Eigenvalues of Frobenius acting on the $\ell$-adic cohomology of complete intersections of low degree

Hélène Esnault

Universität Essen, FB6, Mathematik, 45117 Essen, Germany

Received 14 April 2003; accepted after revision 24 July 2003

Abstract

We show that the eigenvalues of Frobenius acting on $\ell$-adic cohomology of a complete intersection of low degree defined over the finite field $\mathbb{F}_q$ modulo the cohomology of the projective space are divisible as algebraic integers by $q^\kappa$, where the natural number $\kappa$ is predicted by the theorem of Ax and Katz (Amer J. Math. 93 (1971) 485–499) on the congruence for the number of rational points. To cite this article: H. Esnault, C. R. Acad. Sci. Paris, Ser. I 337 (2003).

© 2003 Académie des sciences. Published by Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

Résumé


© 2003 Académie des sciences. Published by Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

Version française abrégée

Si $X \subset \mathbb{P}^n$ est une intersection complète de degrés $d_1 \geq \cdots \geq d_r$ définie sur le corps fini $\mathbb{F}_q$, et si $\kappa$ est le maximum de 0 et de la partie entière du nombre rationnel $(n - d_2 - \cdots - d_r)/d_1$, nous montrons que les valeurs propres de Frobenius agissant sur la cohomologie $\ell$-adique relative $H^i(\overline{\mathbb{P}^n}, \overline{X}, \mathbb{Q}_\ell)$ sont divisibles par $q^\kappa$ en tant qu’entiers algébriques (Theorem 1.1). Ici, $\overline{\mathbb{P}^n} = \mathbb{P}^n \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$, $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ et $\overline{\mathbb{F}_q}$ est la clôture algébrique de $\mathbb{F}_q$. Si $X$ est lisse, utilisant le théorème de Ax et Katz [12], c’est une conséquence immédiate de la formule des traces de Grothendieck–Lefschetz [11] et de ce que la cohomologie primitive de $\overline{X}$ est concentrée en dimension moitié [3]. Si $X$ est singulière, cela n’est plus le cas. Si $\overline{\mathbb{F}_q}$ est remplacé par un corps $k$ de caractéristique 0, l’assertion qui correspond dans la philosophie motivique au Theorem 1.1 est que le type de Hodge est $\geq \kappa$ sous les mêmes hypothèses de degré et d’intersection complète, et est prouvée dans [7] pour les hypersurfaces et [9] en général.

E-mail address: esnault@uni-essen.de (H. Esnault).

1631-073X/$ – see front matter © 2003 Académie des sciences. Published by Éditions scientifiques et médicales Elsevier SAS. All rights reserved.
doi:10.1016/S1631-073X(03)00370-4
1. Introduction

If \( X \subset \mathbb{P}^n \) is a projective variety defined over the finite field \( k = \mathbb{F}_q \) by equations of degrees \( d_1 \geq d_2 \geq \cdots \geq d_r \), then the fundamental theorem of Ax and Katz [12] asserts that the number of rational points of \( X \) fulfills

\[
|X(\mathbb{F}_q)| \equiv |\mathbb{P}^n(\mathbb{F}_q)| \mod q^\kappa.
\]

If \( X \) is smooth, this is equivalent to saying that the eigenvalues of the Frobenius action on \( H^i(\mathbb{F}_q, X, \mathbb{Q}_\ell) \) for all \( i \) are of the shape

\[
q^\kappa \cdot \text{algebraic integer}.
\]

Let \( \zeta(U, t) \) be the zeta function of \( U \) defined by its logarithmic derivative

\[
\frac{\zeta'(U, t)}{\zeta(U, t)} = \sum_{s \geq 1} |U(\mathbb{F}_{q^s})| t^{s-1}.
\]

By the theorem of Dwork [8], we know that \( \zeta(U, t) \) is a rational function

\[
\zeta(X, t) \in \mathbb{Q}(t),
\]


\[
\zeta(U, t) = \prod_{i=0}^{2 \dim(U)} \det(1 - F(t)^{-i})^{(-1)^{i+1}},
\]

where \( F_i \) is the geometric Frobenius acting on the compactly supported \( \ell \)-adic cohomology \( H^i(U, \mathbb{Q}_\ell) \). For \( X \) smooth, the Weil conjectures [6] assert that the eigenvalues in any complex embedding \( \mathbb{Q}_\ell \subset \mathbb{C} \) of the geometric Frobenius acting on \( H^i(X, \mathbb{Q}_\ell) \) have absolute values \( q^{i/2} \). The conclusion is that there is no possible cancellation between the numerator and the denominator of (5). Thus (1) for all finite field extensions \( \mathbb{F}_{q'_\ell} \supset \mathbb{F}_q \) is equivalent to (2). If we actually assume that \( X \) is not only smooth but also a complete intersection, then we know that only \( H^{n-r}(\overline{X}, \mathbb{Q}_\ell) \) is not equal to \( H^{n-r}(\mathbb{F}_q, \mathbb{Q}_\ell) \) [4]. Thus in this case, we do not need the Weil conjectures to conclude that there is no cancellation.

On the other hand, if we replace \( k \) by a field of characteristic 0, keeping the same degree assumption, then we know that the Hodge type of \( X \), that is the largest integer \( \mu \) such that the Hodge filtration in de Rham cohomology satisfies

\[
F_\mu H^i_{\text{dR}}(U) = H^i_{\text{dR}}(U)
\]

is \( \geq \kappa \) (see [3] in the smooth case, [7] for hypersurfaces, [9] for complete intersections and [10] for the general case). The purpose of this Note is to show

**Theorem 1.1.** Let \( X \subset \mathbb{P}^n \) be a complete intersection defined over the finite field \( k = \mathbb{F}_q \) by equations of degrees \( d_1 \geq d_2 \geq \cdots \geq d_r \). Then the eigenvalues of the geometric Frobenius acting on \( \ell \)-adic cohomology \( H^i(\mathbb{F}_q, \overline{X}, \mathbb{Q}_\ell) \) are of the shape \( q^\kappa \cdot \text{algebraic integer} \) for all \( i \), where \( \kappa \) is the maximum of 0 and of the integral part of \( \frac{2 \dim(U)}{d_i} \).
The proof reduces the assertion to Deligne’s integrality statement [4] which is true for all varieties, without supplementary assumptions. However, the reduction depends heavily on the complete intersection assumption.

2. The proof of Theorem 1.1

If $X$ is a complete intersection of dimension $n - r$, Artin’s vanishing theorem [1], Theorem 3.1, implies

$$H^i(X, \mathbb{Q}_l) / H^i(U^n, \mathbb{Q}_l) = H^{i+1}(X, \mathbb{Q}_l) = 0 \quad \text{for } i < \dim X = n - r. \quad (7)$$

By the Lefschetz trace formula (5), we see, as already observed by Wan (see introduction of [2]), that the theorem of Ax–Katz (1) together with the divisibility assertion for $F_i$ for all $i$ but one, implies the divisibility assertion for the last $F_i$. Thus we just have to show divisibility for $i > n - r$, or equivalently divisibility of the $F_i$ acting on $H^i(\overline{U}, \mathbb{Q}_l) = H^i(\mathbb{P}^n, X, \mathbb{Q}_l)$ for $i > n - r + 1$. We observe that divisibility of the eigenvalues of Frobenius is compatible with base field extension, and consequently we are allowed to enlarge by necessity our ground field $\mathbb{F}_q$ to $\mathbb{F}_q$. Let $A \cong \mathbb{F}_q^{n-1}$ be a linear hyperplane in general position. One has an exact sequence of

$$\cdots \to H^i_A(\mathbb{P}^n, X, \mathbb{Q}_l) \to H^i(\mathbb{P}^n, X, \mathbb{Q}_l) \to H^i(\mathbb{P}^n \setminus A, X \setminus X \cap A, \mathbb{Q}_l) \to \cdots \quad (8)$$

(see, e.g., [13], III, Proposition 1.25 applied to $F = j_!\mathbb{Q}_l$, for $j : \mathbb{P}^n \setminus X \to \mathbb{P}^n$, and $Z = A$, $U = \mathbb{P}^n \setminus A$) which is compatible with the Frobenius action. For $i > n - r + 1$, Artin’s vanishing theorem [1] again implies $H^i(\mathbb{P}^n \setminus A, X \setminus X \cap A, \mathbb{Q}_l) = 0$. Thus we are reduced to proving the assertion for the Frobenius action on $H^i_A(\mathbb{P}^n, X, \mathbb{Q}_l)$ for $i > n - r + 1$. One has

**Theorem 2.1 (P. Deligne).** The Gysin homomorphism

$$H^{i-2}(\overline{A}, X \cap \overline{A}, \mathbb{Q}_l) \xrightarrow{\text{Gysin homomorphism}} H^i_A(\mathbb{P}^n, X, \mathbb{Q}_l)$$

is an isomorphism of Frobenius modules for $A$ in a non-trivial open subset of the dual projective space $(\mathbb{P}^n)^\vee$.

More generally, if $\mathcal{F}$ is an $\ell$-adic sheaf on $\mathbb{P}^n$, then for $A$ in a non-trivial open subset of the dual projective space $(\mathbb{P}^n)^\vee$, the Gysin homomorphism

$$H^{i-2}(A, i^* \mathcal{F})(-1) \to H^i_A(\mathbb{P}^n, \mathcal{F}),$$

where $i : A \to \mathbb{P}^n$ is the closed embedding, is an isomorphism.

**Proof (P. Deligne).** We consider the universal family

$$\iota : A = \left\{(x, A) \in \mathbb{P}^n \times_k (\mathbb{P}^n)^\vee, \ x \in A \right\} \mapsfrom \mathbb{P}^n \times_k (\mathbb{P}^n)^\vee \quad (9)$$

together with the projections

$$\text{pr}_2 : \mathbb{P}^n \times_k (\mathbb{P}^n)^\vee \to (\mathbb{P}^n)^\vee, \quad \text{pr}_1 : \mathbb{P}^n \times_k (\mathbb{P}^n)^\vee \to \mathbb{P}^n. \quad (10)$$

Since $\mathcal{F}$ comes from $\mathbb{P}^n$, and $\mathcal{A} \subset \mathbb{P}^n \times_k (\mathbb{P}^n)^\vee$ is a smooth hypersurface such that $\mathcal{A} \to \mathbb{P}^n$ is smooth, the classical Gysin homomorphism is an isomorphism

$$\iota^* \text{pr}_1^* \mathcal{F}[-2](-1) \cong \iota^! \text{pr}_1^* \mathcal{F}. \quad (11)$$

By [5], Corollary 2.9 applied to $\text{pr}_2$, there is a non-empty open subset $V \subset (\mathbb{P}^n)^\vee$ such that for all $A \in V$ one has the base changes

$$i_A^* (\iota^* \text{pr}_1^* \mathcal{F}[-2](-1)) = i_A^* \mathcal{F}[-2](-1), \quad i_A^* (\iota^! \text{pr}_1^* \mathcal{F}) = i_A^! \mathcal{F}, \quad (12)$$

where $i_A : A \cap q^{-1} A = A \times \{A\} \mapsfrom \mathbb{P}^n \times \{A\}$. Thus (11) and (12) imply that for such a $A$, $i_A^* \mathcal{F}[-2](-1) \cong i_A^! \mathcal{F}$. This concludes the proof. $\Box$
Thus for \( i > n - r + 1 \), and \( n \geq 1 \), (8) becomes

\[
H^{i-2}(\mathbb{A}, X \cap A, \mathbb{Q}_l)(-1) \rightarrow H^i(\mathbb{P}^{n-1}, X, \mathbb{Q}_l)
\]

as a surjection of Frobenius modules. On the other hand, \( X \cap A \) is a complete intersection in \( \mathbb{P}^{n-1} \) of the same degrees \( d_1, \ldots, d_r \) and we have

\[
\kappa(A \cap X) = \max\left(0, \left[ \frac{n - d_2 - \cdots - d_r}{d_1} \right] \right) \geq \max\left(0, \kappa(X) - 1 \right).
\]

with \( \kappa(X) = \max\{0, \left\lfloor \frac{n - d_2 - \cdots - d_r}{d_1} \right\rfloor \} \). Arguing by induction on \( n \), we know that the eigenvalues of Frobenius acting on \( H^i(\mathbb{A}, X \cap A, \mathbb{Q}_l) \) are divisible by \( q^{\kappa(X \cap A)} \) for all \( i \), thus by (13) and (14), those on \( H^i(\mathbb{P}^{n-1}, X, \mathbb{Q}_l) \) by at least \( q^{\kappa(X)} \) for all \( i > n - r + 1 \), thus on all \( i \) by Ax–Katz’ theorem as recalled at the beginning of the section. It remains to start the induction. Note that the formulation allows indeed \( X = \emptyset \), that is \( r > n \). And for \( n = 0 \), one has \( \kappa = 0 \) and the theorem is a (very) easy case of Deligne’s integrality theorem [4], Corollary 5.5.3, saying that the eigenvalues of Frobenius acting on compactly supported cohomology are algebraic integers.

Acknowledgements

I thank Spencer Bloch, Pierre Deligne, Nick Katz, Madhav Nori and Daqing Wan for discussions on this topic. In particular, Spencer Bloch warned that the Gysin Theorem 2.1 needed in the proof has to be justified, and Pierre Deligne provided a proof of it. In our induction, we originally reduced the theorem to [4], Corollary 5.5.3, (ii) which says that \( q \) divides the eigenvalues of Frobenius as algebraic integers on compact cohomology beyond the dimension. Nick Katz observed that without changing the induction, it is enough to reduce to [4], Corollary 5.5.3, which says that the eigenvalues of Frobenius on compact cohomology are algebraic integers. This is an easier statement.

References