We correct the inaccuracies in the proof of Theorem 4.10 of [BE04]. We are thankful to Shishir Agrawal and Lei Fu for alerting us about them.

The functor F is defined in (4.30) of *loc.cit.*. It follows from (4.31) *loc.cit.* that the functor is pro-representable. It follows from (4.34) *loc. cit.* that the functor is pro-smooth. It follows from (4.33) *loc. cit.* that the tangent space is $H^1(X, j_{!*}\mathcal{E}nd(M))$. We first detail this point.

One defines $\hat{X}$ to be the completion along D, and $\hat{U} = \hat{X} \setminus D$ to be the punctured completion. For any scheme Z (formal or not) defined over k, and any k-algebra R, one sets $Z_R = X \times_k \mathrm{Spec}(R)$.

On U one has the abelian category of complexes of coherent sheaves with k-linear differentials. And one has the category of connections on U. An element in $F(k[\epsilon])$ is a connection

$$\mathcal{M} \to (\omega_U \oplus \epsilon\omega_U) \otimes_{\mathcal{O}_{U[\epsilon]}} \mathcal{M} \tag{1}$$

which is thus an extension (skiping the zeroes left and right)

(2)

$$\begin{tikzcd}
\epsilon\mathcal{M} \cong M \arrow[r] \arrow[d] & \mathcal{M} \cong N \arrow[r] \arrow[d] & M \arrow[d] \\
(\omega_U \oplus \epsilon\omega_U) \otimes_{\mathcal{O}_{U[\epsilon]}} \epsilon\mathcal{M} \cong \omega_U \otimes_{\mathcal{O}_U} M \arrow[r] & (\omega_U \oplus \epsilon\omega_U) \otimes_{\mathcal{O}_{U[\epsilon]}} \mathcal{M} \cong \omega^1_U \otimes_{\mathcal{O}_U} N \arrow[r] & \omega_U \otimes_{\mathcal{O}_U}
\end{tikzcd}$$

where N is the U-coherent sheaf, an extension of M by M, associated to $\mathcal{M}$, and (2) has the property that the extension restricted to $\hat{U}$

$$\begin{tikzcd}
\hat{M} \arrow[r] \arrow[d] & \hat{N} \arrow[r] \arrow[d] & \hat{M} \arrow[d] \\
\omega_{\hat{U}} \otimes_{\mathcal{O}_{\hat{U}}} \hat{M} \arrow[r] & \omega^1_{\hat{U}} \otimes \hat{N} \arrow[r] & \omega_{\hat{U}} \otimes_{\mathcal{O}_{\hat{U}}} \hat{M}
\end{tikzcd} \tag{3}$$

splits. Here we used the shorthand $\hat{M} = M|_{\hat{U}}$. Indeed, one has $\hat{N} \cong p^*\hat{M}$ with $p : \hat{U} \otimes_k k[\epsilon] \to \hat{U}$. Thus as a coherent sheaf on $\hat{U} \otimes_k k[\epsilon]$, one has $\hat{N} \cong \hat{M} \oplus \hat{M}$ and the connection is just $\nabla_{\hat{M}}$ factor-wise. This means that (3) splits. And vice-versa, it (3) splits, since the vertical left arrow is the de Rham complex of M, one has $\hat{N} \cong p^*\hat{M}$. Extensions (2) build a finite dimensional k-vector space $H^1_{dR}(U, \mathcal{E}nd(M))$ while extensions in (3) build a finite dimensional k-vector space $H^1_{dR}(\hat{U}, \mathcal{E}nd(M))$. Thus

$$F(k[\epsilon]) = \mathrm{Ker}\big(H^1_{dR}(U, \mathcal{E}nd(M)) \to H^1_{dR}(\hat{U}, \mathcal{E}nd(M))\big). \tag{4}$$

Thus by Remark 4.1 of *loc.cit.* one has

$$F(k[\epsilon]) = H^1(X, j_{!*}\mathcal{E}nd(M)). \tag{5}$$

Let R be the pro-smooth k-algebra pro-representing F. Write $\mathfrak{m} \subset R$ for the maximal ideal. Set $R_n := R/\mathfrak{m}^n$. Let M_n be the universal connection on $U \times \mathrm{Spec}\,(R_n)$. Recall (p.591 of *loc.cit.*) that a pair of good lattices (V, W) for M on U is a pair of vector bundles on X extending M such that $V \subset W$, $\nabla(V) \subset \omega_X(D) \otimes W$ and such that for any $m \geq 1$ the inclusion of complexes $\big(V \to \omega_X(D) \otimes W\big) \hookrightarrow \big(V(mD) \to \omega_X(D) \otimes W(mD)\big)$ is a quasi-isomorphism.

Proposition 1. *For any pair of good lattices (V, W), there exist algebraic vector bundles $\mathcal{V}, \mathcal{W}$ on X_R, together with an algebraic connection $\nabla : \mathcal{V} \to \omega_{X_R/R}(D) \otimes \mathcal{W}$ such that*

0) $\mathcal{V}|_{U_R} = \mathcal{W}|_{U_R}$*;*
1) $\nabla|_{U_{R_n}}$ *is isomorphic to the connection* M_n*;*
2) $\nabla|_X$ *is isomorphic to the connection* $V \to \omega_X(D) \otimes W$*;*
3) $\nabla|_{\hat{U}_R}$ *is isomorphic to* $M|_{\hat{U}} \otimes_k R$*.*

Proof. We have the connections M_n on U_{R_n} and isomorphisms

$$\phi_{n,n+1} : M_{n+1}|_{U_{R_n}} \cong M_n.$$

Denoting again by $\widehat{M}$ the restriction of M to $\hat{U}$, we also have

$$\rho_n : \widehat{M} \otimes_k R_n \cong M_n|_{\hat{U}_{R_n}}. \tag{6}$$

Write

$$\hat{\phi}_{n,n+1} : M_{n+1}|_{\hat{U}_{R_n}} \cong M_n|_{\hat{U}_{R_n}} \tag{7}$$

for the map induced by $\phi_{n,n+1}$. Define

$$\mu_n := \rho_n^{-1} \circ \hat{\phi}_{n,n+1} \circ (\rho_{n+1} \otimes R_n) : \widehat{M} \otimes_k R_n \cong \widehat{M} \otimes_k R_n. \tag{8}$$

These isomorphisms lift, so we may choose

$$\sigma_{n+1} : \widehat{M} \otimes_k R_{n+1} \cong \widehat{M} \otimes_k R_{n+1} \tag{9}$$

lifting μ_n. Now redefine

$$\rho_{n+1,\mathrm{new}} := \rho_{n+1,\mathrm{old}} \circ \sigma_{n+1}. \tag{10}$$

We have

$$\begin{aligned} \hat{\phi}_{n,n+1} \circ (\rho_{n+1,\mathrm{new}} \otimes R_n) &= \hat{\phi}_{n,n+1} \circ (\rho_{n+1,\mathrm{old}} \otimes R_n) \circ \mu_n^{-1} = \\ &\hat{\phi}_{n,n+1} \circ (\rho_{n+1,\mathrm{old}} \otimes R_n) \circ (\rho_{n+1,\mathrm{old}} \otimes R_n)^{-1} \circ \hat{\phi}_{n,n+1}^{-1} \circ \rho_n = \rho_n. \end{aligned} \tag{11}$$

Modifying the ρ_n in this way, we obtain commutative diagrams

$$\begin{array}{ccc} \widehat{M} \otimes_k R_n & \xrightarrow{\rho_{n+1}\otimes R_n} & M_{n+1}|_{\hat{U}_{R_n}} \\ \| & & \downarrow \hat{\phi}_{n,n+1} \\ \widehat{M} \otimes_k R_n & \xrightarrow{\rho_n} & M_n|_{\hat{U}_{R_n}} \end{array} \tag{12}$$

Using ρ_n, we glue M_n and $V \otimes_k R_n$ resp. $W \otimes_k R_n$ to obtain a vector bundle $\mathcal{V}_n$ on X_{R_n}, resp. $\mathcal{W}_n$ on X_{R_n}, together with a connection $\mathcal{V}_n \to \omega_{X_{R_n}/R_n}(D) \otimes \mathcal{W}_n$ restricting to M_n on U_{R_n}. The $\phi_{n,n+1}$ give isomorphisms $\mathcal{V}_{n+1} \otimes R_n \cong \mathcal{V}_n$ and $\mathcal{W}_{n+1} \otimes R_n \cong \mathcal{W}_n$. Thus one obtains a formal connection $(\nabla_n)_n : (\mathcal{V}_n)_n \to (\omega_{X_{R_n}/R_n}(D) \otimes \mathcal{W}_n)_n$ which fulfils 0), 1), 2), 3) with ∇ replaced by $(\nabla_n)_n$.

By Grothendieck formal function theorem ([8], Cor. 5.1.6 in *loc. cit.*) $\mathcal{V}, \mathcal{W}$ are algebraic vector bundles on X_R. Replacing the pair (V, W) by the pair $(V(mD), W(mD))$ for m large enough, we may assume that $H^0(X_R, \mathcal{V})$ spans $\mathcal{V}$, and satisfies base change, that $H^0(X_R, \omega_{X_R}(D) \otimes \mathcal{W})$ spans $\omega_{X_R}(D) \otimes \mathcal{W}$ and satisfies base change. In particular $H^0(X_R, \mathcal{V})$ and $H^0(X_R, \omega_{X_R}(D) \otimes \mathcal{W})$ are projective modules of finite rank over R. The connection defines a R-linear map

$$\partial : H^0(X_R, \mathcal{V}) \to H^0(X_R, \omega_{X_R/R}(D) \otimes \mathcal{W}).$$

From the surjectivity $H^0(X_R, \mathcal{V}) \otimes_R \mathcal{O}_{X_R} \twoheadrightarrow \mathcal{V}$, every local section σ of $\mathcal{V}$ can be written (non-uniquely) as $\sigma = \sum_{\text{finite}} v \otimes \lambda$ with $v \in H^0(X_R, \mathcal{V})$ and $\lambda \in \mathcal{O}_{X_R}$. $(\nabla_n)_n(\sigma) = \sum_{\text{finite}} (\partial(v) \otimes \lambda + v \otimes d(\lambda))$, which is a local algebraic section of $\omega_{X_R}(D) \otimes \mathcal{W}$. This shows that the connection is algebraic, proving 0), 1), 2), 3) at the same time.

□

Proposition 2. *With notations as in Proposition 1, there exists a finitely generated k-algebra $A \subset R$ such that*

1) $\nabla : \mathcal{V} \to \omega_{X_R/R} \otimes \mathcal{W}$ *is defined over A;*
2) *the isomorphism* $\nabla|_{\hat{X}_R}$ *with* $(\mathcal{V}|_{\hat{X}_R} \to \omega_{\hat{X}_R/R}(D) \otimes \mathcal{W}|_{\hat{X}_R})$ *is defined over A;*
3) *the isomorphism* $\nabla|_{\hat{U}_R}$ *with* $(\mathcal{V}|_{\hat{U}_R} \to \omega_{\hat{U}_R/R} \otimes \mathcal{V}|_{\hat{U}_R})$ *is defined over A.*

Proof. By the algebraicity of ∇ from Proposition 1, we deduce 1). Then 2) is the restriction of 1) to $\hat{X}_R$ and 3) to $\hat{U}_R$. This finishes the proof.

□

We set $S = \mathrm{Spec}(A)$ and for any scheme Z (formal or not) over k, we denote by Z_S its base change $Z \times_k S$. We denote by $\nabla_S : \mathcal{V}_S \to \omega_{X_S/S}(D) \otimes \mathcal{W}_S$ the algebraic connection on X_S.

Proposition 3. *The sheaf of $\mathcal{O}_S$-modules* $\mathrm{Hom}_\nabla(\mathcal{V}_{U_S}, M \times_k S)$ *of flat sections is coherent.*

Proof. By Proposition 2 2), 3), the restriction homomorphism

$$\mathrm{Hom}_\nabla(\mathcal{V}_{X_S}, V \times_k S) \to \mathrm{Hom}_\nabla(\mathcal{V}_{U_S}, M \times_k S)$$

is an isomorphism. The left term is equal to

$$\mathrm{Ker}\big(\mathrm{Hom}(\mathcal{V}_{X_S}, V \times_k S) \to \mathrm{Hom}(\mathcal{W}_{X_S}, \omega_X(D) \otimes W \times_k S)\big)$$

where both terms are coherent and the map is $\mathcal{O}_S$-linear. This finishes the proof. $\square$

Proposition 4. *In Proposition 3, we assume now M to be rigid. Then there a torsor $\pi : T \to S$ under the constant groupscheme* $\mathrm{Iso}_\nabla(M, M) \otimes_k S$*, together with an isomorphism between*

$$(\mathcal{V}_T, \mathcal{W}_T, \nabla_T) \cong (V, W, \nabla) \times_k T.$$

If M is irreducible, this is a $\mathbb{G}_m$-torsor.

Proof. By definition, for each geometric point $s \in S\ (k)$, one has an isomorphism between $(\mathcal{V}_s, \mathcal{W}_s, \nabla_s)$ and (V, W, ∇). The Zariski subsheaf of sets $\mathrm{Iso}_\nabla(\mathcal{V}|_{U_S}, M \otimes_k S) \subset \mathrm{Hom}_\nabla(\mathcal{V}|_{U_S}, M \times_k S)$ is acted on on the right by $\mathrm{Iso}_\nabla(M, M) \times_k S$, which endows it with the structure of a torsor. Then T is the total space of the torsor. The existence of the isomorphism then follows from the definition of the torsor. $\square$

We now finish the proof of [BE04, Thm. 4.10]. We have a diagram

$$\begin{tikzcd} & T \arrow[d, "\pi"] \\ \mathrm{Spec}\,(R) \arrow[r, "f"] & S \end{tikzcd} \tag{13}$$

As k has characteristic 0 and π is a torsor under a constant group-scheme, π is smooth. As k is algebraically closed, f lifts to g

$$\begin{tikzcd} & T \arrow[d, "\pi"] \\ \mathrm{Spec}(R) \arrow[ur, "g"] \arrow[r, "f"] & S \end{tikzcd} \tag{14}$$

and thus

$$(\mathcal{V}, \mathcal{W}, \nabla) = g^*\pi^*(\mathcal{V}_A, \mathcal{W}_A, \nabla_A) = (V, W, \nabla) \otimes_k R. \tag{15}$$

By universality and pro-smoothness of R, this shows

$$F(k[\epsilon]) = 0 \tag{16}$$

thus Theorem 4.10 of *loc.cit.* follows from (5).

References

[BE04] Bloch, S., Esnault, H.: *Local Fourier transforms and rigidity for $\mathcal{D}$-modules*, Asian J. Math. **8** 4 (2004), 587–606.

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