0.

Definition 2.1. The category $\mathcal{C}(X/K)$ of flat connections relative to K (or simply of flat connections/K) has for objects, the flat connections $((\mathcal{V}, \nabla),\ \nabla : \mathcal{V} \to \Omega^1_{X/K} \otimes_{\mathcal{O}_X} \mathcal{V})$, where $\mathcal{V}$ is a locally free coherent module of finite type, and for morphisms, the flat morphisms.

It is a rigid abelian tensor category over K, and if we fix a K-rational point $x \in X(K)$, we can endow $\mathcal{C}(X/K)$ with the fiber functor,

$$\mathcal{C}(X/K) \xrightarrow{\omega} \mathrm{Vec}_K, \qquad (\mathcal{V}, \nabla) \longmapsto \omega\big((\mathcal{V}, \nabla)\big) = \mathcal{V}\big|_x =: V, \tag{2.1}$$

with values in the category of finite dimensional K-vector spaces. Thus $\mathcal{C}(X/K)$ becomes a neutral Tannaka category, and by the fundamental Tannaka duality [2, Theorem 2.11], ω defines an equivalence of tensor categories,

$$\mathcal{C}(X/K) \xrightarrow{\omega \cong} \mathrm{Rep}_f\big(G(X/K)\big), \tag{2.2}$$

where $G(X/K)$ is the Tannaka group scheme over K and $\mathrm{Rep}_f(G(X/K))$ is the category of its finite dimensional representations. $G(X/K)$ is a progroup scheme over K which fulfills

$$\begin{aligned} G(X/K) &= \varprojlim_V G(V), \\ G(V) &= \mathrm{Im}\,\big(G(X/K)\big) \quad \text{in } \mathrm{GL}\,\big(\omega\big((\mathcal{V}, \nabla)\big)\big). \end{aligned} \tag{2.3}$$

Let V be an object of $\mathrm{Rep}_f(G(X/K))$. One defines its cohomology $H^i(G(X/K), V)$. Recall from [5, Section 1.4] that if G is a group scheme, its cohomology is defined as the right derived functor of the functor $V \mapsto V^G$ of invariants, and is computed explicitly by cochains. Here $G(X/K)$ is proalgebraic, acts on V via its quotient $G(V)$ and the functor $V \mapsto V^{G(X/K)}$ of invariants factors through $V \mapsto V^{G(V)} = V^{G(X/K)}$. Setting $\mathcal{O}[G(X/K)] := \varinjlim_V \mathcal{O}[G(V)]$ for the K-algebra of functions, with its canonical $G(X/K)$-action, the $G(X/K)$-injective modules are still direct summands of (trivial) $\otimes_K \mathcal{O}[G(X/K)]$, as in [5, Subsection 1.3.10]. In the Ind-category of representations of the proalgebraic group $G(X/K)$, there are enough injective modules, and one defines $H^i(G(X/K), V)$ as the right derived functor to the functor $V \mapsto V^{G(X/K)}$ of invariants. As $V^{G(X/K)} = \mathrm{Hom}_{G(X/K)}(K, V)$, one has, as in [5, equation 1.4.2(1)], that cohomology is also the derived functor $\mathrm{Ext}^i_{G(X/K)}(K, V)$ to $V \mapsto \mathrm{Hom}_{G(X/K)}(K, V)$:

$$H^i\big(G(X/K), V\big) = \mathrm{Ext}^i_{\mathrm{Rep}_f(G(X/K))}(K, V). \tag{2.4}$$

On the other hand, if e is an i-extension of K by V in $\mathrm{Rep}_f(G(X/K))$, via Tannaka duality (2.2), one has an i-extension ϵ of $(\mathcal{O}_X, d)$ by $(\mathcal{V}, \nabla)$ in $\mathcal{C}(X/K)$ with

$$\omega(\epsilon) = e, \tag{2.5}$$

yielding a connecting homomorphism,

$$\delta_\epsilon : H^0_{DR}\big(X, (\mathcal{O}_X, d)\big) \longrightarrow H^i_{DR}\big(X, (\mathcal{V}, \nabla)\big), \tag{2.6}$$

where $H^i_{DR}(X, (\mathcal{V}, \nabla)) := \mathbb{H}^i(X, \Omega^\bullet_{X/K} \otimes_{\mathcal{O}_X} \mathcal{V})$ is the de Rham cohomology of the connection $(\mathcal{V}, \nabla)$. This defines a homomorphism of K-vector spaces,

$$\delta^i(X/K) : H^i\big(G(X/K), V\big) \longrightarrow H^i_{DR}\big(X, (\mathcal{V}, \nabla)\big), \qquad e \longmapsto \delta_\epsilon(1). \tag{2.7}$$

Proposition 2.2. The homomorphism $\delta^i(X/K)$ is an isomorphism for $i = 0, 1$ and is injective for $i = 2$. In particular, if X/K is an affine curve, $H^2(G(X/K), V) = H^2_{DR}(X, (\mathcal{V}, \nabla)) = 0$. Moreover, if X/K is a smooth projective curve of genus $g \geq 1$, then it is an isomorphism for $i = 2$. □

Proof. For $i = 0$, $H^0(G(X/K), V) \subset V$ is the largest trivial $G(X/K)$-subrepresentation. Thus by Tannaka duality (2.2), it corresponds to

$$\big(H^0_{DR}\big(X, (\mathcal{V}, \nabla)\big) \otimes_K \mathcal{O}_X, 1 \otimes d\big), \tag{2.8}$$

which is the largest trivial subconnection of $(\mathcal{V}, \nabla)$, where $\omega((\mathcal{V}, \nabla)) = V$. For $i = 1$, (2.4) says that a class $e \in H^1(G(X/K), V)$ is represented by an extension $e : 0 \to V \to W \to K \to 0$ in $\mathrm{Rep}_f(G(X/K))$ and that two such extensions e, e' yield the same cohomology class if there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \longrightarrow & W & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow & & \downarrow{=} & & \\ 0 & \longrightarrow & V & \longrightarrow & W' & \longrightarrow & K & \longrightarrow & 0 \end{array} \tag{2.9}$$

On the other hand, a class $\epsilon \in H^1_{DR}(X, (\mathcal{V}, \nabla))$, with Cech cocycle $(u_{ij}, v_j) \in \mathcal{C}^1(\mathcal{V}) \times \mathcal{C}^0(\Omega^1_{X/K} \otimes \mathcal{V})$, $\delta(u) = \nabla(u) - \delta(v) = \nabla(v) = 0$, on a Cech covering $\mathcal{U} = \bigcup_i U_i$, is represented by an extension $\epsilon : 0 \to (\mathcal{V}, \nabla) \to (\mathcal{W}, \nabla_{\mathcal{W}}) \to (\mathcal{O}_X, d) \to 0$, with $\mathcal{W}|_{U_i} = (\mathcal{V} \oplus \mathcal{O})|_{U_i}$, $\nabla_{\mathcal{W}}(0 \oplus 1) = v_i$, $\nabla_{\mathcal{W}}|_{\mathcal{V} \oplus 0} = \nabla$. Two such extensions ϵ, ϵ' yield the same cohomology class if and only if there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathcal{V}, \nabla) & \longrightarrow & (\mathcal{W}, \nabla_{\mathcal{W}}) & \longrightarrow & (\mathcal{O}_X, d) & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow & & \downarrow{=} & & \\ 0 & \longrightarrow & (\mathcal{V}, \nabla) & \longrightarrow & (\mathcal{W}', \nabla_{\mathcal{W}'}) & \longrightarrow & (\mathcal{O}_X, d) & \longrightarrow & 0 \end{array} \tag{2.10}$$

Thus Tannaka duality (2.2) yields the result for $i = 1$. We remark that the proof also shows the isomorphism $H^1_{DR}(X, (\mathcal{V}, \nabla)) = H^1(G(X/K), V)$ for $V \in \mathrm{Rep}(G(X/K))$ and $(\mathcal{V}, \nabla)$, the corresponding connection with $\mathcal{V}$ only, quasi-coherent.

We now show injectivity for $i = 2$. Let J be the injective envelope of V in $\mathrm{Rep}(G(X/K))$ and let $(\mathcal{J}, \nabla)$ be the corresponding (not necessarily coherent) connection. As J is injective in $\mathrm{Rep}(G(X/K))$, one has

$$H^1_{DR}\big(X, (\mathcal{J}, \nabla)\big) \cong H^1\big(G(X/K), J\big) = 0. \tag{2.11}$$

Hence, the long exact sequences associated to the exact sequences $0 \to V \to J \to J/V \to 0$ and $0 \to (\mathcal{V}, \nabla) \to (\mathcal{J}, \nabla) \to (\mathcal{V}/\mathcal{J}, \nabla) \to 0$ yield

$$H^2\big(G(X/K), V\big) \cong H^1\big(G(X/K), J/V\big) \cong H^1_{DR}\big(X, (\mathcal{J}/\mathcal{V}, \nabla)\big) \hookrightarrow H^2_{DR}\big(X, (\mathcal{V}, \nabla)\big). \tag{2.12}$$

We now prove the last part of the proposition. Let X/K be a smooth projective curve, and $(\mathcal{V}, \nabla)$ be a connection. Then $H^2_{DR}(X, (\mathcal{V}, \nabla))$ is a Poincaré dual to $H^0_{DR}(X, (\mathcal{V}, \nabla)^\vee)$, where $(\mathcal{V}, \nabla)^\vee$ is the dual connection. The inclusion $(H^0_{DR}(X, (\mathcal{V}, \nabla)^\vee) \otimes_K \mathcal{O}_X, 1 \otimes d) \subset (\mathcal{V}, \nabla)^\vee$ induces an isomorphism on H^0_{DR}, thus the dual projection

$$\begin{aligned} &(\mathcal{V}, \nabla) \twoheadrightarrow \big(H^2_{DR}\big(X, (\mathcal{V}, \nabla)\big) \otimes_K \mathcal{O}_X, 1 \otimes d\big) \cong \oplus_1^h \big(\mathcal{O}_X, d\big), \\ &h = \dim_K H^2_{DR}\big(X, (\mathcal{V}, \nabla)\big), \end{aligned} \tag{2.13}$$

induces an isomorphism on H^2_{DR}. On the other hand, assuming now that $g \geq 1$, there are two classes $\alpha, \beta \in H^1_{DR}(X)$, so that $0 \neq \alpha \cup \beta \in H^2_{DR}(X) = K$. Thus there is a diagram of extensions:

$$\begin{array}{ccccc} \big(\mathcal{O}_X, d\big) & \longrightarrow & (\mathcal{E}, \nabla) & \longrightarrow & \big(\mathcal{O}_X, d\big) \\ \uparrow & & & & \\ (\mathcal{F}, \nabla) & & & & \\ \uparrow & & & & \\ \big(\mathcal{O}_X, d\big) & & & & \end{array} \tag{2.14}$$

in $\mathcal{C}(X/K)$, which corresponds to α for the horizontal extension and β for the vertical one. Denoting by $(\mathcal{E}, \nabla)_0$ the sub of $\oplus_1^h(\mathcal{E}, \nabla)$ which is the inverse image of $(\mathcal{O}_X, d)$ embedded diagonally in $\oplus_1^h(\mathcal{O}_X, d)$, and setting $(\mathcal{F}, \nabla)_0 = \oplus_1^h(\mathcal{F}, \nabla)$, (2.14) induces a 2-extension in $\mathcal{C}(X/K)$:

$$0 \longrightarrow \oplus_1^h(\mathcal{O}_X, d) \longrightarrow (\mathcal{F}, \nabla)_0 \longrightarrow (\mathcal{E}, \nabla)_0 \longrightarrow \big(\mathcal{O}_X, d\big) \longrightarrow 0, \tag{2.15}$$

which has the property that the image of connecting homomorphism $H^0_{DR}(X) \to \oplus_1^h H^2_{DR}(X)$ followed by a projection $\oplus_1^h H^2_{DR}(X) \to H^2_{DR}(X)$ is the fundamental class $\alpha \cup \beta$. Since (2.13) induces an isomorphism on H^2_{DR}, there are connections $(\mathcal{E}_1, \nabla)$ and $(\mathcal{F}_1, \nabla)$ together with a commutative diagram of 2-extensions:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \oplus_1^h(\mathcal{O}_X, d) & \longrightarrow & (\mathcal{F}, \nabla)_0 & \longrightarrow & (\mathcal{E}, \nabla)_0 & \longrightarrow & (\mathcal{O}_X, d) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & =\uparrow & & \\ 0 & \longrightarrow & (\mathcal{V}, \nabla) & \longrightarrow & (\mathcal{F}_1, \nabla) & \longrightarrow & (\mathcal{E}_1, \nabla) & \longrightarrow & (\mathcal{O}_X, d) & \longrightarrow & 0 \end{array} \tag{2.16}$$

in $\mathcal{C}(X/K)$. We apply Tannaka duality to the bottom 2-extension. This yields a 2-extension:

$$0 \longrightarrow V \longrightarrow F_1 \longrightarrow E_1 \longrightarrow K \longrightarrow 0 \tag{2.17}$$

in $\mathrm{Rep}_f(G(X/K))$, which has the property that the composite map

$$\begin{aligned} H^0\big(G(X/K), K\big) &\xrightarrow{\text{connecting hom.}} H^2\big(G(X/K), V\big) \xrightarrow{\delta^2(X/K)} H^2_{DR}\big(X, (\mathcal{V}, \nabla)\big) \\ &\xrightarrow{(2.13)\text{ iso}} \oplus_1^h H^2_{DR}(X) \xrightarrow{\text{proj.}} H^2_{DR}(X) \end{aligned} \tag{2.18}$$

is an isomorphism. This shows that $\delta^2(X/K)$ is surjective and finishes the proof. ■

Remark 2.3. If $K = \mathbb{C}$, then the classical Riemann-Hilbert correspondence establishes an equivalence of Tannaka categories,

$$\mathrm{Rep}_f\left(\pi_1^{top}\big(X(\mathbb{C}), x\big)\right) \cong \mathcal{C}(X/\mathbb{C}), \tag{2.19}$$

for X a complex smooth projective variety. It defines a homomorphism,

$$\pi_1^{top}\big(X(\mathbb{C}), x\big) \longrightarrow G(X/\mathbb{C})(\mathbb{C}), \tag{2.20}$$

with dense image. In particular, if $X/K = \mathbb{P}^1/\mathbb{C}$, then $\pi_1^{\text{top}}(X(\mathbb{C}), x) = 0$, thus

$$H^2\big(G(X/K), K\big) = 0, \quad \text{while } H^2_{DR}\big(X, \big(\mathcal{O}_X, d\big)\big) = K. \tag{2.21}$$

More generally, (2.18) gives a topological hint why the surjectivity on H^2 in Proposition 2.2 is true on a curve of genus ≥ 1. Indeed in this case, the universal covering of the underlying Riemann surface is contractible, thus the Hochshild-Serre spectral sequence degenerates and one has $H^2_{DR}(X/\mathbb{C}, (\mathcal{V}, \nabla)) \cong H^2(\pi_1^{\text{top}}(X(\mathbb{C}), x), V)$.

3 The not necessarily neutral Tannaka category of flat connections

Let $g : X \to \text{Spec}(k)$ be a smooth scheme with k a field of characteristic 0, where g factors through $f : X \to \text{Spec}(K)$ as in Section 2, thus X/K is a smooth geometrically connected variety, and $K \supset k$ is a field extension with $\text{End}_{\mathcal{C}(K/k)}((K, d_{K/k})) = k$. As already mentioned in Section 1, this is equivalent to saying that k is algebraically closed in K. Indeed, $\text{End}_{\mathcal{C}(K/k)}((K, d_{K/k})) = K^{\nabla}$ is the ring of flat sections. We have $\mathcal{C}(X/k)$ as in Definition 2.1. The objects of $\mathcal{C}(X/k)$ are flat connections $((\mathcal{V}, \nabla),\ \nabla : \mathcal{V} \to \Omega^1_{X/k} \otimes \mathcal{V})$, where $\mathcal{V}$ is a coherent locally free sheaf on X/K, and morphisms are flat morphisms. This is a k-linear category. As in (2.1), we fix a K-rational point $x \in X(K)$ which defines a fiber functor

$$\mathcal{C}(X/k) \xrightarrow{\ \omega\ } \text{Vec}_K, \qquad (\mathcal{V}, \nabla) \longmapsto \mathcal{V}\big|_x =: V. \tag{3.1}$$

Thus $\mathcal{C}(X/k)$ becomes a nonneutral Tannaka category when $K \neq k$. By the fundamental Tannaka duality [3, théorème 1.12], there is a groupoid scheme $G(X/k)$ defined over k, acting on $\text{Spec}(K) \times_k \text{Spec}(K)$ so that ω defines an equivalence of tensor categories

$$\mathcal{C}(X/k) \xrightarrow{\ \omega \cong\ } \text{Rep}_f\big(K : G(X/k)\big), \tag{3.2}$$

where $\text{Rep}_f(K : G(X/k))$ denotes the category of finite dimensional K-representations of $G(X/k)$. See the appendix for a summary of the facts on groupoid schemes which will be used in the sequel.

We denote by $G(X/k)^{\Delta}$ the restriction of $G(X/k)$ to the diagonal. Then $G(X/k)^{\Delta}$ is a discrete groupoid scheme over k. The representation category of a discrete groupoid scheme is equivalent to the representation category of the underlying group scheme over

K. The embedding of Tannaka categories $\mathcal{C}(X/k) \xrightarrow{\text{rest}} \mathcal{C}(X/K)$, $(\mathcal{V},\nabla) \mapsto \text{rest}((\mathcal{V},\nabla)) = (\mathcal{V},\nabla/K)$ with compatible fiber functor yields a homomorphism

$$G(X/K) \xrightarrow{\text{rest}} G(X/k)^{\Delta} \subset G(X/k). \tag{3.3}$$

Proposition 3.1. The representation category on finite dimensional K-vector spaces of $H := \text{rest}(G(X/K)) \subset G(X/k)^{\Delta}$ is equivalent to the full subcategory of $\mathcal{C}(X/K)$, the objects of which are subquotients of objects $\text{rest}((\mathcal{V},\nabla))$. □

Proof. Let us denote by $\mathcal{C}$ the full subcategory of $\mathcal{C}(X/K)$, the objects of which are subquotients of the objects $\text{rest}((\mathcal{V},\nabla))$, and by $G(\mathcal{C})$ its Tannaka group scheme. Recall from [**2**, Proposition 2.21(a)] that $G(X/K) \to G(\mathcal{C})$ is faithfully flat if and only if rest is fully faithful, which in our case is trivial, and any subobject in $\mathcal{C}(X/K)$ of an object in $\mathcal{C}$ is an object in $\mathcal{C}$, which is trivial as well in our case. Recall from [**2**, Proposition 2.21(a)] that $G(\mathcal{C}) \to G(X/k)^{\Delta}$ is a closed immersion if and only if any object of $\mathcal{C}$ is a subquotient of an object in $\text{Rep}(G(X/k)^{\Delta})$. But by definition, objects in $\mathcal{C}$ are subquotients of objects in $\mathcal{C}(X/k)$, thus as fortiori of objects in $\text{Rep}(\mathcal{C}(X/k)^{\Delta})$. ■

Proposition 3.2. The homomorphism of group schemes $G(X/K) \to G(X/k)^{\Delta}$ over K is not necessarily injective. □

Proof. We assume that X is an affine curve, and that $k = \mathbb{C}$, $K = \overline{\mathbb{C}(s)}$, where s is a transcendental element. We wish to show that not every connection on X/K is a subquotient of a flat connection on X/k. We consider a rank 1 connection $(\mathcal{L},\nabla)$ on X/K. Its formal completion at a point $y \in \bar{X} \setminus X$, which we assume to be K-rational with local parameter t, is of the shape

$$\begin{aligned} &\mathcal{L} \otimes_{\mathcal{O}_X} K((t)) = K((t)) \cdot e, \\ &(\nabla \otimes_{\mathcal{O}_X} K((t)))(e) = \alpha(t)\frac{dt}{t} \cdot e, \qquad \alpha(t) \in \frac{1}{t^n}K[[t]]. \end{aligned} \tag{3.4}$$

Assume $(\mathcal{L},\nabla)$ is a subquotient of $(\mathcal{V},\nabla)$ on X/k. By the Turrittin-Levelt decomposition (see, e.g., [**1**, Section 5.9]), one has

$$(\mathcal{V},\nabla/K) \otimes_{\mathcal{O}_X} K((t)) = \oplus_i M_i \otimes U_i, \tag{3.5}$$

with U_i nilpotent, M_i irreducible, $\text{Hom}(M_i, M_j) = K \cdot \delta_{ij}$. Thus $(\mathcal{L},\nabla) \otimes_{\mathcal{O}_X} K((t))$ has to be one of the M_i, say M_0; and it is not only a subquotient, but also a subrelative connection.

We write the matrix of the connection in a basis adapted to the decomposition (3.5):

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \frac{dt}{t} + \begin{pmatrix} A & B \\ C & D \end{pmatrix} ds, \tag{3.6}$$

with coefficients of a, b, A, B, C, D in $K((t))$, $a(dt/t)$ describing the connection/K on $M_0 \otimes U_0$, so $a = \alpha \otimes N_0$ with N_0 a sum of nilpotent Jordan blocks, thus, in particular, with $\mathbb{Q}$-coefficients. The integrability condition implies

$$\partial_s a - t\partial_t A = [a, A]. \tag{3.7}$$

This implies

$$\mathrm{Tr}\ \mathrm{Res}_{t=0}\, \partial_s(a)\frac{dt}{t} = 0 \tag{3.8}$$

as $[a, A]$ has trace zero and $(\partial_t A)dt$ has residue zero. Let us denote by $\alpha_0 \in K$ the constant term in the t-expansion of $\alpha(t) \in (1/t^n)K[[t]]$. If

$$\alpha_0 \in K \setminus \mathbb{C}, \tag{3.9}$$

the condition (3.8) is not fulfilled and $(\mathcal{L}, \nabla) \otimes_{\mathcal{O}_X} K((t))$ is not a subconnection of a flat connection/$\mathbb{C}$. Now starting with $(\mathcal{L}, \nabla)$ on X/K, we can always achieve the condition (3.9). We possibly replace X by a smaller affine $X \setminus \Sigma$ so that $\Gamma(\bar{X} \setminus \Sigma, \omega(y))$ contains a differential form η so that $\alpha_0 + (\mathrm{res}_y\, \eta) \in K \setminus \mathbb{C}$, and then replace $(\mathcal{L}, \nabla)$ by $((\mathcal{L}, \nabla) \otimes (\mathcal{O}_{X\setminus\Sigma}, d + \gamma))$ on $X \setminus \Sigma$. This finishes the proof. ∎

Corollary 3.3. The kernel $N = \mathrm{Ker}(G(X/K) \to G(X/k)^{\Delta})$ is not trivial. □

We will show in Theorem 4.7 that N has a nontrivial subgroup with no $\mathbb{G}_a$ quotient.

4 The universal de Rham extension

In this section, the general assumption is as in Section 3 with the extra assumption on the transcendence degree of K over k.

Assumptions are

$$\begin{aligned} & k \text{ field of characteristic } 0, \text{ algebraically closed in } K, \\ & \operatorname{tr}\deg K/k \leq 1, \\ & f : X \longrightarrow \operatorname{Spec}(K) \text{ smooth geometrically connected variety}, \\ & g : X \longrightarrow \operatorname{Spec}(k). \end{aligned} \tag{4.1}$$

Fixing $x \in X(K)$, we have $\mathcal{C}(X/k)$ and its groupoid scheme $G(X/k)$ as in (3.1).

Let $(\mathcal{V}, \nabla)$ be a flat connection on X/k. Recall [6, Section 3], that its Gauss-Manin connection is defined as the connecting homomorphism $H^i_{DR}(X, (\mathcal{V}, \nabla/K)) \xrightarrow{GM} \Omega^1_K \otimes_K H^i_{DR}(X, (\mathcal{V}, \nabla/K))$ on the relative cohomology $H^i_{DR}(X, (\mathcal{V}, \nabla/K)) := \mathbb{H}^i(X, \Omega^\bullet_{X/K} \otimes \mathcal{V})$ of the extension

$$0 \longrightarrow \Omega^1_{K/k} \otimes_K \left(\Omega^{\bullet-1}_{X/K} \otimes_{\mathcal{O}_X} \mathcal{V}\right) \longrightarrow \Omega^\bullet_{X/k} \otimes_{\mathcal{O}_X} \mathcal{V} \longrightarrow \Omega^\bullet_{X/K} \otimes_{\mathcal{O}_X} \mathcal{V} \longrightarrow 0. \tag{4.2}$$

Remark 4.1. As mentioned in [6, (1.2)], the de Rham cohomology $H^i_{DR}(X, (\mathcal{V}, \nabla_{/K}))$ is also the right derived functor to the left exact global section functor

$$\mathrm{MIC}(X/K) \longrightarrow \mathrm{Vec}^{qc}_K, \qquad (\mathcal{V}, \nabla_{/K}) \longmapsto H^0_{DR}\left(X, \left(\mathcal{V}, \nabla_{/K}\right)\right), \tag{4.3}$$

where $\mathrm{MIC}(X/K)$ is the category of quasi-coherent modules endowed with a flat connection on X/K, the morphisms being the flat morphisms, and Vec^{qc}_K is the category of quasi-coherent K-vector spaces (i.e., of possibly infinite dimensional K-vector spaces). This category has enough injectives. One defines $\mathrm{MIC}(X/k)$ as the category of quasi-coherent modules endowed with a flat connection on X/k. This category has enough injectives. Moreover, as tacitly mentioned in [6, Remark 3.1], the restriction functor $\mathrm{MIC}(X/k) \to \mathrm{MIC}(X/K)$, $(\mathcal{V}, \nabla) \mapsto (\mathcal{V}, \nabla/K)$ sends injectives to injectives. Indeed, the sheaf of algebras of differential operators P-D Diff(X/k) (see [6, (1.2)] for the notation) is flat over its sheaf of subalgebras P-D Diff(X/K) as one can choose local coordinates in Zariski local neighborhoods. Therefore, the restriction functor has an exact left adjoint, that is, for M defined/k, with restriction to K denoted by M/K, and N defined/K, one has

$$\mathrm{Hom}_{\text{P-D Diff}(X/k)}\left(\text{P-D Diff}(X/k) \otimes_{\text{P-D Diff}(X/K)} N, M\right) = \mathrm{Hom}_{\text{P-D Diff}(X/K)}(N, M/K). \tag{4.4}$$

This implies that the restriction to K of injective modules/k is injective/K. (We thank N. Katz for explaining to us in more detail his remark.) Thus for $(\mathcal{V}, \nabla)$, an absolute connection, and $(\mathcal{V}, \nabla) \xrightarrow{\text{resolution}} I^\bullet$, an injective resolution in $\mathrm{MIC}(X/k)$, one has

$$H^i_{DR}\big(X, (\mathcal{V}, \nabla/K)\big) = H^i\big(H^0_{DR}(X, I^\bullet/K)\big). \tag{4.5}$$

On the other hand, the restriction of ∇ to $H^0_{DR}(X, I^\bullet/K) \subset \mathcal{V}$ is the Gauss-Manin connection which we denote by GM. This induces the Gauss-Manin connection on $H^i_{DR}(X, (\mathcal{V}, \nabla/K))$, which we still denote by GM. One obtains a commutative diagram

$$\begin{array}{ccc} \mathrm{MIC}(X/k) & \xrightarrow{\text{rest}} & \mathrm{MIC}(X/K) \\ {\scriptstyle H^i_{DR}(/K)}\downarrow & & \downarrow{\scriptstyle H^i_{DR}} \\ \mathrm{MIC}(K/k) & \xrightarrow{\text{rest}} & \mathrm{Vec}_K^{qc} \end{array} \tag{4.6}$$

We still denote by GM the Gauss-Manin connection on the full subcategory $\mathcal{C}(X/k) \subset \mathrm{MIC}(X/k)$. Then (4.6) contains the subcommutative square

$$\begin{array}{ccc} \mathcal{C}(X/k) & \xrightarrow{\text{rest}} & \mathcal{C}(X/K) \\ {\scriptstyle H^i_{DR}(/K)}\downarrow & & \downarrow{\scriptstyle H^i_{DR}} \\ \mathcal{C}(K/k) & \xrightarrow{\text{rest}} & \mathrm{Vec}_K \end{array} \tag{4.7}$$

Theorem 4.2. Let the assumptions be as in (4.1) and let $(\mathcal{V}, \nabla)$ be an object in $\mathcal{C}(X/k)$. Then there is an extension in $\mathcal{C}(X/k)$:

$$0 \longrightarrow (\mathcal{V}, \nabla) \longrightarrow (\mathcal{W}, \nabla) \longrightarrow \big(H^1_{DR}(X, (\mathcal{V}, \nabla/K)) \otimes_K \mathcal{O}_X, \mathrm{GM} \otimes d\big) \longrightarrow 0, \tag{4.8}$$

with the property that the connecting homomorphism

$$\begin{aligned} &H^0_{DR}\big((H^1_{DR}(X, (\mathcal{V}, \nabla/K)) \otimes_K \mathcal{O}_X, (\mathrm{GM} \otimes d)/K)\big) \\ &\quad = H^1_{DR}\big(X, (\mathcal{V}, \nabla/K)\big) \xrightarrow{\text{connecting}} H^1_{DR}\big(X, (\mathcal{V}, \nabla/K)\big) \end{aligned} \tag{4.9}$$

is the identity. □

Proof. The K-vector space $\mathcal{H}om(H^1_{DR}(X, (\mathcal{V}, \nabla/K)), H^1_{DR}(X, (\mathcal{V}, \nabla/K)))$, is endowed with the connection $\mathcal{H}om(GM, GM)$. The identity

$$1 \in \mathbf{Hom}\left(H^1_{DR}(X, (\mathcal{V}, \nabla/K)), H^1_{DR}(X, (\mathcal{V}, \nabla/K))\right) \tag{4.10}$$

is a flat section. We apply (4.2) to the flat connection

$$\mathcal{W} := \mathcal{H}om\left(H^1_{DR}(X, (\mathcal{V}, \nabla/K)) \otimes \mathcal{O}_X, \mathcal{V}\right) \tag{4.11}$$

on X/k. Thus $1 \in H^1_{DR}(X, \mathcal{W}/K)$, with $GM(1) = 0$. Thus 1 lifts to a class

$$\tilde{1} \in H^1_{DR}(X, \mathcal{W}) = H^1_{DR}\left(X, \mathcal{H}om\left(H^1_{DR}(X, (\mathcal{V}, \nabla/K)) \otimes \mathcal{O}_X, \mathcal{V}\right)\right). \tag{4.12}$$

By the standard cocycle defined in the proof of Proposition 2.2 for a class in H^1_{DR}, the class $\tilde{1}$ defines an extension (4.8) with (4.9) being the identity. ■

Corollary 4.3. Let the assumptions be as in (4.1) and let $(\mathcal{V}, \nabla)$ be an object in $\mathcal{C}(X/k)$. Then the restriction homomorphism

$$\begin{aligned} H^1\left(H, \omega((\mathcal{V}, \nabla/K))\right) &\longrightarrow H^1\left(G(X/K), \omega(\mathcal{V}, \nabla/K)\right) \\ &= (\text{Proposition 2.2})\ H^1_{DR}\left(X, (\mathcal{V}, \nabla/K)\right) \end{aligned} \tag{4.13}$$

is an isomorphism. □

Proof. As by Proposition 3.1, $\mathrm{Rep}_f(H)$ is equivalent to a full subcategory of $\mathcal{C}(X/K)$, the homomorphism

$$H^1\left(H, \omega((\mathcal{V}, \nabla/K))\right) \longrightarrow H^1\left(G(X/K), \omega(\mathcal{V}, \nabla/K)\right) \tag{4.14}$$

is injective. If now $e : 0 \to (\mathcal{V}, \nabla/K) \to (\mathcal{V}', \nabla_{/K}) \to (\mathcal{O}_X, d/K) \to 0$ is an extension in $\mathcal{C}(X/K)$ with class $\bar{e} \in H^1_{DR}(X, (\mathcal{V}, \nabla/K))$, then by Theorem 4.2, e is isomorphic to the pull-back of (4.8) via $1 \in \mathcal{O}_X \mapsto \bar{e} \in H^1_{DR}(X, (\mathcal{V}, \nabla/K))$, where (4.8) is now considered as an extension of relative connections on X/K. Consequently, $(\mathcal{V}', \nabla_{/K})$ is a subconnection of an absolute flat connection, thus $(\mathcal{V}', \nabla_{/K})$ is an object of $\mathrm{Rep}_f(H)$. This shows surjectivity and finishes the proof. ■

Definition 4.4. Define $\mathrm{Rep}_f(H) \subset \mathrm{Rep}_f(H)^t \subset \mathrm{Rep}_f(G(X/K))$ which contains $\mathrm{Rep}_f(H)$ and which is thick, that is, whenever two objects are in $\mathrm{Rep}_f(H)^t$, so is any extension.

Lemma 4.5. $\mathrm{Rep}_f(H)^t$ is defined by its objects. One successively defines Obj_n as extensions in $\mathrm{Rep}_f(G(X/K))$ of objects in Obj_{n-1}, with $\mathrm{Obj}_0 = \mathrm{Obj}(\mathrm{Rep}_f(H))$. Then the objects of $\mathrm{Rep}_f(H)^t$ consist of the union of the Obj_n. □

Proof. We just have to show that each full subcategory Obj_n, $n \geq 0$, is a Tannaka subcategory. By definition, Obj_n is obviously closed undertaking the tensor product and dual objects. Since Obj_n is full in $\mathrm{Rep}_f(G(X/K))$, to show that it is an abelian subcategory, it suffices to check that for any V in Obj_n, all its sub- and quotient objects are again in Obj_n. We show this by induction. For $n = 0$, $\mathrm{Obj}_0 = \mathrm{Rep}_f(G)$, the claim follows from the fact that $G(X/K) \to H$ is injective. In the general case, let $0 \to V_1 \to V \to V_0 \to 0$ be the extension defining V, with $V_i \in \mathrm{Obj}_{n-1}$. Let $W \subset V$. By induction, $V_1 \cap W \subset V_1$ and $W/(W \cap V_1) \subset V_0$ are in Obj_{n-1}, hence by definition of Obj_n, we have $W \in \mathrm{Obj}_n$. Similar discussion holds for quotients of V.

Remark 4.6. If H^t is the Tannaka group of $\mathrm{Rep}_f(H)^t$, thus $\mathrm{Rep}_f(H)^t = \mathrm{Rep}_f(H^t)$, then one has

$$G(X/K) \longrightarrow H^t \longrightarrow H, \tag{4.15}$$

with $G(X/K) \to H^t$ surjective since, as in the proof of Proposition 3.1, every subobject in $\mathrm{Rep}_f(G(X/K))$ of an object in $\mathrm{Rep}_f(H^t)$ is in $\mathrm{Rep}_f(H^t)$.

We define the algebraic K-group

$$N^t := \mathrm{Ker}\left(G(X/K) \twoheadrightarrow H^t\right) \subset N = \mathrm{Ker}\left(G(X/K) \twoheadrightarrow H\right). \tag{4.16}$$

One has the following theorem.

Theorem 4.7. N^t is not always trivial, and one has $H^1(N^t, \mathbb{G}_a) = 0$. □

Proof. The example of Proposition 3.2 has rank 1. Since $\mathrm{Rep}_f(H)^t$ is defined by its objects which are extensions in $\mathrm{Rep}_f(G(X/K))$ of objects in H, N^t could only be trivial if the example was an object in $\mathrm{Rep}_f(H)$, which is not. We consider the exact sequence,

$$0 \longrightarrow N^t \longrightarrow G(X/K) \xrightarrow{\iota} H^t \longrightarrow 0, \tag{4.17}$$

of K-proalgebraic groups. We consider the relative connection $(\mathcal{O}_X, d/K)$. As it is the restriction of an absolute connection, $\omega((\mathcal{O}_X, d))$ is certainly a representation of H, and then it is the trivial representation K both for H and for G(X/K). Since $\mathrm{Rep}(H^t)$ is thick in $\mathrm{Rep}(G(X/K))$, ι induces an isomorphism $H^1(H^t, K) \xrightarrow{\iota^*} H^1(G(X/K), K)$. As in the proof of **Proposition 2.2**, this implies the injectivity $H^2(H^t, K) \xrightarrow{\iota^*} H^2(G(X/K), K)$. Looking now at the long cohomology sequence with K-coefficients associated to (**4.17**), we conclude that $H^1(N^t, K) = 0$. This finishes the proof. ■

Remark 4.8. In order to get rid of the assumption on the transcendence degree of K/k being ≤ 1, one has to introduce the category of vertical connections, that is, those connections on X/k, the curvature of which lies in $\Omega^2_K \otimes \mathrm{End}(\mathcal{V})$. This is because the universal extension (**4.8**) is a priori only a vertical connection, which is not necessarily flat. This introduces correspondingly the Tannaka groups H_v, H_v^t, and so forth with the same conclusions as in **Corollary 4.3** and **Theorem 4.7**. Since we do not see any applications of this, we do not detail the construction.

Remark 4.9. If X/K is an affine curve, then the embedding $\mathrm{Rep}_f(H) \xrightarrow{\iota} \mathrm{Rep}_f(G(X/K))$ is thick. This is equivalent to saying that $H^1(H, V) \xrightarrow{\iota^*} H^1(G(X/K), V)$ is an isomorphism for all objects in $\mathrm{Rep}_f(H)$, which we show now. An object of $\mathrm{Rep}_f(H)$ is of the shape $V = V'/V''$ with $V'' \subset V' \subset W$ with $W = \omega((\mathcal{W}, \nabla))$ and $(\mathcal{W}, \nabla)$ an object in $\mathcal{C}(X/k)$.

We first assume $V' = W$. Note that for all $V \in \mathrm{Rep}_f(H)$, one has $H^0(H, V) = H^0(G(X/K), V)$, as $G(X/K) \to H$ is surjective, and $H^2(G(X/K), V) = 0$ by assumption on X and by **Proposition 2.2**. We have the following commutative diagram with exact rows $(G := G(X/K), V_1 = V'', V_0 = W/V_1)$:

$$
\begin{array}{ccccccccc}
H^0(H, V_0) & \longrightarrow & H^1(H, V_1) & \longrightarrow & H^1(H, W) & \longrightarrow & H^1(H, V_0) & \longrightarrow & H^2(H, V_1) \\
{\scriptstyle =}\downarrow & & \downarrow & & {\scriptstyle =}\downarrow & & \downarrow & & \downarrow \\
H^0(G, V_0) & \longrightarrow & H^1(G, V_1) & \longrightarrow & H^1(G, W) & \longrightarrow & H^1(G, V_0) & \longrightarrow & H^2(G, V_1) = 0
\end{array}
\tag{4.18}
$$

By fullness (**Proposition 3.1**), $H^1(H, V_a) \xrightarrow{\iota^*} H^1(G(X/K), V_a)$ is injective, $a = 0, 1$, hence bijective.

Now consider the exact sequence $0 \to V'' \to V' \to V \to 0$ with $V'' \subset V' \subset V/K$, $V \in \mathcal{C}(X/k)$, which also yields a commutative diagram as above. Here we have isomorphisms $H^1(H, V') \cong H^1(G(X/K), V')$ and the same for V''. As $H^1(H, V) \xrightarrow{\iota^*} H^1(G(X/K), V)$ is injective by fullness, then it is surjective as well. This finishes the proof. ■

5 The exact sequence of groupoids

The assumptions in this section are the same as in Section 3: $g : X \to \mathrm{Spec}(k)$ is a smooth scheme with k a field of characteristic 0, with factorization $f : X \to \mathrm{Spec}(K)$, which makes X a smooth, geometrically connected variety over the extension $K \supset k$. We assume $\mathrm{End}_{\mathcal{C}(K/k)}((K, d_{K/k})) = k$. Fixing $x \in X(K)$, we have $\mathcal{C}(X/k)$ and its groupoid scheme $G(X/k)$, as in Section 3. See also the appendix.

We recall that $G(X/k)^{\Delta}$ is the discrete groupoid scheme, pull back of $G(X/k)$ over the diagonal $\Delta \to \mathrm{Spec}(K) \times_k \mathrm{Spec}(K)$. We define similarly $G(K/k)^{\Delta}$, pull back of $G(K/k)$ over the diagonal $\Delta \to \mathrm{Spec}(K) \times_k \mathrm{Spec}(K)$. Both $G(X/k)^{\Delta}$ and $G(K/k)^{\Delta}$, as K-schemes, are K-algebraic groups.

Lemma 5.1. The homomorphism of groupoid schemes,

$$G(X/k) \xrightarrow{f^*} G(K/k), \tag{5.1}$$

is surjective, and induces a surjective homomorphism

$$G(X/k)^{\Delta} \xrightarrow{f^*} G(K/k)^{\Delta}, \tag{5.2}$$

of algebraic groups. □

Proof. The composite map $G(K/k) \xrightarrow{x^*} G(X/k) \xrightarrow{f^*} G(K/k)$ is the identity. ■

We define the progroup scheme over K:

$$L = \mathrm{Ker}\left(G(X/k)^{\Delta} \xrightarrow{f^*} G(K/k)^{\Delta}\right). \tag{5.3}$$

Since the composite of functors,

$$\mathcal{C}(K/k) \xrightarrow{f^*} \mathcal{C}(X/k) \xrightarrow{\mathrm{rest}} \mathcal{C}(X/K), \tag{5.4}$$

sends any object to a finite sum of the trivial object, the composite map of groupoid schemes,

$$G(X/K) \xrightarrow{\mathrm{rest}} G(X/k) \xrightarrow{f^*} G(K/k), \tag{5.5}$$

fulfills

$$\mathrm{rest}\left(G(X/K)\right) = H \subset L. \tag{5.6}$$

Recall from **Section 3** that $\mathrm{Rep}_f(K : G(X/K))$ denotes the category of finite dimensional K-representations of $G(X/k)$ (see the appendix for details).

Lemma 5.2 (key lemma)**.** Let $V \in \mathrm{Obj}(\mathrm{Rep}_f(K : G(X/k)))$. Then

$$H^0(L, V) = H^0\big(G(X/K), V\big). \tag{5.7}$$

□

Proof. It is clear that

$$H^0(L, V) \subset H^0\big(G(X/K), V\big). \tag{5.8}$$

We wish to show surjectivity. We first show the following claim.

Claim 5.3.

$$f^* H^0_{DR}\big(X, (\mathcal{V}, \nabla/K)\big) H^0_{DR}\big(X, (\mathcal{V}, \nabla/K)\big) \otimes_K \mathcal{O}_X \subset \mathcal{V} \tag{5.9}$$

is the largest subbundle which is stabilized by ∇ and on which

$$\nabla = f^*\delta, \quad \text{with } \delta = \nabla|_{H^0_{DR}(X,(\mathcal{V},\nabla/K))} \quad \text{so } \big(H^0_{DR}\big(X, (\mathcal{V}, \nabla/K)\big), \delta\big) \in \mathrm{Obj}\,\big(\mathcal{C}(K/k)\big). \tag{5.10}$$

□

Proof. Indeed, by flatness of ∇, the composite map

$$H^0_{DR}\big(X, (\mathcal{V}, \nabla/K)\big) \xrightarrow{\nabla} \Omega^1_K \otimes \mathcal{V} \xrightarrow{\nabla} \Omega^1_K \otimes \Omega^1_{X/K} \otimes \mathcal{V} \tag{5.11}$$

is vanishing. On the other hand, one has

$$\Omega^1_K \otimes \mathcal{V} \xrightarrow{\nabla = 1_{\Omega^1_K} \otimes (\nabla/K)} \Omega^1_K \otimes \Omega^1_{X/K} \otimes \mathcal{V}. \tag{5.12}$$

One concludes that

$$\nabla\big(H^0\big(X, (\mathcal{V}, \nabla/K)\big)\big) \subset \Omega^1_K \otimes_K \mathrm{Ker}(\nabla/K) = \Omega^1_K \otimes H^0\big(X, (\mathcal{V}, \nabla/K)\big). \tag{5.13}$$

Consequently, $H^0(X, (\mathcal{V}, \nabla/K)) \otimes_K \mathcal{O}_X \subset \mathcal{V}$ is stabilized by ∇ and lies in the largest subbundle on which ∇ is of the shape $f^*\delta$. On the other hand, it has to be the largest such, as any other $\mathcal{W} \subset \mathcal{V}$ would have the property that $(\nabla/K)|_{\mathcal{W}}$ is generated by flat sections. ■

Claim 5.3 shows that $H^0(G(X/K), V)$ is a $G(X/k)$-representation on which $G(X/k)$ acts via its quotient $G(K/k)$, thus $H^0(G(X/K), V) \subset H^0(L, V)$. This finishes the proof. ■

Corollary 5.4. If V_i, $i = 1, 2$ are objects of $\mathrm{Rep}_f(K : G(X/k))$, with restrictions V_i/K as objects of $\mathrm{Rep}_f(G(X/K))$, then

$$\mathrm{Hom}_L\left(V_1, V_2\right) = \mathrm{Hom}_{G(X/K)}\left(V_1, V_2\right). \tag{5.14}$$

□

Proof. We just set $V = V_1^\vee \otimes V_2$ and apply $\mathrm{Hom}(V_1, V_2) = H^0(V)$ in the corresponding category together with Lemma 5.2. ■

Lemma 5.5. Let N be a normal subgroup of a proalgebraic group G over $\mathrm{Spec}(K)$. Then any finite dimensional representation of N is a quotient of a finite dimensional representation of G considered as a representation of N. Consequently any finite dimensional representation of N can be embedded into a finite dimensional representation of G considered as representation of N. □

Proof. Let

$$\mathrm{Ind} : \mathrm{Rep}(N) \longrightarrow \mathrm{Rep}(G) \tag{5.15}$$

be the induced representation functor, it is the right adjoint functor to the restriction functor

$$\mathrm{Res} : \mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(N), \tag{5.16}$$

that is, one has a functorial isomorphism

$$\mathrm{Hom}_G\left(V, \mathrm{Ind}(W)\right) \xrightarrow{\cong} \mathrm{Hom}_N\left(\mathrm{Res}(V), W\right). \tag{5.17}$$

Let $A := G/N$ be the quotient group. It is well known that $\mathcal{O}(G)$ is faithfully flat over its subalgebra $\mathcal{O}(A)$ [9, Chapter 13]. This implies that the functor Ind is faithfully exact [8, Chapter 2].

Setting $V = \mathrm{Ind}(W)$ in (5.17), one obtains a canonical map $u_W : \mathrm{Ind}(W) \to W$ in $\mathrm{Rep}(N)$ which gives back the isomorphism in (5.17) as follows: $\mathrm{Hom}_G(V, \mathrm{Ind}(W)) \ni h \mapsto u_W \circ h \in \mathrm{Hom}_N(\mathrm{Res}(V), W)$. The map u_W is nonzero whenever W is nonzero. Indeed, since Ind is faithfully exact, $\mathrm{Ind}(W)$ is nonzero whenever W is nonzero. Thus, if $u_W = 0$, then both sides of (5.17) are zero for any V. On the other hand, for $V = \mathrm{Ind}(W)$, the right-hand side contains the identity map. This shows that u_W cannot vanish.

We want to show that u_W is always surjective. Let $U = \mathrm{Im}(u_W)$ and let $T = W/U$. We have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ind}(U) & \longrightarrow & \mathrm{Ind}(W) & \longrightarrow & \mathrm{Ind}(T) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U & \longrightarrow & W & \longrightarrow & T & \longrightarrow & 0 \end{array} \tag{5.18}$$

The composition $\mathrm{Ind}(W) \twoheadrightarrow \mathrm{Ind}(T) \to T$ is 0, therefore $\mathrm{Ind}(T) \to T$ is a zero map, implying $T = 0$.

Since $\mathrm{Ind}(W)$ is a union of its finite dimensional subrepresentations, we can therefore find a finite dimensional G-subrepresentation $W_0(W)$ of $\mathrm{Ind}(W)$, which still maps surjectively on W. In order to obtain the statement on the embedding of W, we dualize $W_0(W^\vee) \twoheadrightarrow W^\vee$. ■

Lemma 5.6. Let G^Δ be the diagonal subgroup of a k-groupoid G acting transitively upon Spec(K). Then any finite dimensional representation of G^Δ is a quotient of a finite dimensional representation of G considered as a representation of G^Δ. Consequently, any finite dimensional representation of G^Δ can be embedded into a finite dimensional representation of G considered as a representation of G^Δ. □

Proof. The proof of this lemma will be given in Section 6.4. ■

In our situation, we have the normal group L in the group $G(X/k)^\Delta$, which is the diagonal subgroup of the groupoid $G(X/k)$. Thus, we have the following corollary.

Corollary 5.7. Every finite dimensional representation of L can be embedded into the restriction to L of a finite dimensional representation of $G(X/k)$, consequently, it can be represented as a quotient of the restriction to L of a finite dimensional representation of $G(X/k)$. □

We are now in the position to prove the following theorem.

Theorem 5.8. The category $\mathrm{Rep}_f(L)$ is equivalent to the full subcategory $\mathcal{C}$ of $\mathcal{C}(X/K)$, the objects of which are subobjects as well as quotient objects of the restriction to K of an absolute connection. □

Remark 5.9. The objects of $\mathcal{C}$ are of the shape $\mathrm{Im}(\varphi)$, where $\varphi \in \mathrm{Hom}(V_1/K, V_2/K)$, with $V_i = \omega((\mathcal{V}_i, \nabla_i))$, $(\mathcal{V}_i, \nabla_i)$ are objects of $\mathcal{C}(X/k)$ and φ a morphism in $\mathcal{C}(X/K)$.

Proof of Theorem 5.8. Let us first remark that by definition, $\mathcal{C}$ is a full subcategory of $\mathcal{C}(X/K)$, which is trivially closed undertaking the tensor product. We do not know yet whether it is an abelian subcategory.

We denote by $\mathcal{Q} : \mathrm{Rep}_f(L) \to \mathcal{C}(X/K)$ the functor defined by the homomorphism $q : G(X/K) \to L$. By Corollary 5.7 the image of $\mathcal{Q}$ lies in $\mathcal{C}$.

Being a tensor functor, $\mathcal{Q}$ is faithful. We show that it is also full. Let U_0, U_1 be objects in $\mathrm{Rep}_f(L)$ and $\phi : \mathcal{Q}(U_0) \to \mathcal{Q}(U_1)$ a $\mathcal{C}$-morphism, that is, ϕ is a K-linear map $U_0 \to U_1$, which is only $G(X/K)$-linear where the actions of $G(X/K)$ is induced from the homomorphism $q : G(X/K) \to L$. It is to show that ϕ is in fact L-linear.

By Corollary 5.7 there are L-linear morphisms $\pi : V_0 \twoheadrightarrow U_0$ and $\iota : U_1 \hookrightarrow V_1$, where V_0 and V_1 are objects in $\mathrm{Rep}_f(K : G(X/k))$. One has $U_i = \omega((\mathcal{U}_i, \nabla_i))$ for relative connections $(\mathcal{U}_i, \nabla_i) \in \mathrm{Obj}(\mathcal{C}(X/K))$, $V_i = \omega((\mathcal{V}_i, \nabla_i))$ for absolute connections $(\mathcal{V}_i, \nabla_i) \in \mathrm{Obj}(\mathcal{C}(X/k))$ with $(\mathcal{V}_0, \nabla_0/K) \xrightarrow{\pi \twoheadrightarrow} (\mathcal{U}_0, \nabla)$, $(\mathcal{U}_1, \nabla_1) \xrightarrow{\iota \hookrightarrow} (\mathcal{V}_1, \nabla_1/K)$. We set $\psi = \iota\phi\pi$:

$$\begin{array}{ccc} U_0 & \xrightarrow{\phi} & V_0 \\ {\scriptstyle \pi\,\mathrm{surj}}\uparrow & & \downarrow{\scriptstyle \iota\,\mathrm{inj}} \\ V_0 & \xrightarrow[\psi]{} & V_1 \end{array} \tag{5.19}$$

By Corollary 5.4, the map ψ is L-linear. This implies that ϕ is L-linear as well.

We now show that each object of $\mathcal{C}$ is isomorphic to the image under $\mathcal{Q}$ of a representation of L. An object of $\mathcal{C}$ has the form $\mathrm{Im}(\varphi)$, where $\varphi : V_0/K \to V_1/K$, as in Remark 5.9. By the above discussion, φ is also in the image of $\mathcal{Q}$, hence so is $\mathrm{Im}(\varphi)$.

Thus the functor $\mathcal{Q} : \mathrm{Rep}_f(L) \to \mathcal{C}$ is fully faithful and each object of $\mathcal{C}$ is isomorphic to the image of an object of $\mathrm{Rep}_f(L)$. This shows that $\mathcal{C}$ is a tensor subcategory in $\mathcal{C}(X/K)$ and $\mathcal{Q}$ is an equivalence. ■

On the other hand, one has the following theorem.

Theorem 5.10 (Deligne). Let $(\mathcal{L}, \nabla)$ be an object in $\mathcal{C}(X/K)$, and assume there is an object $(\mathcal{V}, \nabla_{\mathcal{V}}) \in \mathcal{C}(X/k)$ so that in $\mathcal{C}(X/K)$, one has an injection $(\mathcal{L}, \nabla) \subset (\mathcal{V}, \nabla_{\mathcal{V}}/K)$. Then there is an object $(\mathcal{W}, \nabla_{\mathcal{W}}) \in \mathcal{C}(X/k)$ so that in $\mathcal{C}(X/K)$, one has a surjection $(\mathcal{W}, \nabla_{\mathcal{W}}/K) \twoheadrightarrow (\mathcal{L}, \nabla)$. □

Proof. Variant of Deligne's proof. We first assume that $(\mathcal{L}, \nabla)$ is of rank 1. Then we define the $(\mathcal{L}, \nabla)$-isotypical component $(\mathcal{W}, \nabla)$ of $(\mathcal{V}, \nabla_{\mathcal{V}}/K)$ as follows. Set $(\mathcal{V}', \nabla) = (\mathcal{V}, \nabla_{\mathcal{V}}/K) \otimes (\mathcal{L}, \nabla)^{\vee}$, which is an object in $\mathcal{C}(X/K)$. Then the inclusion $(\mathcal{L}, \nabla) \subset (\mathcal{V}, \nabla_{\mathcal{V}}/K)$ corresponds to a nontrivial section in $H^0_{DR}(X, (\mathcal{V}', \nabla))$. Set $\mathcal{V}_1 = \mathcal{V}'/(H^0_{DR}(X, (\mathcal{V}', \nabla)) \otimes \mathcal{O}_X)$, with induced connection relative to K. If $H^0_{DR}(X, (\mathcal{V}_1, \text{induced connection}/K)) = 0$, then one defines

$(\mathcal{W}, \nabla) = H^0_{DR}(X, (\mathcal{V}', \nabla)) \otimes (\mathcal{L}, \nabla)$. If not, define $\mathcal{V}'_1$ to be the inverse image of $H^0_{DR}(X, (\mathcal{V}_1, \nabla)) \otimes \mathcal{O}_X$ in $\mathcal{V}'$. The relative connection on $\mathcal{V}'$ induced from $\nabla_{\mathcal{V}}/K$ stabilizes $\mathcal{V}'_1$. Define $\mathcal{V}_2 = \mathcal{V}'/\mathcal{V}'_1$ with its induced connection. If $H^0_{DR}(X, (\mathcal{V}_2, \nabla)) = 0$, one defines, similarly as before, $(\mathcal{W}, \nabla) = (\mathcal{V}'_1$, induced connection$) \otimes (\mathcal{L}, \nabla)$. If not, we go on. So $(\mathcal{W}, \nabla)$ is the largest subconnection of $(\mathcal{V}, \nabla/K)$, which is a successive extension of $(\mathcal{L}, \nabla)$ by itself. In particular, $(\mathcal{L}, \nabla)$ is a quotient of $(\mathcal{W}, \nabla)$ as well.

On the other hand, as $\nabla_{\mathcal{V}}/K$ stabilizes $\mathcal{W}$, one has $\nabla_{\mathcal{V}}(\mathcal{W}) \subset (\Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{V})' \subset \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{V}$, where $(\Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{V})'$ denotes the inverse image of $\Omega^1_{X/K} \otimes_{\mathcal{O}_X} \mathcal{W}$ via the projection $\Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{V} \to \Omega^1_{X/K} \otimes_{\mathcal{O}_X} \mathcal{V}$. Consequently, the composite map $\mathcal{W} \to \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{V} \to \Omega^1_{X/k} \otimes_{\mathcal{O}_X} (\mathcal{V}/\mathcal{W})$, which is $\mathcal{O}_X$-linear, has values in $f^{-1}(\Omega^1_{K/k}) \otimes_K (\mathcal{V}/\mathcal{W})$. We denote by $\ell : \mathcal{W} \to f^{-1}(\Omega^1_{K/k}) \otimes_K (\mathcal{V}/\mathcal{W})$ the composite map. Integrability of $\nabla_{\mathcal{V}}$ implies that the diagram

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\ \ell\ } & f^{-1}\left(\Omega^1_{K/k}\right) \otimes_{f^{-1}(K)} (\mathcal{V}/\mathcal{W}) \\ \downarrow{\scriptstyle \nabla_{\mathcal{V}}/K} & & \downarrow{\scriptstyle 1_{\Omega^1_{K/k}} \otimes (\nabla_{\mathcal{V}}/K)|_{(\mathcal{V}/\mathcal{W})}} \\ \Omega^1_{X/K} \otimes_{\mathcal{O}_X} \mathcal{W} & \xrightarrow{\ \ell\ } & f^{-1}\left(\Omega^1_{K/k}\right) \otimes_{f^{-1}(K)} \Omega^1_{X/K} \otimes_{\mathcal{O}_X} (\mathcal{V}/\mathcal{W}) \end{array} \tag{5.20}$$

is commutative.

The right vertical map is the tensor product of the K-vector space $\Omega^1_{K/k}$ with the relative connection $(\mathcal{V}/\mathcal{W}) \xrightarrow{(\nabla_{\mathcal{V}}/K)|_{(\mathcal{V}/\mathcal{W})}} \Omega^1_{X/K} \otimes_{\mathcal{O}_X} (\mathcal{V}/\mathcal{W})$. Thus $(f^{-1}(\Omega^1_{K/k}) \otimes_{f^{-1}(K)} (\mathcal{V}/\mathcal{W}), 1_{\Omega^1_{K/k}} \otimes (\nabla_{\mathcal{V}}/K)|_{(\mathcal{V}/\mathcal{W})})$ is an object in the Ind-category spanned by $\mathcal{C}(X/K)$, or simply in $\mathcal{C}(X/K)$ if the transcendence degree of K/k is finite, and ℓ is a morphism in this category. Consequently, $\ell(\mathcal{W})$ is a subobject in $\mathcal{C}(X/K)$. This implies that $\ell(\mathcal{L})$ is a subobject in $\mathcal{C}(X/K)$ as well, and the projection $\mathrm{pr} \circ \ell(\mathcal{L})$ in all $((\mathcal{V}/\mathcal{W}), (\nabla_{\mathcal{V}}/K)|_{(\mathcal{V}/\mathcal{W})})$ obtained by K-linear projection $\Omega^1_{K/k} \to K$ is a subobject in $\mathcal{C}(X/K)$ as well. By definition of $\mathcal{W}$, this implies that $\mathrm{pr} \circ \ell(\mathcal{L}) = 0$ for all such projections, thus $\ell(\mathcal{L}) = 0$, thus $\ell(\mathcal{W}) = 0$. We conclude that $\nabla_{\mathcal{V}}$ stabilizes $\mathcal{W}$. We define $\nabla_{\mathcal{W}} = \nabla_{\mathcal{V}}|_{\mathcal{W}}$. This finishes the proof in this case.

It remains to consider the case when $(\mathcal{L}, \nabla)$ has a higher rank r. We write

$$(\mathcal{L}, \nabla) = \det(\mathcal{L}, \nabla) \otimes \wedge^{r-1}(\mathcal{L}, \nabla)^\vee, \tag{5.21}$$

which shows the existence of a surjective map

$$\left(\mathcal{W}', \nabla'/K\right) \otimes \wedge^{r-1}\left(\mathcal{V}, \nabla_{\mathcal{V}}/K\right)^\vee \longrightarrow\!\!\!\!\!\rightarrow \ (\mathcal{L}, \nabla) \subset \left(\mathcal{V}, \nabla_{\mathcal{V}}/K\right), \tag{5.22}$$

where $(\mathcal{W}', \nabla')$ is the object in $\mathcal{C}(X/k)$ constructed in the rank 1 case for $\det(\mathcal{L}, \nabla)$. We set $(\mathcal{W}, \nabla_{\mathcal{W}}) = (\mathcal{W}', \nabla') \otimes \wedge^{r-1}(\mathcal{V}, \nabla_{\mathcal{V}})^\vee$. This finishes the proof. ■

Theorems 5.8 and 5.10, together with Proposition 3.1, allow us now to conclude the following theorem.

Theorem 5.11. Assume as usual that $\mathrm{End}_{\mathcal{C}(K/k)}((K, d_{K/k})) = k$. Then $H = L$, that is the sequence of groupoid schemes,

$$G(X/K) \longrightarrow G(X/k) \xrightarrow{f^*} G(K/k) \longrightarrow 0, \tag{5.23}$$

is exact, in the sense that one sees the k-groupoid scheme, $L^\delta = \mathrm{Ker}(f^*)$, as a K-group scheme and then $\mathrm{Im}(G(X/K)) = \mathrm{Ker}(f^*)$. □

Proof. Since both categories $\mathrm{Rep}_f(H) \supset \mathrm{Rep}_f(L)$ are full subcategories of $\mathrm{Rep}_f(G(X/K))$, it is enough to identify their objects. If V is an object of $\mathrm{Rep}_f(H)$, then there are $E_1 \subset E \subset F$ in $\mathrm{Rep}_f(G(X/K))$, with $V = E/E_1$ and $F = \omega(\mathcal{F}, \nabla/K)$, with $(\mathcal{F}, \nabla)$ an object in $\mathcal{C}(X/k)$. By Theorem 5.10, the relative connection $(\mathcal{E}, \nabla)$ with $\omega((\mathcal{E}, \nabla)) = E$ is the quotient in $\mathcal{C}(X/K)$ of an absolute connection, so the connection $(\mathcal{V}, \nabla)$ with $V = \omega((\mathcal{V}, \nabla))$ is the quotient in $\mathcal{C}(X/K)$ of an absolute connection as well. Applying this result to $V^\vee$, one concludes that $(\mathcal{V}, \nabla)$ is also a subconnection in $\mathcal{C}(X/K)$ of an object in $\mathcal{C}(X/k)$. This finishes the proof. ■

We now prove the theorem which was one motivation behind the paper.

Theorem 5.12. Assume as usual that $\mathrm{End}_{\mathcal{C}(K/k)}((K, d_{K/k})) = k$ and that the transcendence degree of K/k is ≤ 1. Let $V \in \mathrm{Obj}(\mathrm{Rep}_f(K : G(X/k)))$. Then

$$H^i(L, V) = H^i\big(G(X/K), V\big) \quad \text{for } i = 0, 1. \tag{5.24}$$

□

Proof. This is an immediate consequence of Theorem 5.11 together with Corollary 4.3. ■

Corollary 5.13. Under the assumptions of Theorem 5.12, let $V = \omega((\mathcal{V}, \nabla))$ be an object of $\mathrm{Rep}_f(K : G(X/k))$. Then the extension defined in (4.8) yields an extension in $\mathrm{Rep}_f(K : G(X/k))$:

$$0 \longrightarrow V \longrightarrow \omega(\mathcal{W}, \nabla) \longrightarrow H^1_{DR}\big(X, (\mathcal{V}, \nabla/K)\big) = H^1(L, V) \longrightarrow 0, \tag{5.25}$$

with the property that the connecting homomorphism,

$$H^0\big(L, H^1(L, V)\big) = H^1(L, V) \xrightarrow{\text{connecting}} H^1(L, V), \tag{5.26}$$

is the identity. □

6 The Gauss-Manin connection from the Tannaka viewpoint

As usual, we consider an absolute connection $(\mathcal{V}, \nabla) \in \mathrm{Obj}(\mathcal{C}(X/k))$ together with its fiber functor $V = \mathcal{V}|_x \in \mathrm{Obj}(\mathrm{Vec}_K)$. The finite dimensional K-vector space $H^0(L, V)$ is a $G(K/k)$-representation in a natural way. Indeed, for $(a, b) : T \to \mathrm{Spec}(K) \times_k \mathrm{Spec}(K)$ and $g_{ab} \in G(K/k)(T)_{ab}$, consider $\widetilde{g}_{ab} \in G(X/k)(T)$ a preimage. Then (see **Appendix A.3**) $\widetilde{g}_{ab}^{-1} \circ \widetilde{g}_{ab} : a^*V \to b^*V \to a^*V$ is the identity on $a^*(V^L)$ as $\widetilde{g}_{ab}^{-1} \circ \widetilde{g}_{ab} \in L(T)_{aa}$. Thus the lifting $\widetilde{g}_{ab}$ yields a well-defined action of $G(K/k)$ on $H^0(L, V)$.

One considers the following diagram of functors:

$$\begin{array}{ccc} \mathrm{Rep}\big(K : G(X/k)\big) & \xrightarrow{H^0(L,V)} & \mathrm{Rep}\big(K : G(K/k)\big) \\ \big\downarrow & & \big\downarrow \\ \mathrm{MIC}(X/k) & \xrightarrow[H^0_{\mathrm{DR}}(X,(\mathcal{V},\nabla/K))]{} & \mathrm{MIC}(K/k) \end{array} \tag{6.1}$$

According to **Lemma 5.2**, the canonical morphism

$$H^0(L, V) \longrightarrow H^0_{\mathrm{DR}}\big(X/K, (\mathcal{V}, \nabla/K)\big) \tag{6.2}$$

is an isomorphism. Thus the above diagram is commutative. As a consequence, we obtain, canonical morphisms

$$\mathbf{R}^n_{G(X/k)} H^0(L, V) \longrightarrow \mathbf{R}^n_{\mathrm{MIC}} H^0_{\mathrm{DR}}\big(X, (\mathcal{V}, \nabla/K)\big), \tag{6.3}$$

where, on the left-hand side, the derived functor is taken in $\mathrm{Rep}(K : G(X/k))$ and on the right-hand side, the derived functor is taken in $\mathrm{MIC}(X/k)$. From **Remark 4.1** and the commutative diagram (**4.6**), we know that the right-hand side is the nth relative de Rham cohomology,

$$H^n_{\mathrm{DR}}\big(X, (\mathcal{V}, \nabla/K)\big) = \mathbf{R}^n_{\mathrm{MIC}} H^0_{\mathrm{DR}}\big(X, (\mathcal{V}, \nabla/K)\big), \tag{6.4}$$

equipped with the Gauss-Manin connection. It is a finite dimensional K-vector space, as $\mathcal{V}$ if of finite rank. Thus $H^n_{\mathrm{DR}}(X, (\mathcal{V}, \nabla/K))$ together with its Gauss-Manin connection is an object of $\mathrm{Rep}_f(K : G(K/k))$ and the homomorphism in (**6.3**) is $G(K/k)$-equivariant.

The group cohomology $H^i(L,V)$ for any L-representation V is defined as the right derived functor of the functor

$$\mathrm{Rep}(L) \xrightarrow{H^0(L,V)} \mathrm{Vec}_K^{qc}\,. \tag{6.5}$$

In case V is the restriction to L of a representation of $G(X/k)$, there exists a canonical homomorphism,

$$\mathbf{R}^n_{G(X/k)} H^0(L,V) \longrightarrow H^n(L,V), \tag{6.6}$$

defined by constructing a map from an injective resolution of V in $\mathrm{Rep}(K : G(X/k))$ to an injective resolution in $\mathrm{Rep}(L)$ (see the appendix, Lemma A.1).

Proposition 6.1. The canonical homomorphism

$$\mathbf{R}^n_{G(X/k)} H^0(L,V) \longrightarrow H^n(L,V) \tag{6.7}$$

is an isomorphism. Consequently, it induces a representation of $G(K/k)$ on $H^n(L,V)$ which has the property that the canonical homomorphism

$$H^i(L,V) \longrightarrow H^i_{DR}\big(X, (\mathcal{V}, \nabla/K)\big) \tag{6.8}$$

is $G(K/k)$-equivariant. □

Proof. According to the discussion above, it suffices to show that a representation of $G(X/k)$, which is injective (as an object in $\mathrm{Rep}(K : G(X/k))$), remains injective when considered as a representation of L. The proof is based on the following lemma which will be proved in the rest of the section.

Set $G := G(X/k)$. Let $\mathcal{O}(G)$ be the ring of regular functions on G. There is a natural action of G on $\mathcal{O}(G)$ called the left regular action, see the appendix.

Lemma 6.2. The G-representation $\mathcal{O}(G)$ restricted to G^Δ is injective in the category $\mathrm{Rep}(G^\Delta)$. □

Let us first assume this lemma. Since L is normal in G^Δ, $\mathcal{O}(G^\Delta)$ is injective as an L-representation. Indeed, $\mathcal{O}(G^\Delta)$ is faithfully flat over $\mathcal{O}(G^\Delta/L)$, [9, Chapter 16], hence by [8, Theorem 1], it is injective as an L-representation. Therefore any injective G^Δ-representation, being direct summand of a direct sum of copies of $\mathcal{O}(G^\Delta)$, remains injective when considered as an L-representation. Thus $\mathcal{O}(G)$ is also an injective L-representation.

According to Lemma A.2 in the appendix, we have the following resolution of V in Rep(K : G):

$$\begin{array}{ccc} V \otimes_t \mathcal{O}(G) & \longrightarrow V \otimes_t \mathcal{O}(G) \otimes_t \mathcal{O}(G) \longrightarrow \cdots \\ \cong \big\uparrow & \\ V & \end{array} \tag{6.9}$$

where the tensor product is taken over K and the action of K on $\mathcal{O}(G)$, indicated by the subscript $_t$, is induced from the map $t : G \to \mathrm{Spec}(K)$. On each term $V \otimes \mathcal{O}(G) \otimes \mathcal{O}(G) \otimes \cdots \otimes \mathcal{O}(G)$ of the complex, G acts by its action on the last tensor term. Hence, as a complex of G^{Δ}-modules, it is a resolution of V by injective G^{Δ}-modules.

As (6.9) is an injective resolution of V both in Rep(K : G) and in $\mathrm{Rep}(G^{\Delta})$, its cohomology computes $\mathbf{R}^* H^0(L, V)$ as well as $H^n(L, V)$. This shows Proposition 6.1. ■

The rest of this section is devoted to the proof of Lemma 6.2.

6.1 The algebra $\mathcal{O}(G^{\Delta})$

We refer to the appendix for the properties of $\mathcal{O}(G)$ and $\mathcal{O}(G^{\Delta})$.

By definition of G^{Δ}, we have

$$\mathcal{O}\big(G^{\Delta}\big) \cong \mathcal{O}(G) \otimes_{K \otimes_k K} K, \tag{6.10}$$

where $K \otimes_k K \to K$ is the product map. Then $J := \mathrm{Ker}(K \otimes_k K \to K)$ is generated by elements of the form $\lambda \otimes 1 - 1 \otimes \lambda$, $\lambda \in K$. Since $\mathcal{O}(G)$ is faithful over $K \otimes_k K$, tensoring the exact sequence $0 \to J \to K \otimes_k K \to K \to 0$ with $\mathcal{O}(G)$ over $K \otimes_k K$, one obtains an exact sequence

$$0 \longrightarrow J\mathcal{O}(G) \longrightarrow \mathcal{O}(G) \xrightarrow{\pi} \mathcal{O}\big(G^{\Delta}\big) \longrightarrow 0. \tag{6.11}$$

That is, we can identify $J \otimes_{K \otimes_k K} \mathcal{O}(G)$ with its image $J\mathcal{O}(G)$ in $\mathcal{O}(G)$.

6.2 The functor Ind

For any representation $W \in \mathrm{Rep}(G^{\Delta})$, define $\mathrm{Ind}(W)$ to be

$$\mathrm{Ind}(W) := \big(W \otimes_t \mathcal{O}(G)\big)^{G^{\Delta}}, \tag{6.12}$$

where G^{Δ} acts on W as usual and on $\mathcal{O}(G)$ through the right regular action of G on $\mathcal{O}(G)$ (i.e., $\mathcal{O}(G)$ is a right G-module). On this invariant space, G acts through the left regular action on $\mathcal{O}(G)$. Thus Ind is a functor $\mathrm{Rep}(G^{\Delta}) \to \mathrm{Rep}(K : G)$.

The space $\mathrm{Ind}(W)$ can also be given as the equalizer of the maps

$$\begin{aligned} p &: W \otimes_t \mathcal{O}(G) \xrightarrow{\rho_W \otimes \mathrm{id}} W \otimes \mathcal{O}\big(G^{\Delta}\big) \otimes_t \mathcal{O}(G), \\ q &: W \otimes_t \mathcal{O}(G) \xrightarrow{\mathrm{id} \otimes \Delta} W \otimes_t \mathcal{O}(G)\, {}_s\!\otimes_t \mathcal{O}(G) \xrightarrow{\pi} W \otimes \mathcal{O}\big(G^{\Delta}\big) \otimes_t \mathcal{O}(G), \end{aligned} \tag{6.13}$$

where $\rho_W : W \to W \otimes \mathcal{O}(G^{\Delta})$ is the coaction of $\mathcal{O}(G^{\Delta})$ on W, Δ is the coproduct on $\mathcal{O}(G)$.

Lemma 6.3. There exists a functorial isomorphism

$$\mathrm{Hom}_G\,\big(V, \mathrm{Ind}(W)\big) \cong \mathrm{Hom}_{G^{\Delta}}(V, W), \tag{6.14}$$

$V \in \mathrm{Rep}(K : G)$, $W \in \mathrm{Rep}(G^{\Delta})$, that is, Ind is the right adjoint to the functor restricting G-representations to G^{Δ}. □

Proof. The map is given by composing with the canonical projection $\mathrm{Ind}(W) \to W$, $v \otimes h \mapsto v\varepsilon(h)$. The converse map is given by $f \mapsto (f \otimes \mathrm{id})\rho_W$. ■

Lemma 6.4. The functor Ind is exact if and only if $\mathcal{O}(G)$ is injective as a G^{Δ}-module. □

Proof. Since G^{Δ} is a group scheme over a field K, its representations are a union of their subrepresentations of finite dimension over K. Therefore the injectivity of $\mathcal{O}(G)$ requires only to be checked on finite dimensional representations of G^{Δ}. For such a representation W, we have

$$\mathrm{Hom}_{G^{\Delta}}\,\big(W, \mathcal{O}(G)\big) \cong \mathrm{Hom}_{G^{\Delta}}\,\big(K, W^* \otimes \mathcal{O}(G)\big)\,\mathrm{Ind}\,\big(W^*\big). \tag{6.15}$$

Since the dualizing functor $(-)^*$ and the functor tensoring over K are exact, the claim follows. ■

Let us use the following notation of Sweedler for the coproduct on $\mathcal{O}(G)$:

$$\Delta(g) = \sum_{(g)} g_{(1)} \otimes g_{(2)}. \tag{6.16}$$

Lemma 6.5. The following map:

$$\begin{aligned} \varphi &: \mathcal{O}(G) \otimes_{K \otimes_k K} \mathcal{O}(G) \longrightarrow \mathcal{O}\big(G^{\Delta}\big) \otimes_{K\ t} \mathcal{O}(G), \\ g \otimes h &\longmapsto \sum_{(g)} \pi(g_{(1)}) \otimes g_{(2)} h, \end{aligned} \tag{6.17}$$

is an isomorphism, where π is defined in formula (6.11). □

Proof. We define the inverse map to this map. Let

$$\bar{\psi} : \mathcal{O}(G)\,{}_s\!\otimes_t \mathcal{O}(G) \longrightarrow \mathcal{O}(G) \otimes_{K\otimes_k K} \mathcal{O}(G) \tag{6.18}$$

be the map that maps $g \otimes h \mapsto \sum_{(g)} g_{(1)} \otimes \iota(g_{(2)})h$. We have for $\lambda \in K$ and for $t, s : K \to \mathcal{O}(G)$

$$\begin{aligned}
\bar{\psi}\big(t(\lambda)g\,{}_s\!\otimes_t h\big) &= \sum_{(g)} g_{(1)} \otimes \iota\big(t(\lambda)g_{(2)}\big)h \\
&= \sum_{(g)} g_{(1)} \otimes_{K\otimes_k K} s(\lambda)\iota\big(g_{(2)}\big)h \quad \text{by (A.13)} \\
&= s(\lambda) \sum_{(g)} g_{(1)} \otimes_{K\otimes_k K} \iota\big(g_{(2)}\big)h \\
&= \bar{\psi}\big(s(\lambda)g\,{}_s\!\otimes_t h\big).
\end{aligned} \tag{6.19}$$

Thus $\bar{\psi}$ maps $J\mathcal{O}(G)\,{}_s\!\otimes_t \mathcal{O}(G)$ to 0, hence factors through a map $\psi : \mathcal{O}(G^\Delta) \otimes_t \mathcal{O}(G) \to \mathcal{O}(G) \otimes_{K\otimes_k K} \mathcal{O}(G)$. Checking $\varphi\psi = \mathrm{id}$, $\psi\varphi = \mathrm{id}$ can be easily done using the property (A.14) of ι. ■

Corollary 6.6. For any $W \in \mathrm{Rep}(G^\Delta)$, one has the following isomorphism:

$$\begin{aligned}
&\Phi : \mathrm{Ind}(W) \otimes_{K\otimes_k K} \mathcal{O}(G) \cong W \otimes_t \mathcal{O}(G), \\
&\Phi(w \otimes g \otimes h) = w \otimes gh.
\end{aligned} \tag{6.20}$$

The inverse is given by

$$\begin{aligned}
&W \otimes_t \mathcal{O}(G) \longrightarrow W \otimes \mathcal{O}\big(G^\Delta\big) \otimes \mathcal{O}(G) \xrightarrow{\Psi} W \otimes_t \mathcal{O}(G) \otimes_{K\otimes_k K} \mathcal{O}(G), \\
&\Psi = \big(\mathrm{id}_W \otimes \psi\big)\big(\rho_W \otimes \mathrm{id}\big).
\end{aligned} \tag{6.21}$$

□

Proof. Tensoring the isomorphism in (6.17) with W and applying the functor $(-)^{G^\Delta}$, we obtain Φ. ■

6.3 Proof of Lemma 6.2

According to Lemma 6.4, it suffices to show the exactness of Ind. According to Corollary 6.6, the functor

$$\mathrm{Ind}(-) \otimes_{K\otimes_k K} \mathcal{O}(G) \cong (-) \otimes_t \mathcal{O}(G), \tag{6.22}$$

hence is exact. Since $\mathcal{O}(G)$ is faithfully flat over $K \otimes_k K$, Ind is faithfully exact.

Remark 6.7. The above proof for groupoid schemes is inspired by Takeuchi's proof [8] for the case of group schemes. In particular, it shows that the functor $\mathrm{Ind} : \mathrm{Rep}(G^{\Delta}) \to \mathrm{Rep}(K : G)$ is faithfully exact.

6.4 Proof of Lemma 5.6

Let W be a finite dimensional representation of G^{Δ} and $u_W : \mathrm{Ind}(W) \to W$ be the canonical map in $\mathrm{Rep}(G^{\Delta})$ (i.e., the map that corresponds to the identity map $\mathrm{id} : \mathrm{Ind}(W) \to \mathrm{Ind}(W)$ through the isomorphism in (6.14) for $V = \mathrm{Ind}(W)$). Then, as in the proof of Lemma 5.5, the faithful exactness of $\mathrm{Ind}(W)$ (Remark 6.7) implies that u_W is surjective. Thus, we can find a finite dimensional G-subrepresentation $W_0(W)$ of $\mathrm{Ind}(W)$, which maps surjectively on W. In order to obtain the statement on the embedding for W, one writes $W = (W^{\vee})^{\vee}$, applies the surjectivity $W_0(W^{\vee}) \twoheadrightarrow W^{\vee}$ we just constructed, and dualizes to $W \hookrightarrow (W_0(W^{\vee}))^{\vee}$.

Appendix

Groupoid schemes

In this appendix, we briefly recall the notions of affine groupoids and their representations which are used in the paper. Our reference is [3, Section 3].

A.1 Groupoid schemes

We fix a field k. By a k-affine scheme we mean the spectrum of a k-algebra (not necessarily finitely generated over k). Let S/k be a k-affine scheme. With this terminology, S can be taken to be the spectrum of a field extension $S = K \supset k$. An affine k-groupoid scheme acting on S is a k-affine scheme G together with two morphisms (the source and the target maps), $s, t : G \to S$, satisfying the following axioms.

(i) There exists a map $m : G\,{}_s\!\times_t G \to G$ called the product of G, satisfying the following associativity property:

$$m(m\,{}_s\!\times_t \mathrm{id}_G) = m(\mathrm{id}_G\,{}_s\!\times_t m). \tag{A.1}$$

(ii) There exists a map $\varepsilon : S \to G$ called the unit element map, satisfying the following property:

$$m(\varepsilon\,{}_s\!\times_t \mathrm{id}_G) = m(\mathrm{id}_G\,{}_s\!\times_t m) = \mathrm{id}_G\,. \tag{A.2}$$

(iii) There exists a map $\iota : G \to G$, called the inverse map, satisfying the following properties:

$$\begin{aligned} &\iota \circ s = t; \qquad \iota \circ t = s, \\ &m(\iota\ {}_s\!\times_t \mathrm{id}_G) = \varepsilon \circ s, \qquad m(\mathrm{id}_G\ {}_s\!\times_t \iota) = \varepsilon \circ t, \end{aligned} \tag{A.3}$$

where ${}_s\!\times_t$ denotes the fiber product over S with respect to the maps s and t.

Let T be a k-scheme. By definition, the category $(S(T), G(T))$ has for objects, the morphisms $T \to S$, and for morphisms between two objects $a, b : T \to S$, the morphisms $\phi : T \to G$ satisfying

$$(a, b) = (s, t)\phi : T \longrightarrow S \times S. \tag{A.4}$$

The axioms (A.1)–(A.3) for G imply that this category is a groupoid. Note that the set of all morphisms of $(S(T), G(T))$ is precisely $G(T) = \mathrm{Hom}_k(T, G)$.

A groupoid scheme G acting on S is said to be acting transitively if $(s, t) : G \to S \times S$ is a faithfully flat map. A groupoid scheme G acting on S is called discrete if the structure map (s, t) factors through the diagonal map $\Delta : S \to S \times S$ and a map $u : G \to S$. In this case, G equipped with u is an S-group scheme.

For example, define G^Δ as the pull-back of G along the diagonal map $\Delta : S \times S$:

$$\begin{array}{ccc} G^\Delta & \longrightarrow & G \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ \Delta & \xrightarrow{\text{diagonal}} & S \times_k S \end{array} \tag{A.5}$$

Then G^Δ is a discrete S-groupoid scheme, which is a subgroupoid scheme in G.

Another simple example is S, which is a groupoid acting on itself by means of the diagonal map.

A.2 Homomorphisms

A morphism of k-groupoid schemes acting on a k-scheme S is a morphism of the underlying k-schemes which is compatible with all structure maps. For instance, the unit element map $\varepsilon : S \to G$ is a morphism of groupoid schemes.

For two homomorphisms of groupoid schemes $G_i \to G$, $i = 1, 2$, there exists an obvious structure of groupoid scheme on $G_1 \times_G G_2$. In particular, we define the kernel of a homomorphism $f : G_1 \to G$ as the fiber product $\ker f := S \times_G G_1$. It is easy to see that $\ker f$ is a discrete groupoid scheme, defining a group scheme over S. Assuming that G_1 and G act transitively on S, then, by taking the fiber product with S over $S \times S$, that is, taking the diagonal group schemes, we see that $\ker f$ is isomorphic to the kernel of the homomorphism $G_1^\Delta \to G^\Delta$ of group schemes:

$$\begin{array}{ccccc} \ker f & \longrightarrow & G_1 & \xrightarrow{f} & G \\ \| & & \uparrow & & \uparrow \\ \ker f^\Delta & \longrightarrow & G_1^\Delta & \xrightarrow[f^\Delta]{} & G \end{array} \tag{A.6}$$

A.3 Representation

Let V be a quasi-coherent sheaf on S. A representation of G in V is an operation ρ, that assigns to each k-schema T and each morphism $\phi : T \to G$ a T-isomorphism,

$$\rho(\phi) : a^*V \longrightarrow b^*V, \tag{A.7}$$

where $(a, b) = (s, t)\phi$, the source and the target of ϕ, and a^* (resp., b^*) denotes the pull-back of V along a (resp., b). One requires that this operation be compatible with the composition law of the groupoid $(S(T), G(T))$ and with the base change. The latter means: for any morphism $r : T' \to T$,

$$\rho\big(r^*\phi\big) = r^*\rho(\phi). \tag{A.8}$$

In particular, one has the trivial representation of G in $R = \mathcal{O}_S$, where all morphisms $\rho(\phi)$ are identity morphisms.

A.4 Tannaka duality

Assume that G acts transitively on S, then representations of G form an abelian category which is closed undertaking the tensor product. We denote this category by $\mathrm{Rep}(S : G)$. We denote the full subcategory of $\mathrm{Rep}(S : G)$ of those representations which are of finite rank as sheaf on S by $\mathrm{Rep}_f(S : G)$. Each object of $\mathrm{Rep}_f(S : G)$ is locally free when considered as sheaf on S, and each object of $\mathrm{Rep}(S : G)$ is a filtered union of its finite rank subrepresentations. Using the inverse map, to each representation in a coherent locally free $\mathcal{O}_S$-module, one can define a representation in the dual coherent sheaf. Finally, for the trivial representation in $\mathcal{O}_S$, the set of endomorphisms is isomorphic to k. See [3, Section 3] for details.

A category with the above properties is called a tensor category over k. Conversely, for any tensor category $\mathcal{C}$ over k with a fiber functor to the category $\mathrm{Qcoh}(S)$ of quasi-coherent sheaves over S, one can construct a groupoid scheme G acting transitively on S, such that the fiber functor factors becomes an equivalence of tensor categories $\mathcal{C} \xrightarrow{\cong} \mathrm{Rep}(S : G)$. This correspondence is in fact a $1-1$ correspondence between tensor categories over k equipped with a fiber functor to $\mathrm{Qcoh}(S)$ and k-groupoids acting transitively over S, known as the Tannaka duality [3, théorème 1.12].

A.5 Representations of discrete groupoids

If G is a groupoid scheme acting discretely over S, then one can easily deduce from the definition that representations of G are in 1-1 correspondence with representations of the underlying S-group scheme. If (ρ, V) is a representation, for all commutative diagrams,

$$\begin{array}{ccc} T & \xrightarrow{g_{(a,a)}} & G \\ {\scriptstyle =}\big\downarrow & & \big\downarrow \\ T & \xrightarrow{(a,a)} & \Delta = S \end{array} \tag{A.9}$$

with T a k-scheme, then $\rho(g_{a,a})$ is an isomorphism from a^*V to itself. But $G(T) = \varnothing$ for $a \neq b$. In other words, ρ induces a representation of G, as an S-group scheme, in V. The converse is also true.

A.6 The function algebra

Let $R := \mathcal{O}(S)$ denote the algebra of regular functions on S. The groupoid structure on G induces the following structures on $\mathcal{O}(G)$. The source and the target map for G induce algebra maps $s, t : R \to \mathcal{O}(G)$. The transitivity of G on S can be rephrased by saying that $\mathcal{O}(G)$ is faithfully flat over $R \otimes_k R$ with respect to the base map $t \otimes_k s : R \otimes_k R \to \mathcal{O}(G)$.

The composition law for G induces an $R \otimes_k R$-algebra map,

$$\Delta : \mathcal{O}(G) \longrightarrow \mathcal{O}(G)\, {}_s{\otimes}_t\, \mathcal{O}(G), \tag{A.10}$$

satisfying $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$. The unit element of G induces an $R \otimes_k R$-algebra map

$$\varepsilon : \mathcal{O}(G) \longrightarrow R, \tag{A.11}$$

where $R \otimes_k R$ acts on R diagonally (i.e., $\lambda \otimes_k \mu\nu = \lambda\mu\nu$). One has

$$(\varepsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id}\,. \tag{A.12}$$

Finally, the operation which consists of taking the inverse in G induces an automorphism ι of $\mathcal{O}(G)$ which interchanges the actions t and s:

$$\iota\big(t(\lambda)s(\mu)h\big) = s(\lambda)t(\mu)\iota(h), \tag{A.13}$$

and satisfies the following equations:

$$m(\iota \otimes \mathrm{id})\Delta = s \circ \varepsilon \qquad m(\mathrm{id} \otimes \iota)\Delta = t \circ \varepsilon. \tag{A.14}$$

A representation ρ of G in V induces a map $\rho : V \to V \otimes_t \mathcal{O}(G)$, called coaction of $\mathcal{O}(G)$ on V, such that

$$\big(\mathrm{id}_V \otimes \Delta\big)\rho = \big(\rho \otimes \mathrm{id}_V\big), \qquad \big(\mathrm{id}_V \otimes \varepsilon\big)\rho = \mathrm{id}_V. \tag{A.15}$$

An R-module equipped with such an action is called $\mathcal{O}(G)$-comodule. Conversely, any coaction of $\mathcal{O}(G)$ on an R-module V defines a representation of G in V. In fact, we have an equivalence between the category of G-representations and the category of $\mathcal{O}(G)$-comodules. The discussion in the previous subsection shows that V is projective over R.

In particular, the coproduct on $\mathcal{O}(G)$ can be considered as a coaction of $\mathcal{O}(G)$ on itself and hence defines a representation of G in H, called the right regular representation.

Lemma A.1. Let G be a k-groupoid scheme acting transitively on S. Consider the function algebra $\mathcal{O}(G)$ as G-representation with respect to the left regular action. Then for any injective R-module V, the G-representation $V \otimes_R \mathcal{O}(G)$, where the action of G is given by the action of $\mathcal{O}(G)$, is an injective object in Rep(R : G). In particular, when $R = K$ is a field, Rep(K : G) has enough injective objects. □

Proof. For any $\mathcal{O}(G)$-comodules U, we have the following functorial isomorphism:

$$\begin{aligned} &\mathrm{Hom}_G\left(U, V \otimes_t \mathcal{O}(G)\right) \cong \mathrm{Hom}_R(U, V), \\ &f \longmapsto (\mathrm{id} \otimes \varepsilon) f. \end{aligned} \tag{A.16}$$

Indeed, the inverse is given by

$$g \longmapsto \rho_V \circ g, \tag{A.17}$$

where ρ_V is the coaction of $\mathcal{O}(G)$ on V.

Finally, if K is a field, all K-vector spaces are injective objects. ■

Lemma A.2. Assume that $R = K$ is a field. Let $V \in \mathrm{Rep}(K : G)$. Then the following complex is a resolution of V in $\mathrm{Rep}(S : G)$:

$$\begin{array}{ccccc} V \otimes_t \mathcal{O}(G) & \longrightarrow & V \otimes_t \mathcal{O}(G) \otimes_t \mathcal{O}(G) & \longrightarrow & \cdots \\ \cong \uparrow & & & & \\ V & & & & \end{array} \tag{A.18}$$

where the tensor product is taken over K and the index t specifies the action t of K. □

Proof. This can be done exactly as in the case of group schemes over a field [5, Chapter 4], and will be omitted. ■

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Hélène Esnault: Mathematik, Universität Duisburg-Essen, 45117 Essen, Germany
E-mail address: esnault@uni-essen.de

Phùng Hô Hai: Mathematik, Universität Duisburg-Essen, 45117 Essen, Germany
E-mail address: ho-hai.phung@uni-essen.de