KÜNNETH PROJECTORS AND CORRESPONDENCES ON OPEN VARIETIES

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Abstract. We consider correspondences on smooth quasiprojective varieties U. An algebraic cycle inducing the Künneth projector onto $H^1(U)$ is constructed. Assuming normal crossings at infinity, the existence of relative motivic cohomology is shown to imply the independence of ℓ for traces of open correspondences.

1. Introduction

Let X be a smooth, projective algebraic variety over an algebraically closed field k, and let $H^*(X)$ denote a Weil cohomology theory. The existence of algebraic cycles on $X \times X$ inducing as correspondences the various Künneth projectors $\pi^i : H^*(X) \to H^i(X)$ is one of the standard conjectures of Grothendieck, [11], [12]. It is known in general only for the cases $i = 0, 1, 2d-1, 2d$ where $d = \dim X$. The purpose of this note is to consider correspondences on smooth quasi-projective varieties U. In the first section we prove the existence of an "algebraic" Künneth projector $\pi^1 : H^*(U) \to H^1(U)$ assuming that U admits a smooth, projective completion X. The word algebraic is placed in quotes here because in fact the algebraic cycle on $X \times U$ inducing π^1 is not, as one might imagine, trivialized on $(X \setminus U) \times U$. It is only partially trivialized. This partial trivialization is sufficient to define a class in $H_c^{2d-1}(U) \otimes H^1(U)$ giving the desired projection. Of course, our cycle on $X \times U$ will be trivialized on $(X \setminus U) \times V$ for $V \subset U$ suitably small nonempty open, but our method does not in any obvious way yield a full trivialization on $(X \setminus U) \times U$. We finish this first section with some comments on π^i for $i > 1$ and some speculation, mostly coming from discussions with A. Beilinson, on how these ideas might be applied to study the Milnor conjecture that the Galois cohomology ring of the function field $H^*(k(X), \mathbb{Z}/n\mathbb{Z})$ is generated by H^1.

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In the last section, we use the existence of relative motivic cohomology [13] to prove an integrality and independence of ℓ result for the trace of an algebraic correspondence Γ on $U \times U$. We are endebted to G. Laumon for pointing out that one may endeavor to prove this using results already in the literature([5], [15], [16], and [8]) by reduction mod p and composition with a high power of Frobenius. Our objective in what follows is to show how techniques in motivic cohomology can apply to such questions, at least when the divisor at infinity $D = X \setminus U$ is a normal crossings divisor.

When the Zariski closure of the correspondence stabilizes the various strata D_I at infinity (e.g. when the correspondence is the graph of Frobenius) then the trace on $H^*(U)$ is realized as an alternating sum of traces on $H^*(D_I)$. When in addition all the intersections with the diagonals are transverse, the contribution to the alternating sum coming from points lying off U cancels, and the trace on $H^*(U)$ is just the sum of the fixed points on U.

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2. The first Künneth component

Let k be an algebraically closed field. We work in the category of algebraic varieties over k. $H^*(X)$ will denote étale cohomology with $\mathbb{Q}_\ell$-coefficients for some ℓ prime to the characteristic of k. If $k = \mathbb{C}$, we take Betti cohomology with $\mathbb{Q}$-coefficients.

Let C be a smooth, complete curve over k, and let $\delta \subset C$ be a nonempty finite set of reduced points. Let $J(C)$ be the Jacobian of C, and let $J(C, \delta)$ be the semiabelian variety which represents the functor

$$X \mapsto \{(\mathcal{L}, \phi) \mid \mathcal{L} \text{ line bundle on } C \times X, \ \deg \mathcal{L}|_{C \times k(X)} = 0 \\ \phi : \mathcal{L}|_{\delta \times X} \cong \mathcal{O}_{\delta \times X}\}/ \cong, \tag{2.1}$$

where the equivalence relation $\cong$ consists of the line bundle isomorphisms commuting with ϕ. There is an exact sequence

$$0 \to T \to J(C, \delta) \to J(C) \to 0 \tag{2.2}$$

where T is the torus $\Gamma(\delta, \mathcal{O}^\times)/\Gamma(C, \mathcal{O}^\times)$. By abuse of notation we shall write $J(C, \delta)$ rather than $J(C, \delta)(k)$. We can identify the character group $\mathrm{Hom}(T, \mathbb{G}_m)$ with $\mathrm{Div}^0_\delta(C)$, the group of 0-cycles of degree 0 supported on δ. A split subgroup $\Delta \subset \mathrm{Div}^0_\delta(C)$ corresponds to a quotient $T \twoheadrightarrow T_\Delta = T/\ker \Delta$, where $\ker \Delta \subset T$ is the subtorus killed by all characters in Δ. We may push out (2.2) and define $J(C,\Delta) := J(C,\delta)/\ker\Delta$:

$$0 \to T_\Delta \to J(C,\Delta) \to J(C) \to 0. \tag{2.3}$$

The functor represented by $J(C,\Delta)$ is the following quotient of (2.1)

$$\begin{aligned} X \mapsto \{(\mathcal{L},\phi) \mid \mathcal{L} \text{ line bundle on } X\times C,\ \deg \mathcal{L}|_{k(X)\times C} = 0 \\ \phi : \otimes_i \mathcal{L}^{\otimes n_i}|_{X\times\{c_i\}} \cong \mathcal{O}_X \text{ for all } \sum n_i c_i \in \Delta\}/\cong . \end{aligned} \tag{2.4}$$

These trivializations should be compatible in an evident way with the group law on Δ.

Lemma 2.1. *We write $H^1(C,\delta) = H^1_c(C\setminus\delta)$. Define*

$$H^1(C,\Delta) := (H^1(C,\delta)/\Delta^\perp)\otimes\mathbb{Q}_\ell,$$

where

$$\Delta^\perp\otimes\mathbb{Q}_\ell \subset \mathbb{Q}_\ell[\delta]/\mathbb{Q}_\ell \subset H^1(C,\delta;\mathbb{Q}_\ell)$$

is perpendicular to $\Delta \subset \mathrm{Div}^0_\delta(C)$ under the evident coordinatewise duality. There is a well defined first Chern class $c_1(\mathcal{L}_\Delta)$ of the universal Poincaré bundle $\mathcal{L}_\Delta$ on $J(C,\Delta)\times C$ which lies in $H^1(J(C,\Delta))(1)\otimes H^1(C,\Delta)$.

Proof. Let $I_\delta \subset \mathcal{O}_{J(C,\Delta)\times C}$ be the ideal of $J(C,\Delta)\times\delta$. Let $\pi : C \to C'$ be the singular curve obtained from C by gluing all the points of δ to a single point $\delta' \in C'$. Define $M_\Delta \subset (1\times\pi)_*(\mathcal{O}^\times_{J(C,\Delta)\times C})/k^\times$ to be the pullback as indicated:

$$\begin{array}{ccccccccc} 0 & \to & (1\times\pi)_*(1+I_\delta) & \to & M_\Delta & \to & (\ker\Delta)_{J(C,\Delta)\times\delta'} & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & (1\times\pi)_*(1+I_\delta) & \to & \frac{(1\times\pi)_*\mathcal{O}^\times_{J(C,\Delta)\times C}}{k^\times} & \to & \frac{(1\times\pi)_*\mathcal{O}^\times_{J(C,\Delta)\times\delta}}{k^\times} & \to & 0. \end{array} \tag{2.5}$$

Using thar $R^1(1\times\pi)_*\mathbb{G}_m = (0)$, it is straightforward to check that pairs $(\mathcal{L},\phi)$ in (2.4) with $X = J(C,\Delta)$ correspond to M_Δ torsors on $J(C,\Delta)\times C'$. In particular, we have a class $[\mathcal{L}_\Delta] \in H^1(J(C,\Delta)\times C', M_\Delta)$ corresponding to the Poincaré bundle.

One gets a diagram of Kummer sequences of sheaves on $J(C,\Delta)\times C'$ (Here $j : C \setminus \delta \hookrightarrow C$)

(2.6)
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (1\times\pi)_*(1\times j)_!\mu_{\ell^n} & \to & (1\times\pi)_*(1+I_\delta) & \xrightarrow{\ell^n} & (1\times\pi)_*(1+I_\delta) & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M_{\Delta,\ell^n} & \to & M_\Delta & \xrightarrow{\ell^n} & M_\Delta & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & ((\ker\Delta)_{J(C,\Delta)\times\delta'})_{\ell^n} & \to & (\ker\Delta)_{J(C,\Delta)\times\delta'} & \xrightarrow{\ell^n} & (\ker\Delta)_{J(C,\Delta)\times\delta'} & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We have $[\mathcal{L}_\Delta] \in H^1(J(C,\Delta)\times C', M_\Delta)$ and so by the Kummer coboundary, $c_1(\mathcal{L}_\Delta) \in \varprojlim_n H^2(J(C,\Delta)\times C', M_{\Delta,\ell^n})$. But $M_{\Delta,\ell^n} \cong \mathbb{Z}/\ell^n_{J(C,\Delta)} \boxtimes \psi_{\ell^n}$, where ψ_{ℓ^n} fits into an exact sequence of sheaves on C'

$$0 \to \pi_* j_! \mu_{\ell^n} \to \psi_{\ell^n} \to (\ker\Delta)_{\delta',\ell^n} \to 0. \tag{2.7}$$

We can identify $\Delta^\perp \otimes \mu_{\ell^n}$ with $(\ker\Delta)_{\ell^n}$. the exact cohomology sequence from (2.7) yields

$$(\ker\Delta)_{\mu_{\ell^n}} \to H^1(C,\delta;\mu_{\ell^n}) \to H^1(C',\psi_{\ell^n}) \to 0. \tag{2.8}$$

Passing to the limit over n, it now follows that we may define $c_1(\mathcal{L}_\Delta) \in H^1(J(C,\Delta)) \otimes H^1(C,\Delta)(1)$ as in the statement of the lemma. (Note that $\mathcal{L}_\Delta$ is trivial on $(0)\times C$. Further we are free to replace $\mathcal{L}_\Delta$ by $\mathcal{L}_\Delta \otimes (\mathcal{M} \boxtimes \mathcal{O}_C)$ for $\mathcal{M}$ a line bundle on $J(C,\Delta)$. We may therefore assume the Künneth components of $c_1(\mathcal{L}_\Delta)$ in degrees $(2,0)$ and $(0,2)$ vanish.) $\square$

Lemma 2.2. *Suppose given a morphism $\rho : X \to J(C)$. Let Ξ be a Cartier divisor on $C\times X$ representing ρ. We assume Ξ is flat over C so we may define a correspondence $\Xi_* : \mathrm{Div}(C) \to \mathrm{Div}(X)$. Let $U \subset X$ be nonemtpy open in X. Then there exists a lifting $\rho_{U,\Delta} : U \to J(C,\Delta)$ of ρ if and only if $(\Xi|_{C\times U})_*(\Delta) \subset \mathrm{Div}(U)$ consists of principal divisors. The set of such liftings is a torsor under $\mathrm{Hom}(\Delta, \Gamma(U, \mathcal{O}_U^\times))$.*

Proof. Choose a basis $z_i = \sum_j n_{ij} c_j$ for the free abelian group Δ. Write $\mathcal{O}_{C\times X}(\Xi)_{z_i\times X} := \otimes_j \mathcal{O}_{C\times X}(\Xi)^{\otimes n_{ij}}|_{\{c_j\}\times X}$. The assumption that $\Xi_*(\Delta)$ consists of principal divisors is precisely the assumption that all the line bundles $\mathcal{O}_{C\times X}(\Xi)_{z_i\times X}|_U$ are trivial. The choice of the trivializations for a basis of Δ yields the choice of the desired lifting $\rho_{U,\Delta}$. $\square$

Lemma 2.3. *Assume X is a smooth variety, and let $\rho : X \to J(C)$ be as above. Suppose $U \subset X$ is a dense open set such that $\rho|_U$ admits a lifting $\rho_{U,\Delta} : U \to J(C,\Delta)$. Let $\mathrm{Div}^0_{X\setminus U}(X)$ be the free abelian group on Cartier divisors supported on $X \setminus U$ which are homologous to 0 on X. Then we get a commutative diagram on cohomology*

$$\begin{array}{ccccccccc} 0 & \to & H^1(J(C)) & \to & H^1(J(C,\Delta)) & \to & \Delta\otimes\mathbb{Q}_\ell(-1) & \to & 0 \\ & & \downarrow{\scriptstyle \rho^*} & & \downarrow{\scriptstyle \rho^*_{U,\Delta}} & & \downarrow{\scriptstyle a} & & \\ 0 & \to & H^1(X) & \to & H^1(U) & \to & \mathrm{Div}^0_{X\setminus U}(X)\otimes\mathbb{Q}_\ell(-1) & \to & 0. \end{array} \tag{2.9}$$

Proof. The left hand square is commutative by functoriality. That the cokernels on the top and bottom row are as indicated follows on the top row from the Leray spectral sequence for the projection $\pi : J(C,\Delta) \to J(C)$ and on the bottom from the localization sequence which may be written

$$0 \to H^1(X) \to H^1(U) \to H^2_{X\setminus U}(X) \to H^2(X). \tag{2.10}$$

The identification $H^2_{X\setminus U}(X) \cong \mathrm{Div}_{X\setminus U}(X)\otimes\mathbb{Q}_\ell(-1)$ is saying that by purity, the Gysin homomorphism is an isomorphism. $\square$

Remark 2.4. (i) Fixing $\rho_{U,\Delta}$ amounts to fixing trivializations of the restriction $\mathcal{O}_{C\times X}(\Xi)_{z_i\times X}|_U$ as above. Such trivializations exhibit

$$\mathcal{O}_{C\times X}(\Xi)_{z_i\times X} \cong \mathcal{O}_X(D_i)$$

for some divisor D_i with support on $X \setminus U$. The map labeled a in (2.9) sends $z_i \mapsto D_i$.
(ii) The diagram

$$\begin{array}{ccc} \Delta & \longrightarrow & J(C) \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \rho^*} \\ \mathrm{Div}^0_{X\setminus U}(X) & \longrightarrow & \mathrm{Pic}^0(X) \end{array} \tag{2.11}$$

is commutative, where the horizontal arrows are cycle classes. Indeed, both a and ρ^* are defined by the divisor on $C\times X$. Note that a depends on the choice of $\rho_{U,\Delta}$ but only up to rational equivalence.

Now suppose X is smooth, projective, of dimension d. Let $U \subset X$ be a dense open subset. Write $X\setminus U = D\cup Z$ where $D \subset X$ is a divisor and $\mathrm{codim}(Z \subset X) \geq 2$. We have $H^1(X \setminus D) \cong H^1(U)$. Since we are interested in $H^1(U)$, we may assume $U = X \setminus D$ is the complement of a divisor.

Let $i : C \hookrightarrow X$ be a general linear space section of dimension 1, and let $\delta = C \cap D$. We may choose $\rho : X \to J(C)$ such that the composition

$$\mathrm{Pic}^0(X) \xrightarrow{i^*} J(C) \xrightarrow{\rho^*} \mathrm{Pic}^0(X) \tag{2.12}$$

is multiplication by an integer $N \neq 0$. Indeed, let H be a very ample line bundle so that C is the $(d-1)$-fold product of general sections of H. Intersection with H yields an isogeny $\mathrm{Pic}^0(X) \to \mathrm{Alb}(X)$, which defines an inverse isogeny $\mathrm{Alb}(X) \to \mathrm{Pic}^0(X)$ of degree N. We pull back the Poincaré bundle from $J(C) \times J(C)$ to $C \times X$ via the composite map $C \times X \to J(C) \times \mathrm{Alb}(X) \to J(C) \times \mathrm{Pic}^0(X) \to J(C) \times J(C)$, where the first map is the cycle map, the second one is $1\times$isogeny, the third one is $1\times$ restriction. We define $\mathcal{O}_{C\times X}(\Xi)$ to be the inverse image of the Poincaré bundle. The morphism $\rho : X \to J(C)$ is the correspondence $x \mapsto \mathcal{O}_{C\times X}(\Xi)|_{C\times\{x\}}$ and does not depend on the choice of the section Ξ.

Consider the diagram

(2.13)
$$\begin{array}{ccccccc} 0 \to & \mathbb{Q}_\ell[\delta]/\mathbb{Q}_\ell & \to & H^1_c(C \setminus \delta) & \to & H^1(C) & \to 0 \\ & \downarrow b & & \downarrow i_* & & \downarrow i_* & \\ 0 \to & \frac{H^{2d-2}(D)}{H^{2d-2}(X)}(d-1) & \to & H^{2d-1}_c(U)(d-1) & \to & H^{2d-1}(X)(d-1) & \end{array}$$

Here, the rows are long exact sequences associated to restriction to closed subsets, and the vertical arrows are Gysin maps. The map b can be described as follows. The $\mathbb{Q}_\ell$-vector space $H^{2d-2}(D)(d-1)$ has basis the irreducible components of D, and $b(x)$ is the basis element $[D_x]$ associated to the unique component D_x of D containing x. We have dual exact sequences (defining $\mathrm{Div}^0_D(X)$)

$$0 \to \mathrm{Div}^0_D(X) \to H_{2d-2}(D)(1-d) \to H^2(X)(1) \tag{2.14}$$

$$H^{2d-2}(X)(d-1) \to H^{2d-2}(D)(d-1) \to \frac{H^{2d-2}(D)}{H^{2d-2}(X)}(d-1) \to 0.$$

If we view $\mathbb{Q}_\ell[\delta]$ and $H^{2d-2}(D)(d-1)$ as endowed with symmetric pairings with orthonormal bases the points $x \in \delta$ and the cohomology classes of irreducible components $D_i \subset D$, then b is adjoint to the map $D_i \mapsto D_i \cdot \delta$. We conclude

Lemma 2.5. *Define* $\mathrm{Div}^0_D(X)$ *to be the* $\mathbb{Q}_\ell$*-vector space spanned by divisors on* X *supported on* D *and homologous to* 0 *on* X*. Define* $\Delta \subset \mathrm{Div}^0_\delta(C)$ *to be the image of* $\mathrm{Div}^0_D(X)$ *under pullback* i^**. Then* *there is a commutative diagram*

(2.15)

$$\begin{array}{ccccccc} 0 \to & \mathbb{Q}_\ell[\delta]/\Delta^\perp & \to & H^1_c(C,\Delta) & \to & H^1(C) & \to 0 \\ & \downarrow b & & \downarrow i_* & & \downarrow i_* & \\ 0 \to & \frac{H^{2d-2}(D)}{H^{2d-2}(X)}(d-1) & \to & H^{2d-1}_c(U)(d-1) & \to & H^{2d-1}(X)(d-1). & \end{array}$$

Proof. The map b is dual to the restriction map $\mathrm{Div}^0_D(X) \xrightarrow{i^*} \mathrm{Div}^0_\delta(C)$. By definition $\Delta^\perp$ is orthogonal to the image of i^*, i.e. $\Delta^\perp = \ker b$. $\square$

Lemma 2.6. *Let $\Delta = i^*(\mathrm{Div}^0_D(X)) \subset \mathrm{Div}^0_\delta(C)$ be as in Lemma 2.5. Then ρ defined in* (2.12) *lifts to some $\rho_{U,\Delta} : U \to J(C,\Delta)$.*

Proof. The correspondence defined by $\mathcal{O}_{C\times X}(\Xi)$ in (2.12) carries $\mathcal{O}_C(z)$ for $z \in \Delta = i^*(\mathrm{Div}^0_D(X))$ to line bundles in $\mathrm{Pic}^0(X)$, the classes of which fall in the image of $\rho^* i^*(\mathrm{Div}^0_D(X)) \equiv N \cdot \mathrm{Div}^0_D(X)$ in $\mathrm{Pic}^0(X)$. To be more precise, let D_p be a basis for $\mathrm{Div}^0_D(X)$, and set $z_p = i^* D_p$. This is a basis of Δ. Then $\mathcal{O}_{C\times X}(\Xi)|_{z_p\times X} = \mathcal{O}_X(D_p)$. Thus choose $\rho_{U,\Delta}$ in Lemma 2.2 using this trivialization on U. $\square$

Using the Lemmas 2.1, 2.5, 2.6 together with (2.9), we pull back

$$(2.16)\quad c_1(\mathcal{L}_\Delta) \in H^1(C,\Delta)\otimes H^1(J(C,\Delta))(1) \xrightarrow{i_*\otimes\rho^*_{U,\Delta}} H^{2d-1}_c(U)(d)\otimes H^1(U) \cong H^1(U)^\vee \otimes H^1(U)$$

and define a correspondence $\Phi : H^1(U) \to H^1(U)$.

Lemma 2.7. *The map Φ is the multiplication by N.*

Proof. We consider Φ. It acts on $H^1(U)$, comptibly with the exact sequence

$$(2.17)\qquad 0 \to H^1(X) \to H^1(U) \to \mathrm{Div}^0_D(X)(-1) \to 0$$

By definition of $\rho_{U,\Delta}$, it is equal to $N\cdot\mathrm{Id}$ on $H^1(X)$ and on $\mathrm{Div}^0_D(X)(-1)$. Thus $\Phi - N\cdot\mathrm{Id}$ is a correspondence from $\mathrm{Div}^0_D(X)(-1)$ to $H^1(X)$. We use purity in the sense of Deligne. There is no nontrivial correspondence $\mathrm{Div}^0_D(X)(-1) \to H^1(X)$. If $k = \mathbb{C}$ and we consider Betti cohomology, $\mathrm{Div}^0_D(X)(-1)$ is pure of weight 2 while $H^1(X)$ is pure of weight 1. If k is the algebraic closure of a finite field, we have the same conclusion. Otherwise, all the objects used are defined over a finitely generated field k over a finite field k_0. By Cebotarev theorem, the Galois group of k/k_0 is generated by Frobenii, so we may make sense of the notion of weight for $H^1(U)$. We conclude as in the complex case. $\square$

We now express in terms of cycles the trivialization of $\mathcal{O}_{C\times X}(\Xi)_{p\times X} = \mathcal{O}_X(D_p)$ used in the proof of Lemma 2.6.

Theorem 2.8. *With notation as above, there exists a cycle Γ on $X\times U$ of dimension $d = \dim X$ together with rational functions f_μ on X for each divisor μ homologically equivalent to 0 on X and supported on $D = X_U$ such that $pr_{2*}(\Gamma\cdot(\mu\times U)) = (f_\mu)$. The data $(\Gamma, \{f_\mu\})$ define a class in $H^{2d-1}_c(U)\otimes H^1(U)$ which gives the identity map on $H^1(U)$.*

We close this section with a comment about Künneth projectors $\pi^i : H^*(U)\to H^i(U)$ for $i>1$. We consider the somewhat weaker question of the existence of an algebraic projector when we localize at the generic point of the target, i.e. we consider $H^*(U)\to H^i(U)\to \varinjlim_{V\subset U} H^i(V)$. We assume $U = X\setminus D$ with X smooth, projective, and D a Cartier divisor.

Proposition 2.9. *Let $n<\dim X$ be an integer. Let $Y\subset X$ be a multiple hyperplane section of dimension n which is general with respect to D. Write $\delta = Y\cap D$. Then the restriction map*

$$H^{n+1}_D(X)\to H^{n+1}_\delta(Y) \tag{2.18}$$

is injective.

Proof. Let $d = \dim(X)$. By duality, we have to show surjectivity of the Gysin map $H^{n-1}(\delta)\to H^{2d-(n+1)}(D)(d-n)$. More generally, one has

Theorem 2.10 (P. Deligne)**.** *Let $\mathcal{F}$ be a ℓ-adic sheaf on $\mathbb{P}^N$. Then there exists a non-empty open set $U\subset(\mathbb{P}^N)^\vee$ such that for $\iota : A\hookrightarrow\mathbb{P}^N$ a hyperplane section corresponding to a point of U, the Gysin homomorphism*

$$H^{i-2}(A,\iota^*\mathcal{F})(-1)\to H^i_A(\mathbb{P}^N,\mathcal{F}),$$

is an isomorphism for all i. In particular, if $V\subset\mathbb{P}^N$ is a projective variety, then the Gysin homomorphism $H^i(A\cap V)\to H^{i+2}(V)(1)$ is an isomorphism for $i>\dim(A\cap V)$ and surjective for $i=\dim(A\cap V)$ for a non-empty open set of A.

The proof of the general theorem is written in [7], Theorem 2.1. Applied to $\mathcal{F} = a_*\mathbb{Q}_\ell$, where $a : V\to\mathbb{P}^N$ is the projective embedding, it shows that the Gysin isomorphism $H^i(A\cap V)\to H^{i+2}_{A\cap V}(V)(1)$ is an isomorphism. Then the application follows from Artin's vanishing theorem $H^i(V\setminus(A\cap V)) = 0$ for $i>\dim(V)$. $\square$

Let L be the Lefschetz operator on $H^*(X)$. One of the standard conjectures ($B(X,L)$ in [12]) is the existence of an algebraic correspondence Λ which is a "weak inverse" to L. Assume now that this standard conjecture B is true for X and for all smooth linear space sections $Y \subset X$. The strong Lefschetz theorem implies that $L^{d-n} : H^n(X) \overset{\cong}{\to} H^{2d-n}(X)(d-n)$. Assuming $B(X,L)$, $\Lambda^{d-n} = (L^{d-n})^{-1} : H^{2d-n}(X)(d-n) \cong H^n(X)$. Write $P = \Lambda^{d-n}|_{Y\times X}$. It is easy to check that the composition

$$H^n(X) \xrightarrow{i^*} H^n(Y) \xrightarrow{P} H^n(X) \tag{2.19}$$

is the identity, so $H^n(Y) = \mathrm{Image}(i^*) \oplus \ker(P)$. Consider the diagram

$$\begin{array}{ccccc} H^n(X) & \to & H^n(U) & \xrightarrow{a} & H_D^{n+1}(X) \\ \downarrow{\scriptstyle i^*} & & \downarrow & & \downarrow{\scriptstyle b} \\ H^n(Y) & \xrightarrow{d} & H^n(Y \setminus \delta) & \xrightarrow{c} & H_\delta^{n+1}(Y) \end{array} \tag{2.20}$$

Define

$$H^n(Y\setminus\delta)^0 = \{x \in H^n(Y\setminus\delta) \mid c(x) \in \mathrm{Im}(b\circ a)\}. \tag{2.21}$$

As a consequence of proposition 2.9 and (2.20) we see that $H^n(U) \twoheadrightarrow H^n(Y\setminus\delta)^0/d(\ker(P))$, and the kernel of this map is the image in $H^n(U)$ of elements $x \in H^n(X)$ such that $i^*x \in \ker(P) \oplus \mathrm{Image}(H^n_\delta(Y) \to H^n(Y))$. For such an x, it will necessarily be the case that $x = P(i^*x)$ is supported on a proper closed subset of X. In particular, for some $V \subset U$ open dense, P will induce a map

$$P_U : H^n(Y\setminus\delta)^0 \to H^n(V). \tag{2.22}$$

The map $i^* : H^n(U) \to H^n(Y\setminus\delta)^0$ dualizes to $i_* : (H^n(Y\setminus\delta)^0)^\vee \to H_c^{2d-n}(U)$, so we may define

$$(i_*\otimes P_U) : H^n(Y\setminus\delta)^0)^\vee \otimes H^n(Y\setminus\delta)^0 \to H_c^{2d-n}(U)\otimes H^n(V). \tag{2.23}$$

Let $T \subset H^n(Y\setminus\delta)^0$ be the subgroup of cohomology classes supported in codimension 1. Assuming inductively that we are able to define an algebraic correspondence on Y which carries a class

$$\gamma \in (H^n(Y\setminus\delta)^0)^\vee \otimes (H^n(Y\setminus\delta)^0/T) \tag{2.24}$$

corresponding to the evident map $H^n(Y\setminus\delta)^0 \twoheadrightarrow H^n(Y\setminus\delta)^0/T$, it would follow since $P_U(T) \subset \ker(H^n(V) \to \varinjlim_{V\subset U} H^n(V))$ that we could view

$$(i_*\otimes P_U)(\gamma) \in H_c^{2d-n}(U) \otimes \varinjlim_{V\subset U} H^n(V). \tag{2.25}$$

This correspondence would have the desired properties.

One interest in pursuing this line of investigation concerns the Milnor conjecture that the Galois cohomology with $\mathbb{Z}/\ell\mathbb{Z}$-coefficients prime to the residue characteristic is generated as an algebra by H^1. There is a geometric proof of this result in top degree [2], so, for example, elements in $H^n(Y \setminus \delta)$ lie in the subalgebra generated by H^1 after localization. If P_U exists as an algebraic correspondence, then using the existence of a norm in Milnor K-theory, one could show that the Milnor conjecture was true for $H^*(k(X), \mathbb{Z}/\ell\mathbb{Z})$ for almost all ℓ. (The condition on ℓ arises because the standard conjectures only make sense after tensoring with $\mathbb{Q}$.) Here, the idea that cohomology classes in degree n might come by correspondence from an algebraic variety of dimension $\leq n$ was suggested to us by Alexander Beilinson.

3. Open correspondences

The aim of this section is to give a simple motivic proof of the independence of ℓ or of a complex embedding of a ground field k of the trace for open correspondences. If we assume that k is finite, then, as conjectured by Deligne, high Frobenius power twists move the correspondence to a general position correspondence and the local factors have been computed in [5], [15], [16], [8]. Surely in this case the simple observations which follow are weaker.

We consider open correspondences. This means the following. Let X be a smooth projective variety of dim d over an algebraically closed field k, and let $U \subset X$ be a nontrivial open subvariety, with complement $D = X \setminus U$. One considers codim d cycles $\Gamma \subset U \times U$ which have the property that they induce a correspondence $\Gamma_* : H^i(U) \to H^i(U)$ or equivalently $H^i_c(U) \to H^i_c(U)$ for all i. Here cohomology is étale $\mathbb{Q}_\ell$ cohomology or Betti cohomology if $k = \mathbb{C}$ and we denote by p_i : $X \times X \to X$ the two projections. We write

$$\Gamma = \sum n_j \Gamma_j \tag{3.1}$$

where Γ_j is irreducible, $n_j \in \mathbb{Z}$ and define

$$\bar{\Gamma} := \sum n_j \bar{\Gamma}_j, \ \Gamma_j \subset U \times U \tag{3.2}$$

where $^-$ is the Zariski closure in $X \times X$. We will use the following facts ([4], Théorème 2.9).

Proposition 3.1. *Let $q : \Gamma \to U$ be a proper map of quasi-projective varieties. Then one has a pull-back map*

$$q^* : H^i_c(U) \to H^i_c(\Gamma). \tag{3.3}$$

Let

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{\iota} & \Gamma' \\
{\scriptstyle p}\downarrow & & {\scriptstyle p'}\downarrow \\
U & \xrightarrow{=} & U
\end{array}
\tag{3.4}
$$

be a commutative diagram of quasi-projective varieties of dimension d, with ι an open embedding, p' proper and U smooth. Then there is are push-down maps p_*, p'_* making the following diagram commutative

$$
\begin{array}{ccc}
H^i_c(\Gamma) & \xrightarrow{\iota^*} & H^i_c(\Gamma') \\
{\scriptstyle p_*}\downarrow & & {\scriptstyle p'_*}\downarrow \\
H^i_c(U) & \xrightarrow{=} & H^i_c(U)
\end{array}
\tag{3.5}
$$

Definition 3.2. If $p_2|_{(\Gamma)_j} : \Gamma_j \to U$ is proper for all j, or equivalently $\Gamma_j \subset X \times U$ is Zariski closed or equivalently if

$$\bar{\Gamma}_j \cap (D \times X) \subset X \times D \ \forall\ j, \tag{3.6}$$

then one defines

$$(\Gamma_j)_* : H^i_c(U) \xrightarrow{p_2^*} H^i_c(\Gamma_j) \xrightarrow{(p_1)_*} H^i_c(U), \tag{3.7}$$

and call it the open correspondence defined by Γ_j. The correspondence defined by Γ is then by definition $\Gamma_* = \sum n_j(\Gamma_j)_*$.

(The ordering (p_1, p_2) here is chosen as in [10].)

Remark 3.3. If we compare this condition to the one yielding to Künneth correspondences in section 2, then it is much stronger. Indeed, Theorem 2.8 yields a cycle Γ which meets physically $D \times U$, but cohomologically it washs out, while in this section we handle the case where there is physically no intersection.

Remark 3.4. We have $\Gamma_j = \bar{\Gamma}_j \backslash (X \times D)$ and we set $\Gamma'_j = \bar{\Gamma}_j \backslash (D \times X)$. Thus $p_1|_{\Gamma'_j} : \Gamma'_j \to U$ is projective. One has the following commutative diagram

$$
\begin{array}{ccccccc}
H^i_c(U) & \xrightarrow{p_2^*} & H^i_c(\Gamma_j) & \xrightarrow{\iota^*} & H^i_c(\Gamma'_j) & \xrightarrow{(p_1)_*} & H^i_c(U) \\
{\scriptstyle =}\downarrow & & \downarrow & & \downarrow & & {\scriptstyle =}\downarrow \\
H^i_c(U) & \to & H^{i+2d}_{c,\Gamma_j}(X \times U, d) & & H^{i+2d}_{c,\Gamma'_j}(U \times X, d) & \to & H^i_c(U) \\
{\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & \downarrow & & {\scriptstyle =}\downarrow \\
H^i_c(U) & \xrightarrow{p_2^*} & H^{i+2d}_{c,\Gamma_j}(U \times U, d) & \xrightarrow{\iota^*} & H^{i+2d}_{c,\Gamma'_j}(U \times X, d) & \xrightarrow{(p_1)_*} & H^i_c(U)
\end{array}
\tag{3.8}
$$

with factorization of the upper horizontal composition of maps, which is $\Gamma_{j,*}$, through the lower horizontal composition of maps. So from far left to far right on the bottom line, this is the correspondence $\Gamma_{j,*}$.

We assume now that we have the assumption as in Definition 3.2 and we wish to give conditions under which one can compute the trace of Γ_* which is defined by

$$\mathrm{Tr}(\Gamma_*) := \sum_{i=0}^{2d}(-1)^i \mathrm{Tr}(\Gamma_*|_{H^i_c(U)}). \tag{3.9}$$

As it stands, the trace of Γ_* depends a priori on ℓ, or, for varieties defined over a field k of characteristic 0, and Betti cohomology taken with respect to a complex embedding $\iota : k \to \mathbb{C}$, it depends on ι. One has

Theorem 3.5. *Let X be a smooth projective variety of dimension d defined over a field k, together with a strict normal crossings divisor $D \subset X$ of open complement $U = X \setminus D$. Let $\Gamma \subset U \times U$ be a dimension d cycle defining an open correspondence Γ_* on ℓ-adic cohomology or Betti cohomology as in Definition 3.2. Then $\mathrm{Tr}(\Gamma_*)$ does not depend on ℓ in ℓ-adic cohomology or on the complex embedding of k in Betti cohomology.*

Proof. We use the relative motivic cohomology $H^{2d}_M(X\times U, D\times U, \mathbb{Z}(d))$, as defined in [13], chapter 4, 2.2 and p. 209. The group $H^m_M(X \times U, D \times U, \mathbb{Z}(n))$ is the homology $H_{2n-m}(\mathcal{Z}^n(X \times U, D \times U, *))$, where $\mathcal{Z}^n(X \times U, D \times U, *)$ is the single complex associated to the double higher Chow cycle complex

$$\begin{array}{ccccc}
\cdots & & \cdots & & \cdots \\
\partial\big\downarrow & & \partial\big\downarrow & & \big\downarrow \\
\mathcal{Z}^n(X\times U, 1) & \xrightarrow{\mathrm{rest}} & \mathcal{Z}^n(D^{(1)}\times U, 1) & \xrightarrow{\mathrm{rest}} & \mathcal{Z}^n(D^{(2)}\times U, 1) \\
\partial\big\downarrow & & \partial\big\downarrow & & \big\downarrow \\
\mathcal{Z}^n(X\times U, 0) & \xrightarrow{\mathrm{rest}} & \mathcal{Z}^n(D^{(1)}\times U, 0) & \xrightarrow{\mathrm{rest}} & \mathcal{Z}^n(D^{(2)}\times U, 0).
\end{array} \tag{3.10}$$

Here $D^{(a)}$ is the normalization of all the strata of codimension a, $\mathcal{Z}^n(D^{(a)} \times U, b)$ is a group of cycles on $D^{(a)} \times U \times S^b$ where $S^\bullet$ is the cosimplicial scheme $S^n = \mathrm{Spec}\,(k[t_0, \ldots, t_n]/(\sum_{i=0}^n t_i - 1))$ with face maps $S^n \hookrightarrow S^{n+1}$ defined by $t_i = 0$. More precisely, $\mathcal{Z}^n(D^{(a)} \times U, b)$ is generated by the codimension n subvarieties $Z \subset D^{(a)} \times U \times S^b$ such that, for each face F of S^b, and each irreducible component $F' \subset D^{(a)}$ of the strata of D we have $\mathrm{codim}_{F'\times U\times F}(Z \cap (F' \times U \times F)) \geq n$. The horizontal restriction maps are the intersection with the smaller strata, the vertical ∂'s are the boundary maps.

This relative motivic cohomology acts as correspondences on $H^*_c(U)$, where $H^*_c(U)$ is ℓ-adic or Betti cohomology ([3], section 4). Let us write $\Gamma = \sum n_j \Gamma_j$. By Definition 3.2, one has $\Gamma_j \subset X \times U$ closed with $\Gamma_j \cap (D \times U) = \emptyset$, thus in particular, $\Gamma \in \mathcal{Z}^d(X \times U, 0)$ with $\mathrm{rest}(\Gamma) = 0$ in $\mathcal{Z}^d(D^{(1)} \times U, 0)$, thus it defines a class

$$[\Gamma] \in H^{2d}_M(X \times U, D \times U, d). \tag{3.11}$$

Similarly, we consider the restriction $\Delta_U \subset U \times X$ of the diagonal $\Delta \subset X \times X$. This defines a class in $\mathcal{Z}^d(U \times X, 0)$. As $\mathrm{rest}(\Gamma_U) = 0$ in $\mathcal{Z}^d(U \times D^{(1)}, 0)$, it defines a class

$$[\Delta_U] \in H^{2d}_M(U \times X, U \times D, d). \tag{3.12}$$

We want to pair $[\Gamma]$ with $[\Delta_U]$. We argue using M. Levine's work. Let Y be a N-dimensional smooth projective variety defined over k, with two strict normal crossings divisors A, B so that $A+B$ is a strict normal crossings divisor. By [13], Chapter IV, lemma 2.3.5 and lemma 2.3.6, the motive $M(Y \setminus A, B \setminus B \cap A)$ is dual to the motive $M(Y \setminus B, A \setminus B \cap A)$. It yields a cup product

$$H^a_M(Y \setminus A, B \setminus B \cap A, b) \times H^{2N-a}_M(Y \setminus B, A \setminus A \cap B, N-b) \to \mathbb{Z}. \tag{3.13}$$

This cup product is compatible with the cup product in ℓ-adic or Betti cohomology. We apply this to

$$Y = X \times X,\ A = D \times X,\ B = X \times D \tag{3.14}$$

so we can cup $[\Delta_U] \in H^{2d}_M(Y \setminus A, B \setminus B \cap A, d)$ and $[\Gamma] \in H^{2d}_M(Y \setminus B, A \setminus A \cap B, d)$

$$[\Delta_U] \cup [\Gamma] \in \mathbb{Z}. \tag{3.15}$$

The theorem is then the consequence of the following proposition.

Proposition 3.6.

$$\mathrm{Tr}(\Gamma_*) = [\Delta_U] \cup [\Gamma] \in \mathbb{Z}.$$

Proof. By the compatibility of the cup product (3.13) with cohomology, we just have to prove the proposition with $[\Delta_U]$ and $[\Gamma]$ replaced by their classes $\mathrm{cl}(\Delta_U) \in H^{2d}(U \times X, U \times D, d)$ and $\mathrm{cl}(\Gamma) \in H^{2d}(X \times U, D \times U, d)$ in cohomology. We may assume that $\Gamma \subset X \times U$ is irreducible. On the other hand, by (3.8), the map

$$H^i_c(\Gamma) \xrightarrow{p_{1*}} H^i_c(U) \tag{3.16}$$

factors through

$$H^i_c(\Gamma) \xrightarrow{\text{Gysin}} H^{i+2d}_{\Gamma,c}(X\times U,d) = H^{i+2d}_{\Gamma,c}(U\times U,d) \tag{3.17}$$
$$\to H^{i+2d}_c(U\times U,d) \to H^{i+2d}_c(U\times X,d) \xrightarrow{p_{1*}} H^i_c(U)$$

so the correspondence

$$\Gamma_* : H^i_c(U) \xrightarrow{p_2^*} H^i_c(\Gamma) \xrightarrow{p_{1*}} H^i_c(U) \tag{3.18}$$

is just defined on $\alpha \in H^i_c(U)$ by

$$p_2^*(\alpha)\cup \mathrm{cl}(\Gamma) \in H^{2d}(X\times U, D\times U, d)) \subset H^{i+2d}_c(U\times X,d) \tag{3.19}$$

followed by p_{1*}. Now we argue as in the classical case. Let e^i_a be a basis of $H^i_c(U)$, and $(e^i_a)^\vee$ be its dual basis in $H^{2d-i}(U)(d)$. Write $\mathrm{cl}(\Gamma) = \sum_i\sum_a f^i_a \otimes (e^i_a)^\vee, f^i_a = \sum f^i_{ab}e^i_b \in H^i_c(U)$. So $\Gamma_*(e^i_a) = \sum_b f^i_{ab}$, and $\mathrm{Tr}(\Gamma_*) = \sum(-1)^i\sum_a\sum_b f^i_{ab}$. On the other hand, one has $\mathrm{cl}(\Delta_U) = \sum_i\sum_a (e^i_a)^\vee \otimes (e^i_a)$. Thus $\mathrm{cl}(\Delta_U\cup\Gamma) = \sum(-1)^i\sum_a\sum_b f^i_{ab}$.

□

The proposition finishes the proof of the theorem. □

The rest of the section is devoted to giving a concrete expression for (3.9) under stronger geometric assumptions on Γ.

We define

$$X^o = X \setminus \cup_{i<j}(D_i\cap D_j), \tag{3.20}$$
$$\Gamma^o = \bar\Gamma \cap (X^o\times X^o),\ (\Gamma')^o = \Gamma'\cap(X^o\times X^o),\ D^o = X^o\cap D$$

and similarly for the components $_j$. One has the following compatibility.

Lemma 3.7. *One has a commutative diagram*

$$\begin{array}{ccc} H^i_c(U) & \to & H^i(X^o) \\ {\scriptstyle p_2^*}\downarrow & & {\scriptstyle p_2^*}\downarrow \\ H^i_c(\Gamma_j) & \to & H^i(\Gamma^o_j) \\ {\scriptstyle \iota_*}\downarrow & & {\scriptstyle \iota^*}\downarrow \\ H^i_c((\Gamma_j)') & \to & H^i((\Gamma')^o_j) \\ {\scriptstyle p_{1*}}\downarrow & & {\scriptstyle p_{1*}}\downarrow \\ H^i_c(U) & \to & H^i(X^o) \end{array}$$

where the composition of the left vertical arrows is the correspondence $(\Gamma_j)_*$. We deem $(\Gamma^o_j)_*$ the composition of the right vertical arrows.

Proof. Given (3.5) the lemma follows directly from Definition (3.2). □

We remark that the Grothendieck-Lefschetz trace formula allows to compute the trace of the correspondence $\bar{\Gamma}_*$ on X

$$\mathrm{Tr}(\bar{\Gamma}_*) = \deg(\bar{\Gamma} \cdot \Delta_X) \tag{3.21}$$

Thus, in the corollary, we would like to complete the commutative diagram in an exact sequence of commutative diagrams, so that we can apply the trace formula on all the terms but the one we seek. In the sequel, we give a strong geometric condition under which it is possible.

Definition 3.8. We assume that D is a strict normal crossings divisor. The dim d cycle $\Gamma \subset U \times U$ is said to be in good position with respect to $D \times X$ if the following two conditions are fulfilled.

i) Each $\bar{\Gamma}_j$ cuts each stratum $D_I \times X$ in codim $\geq d$, where $D_I = D_{i_1} \cap \ldots \cap D_{i_r}$ for $I = \{i_1, \ldots, i_r\}$ with $|I| = r$.

ii)
$$\bar{\Gamma}_j \cap (D_I \times X) \subset D_I \times D_I$$
set theoretically.

In this case, for all I we define the cycles

$$Z_{jI} = \bar{\Gamma}_j \cdot (D_I \times X) \subset D_I \times D_I \tag{3.22}$$

Let us be more precise. We denote motivic cohomology by $H^a_M(b)$. We drop the subscript $_j$, thus $\Gamma = \Gamma_j$. This defines

$$Z_I = \sum m_{I,a} Z_{I,a} \tag{3.23}$$

as in (3.14), where the $Z_{I,a}$ are the reduced irreducible components of Z_I. One has the Gysin isomorphisms

$$\oplus_a \mathbb{Q}_\ell[Z_{I,a}] \xrightarrow{\cong} H^{2(d-r)}_{Z_I}(D_I \times D_I, d-r) \xrightarrow{\cong} H^{2d}_{Z_I}(D_I \times X, d) \tag{3.24}$$

This yields the commutative diagram

$$\begin{tikzcd}
\oplus_a \mathbb{Q} \cdot [Z_{I,a}] \arrow[r, "\otimes_{\mathbb{Q}} \mathbb{Q}_\ell"] \arrow[d, "\cong"'] & \oplus_a \mathbb{Q}_\ell \cdot [Z_{I,a}] \arrow[d, "\cong"] \\
H^{2(d-r)}_{M,Z_I}(D_I \times D_I, d-r) \arrow[r, "\otimes_{\mathbb{Q}} \mathbb{Q}_\ell"] \arrow[d, "\cong"'] & H^{2(d-r)}_{Z_I}(D_I \times D_I, d-r) \arrow[d, "\cong"] \\
H^{2d}_{M,Z_I}(D_I \times X, d) \arrow[r, "\otimes_{\mathbb{Q}} \mathbb{Q}_\ell"] & H^{2d}_{Z_I}(D_I \times X, d)
\end{tikzcd} \tag{3.25}$$

So we conclude that Z_I is a well defined cycle

$$Z_I \in H^{2(d-r)}_{M,Z_I}(D_I \times D_I, d-r) \to H^{2(d-r)}_M(D_I \times D_I, d-r), \tag{3.26}$$

which defines a correspondence

(3.27)
$$(Z_I^o)_* = \sum_a m_{I,a}(Z_{I,a})_* : H^i_c(D_I \setminus \cup_{i\notin I} D_{I,i}) \to H^i_c(D_I \setminus \cup_{i \notin I} D_{I,i})$$
$$Z_I^o = Z_I \cap (D_I \setminus \cup_{i\notin I} D_{I,i} \times D_I \setminus \cup_{i\notin I} D_{I,i})$$

by the Definition 3.2. We use the notations from (3.20), setting $\Gamma = \Gamma_j$, together with

$$Z = \cup_i \Gamma \cap (D_i^o \times D_i^o). \tag{3.28}$$

Lemma 3.9. *One has a commutative diagram*

$$\begin{tikzcd}
H_c^{i-1}(D^o) \arrow[r] \arrow[d, "p_2^*"'] & H_c^i(U) \arrow[d, "p_2^*"'] \\
 & H_c^i(\Gamma) \arrow[d, "\iota^*"'] \\
H_c^{i-1}(Z) \arrow[uu, leftarrow] \arrow[r] \arrow[d, "(p_1)_*"'] & H_c^i(\Gamma') \arrow[d, "(p_1)_*"'] \\
H_c^{i-1}(D^o) \arrow[r] & H_c^i(U)
\end{tikzcd}$$

Proof. Indeed, by assumption, one has $\Gamma' \subset X^o \times U$, thus, with the notations of (3.20), one has $\Gamma' = \Gamma^o \setminus Z$.

□

In order to have a unified notation, we set for $|I| = 0$

$$Z_I = \bar{\Gamma}, \ \Delta_I = \Delta_X. \tag{3.29}$$

One has

Theorem 3.10. *Let X be a smooth projective variety over an algebraically closed field k, $D \subset X$ be a strict normal crossing divisor, with complement $U = X \setminus D$, and $\Gamma \subset U \times U$ be a dim d correspondence, with $p_2|_\Gamma : \Gamma \to U$ proper, and in good position with respect to $D \times X$ in the sense of Definition 3.2. Then one has*

$$\mathrm{Tr}(\Gamma_*) = \sum_{r=0}^{d} (-1)^r \sum_{|I|=r} \deg(Z_I \cdot \Delta_I).$$

Proof. We consider the long exact sequence

$$\ldots \to H_c^{i-1}(D^o) \to H_c^i(U) \to H_c^i(X^o) \to \ldots \tag{3.30}$$

and apply Lemma 3.7 and Lemma 3.9. We find that the trace on U is the sum of the traces on X^o and on D^o. If D was smooth, this would finish the proof applying Grothendieck formula (3.21). If there are higher codimensional strata, we argue as follows. We know the trace on D_i^o by induction on the dimension. We have to understand the trace on X^o. We define $X^{oo} = X \setminus \cup_{i<j<k}(D_i \cap D_j \cap D_k)$ and redo Lemma 3.7 with U replaced by X^o and X^o replaced by X^{oo}. Then we redo Lemma 3.9 with U replaced by X^o and $D^o = X^o \setminus U$ replaced by $X^{oo} \setminus X^o$. Then we redo (3.30) with D^o replaced by $X^{oo} \setminus X^o$, U replaced by X^o and X^o replaced by X^{oo}. Again this shifts the trace computation to X^{oo}. We continue like this till the highest codimensional strata and finish with the trace on X, for which we of course apply Grothendieck formula (3.21).

□

We now give a scheme-theoretic condition under which the expression given in Theorem 3.10 depends only on local contributions in U. This condition is inspired by [10], Lemma 2.3.1.

Definition 3.11. We assume that D is a strict normal crossings divisor. The dim d cycle $\Gamma \subset U \times U$ is said to be in scheme theoretic good position with respect to $D \times X$ if it is in good position in the sense of Definition 3.8 and ii) is replaced by

ii')

$$\bar{\Gamma}_j \cap (D_I \times X) \subset D_I \times D_I$$

scheme theoretically, that is all the intersections multiplicities are equal to 1.

Proposition 3.12. *Let X be a smooth projective variety over an algebraically closed field k, $D \subset X$ be a strict normal crossing divisor, with complement $U = X \setminus D$, and $\Gamma \subset U \times U$ be a dim d correspondence, with $p_2|_\Gamma : \Gamma \to U$ proper, and in scheme theoretic good position with respect to $D \times X$ in the sense of Definition 3.11. We assume moreover that $\bar{\Gamma}_j$ and $\Delta|_U$ cut transversally. Then* $\mathrm{Tr}(\Gamma_*) = \deg(\Delta_U \cdot \Gamma)$.

Proof. Due to the good position assumption, all intersection multiplicities are 1 and the contributions lying on $(D \times X) \cup (X \times D)$ cancel in Theorem 3.10.

□

Example 3.13. One case where the conditions of Proposition 3.12 hold is in characteristic p when Γ is the graph of Frobenius. In this case, of course, the result is known by other methods.

Example 3.14. This example is inspired by [10], Remark 2.3.6. We take $X = \mathbb{P}^1, D = \{\infty\}, U = \mathbb{A}^1, \Gamma = \Gamma_{pq} = \{x^p - y^q = 0\} \subset \mathbb{A}^1 \times \mathbb{A}^1$. Then Γ defines an open correspondence with $\mathrm{Tr}(\Gamma_*) = p$. On the other hand, one has

$$\deg(\Gamma_{pq} \cdot \Delta_U) = \dim k[t]/(t^p - t^q) = \begin{cases} \max(p,q) & \text{if } p \neq q \\ \infty & \text{if } p = q \end{cases} \tag{3.31}$$

and one has

$$\deg(\bar{\Gamma}_{pq} \cdot (\infty \times \mathbb{P}^1)) = q, \ \deg(\bar{\Gamma}_{pq} \cdot (\mathbb{P}^1 \times \infty)) = p. \tag{3.32}$$

Thus Γ_{pq} is in scheme theoretic good position with respect to $\infty \times \mathbb{P}^1$ if and only if $p > q$, and is always in good position with respect to $\infty \times \mathbb{P}^1$. Since $\deg(\Delta_{\mathbb{P}^1} \cdot \bar{\Gamma}_{pq}) = \deg(\mathcal{O}(1,1) \cdot \mathcal{O}(p,q)) = p + q$, we see exactly how the formula of Theorem 3.10 works both in Theorem 3.10 and in Proposition 3.12.

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