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Chern Classes of Gauss–Manin Bundles of Weight 1 Vanish

HÉLÈNE ESNAULT and ECKART VIEHWEG
Universität Essen, FB6, Mathematik, 45117 Essen, Germany.
e-mail: {*esnault;viehweg*}*@uni-essen.de*

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Abstract. We show that the Chern character of a variation of polarized Hodge structures of weight one with nilpotent residues at ∞ dies up to torsion in the Chow ring, except in codimension 0.

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1. Introduction

Let $f: X \to B$ be a proper smooth family over a smooth base, defined over a field k of characteristic 0. Then the Gauss–Manin bundles $\mathcal{H}^i := R^i f_* \Omega^\bullet_{X/B}$ are endowed with the Gauss–Manin connection, and thereby, e.g. by Chern–Weil theory, their Chern classes die in de Rham cohomology. By Griffiths' fundamental theorem [9], the Gauss–Manin connection is regular singular, and thinking of $k = \mathbb{C}$, and choosing a smooth compactification $\overline{B}$ of B with $T = \overline{B} \setminus B$ a normal crossing divisor, the underlying local monodromies around the components of the divisor T are quasi-unipotent. If they are unipotent, then Deligne's extension $\overline{\mathcal{H}}^i$ ([4]) has nilpotent residues and an Atiyah class computation ([6], appendix B) shows that the Chern classes of $\overline{\mathcal{H}}^i$ are also zero in de Rham cohomology.

On the other hand, Mumford ([14]) remarked that the Grothendieck–Riemann–Roch theorem applied to the structure sheaf and the dualizing sheaf of a family of curves f, yields vanishing, up to torsion, of the Chern classes of $\mathcal{H}^1$ in the Chow ring, which is much stronger information than the vanishing in de Rham cohomology. If one compactifies f as a semistable family of curves $\bar{f}: \overline{X} \to \overline{B}$, Deligne's extension $\overline{\mathcal{H}}^1$ is simply $R^1 \bar{f}_* \Omega^\bullet_{\overline{X}/\overline{B}}(\log(\bar{f}^{-1}(T))$ and again, the Grothendieck–Riemann–Roch theorem allows us to conclude that the Chern classes of $\overline{\mathcal{H}}^1$ are zero, up to torsion, in the Chow ring as well ([14]). This led the first author to wonder whether Gauss–Manin bundles in general can yield nontrivial algebraic cycles in Chow groups (see [7]). For example, it is proven in [2] that the algebraic Chern–Simons invariants of Gauss–Manin bundles in characteristic $p > 0$ always die (up to torsion).

The main theorem of this article is

THEOREM 1.1. *Let B be a smooth complex variety, with a smooth compactification $\overline{B}$ such that $T := \overline{B} \setminus B$ is a divisor with normal crossings. Let $\mathcal{H}^1$ be a variation of polarized pure Hodge structures of weight* 1, *with unipotent local monodromies along the components of T, and let $\overline{\mathcal{H}}^1$ be its Deligne extension with nilpotent residues. Then* $\mathrm{ch}(\overline{\mathcal{H}}^1) \in CH^0(\overline{B}) \otimes \mathbb{Q}$.

The 2-step F filtration on $\mathcal{H}$ extends to a 2-step filtration $\bar{F}$ on the Deligne's extension (see [10], for example). $\mathbb{E} = \bar{F}^1$ and $\mathbb{E}^\vee = \overline{\mathcal{H}}^1/\bar{F}^1$ are locally free and dual to each other, hence the assertion in Theorem 1.1 is equivalent to

$$\mathrm{ch}_{2\ell}(\mathbb{E}) = \mathrm{ch}_{2\ell}(\mathbb{E}^\vee) = 0 \in CH^{2\ell}(\overline{B}) \otimes \mathbb{Q}, \quad \text{for } \ell \geqslant 1.$$

Any variation of polarized pure Hodge structures of weight 1 is the Gauss–Manin bundle of a family $f: X \to B$ of Abelian varieties ([5]). Thus, Theorem 1.1 may be reformulated over any field of characteristic 0 in the following way:

THEOREM 1.2. *Let $f: X \to B$ be a smooth polarized family of Abelian varieties defined over a field k of characteristic* 0, *with B smooth, and let $\overline{B}$ be a smooth compactification with $T := \overline{B} \setminus B$ a normal crossing divisor. Assume that the first Gauss–Manin bundle $\mathcal{H}^1 = R^1 f_* \Omega^\bullet_{X/B}$ has unipotent local monodromies along the components of T, and let $\overline{\mathcal{H}}^1$ be its Deligne extension with nilpotent residues. Then* $\mathrm{ch}(\overline{\mathcal{H}}^1) \in CH^0(\overline{B}) \otimes \mathbb{Q}$.

For principally polarized families of abelian varieties a version of Theorem 1.2 holds over any field of characteristic $p \neq 2$ (Theorem 5.1), replacing the Chern character of the alternating sum of the Gauss–Manin bundles by the Chern character of the alternating sum of the cohomology of the relative differential forms with log poles. The latter only exist for a certain compactified family of principally polarized Abelian varieties (see Theorem 3.1), so we postpone the precise formulation.

Applying the Grothendieck–Riemann–Roch theorem to powers of a principal polarization in a family of Abelian varieties, Van der Geer proved the vanishing of $\mathrm{ch}_\ell(\mathcal{H}^1)$ in $CH^\ell(B) \otimes \mathbb{Q}$ for $\ell > 0$ ([15]). A reduction to Mumford's curve case, via the Abel–Jacobi map for genus $\leqslant 3$ ([16]) and via the Abel–Prym map for $g = 4, 5$ ([12]), yields the same vanishing for $\overline{\mathcal{H}}^1$.

On the other hand, Mumford's argument for a family of Abelian varieties immediately shows that the alternating sum of the Gauss–Manin bundles $\mathcal{H}^i$ has vanishing Chern classes, up to torsion. The problem is therefore to find a way to separate the different weights. Van der Geer's argument seems to be a detour. De Rham cohomology is not coherent cohomology, yet his proof relies of the remarkable identity Td $\mathbb{E}^\vee = \mathrm{ch}\,(f_*L)$, where $\mathbb{E}^\vee$ denotes again the Hodge bundle and where L is a principal polarization. This identity does not extend across the boundary, as one easily checks for families of elliptic curves.

In this note, we present a way to separate the different weights in the spirit of de Rham cohomology. We mod out the family of Abelian varieties by the (-1) action, which has the effect to separate the even from the odd weights. Then we observe that vanishing for the sum of the Gauss–Manin bundles in even weights is equivalent to vanishing for $\mathcal{H}^1$ (Lemma 2.1).

The next step is to extend the quotient across the boundary, keeping track both of the Gauss–Manin bundles and of the Riemann–Roch theorem. To this aim, one has to consider a compactification of the universal Abelian variety with level $n \geqslant 3$ structure over the moduli space $\mathcal{A}_{g,n}$, and to extend the (-1) action, controlling the fixed points. This is performed by a careful study of [8] (see Theorem 3.1). In order to prove Theorem 1.1, one may replace $\overline{B}$ by any $\overline{S}$, generically finite over $\overline{B}$. Doing so, we may assume that our family is the one constructed in Theorem 3.1.

One then has to understand the effect of the singular fibres of this non-smooth family in the application of the Riemann–Roch–Grothendieck theorem. This is Theorem 4.1, which is perhaps of independent interest. The philosophy of this log version is that most of the extra terms one has in the Riemann–Roch formula are killed by the existence of the Poincaré residues for the relative log 1-forms. The other ones die when one assumes that the relative 1-log forms essentially come from the base of the family.

2. A Numerical Computation

Let $\mathcal{H}$ be a bundle of rank $2g$ over a smooth variety S, which is an extension of a bundle $\mathbb{E}^\vee$ by its dual $\mathbb{E}$.

LEMMA 2.1. *If*

$$\mathrm{ch}\left(\sum_{i=1}^{g} \wedge^{2i}\mathcal{H}\right) \in CH^0(S) \otimes \mathbb{Q},$$

and

$$\mathrm{ch}\left(\sum_{i=0}^{g-1} \wedge^{2i+1}\mathcal{H}\right) = 0 \in CH^\bullet(S) \otimes \mathbb{Q},$$

then one has $\mathrm{ch}(\mathcal{H}) \in CH^0(S) \otimes \mathbb{Q}$.

Proof. Let $K(S)$ denote the K-group of vector bundles on S. Setting as usual (see [11], [3], for example)

$$\lambda_t(\mathcal{H}) = \sum_{i=0}^{2g} \lambda^i(\mathcal{H})t^i \in K(S)[[t]],$$

with $\lambda^i(\mathcal{H}) = [\wedge^i \mathcal{H}] \in K(S)$, and denoting by e_i, $i = 1, \ldots, g$, the Chern roots of $\mathbb{E}$, one has

$$\lambda_t(\mathcal{H}) = \lambda_t(\mathbb{E}) \cdot \lambda_t(\mathbb{E}^\vee) = \prod_{i=1}^{g} (1 + e_i t)(1 + e_i^\vee t).$$

Thus

$$\mathrm{ch}(\lambda_t(\mathcal{H})) = \prod_{i=1}^{g} (1 + e^{a_i} t)(1 + e^{-a_i} t) \in CH^\bullet(S) \otimes \mathbb{Q}$$

with $a_i = c_1(e_i)$. We set

$$\begin{aligned}
\mathrm{ch}^{\mathrm{even}}(\wedge\mathcal{H}) &:= \tfrac{1}{2}(\mathrm{ch}\lambda_1(\mathcal{H}) + \mathrm{ch}\lambda_{-1}(\mathcal{H})) \\
&= \frac{1}{2}\prod_{i=1}^{g}(1 + e^{a_i})(1 + e^{-a_i}) + \frac{1}{2}\prod_{i=1}^{g}(1 - e^{a_i})(1 - e^{-a_i}), \\
\mathrm{ch}^{\mathrm{odd}}(\wedge\mathcal{H}) &:= \tfrac{1}{2}(\mathrm{ch}\lambda_1(\mathcal{H}) - \mathrm{ch}\lambda_{-1}(\mathcal{H})) \\
&= \frac{1}{2}\prod_{i=1}^{g}(1 + e^{a_i})(1 + e^{-a_i}) - \frac{1}{2}\prod_{i=1}^{g}(1 - e^{a_i})(1 - e^{-a_i}).
\end{aligned}$$

Thus the assumption is equivalent to

$$\prod_{i=1}^{g}(1 + e^{a_i})(1 + e^{-a_i}) = 0 \in CH^{\geqslant 1}(S) \otimes \mathbb{Q},$$

$$\prod_{i=1}^{g}(1 - e^{a_i})(1 - e^{-a_i}) = 0 \in CH^{\bullet}(S) \otimes \mathbb{Q}.$$

The first relation reads

$$\prod_{i=1}^{g}(1 + e^{a_i})^2 e^{-a_i} (= 2^{2g}) \in CH^0(S) \otimes \mathbb{Q}$$

or, equivalently,

$$-\sum_{i=1}^{g} a_i + 2\sum_{i=1}^{g} \log(1 + e^{a_i}) \in CH^0(S) \otimes \mathbb{Q}. \tag{2.1}$$

Setting $\psi(t) = \log(1 + e^t)$, one has

$$\psi'(t) = \frac{e^t}{1 + e^t} = 1 - \varphi(t),$$

with

$$\varphi(t) = \frac{1}{1 + e^t} = \frac{1}{2} \sum_{n=0}^{\infty} E_n(0) \frac{t^n}{n!},$$

where the $E_n(0)$ are the Euler numbers at 0 (see [1], 1.14, (2)).

Vanishing of $\mathrm{ch}(\mathcal{H})$ in $CH^{\geqslant 1}(S) \otimes \mathbb{Q}$ is equivalent to vanishing of

$$\mathrm{ch}_{2\ell}(\mathbb{E}) = \sum_{i=1}^{g} \frac{a_i^{2\ell}}{(2\ell)!} \in CH^{2\ell}(S) \otimes \mathbb{Q}, \quad \ell \geqslant 1.$$

Thus by (2.1) it is equivalent to the assertion that the coefficients of $t^{2\ell}$ in the power series expansion of $\psi(t)$ are nonzero for $\ell > 0$, or equivalently that none of the odd coefficients of the expansion of $\varphi(t)$ is vanishing, that is that $E_{2n-1}(0) \neq 0$ for all $n \geqslant 1$. By [1], 1.14, (7), one has

$$E_{2n-1}(0) = \frac{2(1 - 2^{2n})}{2n} B_{2n}(0),$$

where $B_n(0)$ are the Bernoulli numbers at 0, and by [1], 1.13, (16), $B_{2n}(0) \neq 0$ for all $n \geqslant 1$. This concludes the proof. □

3. The Geometry of the Compactified Family of Abelian Varieties

In this section, we extract from [8] the necessary geometric information in order to find a model for the compactification of a family of Abelian varieties, which will allow us to apply in Section 5 a log version of the Grothendieck–Riemann–Roch theorem.

We use the following notations. Fixing the level n, we denote by $S = \mathcal{A}_{g,n}$ the moduli stack of Abelian varieties with level n structure ([13]). For $n \geqslant 3$, not divisible by the characteristic of k, S is a scheme and it carries a universal family $f: X \to S$ of Abelian varieties. We consider one of the compactifications $\bar{f}: \overline{X} \to \overline{S}$ described on p. 195 of [8].

More precisely, the compactification $S \subset \overline{S}$ is determined by a certain polyhedral cone decomposition $\{\sigma_\alpha\}$ of $C(N)$, where N is a free Abelian group of rank g, $B(N)$ is the space of integer valued symmetric bilinear forms and $C(N) \subset B(N_{\mathbb{R}})$ is the convex cone of all positive semi-definite symmetric bilinear forms whose radicals $\mathrm{Ker}(b: N_{\mathbb{R}} \to N_{\mathbb{R}}^{\vee})$, are defined over $\mathbb{Q}$ (p. 96, 2.1). The interior $C^{\circ}(N)$ consists of the positive definite forms.

Then, $\overline{S}$ being chosen, one considers $\tilde{B}(N) = B(N) \times N^\vee$. A compactification $\bar{f}: \overline{X} \to \overline{S}$ is determined by the choice of a polyhedral cone decomposition $\{\tau_\beta\}$ of the cone

$$\tilde{C}(N) = \{(b, \ell), \ell = 0 \text{ on } \operatorname{Ker} b\} \subset C(N) \times N_{\mathbb{R}}^\vee,$$
$$\subset \tilde{B}(N_{\mathbb{R}}) = B(N_{\mathbb{R}}) \times N_{\mathbb{R}}^\vee$$

(p. 195, last section). For the existence of $\bar{f}$ one needs (p. 196, 1.3(v)) that any τ_β maps into a σ_α. Recall moreover that one of the conditions on the polyhedral cone decompositions requires $\{\sigma_\alpha\}$ to be $GL(N)$-invariant (p. 96, 2.2), $\{\tau_\beta\}$ to be $GL(N) \ltimes N$-invariant (p. 196, 1.3), and that there are finitely many orbits.

As is underlined on p. 195, l.1, the family $f: X \to S$ does not necessarily extend to a semi-Abelian group scheme $G \to \overline{S}$ embedded in $\bar{f}: \overline{X} \to \overline{S}$. Yet, Remark 1.4 p. 197 asserts that it is possible to further refine the cone decompositions in such a way that the natural section of $\tilde{B}(N) \to B(N)$ respects the cone decomposition, which means that any $\sigma_\alpha \times \{0\}$ is precisely one of the τ_β. This guarantees that $f: X \to S$ extends to a semi-abelian group scheme $G \to \overline{S}$ embedded in $\bar{f}: \overline{X} \to \overline{S}$.

We set $T := \overline{S} \setminus S,\ Y := \overline{X} \setminus X$.

Refining, we may assume $\{\tau_\beta\}$ and $\{\sigma_\alpha\}$ to be smooth (p. 96, 2.3 and p. 98, (iii)) and $\{\sigma_\alpha\}$ to satisfy the condition (ii) on p. 97. In particular, this says that both $\overline{S},\ \overline{X}$ are smooth, that $T,\ Y$ are normal crossings divisors and that the components of T are nonsingular (p. 118, 5.8, a).

For $n \geqslant 3$, it is explained on pp. 172 and 173 how to refine a given polyhedral smooth cone decomposition to force $\overline{S}$ to be a projective and smooth scheme. We remark, that via p. 173 (c), whatever projective polyhedral decomposition is chosen to define the compactification, it is always possible to refine it to a finer smooth projective one. Of course, this changes the compactifications of S and X. For the family X, one first quotes Theorem 1.1 p. 195 which yields $\bar{f}: \overline{X} \to \overline{S}$ as a morphism of algebraic stacks. However, by p. 207, l.4 to 8, we know that $\overline{X}$ is a smooth projective variety and that $\bar{f}$ is then consequently a projective morphism.

We have essentially reached the first part of the following theorem.

THEOREM 3.1 (Faltings-Chai). *Let k be a field containing the nth roots of unity. For n even $\geqslant 4$ and not divisible by the characteristic of k, there is a compactification $\bar{f}: \overline{X} \to \overline{S}$ of the universal family of principally polarized Abelian varieties of genus g with level n structure, with the following properties*:

(1) *$\overline{X}$ and $\overline{S}$ are smooth projective varieties.*
(2) *T, Y are normal crossings divisors with smooth irreducible components.*
(3) *The sheaf of relative 1-forms $\Omega^1_{\overline{X}/\overline{S}}(\log Y)$ with logarithmic poles along Y is locally free.*
(4) *The Hodge bundle $\mathbb{E} = \bar{f}_*\Omega^1_{\overline{X}/\overline{S}}(\log Y)$ is locally free.*
(5) *One has $f^*\mathbb{E} = \Omega^1_{\overline{X}/\overline{S}}(\log Y)$.*

(6) *One has* $R^q \bar{f}_*(\Omega^p_{\overline{X}/\overline{S}}(\log Y)) = \wedge^q \mathbb{E}^\vee \otimes \wedge^p \mathbb{E}$.
(7) *When the ground field* k *has characteristic* 0, *the Gauss–Manin sheaf* $R^q \bar{f}_* \Omega^\bullet_{\overline{X}/\overline{S}}(\log Y) =: \overline{\mathcal{H}}^q$ *is Deligne's extension* [4] *of its restriction to S. In particular, it is locally free. The residues of the Gauss–Manin connection are nilpotent.*
(8) $f: X \to S$ *extends to a semi-Abelian group scheme* $G \to \overline{S}$ *embedded into* $\bar{f}: \overline{X} \to \overline{S}$.
(9) *The level n-structure sections* S_i *of* $f: X \to S$ *extend to disjoint sections* $\overline{S_i}$ *of* $\bar{f}: \overline{X} \to \overline{S}$.
(10) *The* $\iota := (-1): X \to X$ *involution over S extends to an involution, still denoted by* $\iota: \overline{X} \to \overline{X}$ *over* $\overline{S}$.
(11) *The fixed points of* ι *lie in* $\cup \overline{S_i}$.

Proof. (1), (8) have already been discussed, as well as part of (2). For (3), (4), (5), (6), we refer to Theorem 1.1, p. 195. For (7), we know (p. 218(4)) that the Hodge to de Rham spectral sequence degenerates, which implies via 6. that $\overline{\mathcal{H}}^i$ is locally free. On the other hand, restricting to a generic curve $\overline{C}$ in $\overline{S}$ intersecting T in general points $\overline{C} \setminus C$, one obtains a family $h: \overline{W} \to \overline{C}$, the fibres of which are all normal crossings divisors. As is well known (see [10], p. 130, for example), $\overline{\mathcal{H}}^i$ is the Deligne extension of $\overline{\mathcal{H}}^i|_C$. On the other hand, (8) implies that the Gauss–Manin connection on $\overline{\mathcal{H}}^i|_C$ has nilpotent residues.

For (9), (10), (11) and to see that the components of Y can be assumed to be smooth, we need a more precise discussion of the polyhedral cone decomposition and its relation with the compactification. Obviously the three properties hold true over S, hence extending them to the boundary is a local question. As on p. 207(2), we can even replace $\overline{S}$ by the formal completion along a certain stratum $\mathfrak{Z}$ and $\bar{f}: \overline{X} \to \overline{S}$ by the pullback family. Doing so, we are allowed to use the description of $\bar{f}$ given on pp. 201 and 203.

Recall that the toroidal embeddings $\bar{F} \to \overline{E}$ given by the polyhedral cone decompositions are stratified by locally closed subschemes $\mathfrak{Z}_{\tau_\beta}$ and $\mathfrak{Z}_{\sigma_\alpha}$ (p. 100, 2.5(iv)). The $\mathfrak{Z}_{\sigma_\alpha}$ and $\mathfrak{Z}_{\tau_\beta}$ are orbits under the torus action, and their codimension is equal to the dimension of the $\mathbb{R}$-vectorspace spanned by σ_α or τ_β, respectively.

If the fibre G_0 of G over the general point of the stratum $\mathfrak{Z}$ is a torus, then the formal completions along the stratum, together with the pullback of $\bar{f}: \overline{X} \to \overline{S}$, are obtained by restricting $\bar{F}$ to the formal completion of $\overline{E}$ along a stratum $\mathfrak{Z}_{\sigma_\alpha}$, with $\sigma_\alpha \subset C^\circ(N)$, and by taking the quotient by N (p. 201, last section). In general, G_0 will be an extension of an Abelian variety A of dimension $g - r$ by a torus. As sketched on pp. 202 and 203, one has to replace N by an r-dimensional quotient lattice N_ξ in this case. In particular, one may again assume that σ_α lies in the interior of the cone $C(N_\xi)$. Since the combinatorial description remains the same, we drop the $_\xi$.

By p. 100, 2.5(i) and (ix), the category of rational partial polyhedral cone decompositions is equivalent to the category of torus embeddings. Thus composing a given torus embedding with $\iota = (-1)$ corresponds to the action of ι on the data giving the polyhedral cone decomposition. So we just have to verify that ι respect those data. ι acts on N by multiplication with (-1), hence the action is trivial on $B(N)$, and is (-1) on $N^\vee$. An element $\mu \in N$ acts on $(b, \ell) \in \tilde{B}(N) = B(N) \times N$ by

$$(b, \ell) \mapsto (b, \ell + b(\mu, \ \ )),$$

while $\gamma \in GL(N)$ acts via

$$(b, \ell) \mapsto \big(b \circ \big(\gamma^{-1}, \gamma^{-1}\big), \ell \circ \gamma^{-1}\big)$$

(p. 196, first section). In particular,

$$(-\mathrm{Id}, 0) \in GL(N) \ltimes N$$

maps

$$(b, \ell) \in B(N) \times N^\vee$$

to $(b, -\ell)$. Since the polyhedral cone decompositions are invariant under $GL(N) \ltimes N$, it is invariant under ι.

As for (9), we just remark that a morphism from $\overline{S}$ to the corresponding compactification $\overline{S}_1$ of the moduli stack $\mathcal{A}_{g,1}$ is defined by multiplication with n on the torus (p. 130, 6.7(6)). In different terms, one keeps the cone decompositions in $B(N_{\mathbb{R}})$ and $\tilde{B}(N_{\mathbb{R}}) \times N_{\mathbb{R}}$, but changes the integral structure by multiplication with n on N. To see that the closure of the sections of $f\colon X \to S$ of order n in $\overline{X}$ are disjoint sections of $\bar{f}$, it is again sufficient to consider the pullback of $\bar{f}$ to formal completions of the strata in $\overline{S}$. By p. 202, first section, the n-torsion points are the pull-back of the zero-section of the semi-Abelian group scheme over the formal completion.

As already seen, ι acts trivially on the cone $\sigma_\alpha \subset B(N_{\mathbb{R}})$, and the fixed points under the ι involution lie in strata $\mathfrak{Z}_{\tau_\beta}$ of $\bar{F}$, the N-orbit of which are invariant under ι. We assumed that $\{\tau_\beta\}$ is smooth, that is each cone τ_β is generated by a partial $\mathbb{Z}$-basis

$$\big((b_1, \ell_1), \ldots, (b_r, \ell_r)\big) \quad \text{of} \quad B(N) \times N^\vee.$$

Thus one has a $\mu \in N$ with

$$(b_i, -\ell_i) = (b_{j(i)}, \ell_{j(i)} + b_{j(i)}(\mu, \ \ ))$$

for $i = 1, \ldots, r$. We have taken an even level n. Thus $N = 2 \cdot N'$ for another integral structure, and one has $\mu = 2 \cdot \mu'$ for a $\mu' \in N'$. We obtain

$$b_i = b_{j(i)} \quad \text{and} \quad \ell_i + \ell_{j(i)} = 2b_{j(i)}(\mu', \ \ ).$$

Twisting the free $\mathbb{Z}$-module $B(N) \times N^\vee$ by $\mathbb{Z}/2$ over $\mathbb{Z}$, the basis elements (b_i, ℓ_i) and $(b_{j(i)}, \ell_{j(i)})$ become equal, which implies that $j(i) = i$. This in turn implies that $2\ell_i = 2b_i(\mu', \ )$, and shows that the dimension of the subspace of $B(N) \times N^\vee$ generated by τ_β is the same as the dimension of its image in $B(N)$. This finally implies that $\operatorname{codim}(\mathfrak{Z}_{\tau_\beta})$ is equal to the codimension of its image in $\overline{E}$. So the fixed points of ι all lie in the smooth locus of $\bar{f}$.

We verified all the conditions stated in Theorem 3.1, except that Y can still have singular components. However, since (1), (3)–(11) are compatible with the blowing up of nonsingular strata of the singular locus of Y_{red}, this last point can be achieved. □

4. A Log Version of the Grothendieck–Riemann–Roch Theorem

In this section, we show that the Grothendieck–Riemann–Roch theorem extends to a log version.

THEOREM 4.1. *Let $\bar{f}:\overline{X} \to \overline{S}$ be a projective morphism of relative dimension g over a field, with $\overline{X}$, $\overline{S}$ smooth, compactifying the smooth projective morphism $f: X \to S$, with the following properties*:

(1) *Both $T = \overline{S}\backslash S$ and $Y := (\bar{f}^*(T))_{\mathrm{red}}$ are normal crossings divisors with smooth irreducible components.*

(2) *The sheaves $\Omega^i_{\overline{X}/\overline{S}}(\log Y)$ are locally free.*

(3) *There are cycles $W \in CH^g(\overline{X}) \otimes \mathbb{Q}$, $\xi \in CH^g(\overline{S}) \otimes \mathbb{Q}$, such that*
$$c_g\big(\Omega^1_{\overline{X}/\overline{S}}(\log Y)\big) = \bar{f}^*(\xi) + W \in CH^g(\overline{X}) \otimes \mathbb{Q},$$
with the property $W \cdot Y_i = 0 \in CH^{g+1}(\overline{X}) \otimes \mathbb{Q}$ for all irreducible components of $Z := \bar{f}^*(T) - Y$.

Then one has
$$\sum_i (-1)^i \mathrm{ch}\bigg(\sum_j R^j \bar{f}_* \Omega^{i-j}_{\overline{X}/\overline{S}}(\log Y)\bigg) \in CH^0(\overline{S}) \otimes \mathbb{Q}.$$

Remark. 4.2. (i) Theorem 4.1 applies in particular when $Z = \emptyset$, that is when the fibers have no multiplicities. In this case, (3) is automatically fulfilled and, as we will see, the proof does not require any combinatorics.

(ii) For $\bar{f}:\overline{X} \to \overline{S}$ semistable in codimension one, Z will turn out to be the union of all components of the boundary, each of which with image in $\overline{S}$ of codimension larger than or equal to 2.

Proof. We apply the Grothendieck–Riemann–Roch theorem [3] to the alternating sum of the sheaves $\Omega^i_{\overline{X}/\overline{S}}(\log Y)$. It yields

$$\sum_i (-1)^i \mathrm{ch}\bigg(\sum_j R^j \bar{f}_* \Omega^{i-j}_{\overline{X}/\overline{S}}(\log Y)\bigg) = \bar{f}_*\bigg(\mathrm{Todd}(T_{\overline{X}/\overline{S}}) \cdot \bigg(\sum_i (-1)^i \mathrm{ch}\big(\Omega^i_{\overline{X}/\overline{S}}(\log Y)\big)\bigg)\bigg).$$

We consider the residue sequences

$$0 \to \bar{f}^*\Omega^1_{\overline{S}} \to \bar{f}^*\Omega^1_{\overline{S}}(\log T) \to \mathcal{O}_{\tilde{T}} \to 0,$$
$$0 \to \Omega^1_{\overline{X}} \to \Omega^1_{\overline{X}}(\log Y) \to \mathcal{O}_{\tilde{Y}} \to 0,$$

where, in order to simplify notations, we have set

$$\tilde{T}_j = \bar{f}^* T_j \quad \text{and} \quad \mathcal{O}_{\tilde{T}} = \bigoplus_j \mathcal{O}_{\tilde{T}_j},$$

for the irreducible components T_j of T, and similarly $\mathcal{O}_{\tilde{Y}} = \oplus_i \mathcal{O}_{Y_i}$. The pullback of differential forms

$$\bar{f}^*\Omega^1_{\overline{S}}(\log T) \to \Omega^1_{\overline{X}}(\log Y) \tag{4.1}$$

induces a natural map $\mathcal{O}_{\tilde{T}} \to \mathcal{O}_{\tilde{Y}}$.

Multiplicativity of the Todd class implies

$$\begin{aligned}&\mathrm{Todd}(T_{\overline{X}}) \cdot \mathrm{Todd}(\bar{f}^* T_{\overline{S}})^{-1}\\ &\quad = \mathrm{Todd}(T_{\overline{X}/\overline{S}}(\log Y)) \cdot \prod_{q \geqslant 1} \mathrm{Todd}(\mathcal{E}xt^q(\mathcal{O}_{\tilde{Y}}, \mathcal{O}_{\overline{X}}))^{(-1)^{q+1}} \cdot\\ &\qquad \cdot \prod_{q \geqslant 1} \mathrm{Todd}(\mathcal{E}xt^q(\mathcal{O}_{\tilde{T}}, \mathcal{O}_{\overline{X}}))^{(-1)^q}.\end{aligned}$$

We define $\mathcal{B}$ and $\mathcal{C}$ via the exact sequences

$$0 \to \mathcal{B} \to \mathcal{O}_{\tilde{Y}} \to \mathcal{O}_{\tilde{Y}}/\mathcal{O}_{\tilde{T}} \to 0,$$
$$0 \to \mathcal{C} \to \mathcal{O}_{\tilde{T}} \to \mathcal{B} \to 0.$$

Using again multiplicativity, one obtains

$$\begin{aligned}&\mathrm{Todd}(T_{\overline{X}}) \cdot \mathrm{Todd}(\bar{f}^* T_{\overline{S}})^{-1}\\ &\quad = \mathrm{Todd}(T_{\overline{X}/\overline{S}}(\log Y)) \cdot \prod_{q \geqslant 1} \mathrm{Todd}(\mathcal{E}xt^q(\mathcal{O}_{\tilde{Y}}/\mathcal{O}_{\tilde{T}}, \mathcal{O}_{\overline{X}}))^{(-1)^{q+1}} \cdot\\ &\qquad \cdot \prod_{q \geqslant 1} \mathrm{Todd}(\mathcal{E}xt^q(\mathcal{C}, \mathcal{O}_{\overline{X}}))^{(-1)^q}.\end{aligned}$$

On the other hand, one has the well-known relation [11]

$$\begin{aligned}&\mathrm{Todd}(T_{\overline{X}/\overline{S}}(\log Y)) \cdot \Big(\sum (-1)^i \mathrm{ch}\big(\Omega^i_{\overline{X}/\overline{S}}(\log Y)\big)\Big)\\ &\quad = (-1)^g c_g(\Omega^1_{\overline{X}/\overline{S}}(\log Y)).\end{aligned}$$

Let us write

$$\prod_{q \geqslant 1} \mathrm{Todd}(\mathcal{E}xt^q(\mathcal{O}_{\tilde{Y}}/\mathcal{O}_{\tilde{T}}, \mathcal{O}_{\overline{X}}))^{(-1)^{q+1}} = 1 + \mathrm{err}$$

with err $\in CH^\bullet(\overline{X})$. The sheaf $\mathcal{O}_{\tilde{Y}}/\mathcal{O}_{\tilde{T}}$ is a direct sum of structure sheaves $\mathcal{O}_{Y_{i_1}\cap\ldots\cap Y_{i_\ell}}$.

The Poincaré residue map $\Omega^1_{\overline{X}/\overline{S}}(\log Y) \twoheadrightarrow \mathcal{O}_{Y_{i_1}\cap\ldots\cap Y_{i_\ell}}$ is surjective, hence

$$c_g\big(\Omega^1_{\overline{X}/\overline{S}}(\log Y)\big)\cdot(Y_{i_1}\cap\ldots\cap Y_{i_\ell}) = 0 \in CH^{g+\ell}(\overline{X}).$$

Since err is a sum of terms supported in those strata $Y_{i_1}\cap\ldots\cap Y_{i_\ell}$, we conclude

$$\begin{aligned} &c_g\big(\Omega^1_{\overline{X}/\overline{S}}(\log Y)\big)\cdot\prod_{q\geqslant 1}\mathrm{Todd}(\mathcal{E}xt^q(\mathcal{O}_{\tilde{Y}}/\mathcal{O}_{\tilde{T}},\mathcal{O}_{\overline{X}}))^{(-1)^{q+1}}\\ &\quad = c_g\big(\Omega^1_{\overline{X}/\overline{S}}(\log Y)\big).\end{aligned}$$

Thus

$$\begin{aligned} &\bar{f}_*\mathrm{Todd}(T_{\overline{X}/\overline{S}})\cdot\Big(\sum_i(-1)^i\mathrm{ch}\big(\Omega^i_{\overline{X}/\overline{S}}(\log Y)\big)\Big)\\ &\quad = (-1)^g\bar{f}_*\bigg(c_g\big(\Omega^1_{\overline{X}/\overline{S}}(\log Y)\big)\cdot\prod_{q\geqslant 1}\mathrm{Todd}(\mathcal{E}xt^q(\mathcal{C},\mathcal{O}_{\overline{X}}))^{(-1)^q}\bigg). \end{aligned} \tag{4.2}$$

If $Z=\emptyset$, the sheaf $\mathcal{C}$ is zero, hence $\mathrm{Todd}(\mathcal{E}xt^q(\mathcal{C},\mathcal{O}_{\overline{X}}))=1$. This finishes the proof of Remark 4.2(i).

For the general case, we will express the right-hand side of (4.2) as the direct image of terms of the form $(-1)^g c_g(\Omega^1_{\overline{X}/\overline{S}}(\log Y))\cdot(1+w)$, where $w\in CH^{\geqslant 1}(\overline{X})$ is a linear combination of products $Y_{i_1}\cdots Y_{i_\ell}$. For some of those, one can again use the Poincaré residue for relative differential forms, to show that

$$(-1)^g c_g\big(\Omega^1_{\overline{X}/\overline{S}}(\log Y)\big)\cdot Y_{i_1}\cdots Y_{i_\ell}=0.$$

For the remaining ones, we will show that one of the Y_{i_ν} must be a component of Z, and the assumption (3) in Theorem 4.1 will allow us to show that

$$\bar{f}_*(-1)^g c_g\big(\Omega^1_{\overline{X}/\overline{S}}(\log Y)\big)\cdot Y_{i_1}\cdots Y_{i_\ell}=0.$$

So we have to study the residue map and the sheaf $\mathcal{O}_{\tilde{Y}}/\mathcal{O}_{\tilde{T}}$ more precisely. Let us first fix notations. We write

$$Y=\sum_{i\in\mathcal{I}}Y_i,\qquad T=\sum_{j\in\mathcal{J}}T_j,$$

where the Y_i, T_j are prime divisors. For $I\subset\mathcal{I}$, $J\subset\mathcal{J}$, we define

$$Y_I=\bigcap_{i\in I}Y_i,\qquad T_J=\bigcap_{j\in J}T_j.$$

Since $\Omega^1_{\overline{X}/\overline{S}}(\log Y)$ is locally free, $\bar{f}$ sends strata to strata. Indeed, choose J maximal with the property that $T_J\supset\bar{f}(Y_I)$. If $Y_I\to T_J$ is not surjective, we find a local parameter t on $\overline{S}$, in a general point of $\bar{f}(Y_I)$, such that $\bar{f}(Y_I)$ lies in the zero locus of t but not in T_J. As $\mathcal{O}_{\overline{S}}\,\mathrm{d}t$ is locally split in $\Omega^1_{\overline{S}}(\log T)$, and the injection

$$\bar{f}^*\Omega^1_{\overline{S}}(\log T) \subset \Omega^1_{\overline{X}}(\log Y)$$

is locally split, $\mathcal{O}_{\overline{X}}\,\mathrm{d}t$ is locally split in $\Omega^1_{\overline{X}}(\log Y)$ as well. But since by assumption,

$$t = \sum_{i\in I} y_i\alpha_i,\ \ \alpha_i \in \mathcal{O}_{\overline{X}},$$

one finds

$$\mathrm{d}t = \sum_{i\in I} y_i \cdot \left(\mathrm{d}\alpha_i + \frac{\mathrm{d}y_i}{y_i}\alpha_i\right),$$

a contradiction.

This allows to define, for each $I \subset \mathcal{I}$, the index set $J := J(I) \subset \mathcal{J}$ with the property $\bar{f}(Y_I) = T_J$. It yields a $\mathbb{Q}$-linear map

$$\varphi_I\colon \bigoplus_{j\in J}\mathbb{Q}\cdot T_j \to \bigoplus_{i\in I}\mathbb{Q}\cdot Y_i,$$

$$\varphi_I(T_j) = \sum_{i\in I}\nu_i^j Y_i, \quad \text{where } \bar{f}^*T_j = \sum_{i\in\mathcal{I}}\nu_i^j Y_i.$$

One defines

$$\delta_Y = \mathrm{codim}(\mathrm{Im}(\varphi_I)) = |I| - \dim(\mathrm{Im}(\varphi_I)). \tag{4.3}$$

CLAIM 4.3. *One has $\delta_I > 0$ if and only of the residue map*

$$\Omega^1_X(\log Y) \twoheadrightarrow \mathcal{O}_{Y_I}$$

factors through $\Omega^1_{\overline{X}/\overline{S}}(\log Y) \twoheadrightarrow \mathcal{O}_{Y_I}$.

Proof. We fix I, $J = J(I)$ as before. The question being local at generic points of T_J and Y_I, we choose local parameters $\{t_j\}$ and $\{y_i\}$ defining T_j and Y_i locally. Then the equation of the morphism $\bar{f}$ is simply $t_j = \prod_{i\in I} y_i^{\nu_i^j}$. As in the beginning of the proof of Theorem 4.1, by functoriality of the Poincaré residue, the inclusion (4.1) maps

$$\bar{f}^*\Omega^1_{\overline{S}}(\log T) \xrightarrow{\mathrm{res}} \bar{f}^*\left(\bigoplus_{j\in J}\mathcal{O}_{T_j}\right) \quad \text{to} \quad \Omega^1_{\overline{X}}(\log Y) \xrightarrow{\mathrm{res}} \bigoplus_{i\in I}\mathcal{O}_{Y_i}.$$

It induces

$$\bigoplus_{j\in J}\bar{f}^*(\mathcal{O}_{T_j})|_{Y_I} = \bigoplus_J \mathcal{O}_{Y_I} \to \bigoplus_I \mathcal{O}_{Y_I}$$

$$\mathrm{res}\left(\frac{\mathrm{d}t_j}{t_j}\right) \mapsto \sum_{i\in I}\nu_i^j\,\mathrm{res}\left(\frac{\mathrm{d}y_i}{y_i}\right). \qquad \square$$

CLAIM 4.4. *Let $i_1, \dots, i_\mu \in \mathcal{I}$, not necessarily pairwise distinct. Assume that $Y_I \neq \emptyset$ and $\delta_I > 0$, where $I \subset \mathcal{I}$ is the smallest subset of $\mathcal{I}$ containing $\{i_1, \dots, i_\mu\}$. Then the cycle $Y_{i_1} \cdots Y_{i_\mu} \in CH^\mu(\overline{X})$ fulfills*

$$c_g\big(\Omega^1_{\overline{X}/\overline{S}}(\log Y)\big) \cdot Y_{i_1} \cdots Y_{i_\mu} = 0 \in CH^{g+\mu}(\overline{X}).$$

Proof. Renumbering the components, we may write $I = \{i_1, \dots, i_\eta\}$ for a pairwise distinct set of indices $i_1, \dots, i_\eta$. By Claim 4.3, one has

$$c_g\big(\Omega^1_{\overline{X}/\overline{S}}(\log Y)\big) \cdot Y_I = 0,$$

thus, a fortiori, $c_g(\Omega^1_{\overline{X}/\overline{S}}(\log Y)) \cdot Y_I \cdot Y_{i_{\eta+1}} \cdots Y_{i_\mu} = 0$. □

Recall that for the proof of Theorem 4.1, it remains to show that the term

$$(-1)^g \bar{f}_*\left(c_g\big(\Omega^1_{\overline{X}/\overline{S}}(\log Y)\big) \cdot \prod_{q \geqslant 1} \mathrm{Todd}(\mathcal{E}xt^q(\mathcal{C}, \mathcal{O}_{\overline{X}}))^{(-1)^q}\right)$$

in formula (4.2) is equal to $(-1)^g \bar{f}_*(c_g(\Omega^1_{\overline{X}/\overline{S}}(\log Y)))$. Writing

$$\tilde{T}_j = \bar{f}^*(T_j) \quad \text{and} \quad Z_j = \tilde{T}_j - (\tilde{T}_j)_{\mathrm{red}}$$

one has a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_j \mathcal{O}_{Z_j}(-(\tilde{T}_j)_{\mathrm{red}}) & \longrightarrow & \bigoplus_j \mathcal{O}_{\tilde{T}_j} & \longrightarrow & \bigoplus_j \mathcal{O}_{(\tilde{T}_j)_{\mathrm{red}}} & \longrightarrow & 0 \\
 & & \rho_1 \downarrow & & = \downarrow & & \downarrow \rho_2 & & \\
0 & \longrightarrow & \mathcal{C} & \longrightarrow & \bigoplus_j \mathcal{O}_{\tilde{T}_j} & \longrightarrow & \bigoplus_i \mathcal{O}_{Y_i} & &
\end{array}$$

with exact rows. Since ρ_1 is injective $\mathrm{Ker}\,\rho_2 = \mathrm{Coker}\,\rho_1 =: \mathcal{K}$. Thus

$$\begin{aligned}
&\prod_{q \geqslant 1} \mathrm{Todd}(\mathcal{E}xt^q(\mathcal{C}, \mathcal{O}_{\overline{X}}))^{(-1)^q} \\
&\quad = \prod_{q \geqslant 1} \mathrm{Todd}(\mathcal{E}xt^q(\oplus_j \mathcal{O}_{Z_j}(-(\tilde{T}_j)_{\mathrm{red}}, \mathcal{O}_{\overline{X}})))^{(-1)^q} \cdot \\
&\qquad \cdot \prod_{q \geqslant 1} \mathrm{Todd}(\mathcal{E}xt^q(\mathcal{K}, \mathcal{O}_{\overline{X}}))^{(-1)^q}.
\end{aligned}$$

One has

$$\mathcal{E}xt^1(\mathcal{O}_{Z_j}(-(\tilde{T}_j)_{\mathrm{red}}), \mathcal{O}_{\overline{X}}) = \mathcal{O}_{\overline{X}}(\tilde{T}_j)/\mathcal{O}_{\overline{X}}((\tilde{T}_j)_{\mathrm{red}}),$$

while

$$\mathcal{E}xt^q(\mathcal{O}_{Z_j}(-(\tilde{T}_j)_{\mathrm{red}}), \mathcal{O}_{\overline{X}}) = 0, \quad q \geqslant 2.$$

Thus

$$\prod_{q\geqslant 1} \mathrm{Todd}(\mathcal{E}xt^q(\mathcal{O}_{Z_j}(-(\tilde{T}_j)_{\mathrm{red}}, \mathcal{O}_{\overline{X}})))^{(-1)^q} = 1 + v_j,$$

where $v_j \in CH^{\geqslant 1}(\overline{X})$ can be written as a sum of terms $m_{i_1,\ldots,i_\mu} Y_{i_1} \cdots Y_{i_\mu}$, where at least one of the Y_{i_p} lies in Z_j, and $m_{i_1,\ldots,i_\mu} \in \mathbb{Z}$. On the other hand, one has the exact sequence

$$0 \to \mathcal{K} \to \bigoplus_j \mathcal{O}_{(\tilde{T}_j)_{\mathrm{red}}} \to \mathcal{O}_{\tilde{Y}} \to \mathcal{O}_{\tilde{Y}}/\mathcal{O}_{\tilde{T}} \to 0.$$

Let us write

$$Y = \Phi + V \quad \text{and} \quad (\tilde{T}_j)_{\mathrm{red}} = \Phi_j + V_j,$$

where each component of Φ_j maps surjectively onto T_j, where each component of V and V_j maps to a lower-dimensional strata in T, and with $\Phi = \left(\sum_j \Phi_j\right)_{\mathrm{red}}$. Of course $V_j \subset V$ and $V \subset Z$. Denoting as usual by $\tilde{}$ the normalization, one has

$$\begin{aligned}
&\prod_{q\geqslant 1} \mathrm{Todd}(\mathcal{E}xt^q(\mathcal{K}, \mathcal{O}_{\overline{X}}))^{(-1)^q} \\
&\quad = \prod_{q\geqslant 1} \Big[\mathrm{Todd}(\mathcal{E}xt^q(\oplus_j(\mathcal{O}_{\tilde{\Phi}_j} \oplus \mathcal{O}_{\tilde{V}_j}), \mathcal{O}_{\overline{X}}))^{(-1)^q} \cdot \\
&\qquad \cdot \mathrm{Todd}(\mathcal{E}xt^q(\oplus_j((\mathcal{O}_{\tilde{\Phi}_j} \oplus \mathcal{O}_{\tilde{V}_j})/\mathcal{O}_{(\tilde{T}_j)_{\mathrm{red}}}, \mathcal{O}_{\overline{X}})))^{(-1)^{q+1}} \cdot \\
&\qquad \cdot \mathrm{Todd}(\mathcal{E}xt^q(\mathcal{O}_{\tilde{Y}}, \mathcal{O}_{\overline{X}}))^{(-1)^{q+1}} \cdot \mathrm{Todd}(\mathcal{E}xt^q(\mathcal{O}_{\tilde{Y}}/\mathcal{O}_{\tilde{T}}, \mathcal{O}_{\overline{X}}))^{(-1)^q}\Big].
\end{aligned}$$

Next, for $i \in \mathcal{I}$ let us define

$$N(i) = |\{j \subset \mathcal{J};\ Y_i \subset (\tilde{T}_j)_{\mathrm{red}}\}| - 1.$$

Remark that $N(i)$ is zero except when $Y_i \subset V$. One has

$$\begin{aligned}
&\mathrm{Todd}(\mathcal{E}xt^q(\oplus_j(\mathcal{O}_{\tilde{\Phi}_j} \oplus \mathcal{O}_{\tilde{V}_j}), \mathcal{O}_{\overline{X}}))^{(-1)^q} \cdot \mathrm{Todd}(\mathcal{E}xt^q(\mathcal{O}_{\tilde{Y}}, \mathcal{O}_{\overline{X}}))^{(-1)^{q+1}} \\
&\quad = \mathrm{Todd}(\mathcal{E}xt^q(\oplus_{i\in\mathcal{I}}\mathcal{O}_{Y_i}^{\oplus N(i)}, \mathcal{O}_{\overline{X}}))^{(-1)^q}.
\end{aligned}$$

Moreover, one has

$$\mathrm{Todd}(\mathcal{E}xt^q((\mathcal{O}_{\tilde{\Phi}_j} \oplus \mathcal{O}_{\tilde{V}_j})/\mathcal{O}_{(\tilde{T}_j)_{\mathrm{red}}}, \mathcal{O}_{\overline{X}}))^{(-1)^{q+1}} = 1 + w_j,$$

where $w_j \in CH^{\geqslant 1}(\overline{X})$ is the sum of terms $m_{i_1,\ldots,i_\mu} Y_{i_1} \cdots Y_{i_\mu}$, where at least two of the i_p are different.

Assume that $Y_{i_1}, \ldots, Y_{i_\mu}$ are all components of Φ_j. The image of $Y_{\{i_1,\ldots,i_\mu\}}$ is one of the strata of T. Since Y is a normal crossing divisor, $\bar{f}(Y_{\{i_1,\ldots,i_\mu\}}) = T_j$. So $\delta_{\{i_1,\ldots,i_\mu\}} > 0$ and by Claim 4.4 the intersection of $c_g(\Omega^1_{\overline{X}/\overline{S}}(\log Y))$ with the cycle $Y_{i_1} \cdots Y_{i_\mu}$ is zero.

Altogether, one has

$$c_g(\Omega^1_{\overline{X}/\overline{S}}(\log Y)) \cdot \prod_{q\geqslant 1} \mathrm{Todd}(\mathcal{E}xt^q(\mathcal{C}, \mathcal{O}_{\overline{X}}))^{(-1)^q}$$
$$= c_g\big(\Omega^1_{\overline{X}/\overline{S}}(\log Y)\big) \cdot (1+w),$$

where the cycle $w \in CH^{\geqslant 1}(\overline{X})$ lies in the subspace generated by products $Y_{\iota_1} \cdots Y_{\iota_\ell}$ with at least one Y_{ι_ν} contained in Z.

We conclude the proof of Theorem 4.1 applying Proposition 4.5. □

PROPOSITION 4.5. *Let $\bar{f}: \overline{X} \to \overline{S}$ be as in Theorem* 4.1. *Then $\ell \geqslant 1$ irreducible components $Y_{\iota_1}, \ldots, Y_{\iota_\ell}$ of Y, with $Y_{\iota_\ell} \subset |Z|$, fulfill*

$$\bar{f}_*\big(c_g\big(\Omega^1_{\overline{X}/\overline{S}}(\log Y)\big) \cdot Y_{\iota_1} \cdots Y_{\iota_\ell}\big) = 0 \in CH^\ell(\overline{S}) \otimes \mathbb{Q}.$$

By Claim 4.4, we only have to study strata for which $\delta_I = 0$.

LEMMA 4.6. *If $\delta_I = 0$ and $i \in I$ is given, then there is a divisor Γ supported in $T \subset \overline{S}$, there are indices $\ell_1, \ldots, \ell_m \in \mathcal{I} \setminus I$ and multiplicities $\beta_1, \ldots, \beta_m \in \mathbb{Q}$ fulfilling*

$$Y_i \cdot Y_I = \bar{f}^*(\Gamma) \cdot Y_I + \sum_{\nu=1}^{m} \beta_\nu Y_{I\cup\{\ell_\nu\}} \in CH^{1+|I|}(\overline{X}) \otimes \mathbb{Q}.$$

Proof. By formula (4.3), we know that Y_i lies in the image of φ_I. Thus there is a $\mathbb{Q}$-divisor Γ supported on T_J such that $Y_i - \bar{f}^*(\Gamma)$ is supported away of Y_a, $a \in I$. □

LEMMA 4.7. *For $\ell \geqslant 1$ let $\mathfrak{P}_\ell$ be the set of all sets of indices I with $|I| = \mu \leqslant \ell$, with $Y_I \neq \emptyset$, $\delta_I = 0$, and such that there is one $i \in I$ with $Y_i \subset |Z|$. For irreducible components $Y_{\iota_1}, \ldots, Y_{\iota_\ell}$ of Y, with $Y_{\iota_\ell} \subset |Z|$, there exist multiplicities $a_I \in \mathbb{Q}$ for $I \in \mathfrak{P}_\ell$, cycles*

$$\Gamma_I \in CH^{\ell-|I|}(\overline{S}) \otimes \mathbb{Q}, \quad \mathit{with}\ \Gamma_I|_{\overline{S}\setminus T} = 0 \in CH^{\ell-|I|}(\overline{S} \setminus T) \otimes \mathbb{Q},$$

such that

$$c_g\big(\Omega^1_{\overline{X}/\overline{S}}(\log Y)\big) \cdot Y_{\iota_1} \cdots Y_{\iota_\ell} = c_g\big(\Omega^1_{\overline{X}/\overline{S}}(\log Y)\big) \cdot \sum_{I\in\mathfrak{P}_\ell} a_I \bar{f}^*(\Gamma_I) \cdot Y_I. \tag{4.4}$$

Proof. Obviously $\delta_{\{i\}} = 0$, thus $\mathfrak{P}_1 = \{i \in \mathcal{I}, Y_i \subset |Z|\}$. So for $\ell = 1$ there is nothing to show.

We assume now by induction on ℓ that formula (4.4) holds true for ℓ. We want to prove it for $\ell + 1$. It remains to show that for $i \in \mathcal{I}$ and $I \in \mathfrak{P}_\ell$,

$$c_g\big(\Omega^1_{\overline{X}/\overline{S}}(\log Y)\big) \cdot Y_i \cdot Y_I \tag{4.5}$$

has the shape required on the right-hand side of formula (4.4).

If $i \notin I$, then either $\delta_{\{i\}\cup I} = 0$, which implies $\{i\} \cup I \in \mathfrak{P}_{\ell+1}$, or else the expression (4.5) is zero by Claim 4.4.

Thus we assume that $i \in I$. By Lemma 4.6, one has

$$Y_i \cdot Y_I = \bar{f}^*(\Gamma) \cdot Y_I + \sum_{\nu=1}^{m} \beta_\nu Y_{\{\ell_\nu\}\cup I},$$

with $\ell_\nu \notin I$. Thus again, either $\{\ell_\nu\} \cup I \in \mathfrak{P}_{\ell+1}$, in which case we are done, or else

$$c_g\big(\Omega^1_{\overline{X}/\overline{S}}(\log Y)\big) \cdot Y_{\{\ell_\nu\}\cup I} = 0. \qquad \square$$

Proof of Proposition 4.5. The definition of $\mathfrak{P}_\ell$ implies that the restriction to $\overline{X} \setminus |Z|$ of the cycles $Y_I \in CH^{|I|}(\overline{X}) \otimes \mathbb{Q}$ die. In particular one has

$$W \cdot Y_I = 0 \in CH^{g+|I|}(\overline{X}) \otimes \mathbb{Q}.$$

Thus formula (4.4) implies

$$c_g\big(\Omega^1_{\overline{X}/\overline{S}}(\log Y)\big) \cdot Y^\ell = \bar{f}^*(\xi) \cdot \sum_{I\in\mathfrak{P}_\ell} a_I \bar{f}^*(\Gamma_I) \cdot Y_I \in CH^{g+\ell}(\overline{X}) \otimes \mathbb{Q}. \tag{4.6}$$

On the other hand, $\delta_I = 0$ implies that $|I| \leqslant |J(I)|$. This in turn implies that the fiber dimension of $Y_I \twoheadrightarrow T_{J(I)}$ is at least g, thus by projection formula, $\bar{f}_*$ of the right-hand side of formula (4.6) vanishes as well. This concludes the proof of the proposition. $\square$

Theorem 4.1 implies

COROLLARY 4.8. *The assumptions are as in Theorem* 4.1 *and*, *moreover*, *the following conditions are fulfilled*:

(5) *The characteristic of the ground field k is* 0.
(6) *The Hodge to de Rham spectral sequence*
$$E_2^{pq} = R^q \bar{f}_* \Omega^p_{\overline{X}/\overline{S}}(\log Y) \Longrightarrow \overline{\mathcal{H}}^i := R^i \bar{f}_* \Omega^\bullet_{\overline{X}/\overline{S}}(\log Y)$$
degenerates in E_2 *and the Gauss–Manin sheaves* $\overline{\mathcal{H}}^i$ *are locally free*.

Then

$$\sum_i (-1)^i \mathrm{ch}(\overline{\mathcal{H}}^i) \in CH^0(\overline{S}) \otimes \mathbb{Q}.$$

5. Vanishing of the Chern Character of the Gauss–Manin Bundles of Weight One

In this section, we prove Theorem 1.1, which is the main result of this article.

Proof. Let $B \subset \overline{B}$ be as in the theorem. There exists a generically finite morphism $\pi\colon B_1 \to \overline{B}$ together with a morphism $\psi\colon B_1 \to \overline{S}$, where $\overline{S}$ is as in Theorem 3.1, with

$$\pi^*\overline{\mathcal{H}}^1 = \psi^* R^1 \bar{f}_*\big(\Omega^\bullet_{\overline{X}/\overline{S}}(\log Y)\big).$$

Indeed, the polarization of $\mathcal{H}^1$ is not necessarily coming from a principal polarization on the underlying family of Abelian varieties, and also, this family might not have a level n-structure, but both can be achieved after replacing B by a generically finite covering.

Since B_1 and $\overline{B}$ are smooth, the projection formula for π implies that $CH^\bullet(\overline{B})\otimes \mathbb{Q}$ is a subring of $CH^\bullet(B_1) \otimes \mathbb{Q}$. Thus using ψ^* we may assume that $B = \overline{S}$ and

$$\overline{\mathcal{H}}^1 = R^1 \bar{f}_*\big(\Omega^\bullet_{\overline{X}/\overline{S}}(\log Y)\big).$$

By Corollary 4.8, we know

$$\sum_i (-1)^i \mathrm{ch}(\overline{\mathcal{H}}^i) \in CH^0(\overline{S}) \otimes \mathbb{Q}.$$

On the other hand, consider as in Theorem 3.1(11), the involution ι on $\overline{X}$. By loc. cit. (9), (11), the fixed points of ι lie in disjoint sections $\overline{S_\alpha} \subset \overline{X}$ of $\bar{f}$. In particular, setting again $Y = (\bar{f}^*(T))_{\mathrm{red}}$ and $Y+Z = \bar{f}^*(T)$, the sections do not hit the divisor Z. We consider the blow up $a\colon \overline{X}' \to \overline{X}$ of the sections $\overline{S_\alpha}$. Hence $\overline{X}'$ is nonsingular, the divisor $Y' = a^{-1}Y$ is again a normal crossings divisor, and $(\bar{f}\circ a)^*T$ is reduced in a neighborhood of the exceptional divisors $E_\alpha = a^{-1}\overline{S_\alpha}$. ι acts on the relative Zariski tangent space of a section $\overline{S_\alpha}$ by multiplication with -1, hence it induces an action on $\overline{X}'$, again denoted by ι. The restriction of ι to E_α is trivial, and ι acts fixed point free on $\overline{X}' \setminus \cup E_\alpha$. One has

$$R^i(\bar{f}\circ a)_*\big(\Omega^\bullet_{\overline{X}'/\overline{S}}(\log a^{-1}Y)\big) = \overline{\mathcal{H}}^i \oplus \mathcal{T}_i,$$

where $\mathcal{T}_i$ is an algebraically trivial bundle on which ι acts trivially, and $\mathcal{T}_i = 0$ for i odd.

The quotient, $\mathfrak{K} = \overline{X}/(\iota)$ is nonsingular and $h\colon \mathfrak{K} \to \overline{S}$ is a proper family, smooth over $S \subset \overline{S}$. Here $\mathfrak{K}$ stands for Kummer.

The ramification locus $\cup E_\alpha$ of $\overline{X}' \to \mathfrak{K}$ is contained in the smooth locus of $\overline{X}' \to \overline{S}$, hence $h^{-1}(T)$ is a normal crossings divisor, reduced in a neighborhood of the image of $\cup E_\alpha$. So for $\kappa = (h^{-1}(T))_{\mathrm{red}}$ the sheaf $\Omega^1_{\mathfrak{K}/\overline{S}}(\log \kappa)$ remains locally free and

$$c_g\big(\Omega^1_{\mathfrak{K}/\overline{S}}(\log \kappa)\big) = c_g\big(h^*\bar{f}_*\big(\Omega^1_{\overline{X}/\overline{S}}(\log Y)\big)\big)$$

in a neighborhood of the multiple locus of $h^{-1}(T)$. Altogether, $h\colon \mathfrak{K} \to \overline{S}$ satisfies assumptions (1)–(3) in Theorem 4.1, except possibly that the components of κ

might be singular. As at the very end of the proof of Theorem 3.1, we blow up nonsingular strata of κ_{red} to have (1) as well.

The Gauss–Manin bundles $\overline{\mathcal{H}}^i_{\mathfrak{K}} := R^i h_* \Omega^\bullet_{\mathfrak{K}/\overline{S}}(\log \kappa)$ vanish for $i = 2p+1,\ p > 0$, and fulfill

$$\overline{\mathcal{H}}^{2p}_{\mathfrak{K}} = \big(R^{2p}(\bar{f} \circ a)_* \Omega^\bullet_{\overline{X}'/\overline{S}}(\log a^{-1} Y)\big)^{\iota},$$

where $^{\iota}$ means the invariants under ι. Since ι acts trivially on $\overline{\mathcal{H}}^{2p}$, one obtains $\overline{\mathcal{H}}^{2p}_{\mathfrak{K}} = \overline{\mathcal{H}}^{2p} \oplus \mathcal{T}_{2p}$. Corollary 4.8 implies

$$\mathrm{ch}\Big(\sum_{p\geqslant 0} \overline{\mathcal{H}}^{2p}_{\mathfrak{K}}\Big) = \Big(= \mathrm{ch}\Big(\sum_{p\geqslant 0} \overline{\mathcal{H}}^{2p}\Big)\Big) \in CH^0(\overline{S}) \otimes \mathbb{Q}.$$

Lemma 2.1 implies then

$$\mathrm{ch}(\overline{\mathcal{H}}^1) \in CH^0(\overline{S}) \otimes \mathbb{Q}.$$

This concludes the proof. □

THEOREM 5.1 *Let k be a field of characteristic $p \neq 2$. Let $\bar{f}: \overline{X} \to \overline{S}$ be a compactified principally polarized family of Abelian varieties as in Theorem* 3.1. *Then one has*

$$\mathrm{ch}(\mathbb{E}^\vee \oplus \mathbb{E}) = \mathrm{ch}\big(R^1 \bar{f}_* \mathcal{O}_{\overline{X}} \oplus \bar{f}_* \Omega^1_{\overline{X}/\overline{S}}(\log Y)\big) \in CH^0(\overline{S}) \otimes \mathbb{Q},$$

or equivalently

$$\mathrm{ch}_{2\ell}(\mathbb{E}) = \mathrm{ch}_{2\ell}(\mathbb{E}^\vee) = 0 \in CH^{2\ell}(\overline{S}) \otimes \mathbb{Q}, \quad \textit{for } \ell \geqslant 1.$$

Proof. We replace in the proof of Theorem 1.1 the Gauss–Manin bundle $\overline{\mathcal{H}}^i$ by the sum $\sum_j R^j \bar{f}_* \Omega^{i-j}_{\overline{X}/\overline{S}}(\log Y)$ of the Hodge bundles (see Theorem 3.1(6)). □

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