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Determinant Bundle in a Family of Curves, after A. Beilinson and V. Schechtman

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Abstract: Let $\pi : X \to S$ be a smooth projective family of curves over a smooth base S over a field of characteristic 0, together with a bundle E on X. Then A. Beilinson and V. Schechtman define in [1] a beautiful "trace complex" ${}^{\mathrm{tr}}\mathcal{A}_E^\bullet$ on X, the 0^{th} relative cohomology of which describes the Atiyah algebra of the determinant bundle of E on S. Their proof reduces the general case to the acyclic one. In particular, one needs a comparison of $R\pi_*({}^{\mathrm{tr}}\mathcal{A}_F^\bullet)$ for $F = E$ and $F = E(D)$, where D is étale over S (see Theorem 2.3.1, reduction ii) in [1]). In this note, we analyze this reduction in more detail and correct a point.

1. Introduction

Let $\pi : X \to S$ be a smooth projective morphism of relative dimension 1 over a smooth base S over a field k of characteristic 0. One denotes by T_X and the tangent sheaf over k, by $T_{X/S}$ the relative tangent sheaf, and by $\omega_{X/S}$ the relative dualizing sheaf. For an algebraic vector bundle E on X, one writes $E^\circ = E^* \otimes_{\mathcal{O}_X} \omega_{X/S}$. Let $\mathrm{Diff}(E, E)$ (resp. $\mathrm{Diff}(E/S, E/S) \subset \mathrm{Diff}(E, E)$) be the sheaf of first order (resp. relative) differential operators on E and $\epsilon : \mathrm{Diff}(E, E) \to \mathcal{E}\mathrm{nd}(E) \otimes_{\mathcal{O}_X} T_X$ be the symbol map. The Atiyah algebra $\mathcal{A}_E := \{a \in \mathrm{Diff}(E, E) | \epsilon(a) \in \mathrm{id}_E \otimes_{\mathcal{O}_X} T_X\}$ of E is the subalgebra of $\mathrm{Diff}(E, E)$ consisting of the differential operators for which the symbolic part is a homothety. Similarly the relative Atiyah algebra $\mathcal{A}_{E/S} \subset \mathcal{A}_E$ of E consists of those differential operators with symbol in $\mathrm{id}_E \otimes_{\mathcal{O}_X} T_{X/S}$ and $\mathcal{A}_{E,\pi} \subset \mathcal{A}_E$ with symbols in $T_\pi = d\pi^{-1}(\pi^{-1}T_S) \subset T_X$. Let $\Delta \subset X \times_S X$ be the diagonal. Then there is a canonical sheaf isomorphism $\mathrm{Diff}(E/S, E/S) \cong \frac{E \boxtimes E^\circ(2\Delta)}{E \boxtimes E^\circ}$ (see [1], Sect. 2) which is locally written as follows. Let x be a local coordinate of X at a point p, and (x, y) be the induced local coordinates on $X \times_S X$ at (p, p), such that the equation of Δ becomes

$x = y$. Let e_i be a local basis of E, e_j^* be its local dual basis. Then the action of

$$P = \sum_{i,j} e_i \otimes e_j^* \frac{P_{ij}(x, y)}{(x - y)^2} dy$$

on $s = \sum_\ell s_\ell(y) e_\ell$ is

$$P(s) = \sum_i e_i \sum_j (P_{ij}^{(1)}(x, 0) s_j(x) + P_{ij}(x, x) s_j^{(1)}(x, 0)), \tag{1.1}$$

where

$$P_{ij}(x, y) = P_{ij}(x, x) + (y - x) P_{ij}^{(1)}(x, y - x),$$
$$s_j(y) = s_j(x) + (y - x) s_j^{(1)}(x, y - x).$$

Beginning with

$$0 \to \frac{E \boxtimes E^\circ}{E \boxtimes E^\circ(-\Delta)} \to \frac{E \boxtimes E^\circ(2\Delta)}{E \boxtimes E^\circ(-\Delta)} \to \mathrm{Diff}(E/S, E/S) \to 0, \tag{1.2}$$

restricting to $\mathcal{A}_{E/S} \subset \mathrm{Diff}(E, E)$, and pushing forward by the trace map $\frac{E \boxtimes E^\circ}{E \boxtimes E^\circ(-\Delta)} \to \omega_\Delta \cong \omega_{X/S}$, yields an exact sequence

$$0 \to \omega_{X/S} \to {}^{\mathrm{tr}}\mathcal{A}_E^{-1} \xrightarrow{\gamma_E} \mathcal{A}_{E/S} \to 0. \tag{1.3}$$

One defines the trace complex ${}^{\mathrm{tr}}\mathcal{A}^\bullet$ by $\mathcal{A}_{E,\pi}$ for $i = 0$, ${}^{\mathrm{tr}}\mathcal{A}_E^{-1}$ for $i = -1$, $\mathcal{O}_X$ for $i = -2$ and 0 else, with differentials $d^{-1} := \gamma_E$ and d^{-2} equal to the relative Kähler differential (see [1], Sect. 2).

One has an exact sequence of complexes

$$0 \to \Omega^\bullet_{X/S}[2] \to {}^{\mathrm{tr}}\mathcal{A}_E^\bullet \to (T_{X/S} \to T_\pi)[1] \to 0, \tag{1.4}$$

where $\Omega^\bullet_{X/S}$ is the relative de Rham complex of π. Taking relative cohomology, one obtains the exact sequence

$$0 \to \mathcal{O}_S \to R^0\pi_*({}^{\mathrm{tr}}\mathcal{A}_E^\bullet) \to T_S \to 0. \tag{1.5}$$

Furthermore $R^0\pi_*({}^{\mathrm{tr}}\mathcal{A}_E^\bullet)$ is a sheaf of algebras ([1], 1.2.3). One denotes by $\pi({}^{\mathrm{tr}}\mathcal{A}_E^\bullet)$ the sheaf on S together with its algebra structure.

Finally, let $\mathcal{B}_i, i = 1, 2$ be two sheaves of algebras on S, with an exact sequence of sheaves of algebras

$$0 \to \mathcal{O}_S \to \mathcal{B}_i \to T_S \to 0. \tag{1.6}$$

One defines $\mathcal{B}_1 + \mathcal{B}_2$ by taking the subalgebra of $\mathcal{B}_1 \oplus \mathcal{B}_2$, inverse image $\mathcal{B}_1 \times_{T_S} \mathcal{B}_2$ of the diagonal embedding $T_S \to T_S \oplus T_S$, and its push out via the trace map $\mathcal{O}_S \oplus \mathcal{O}_S \to \mathcal{O}_S$.

The aim of this note is to prove

Theorem 1.1. *Let $D \subset X$ be a divisor, étale over S. One has a canonical isomorphism*

$$\pi({}^{\mathrm{tr}}\mathcal{A}_E^\bullet) \cong \pi({}^{\mathrm{tr}}\mathcal{A}_{E(-D)}^\bullet) + \mathcal{A}_{\det \pi_*(E|_D)} \tag{1.7}$$

This is [1] Theorem 2.3.1, ii). We explain in more details the proof given there and correct a point in it.

2. Proof of Theorem 1.1

The proof uses the construction of a complex $\mathcal{L}^\bullet$, together with maps $\mathcal{L}^\bullet \to {}^{\mathrm{tr}}\mathcal{A}_E^\bullet$ and $\mathcal{L}^\bullet \to {}^{\mathrm{tr}}\mathcal{A}_{E(-D)}^\bullet \oplus i_{D*}\mathcal{A}_{E|_D}$ inducing isomorphisms from $R^0\pi_*\mathcal{L}^\bullet$ with the left and the right-hand side of Theorem 1.1. We make the construction of $\mathcal{L}^\bullet$ and the maps explicit, and show that the induced morphisms are surjective, with the same (non-vanishing) kernel.

We first recall the definition of the sub-complex $\mathcal{L}^\bullet \subset {}^{\mathrm{tr}}\mathcal{A}_E^\bullet$ (see [1, Theorem 2.3.1, ii)]): $\mathcal{L}^0 \subset {}^{\mathrm{tr}}\mathcal{A}_E^0$ consists of the differential operators P with $\epsilon(P) \in T_\pi < -D >$, where $T_\pi < -D >= T_\pi \cap T_X < -D >$ and $T_X < -D >= \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^1 < D >, \mathcal{O}_X)$, where $\Omega_X^1 < D >$ denotes the sheaf of 1-forms with log poles along D. In particular, $(d^{-1})^{-1}(\mathcal{L}^0) \subset {}^{\mathrm{tr}}\mathcal{A}_E$ maps to $i_{D*}\mathcal{A}_{E|_D}$, and $\mathcal{L}^{-1} \subset (d^{-1})^{-1}(\mathcal{L}^0)$ is defined as the kernel. Then $\mathcal{L}^{-2} = \mathcal{O}_X$. The product struture on ${}^{\mathrm{tr}}\mathcal{A}_E^\bullet$ is defined in [1], 2.1.1.2, and coincides with the Lie algebra structure on ${}^{\mathrm{tr}}\mathcal{A}_E^0 = \mathcal{A}_{E,\pi}$. Since $\mathcal{L}^{-2} = {}^{\mathrm{tr}}\mathcal{A}_E^{-2}$, to see that the product structure stabilizes $\mathcal{L}^\bullet$, one just has to see that $\mathcal{L}^0 \subset {}^{\mathrm{tr}}\mathcal{A}_E^0$ is a subalgebra, which is obvious, and that $\mathcal{L}^0 \times \mathcal{L}^{-1} \to {}^{\mathrm{tr}}\mathcal{A}_E^{-1}$ takes values in $\mathcal{L}^{-1}$, which is a consequence of Proposition 2.2.

As in Sect. 1, we denote by $\mathcal{A}_{E/S}$ the relative Atiyah algebra of E, with symbolic part $T_{X/S}$ and by $\mathcal{A}_{E,\pi}$ Beilinson's subalgebra of the global Atiyah algebra with symbolic part T_π. If $\iota : F \subset E$ is a vector bundle, isomorphic to E away of D, then one has an injection of differential operators

$$\mathrm{Diff}(E, F) \xrightarrow{i} \mathrm{Diff}(E, E) \tag{2.1}$$

induced by ι on the second argument, and an injection

$$\mathrm{Diff}(E, F) \xrightarrow{j} \mathrm{Diff}(F, F) \tag{2.2}$$

induced by ι on the first argument. One has

Definition 2.1.

$$\mathcal{A}_{(E/S,F/S)} := \mathcal{A}_{E/S} \cap_i \mathrm{Diff}(E, F) \cong \mathcal{A}_{F/S} \cap_j \mathrm{Diff}(E, F)$$

Recall $\gamma_E : {}^{\mathrm{tr}}\mathcal{A}_E^{-1} \to \mathcal{A}_{E/S}$ denotes the map coming from the filtration by the order of poles of $\mathcal{O}_{X\times X}(*\Delta)$ on ${}^{\mathrm{tr}}\mathcal{A}_E^{-1}$. One has

Proposition 2.2.

$$\gamma_E^{-1}(\mathcal{A}_{E/S,E(-D)/S}) \cong \gamma_{E(-D)}^{-1}(\mathcal{A}_{E/S,E(-D)/S}) \cong \mathcal{L}^{-1}.$$

Proof. One considers

$$\begin{aligned} &\frac{E(-D) \boxtimes E^\circ(2\Delta) + E \boxtimes E^\circ}{E \boxtimes E^\circ(-\Delta)} \\ &= \left[\frac{E(-D) \boxtimes E^\circ(2\Delta)}{E(-D) \boxtimes E^\circ(-\Delta)} \oplus \frac{E \boxtimes E^\circ}{E \boxtimes E^\circ(-\Delta)}\right] \Big/ \left[\frac{E(-D) \boxtimes E^\circ}{E(-D) \boxtimes E^\circ(-\Delta)}\right] \end{aligned} \tag{2.3}$$

which, via the natural inclusion to

$$\frac{E \boxtimes E^{\circ}(2\Delta)}{E \boxtimes E^{\circ}(-\Delta)} \tag{2.4}$$

is the inverse image $\gamma_E^{-1}\Big(\mathrm{Diff}(E, E(-D))\Big)$ (here we abuse notations, still denoting by γ_E the map coming from the filtration), and via the map coming from the natural inclusion

$$\frac{E(-D) \boxtimes E^{\circ}(2\Delta)}{E(-D) \boxtimes E^{\circ}(-\Delta)} \to \frac{E(-D) \boxtimes E^{\circ}(D)(2\Delta)}{E(-D) \boxtimes E^{\circ}(D)(-\Delta)} \tag{2.5}$$

and the identification with the first term of the filtration on $\frac{E(-D)\boxtimes E^{\circ}(D)(2\Delta)}{E(-D)\boxtimes E^{\circ}(D)(-\Delta)}$

$$\frac{E \boxtimes E^{\circ}}{E \boxtimes E^{\circ}(-\Delta)} \cong \frac{E(-D) \boxtimes E^{\circ}(D)}{E(-D) \boxtimes E^{\circ}(D)(-\Delta)} \tag{2.6}$$

is the inverse image $\gamma_{E(-D)}^{-1}\Big(\mathrm{Diff}(E, E(-D))\Big)$. $\square$

The filtration induced by the order of poles of $\mathcal{O}_{X\times X}(*D)$ induces the exact sequences

$$0 \to \mathcal{H}\mathrm{om}(E, E(-D)) \to \mathcal{A}_{(E/S, E(-D)/S)} \to T_{X/S}(-D) \to 0, \tag{2.7}$$
$$0 \to \mathcal{E}\mathrm{nd}(E) \to \mathcal{A}_{E/S} \to T_{X/S} \to 0, \tag{2.8}$$
$$0 \to \mathcal{E}\mathrm{nd}(E) \to \mathcal{A}_{E(-D)/S} \to T_{X/S} \to 0. \tag{2.9}$$

Now, as one has an injection $\mathcal{L}^{\bullet} \subset {}^{\mathrm{tr}}\mathcal{A}_E^{\bullet}$ with cokernel $\mathcal{Q}$, and again by looking at the filtration by the order of poles on the sheaf in degree (-1), one obtains

$$\mathcal{Q} \cong \mathcal{E}\mathrm{nd}(E)|_D[1] \tag{2.10}$$

and

Theorem 2.3. *One has an exact sequence*

$$0 \to R^0\pi_*(\mathcal{E}\mathrm{nd}(E)|_D) \to R^0\pi_*(\mathcal{L}^{\bullet}) \to R^0\pi_*({}^{\mathrm{tr}}\mathcal{A}_E^{\bullet}) \to 0.$$

On the other hand, one has an injection $\mathcal{L}^{\bullet} \subset {}^{\mathrm{tr}}\mathcal{A}^{\bullet}_{E(-D)} \oplus i_{D*}\mathcal{A}_{E|_D}$ with cokernel $\mathcal{P}$, and, as $\mathcal{L}^{\bullet}$ injects into ${}^{\mathrm{tr}}\mathcal{A}^{\bullet}_{E(-D)}$, one has an exact sequence

$$0 \to i_{D*}\mathcal{A}_{E|_D}[0] \to \mathcal{P} \to [{}^{\mathrm{tr}}\mathcal{A}^{\bullet}_{E(-D)}/\mathcal{L}^{\bullet}] \to 0, \tag{2.11}$$

where $i_D : D \to X$ is the closed embedding. We see that the induced filtration on the sheaf in degree (-1) of $[{}^{\mathrm{tr}}\mathcal{A}^{\bullet}_{E(-D)}/\mathcal{L}^{\bullet}]$ has graded pieces $(0, \mathcal{E}\mathrm{nd}(E|_D), T_{X/S}|_D)$, whereas the filtration on the sheaf in degree (0) has graded pieces $(0, T_\pi/T_\pi < -D > \cong T_{X/S}|_D)$. This last point comes from the obvious

Lemma 2.4.

$$\{P \in \mathrm{Diff}(E, E), P(E(-D)) \subset E(-D)\} \cong$$
$$\{P \in \mathrm{Diff}(E(-mD), E(-mD)), \epsilon(P) \in \mathcal{E}\mathrm{nd}(E) \otimes T < -D >\}$$

for any $m \in \mathbb{Z}$, where ϵ is the symbol map.

So

Lemma 2.5. $[{}^{\mathrm{tr}}\mathcal{A}^\bullet_{E(-D)}/\mathcal{L}^\bullet]$ *is quasi-isomorphic to* $\mathcal{E}\mathrm{nd}(E|_D)[1]$.

The connecting morphism $R^{-1}\pi_*[{}^{\mathrm{tr}}\mathcal{A}^\bullet_{E(-D)}/\mathcal{L}^\bullet] \to R^0\pi_*(i_{D*}\mathcal{A}_{E|_D})[0]$ is just the natural embedding $\pi_*(\mathcal{E}\mathrm{nd}(E|_D)) \to \pi_*(i_{D*}\mathcal{A}_{E|_D})$ with cokernel $\pi_*\pi|_D^{-1}T_S$. If D is irreducible, one has $\pi_*\pi|_D^{-1}T_S \cong T_S$, and therefore

Proposition 2.6. *If D is irreducible, one has an exact sequence*

$$0 \to R^0\pi_*\mathcal{L}^\bullet \to R^0\pi_*[{}^{\mathrm{tr}}\mathcal{A}^\bullet_{E(-D)}] \oplus R^0\pi_*[i_{D*}\mathcal{A}_{E|_D}] \to T_S \to 0$$

and the image of $R^0\pi_*\mathcal{L}^\bullet$ is obtained from the direct sum by taking the pull back under the diagonal embedding $T_S \to T_S \oplus T_S$.

On the other hand, still assuming D irreducible, one has the exact sequence

$$0 \to \pi_*\mathcal{E}\mathrm{nd}(E|_D) \to R^0\pi_*[i_{D*}\mathcal{A}_{E|_D}] \to T_S \cong \pi_*\pi|_D^{-1}T_S \to 0 \tag{2.12}$$

and the Atiyah algebra $\mathcal{A}_{\det(\pi_*E|_D)}$ is the push out of $R^0\pi_*[i_{D*}\mathcal{A}_{E|_D}]$ by the trace map $\pi_*\mathcal{E}\mathrm{nd}(E|_D) \to \mathcal{O}_S$.

Defining

$$\mathcal{K} := \mathrm{Ker}\Big(\mathcal{O}_S \oplus \pi_*\mathcal{E}\mathrm{nd}(E|_D) \xrightarrow{\mathrm{id}\oplus\mathrm{Tr}} \mathcal{O}_S\Big) \cong \pi_*\mathcal{E}\mathrm{nd}(E|_D), \tag{2.13}$$

one thus obtains

Theorem 2.7. *If D is irreducible, one has an exact sequence*

$$0 \to \mathcal{K} \to R^0\pi_*\mathcal{L}^\bullet \to \pi_*({}^{\mathrm{tr}}\mathcal{A}^\bullet_{E(-D)}) + \mathcal{A}_{\det\pi_*(E|_D)} \to 0.$$

It can be easily shown that the embedding $\pi_*\mathcal{E}\mathrm{nd}(E|_D) \subset R^0\pi_*\mathcal{L}^\bullet$ in Theorems 2.3 and 2.7 is the same embedding of a subsheaf of ideals. It finishes the proof of Theorem 1.1 when D is irreducible. In general, since D is étale over S, its irreducible components are disjoint, thus one proves Theorem 1.1 by adding one component at a time.

References

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