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A Remark on an Algebraic Riemann–Roch Formula for Flat Bundles*

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Abstract. Let $f: X \to S$ be a smooth projective morphism over an algebraically closed field, with X and S regular. When (E, ∇) is a flat bundle over X, then its Gauss–Manin bundles on S have a flat connection and one may ask for a Riemann–Roch formula relating the algebraic Chern–Simons and Cheeger–Simons invariants. We give an answer for $X = Y \times S$, $f =$ projection. The method of proof is inspired by the work of Hitchin and Simpson.

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0. Introduction

Let X be a smooth variety defined over an algebraically closed field k, and let (E, ∇) be a bundle with an integrable connection. Then (E, ∇) carries algebraic classes $c_n(E, \nabla)$ in the subgroup $\mathbb{H}^n(X, \Omega^\infty \mathcal{K}_n)$ of the group of algebraic differential characters $\mathrm{AD}^n(X)$ consisting of the classes mapping to 0 in $H^0(X, \Omega_X^{2n})$. These classes lift the Chern classes $c_n(E, \nabla) \in \mathrm{CH}^n(X)$, the algebraic Chern–Simons invariants $w_n(E, \nabla) \in H^0(X, \mathcal{H}_{DR}^{2n-1})$ for $n \geqslant 2$, as well as the analytic secondary invariants $c_n^{\mathrm{an}}(E, \nabla) \in H^{2n-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(n))$ when $k = \mathbb{C}$ (see [5, 3, 6] and Section 1).

Let now $f: X \to S$ be a smooth projective morphism. Then the de Rham cohomology sheaves $R^j f_*(\Omega^\bullet_{X/S} \otimes E, \nabla)$ carry the (flat) Gauss–Manin connection, still denoted by ∇. Therefore, one can ask for a Riemann–Roch formula relating

$$c_n(\sum(-1)^j (R^j f_*(\Omega^\bullet_{X/S} \otimes E, \nabla), \nabla))$$

and $c_m(E, \nabla)$, as the classes $c_n(E, \nabla)$ verify the Whitney product formula for exact sequences of bundles with compatible (flat) connections. This formula should be compatible with Riemann–Roch–Grothendieck formula in $\mathrm{CH}^n(S)$, and with Bismut–Lott and Bismut formula in $H^{2n-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(n))$ when $k = \mathbb{C}$ ([1, 2]). In this note, we propose an answer in the case $X = Y \times S$ and f is the projection:

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THEOREM 0.0.1. *Let Y be a smooth projective variety of dimension d and S be a smooth variety. Let (E, ∇) be a bundle with a flat connection on $X = Y \times S$. Then*

$$c_n\left(\sum_j (-1)^j (R^j f_*(\Omega^\bullet_{X/S} \otimes E, \nabla), \nabla)\right) = (-1)^d f_* c_n(E, \nabla)|_{c_d(\Omega^1_{X/S})},$$

where f is the projection to S, where the right-hand side means that one takes a zero cycle $\sum_i m_i p_i \subset Y$ representing $c_d(\Omega^1_Y)$, and

$$f_* c_n(E, \nabla)|_{c_d(\Omega^1_{X/S})} = \sum_i m_i c_n(E, \nabla)|_{\{p_i\} \times S} \in \mathbb{H}^n(S, \Omega^\infty \mathcal{K}_n)$$

(*see Section 1*).

For $n = 1$, $\mathbb{H}^1(S, \Omega^\infty \mathcal{K}_1)$ is the group of isomorphism classes of rank one bundles with an integrable connection. Thus the formula for the determinant bundle as a flat bundle is similar to Deligne–Laumon formula for the determinant bundle of a ℓ-adic sheaf E over X with $S = \operatorname{Spec} \mathbb{F}_q$, $(p, \ell) = 1$ ([4, 7]). But here, we treat only the case where f is split, which is certainly of no interest from the ℓ-adic viewpoint. However, we do not know what would be a corresponding formula in higher codimension in the arithmetic case.

When $k = \mathbb{C}$, Theorem 0.0.1 is trivial is one replaces AD^n by $H^{2n-1}_{\mathrm{an}}(\mathbb{C}/\mathbb{Z}(n))$. In fact, the classes of E on X in this group depend only on the underlying local system V, and one knows that V has a filtration $V_i \subset V$ by local subsystems such that $V_i/V_{i-1} = A_i \otimes B_i$, where A_i (resp. B_i) is an irreducible local system on Y (resp. S). Consequently, if S is projective, the algebraic bundle with connection E has a filtration by compatible flat subbundles E_i with $E_i/E_{i-1} = p^* F_i \otimes f^* G_i$, where F_i is flat on Y and G_i is flat on S (p is the projection to S), and Theorem 0.0.1 is a trivial consequence of the projection formula. In general, the algebraic bundle is not uniquely determined by its local system.

The method used is inspired by the work of Hitchin and Simpson. We deform

$$\sum_j (-1)^j (R^j f_*(\Omega^\bullet_{X/S} \otimes E, \nabla), \nabla)$$

to the alternate sum of a flat structure

$$(R^j f_*(\Omega^\bullet_{X/S} \otimes E, \alpha), \nabla_i)$$

defined on the cohomology of a Higgs bundle, where the Higgs structure is defined by a form $\alpha \in H^0(Y, \Omega^1_Y)$, if $H^0(Y, \Omega^1_Y) \neq 0$. If not, one has to introduce poles (see Section 2).

1. Trace of Algebraic Differential Characters

1.1. ALGEBRAIC DIFFERENTIAL CHARACTERS

Let X be a smooth variety, $D \subset X$ be a normal crossing divisor, $\mathcal{K}_n$ be the Zarisky sheaf image of the Zarisky sheaf of Milnor K-theory in $K_n^M(k(x))$. We recall [6], Section 2, that the group

$$\mathrm{AD}^n(X, D) := \mathbb{H}^n(X, \mathcal{K}_n \xrightarrow{d\log} \Omega_X^n(\log\ D) \to \cdots \to \Omega_X^{2n-1}(\log\ D))$$

has a commutative product

$$\mathrm{AD}^m(X, D) \times \mathrm{AD}^n(X, D) \to \mathrm{AD}^{m+n}(X, D),$$

respecting the subgroup

$$\begin{aligned}&\mathbb{H}^n(X, \Omega_X^\infty(\log\ D)\mathcal{K}_n)\\ &\quad := \mathbb{H}^n(X, \mathcal{K}_n \xrightarrow{d\log} \Omega_X^n(\log\ D) \to \cdots)\\ &\quad = \mathrm{Ker}\ (\mathrm{AD}^n(X, D) \to H^0(X, \Omega_X^{2n}(\log\ D))\end{aligned}$$

and compatible with the products in $\mathcal{K}_n$ and $\Omega_X^{\geqslant n}(\log\ D)$. A bundle (E, ∇) with a $\Omega_X^1(\log\ D)$ connection has functorial and additive classes $c_n(E, \nabla) \in \mathrm{AD}^n(X, D)$, lying in $\mathbb{H}^n(X, \Omega_X^\infty(\log\ D)\mathcal{K}_n)$ when $\nabla^2 = 0$.

LEMMA 1.1.1. *Let $X = \mathbb{P}^1 \times S$, with S smooth, $B \subset \mathbb{P}^1$ be a divisor, $D = B \times S$, $p_i, i = 1, 2$ be the projections of X to $\mathbb{P}^1$ and S. Then one has a direct sum decomposition*

$$\begin{aligned}&p_2^* \oplus p_2^* \otimes p_1^*:\\ &\quad \mathbb{H}^n(S, \Omega_S^\infty \mathcal{K}_n) \oplus \mathbb{H}^{n-1}(S, \Omega_S^\infty \mathcal{K}_{n-1}) \otimes \mathbb{H}^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^\infty(\log\ B)\mathcal{K}_1)\\ &\quad \to \mathbb{H}^n(X, \Omega_X^\infty(\log\ D)\mathcal{K}_n).\end{aligned}$$

Proof. One has the Künneth formulae

$$\begin{aligned}&H^\ell(X, \mathcal{K}_n)\\ &\quad = p_2^* H^\ell(S, \mathcal{K}_n) \oplus p_2^* H^{\ell-1}(S, \mathcal{K}_{n-1}) \cup p_1^* H^1(\mathbb{P}^1, \mathcal{K}_1);\\ &\mathbb{H}^\ell(X, \Omega_X^{\geqslant n}(\log\ D))\\ &\quad = p_2^* \mathbb{H}^\ell(S, \Omega_S^{\geqslant n}) \oplus p_2^* \mathbb{H}^\ell(S, \Omega_S^{\geqslant n-1}) \cup p_1^* \mathbb{H}^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\log\ B)) \text{ if } B \neq \emptyset;\\ &\mathbb{H}^\ell(X, \Omega_X^{\geqslant n}(\log\ D))\\ &\quad = p_2^* \mathbb{H}^\ell(S, \Omega_S^{\geqslant n}) \oplus p_2^* \mathbb{H}^{\ell-1}(S, \Omega_S^{\geqslant n-1}) \cup p_1^* H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1) \text{ if } B = \emptyset.\end{aligned}$$

Moreover, in the long exact cohomology sequence associated to the short exact sequence

$$0 \to \Omega_X^{\geqslant n}(\log\ D)[n-1] \to \Omega_X^\infty(\log\ D)\mathcal{K}_n \to \mathcal{K}_n \to 0$$

the map

$$H^{\ell}(X,\mathcal{K}_n) \xrightarrow{d\log} \mathbb{H}^{\ell+n}(X,\Omega_X^{\geqslant n}(\log\ D))$$

respects this direct sum composition. (If $B \neq \emptyset$, the term

$$p_2^* H^{\ell-1}(S,\mathcal{K}_{n-1}) \cup p_1^* H^1(\mathbb{P}^1,\mathcal{K}_1)$$

maps to zero.) This proves the lemma. □

1.2. TRACE

PROPOSITION 1.2.1. *Let $X = Y \times S$, with Y smooth projective, S smooth, and $f: X \to S$ be the projection. Let $\Sigma = \sum m_i p_i$ be a zero cycle in Y, and (E,∇) be a bundle with an integrable connection on X. Then*

$$f_* c_n(E,\nabla)|_{\Sigma\times S} := \sum m_i c_n((E,\nabla)|_{\{p_i\}\times S}) \in \mathbb{H}^n(S,\Omega_S^{\infty}\mathcal{K}_n)$$

does not depend on the choice of the representative Σ in its equivalence class $[\Sigma] \in \mathrm{CH}_0(S)$.

Proof. Let $\Sigma' = \sum m_i p_i'$ be another choice. Then there are rational functions f_i on curves $C_i \subset Y$ such that $\Sigma - \Sigma' = \sum \operatorname{div} f_i$. Therefore, it is sufficient to prove the following: let $\nu: C \to Y$ be the normalization of an irreducible curve $\nu(C) \subset Y$, and $\varphi: C \to \mathbb{P}^1$ be a nontrivial rational function on C. Then

$$c_n((\nu\times \mathrm{id}_S)^*(E,\nabla)|_{\varphi^{-1}(0)\times S}) = c_n((\nu\times \mathrm{id}_S)^*(E,\nabla)|_{\varphi^{-1}(\infty)\times S}).$$

Let $(\mathcal{E},D) := (\nu\times\mathrm{id}_X)^*(E,\nabla)$. Let $B\subset\mathbb{P}^1$ be the ramification locus of φ, $\pi = \varphi\times\mathrm{id}_S: C\times S \to \mathbb{P}^1\times S$. Then $\pi_*(\mathcal{E},D)$ is a bundle with an integrable connection with logarithmic poles along $D = B\times S$, and has classes

$$c_n(\pi_*(\mathcal{E},D)) \in \mathbb{H}^n(\mathbb{P}^1\times S,\Omega_X^{\infty}(\log\ D)\mathcal{K}_n).$$

For $t\notin B$, then

$$c_n(\pi_*(\mathcal{E},D)|_{\{t\}\times S}) = \sum_{s\in\varphi^{-1}(t)} c_n((E,D)|_{\{s\}\times S}) = c_n(\mathcal{E},D)|_{\varphi^{-1}(t)\times S}.$$

For $t\in B$, the residue map

$$\mathrm{res}_{\{t\}\times S}: p_2^*\Omega_S^{n-1}\otimes p_1^*\Omega^1_{\mathbb{P}^1}(\log\ B) \to \Omega^{n-1}_{\{t\}\times S}$$

verifies

$$\mathrm{res}_{\{t\}\times S}\otimes_{\mathcal{O}_{\mathbb{P}^1\times S}}\mathcal{O}_{\{t\}\times S} = \mathrm{Identity}_{\{t\}\times S}.$$

Therefore, there is a canonical splitting

$$\Omega_X^n(\log\ D)|_{\{t\}\times S} = \Omega_S^n \oplus \Omega^{n-1}_{\{t\}\times S}.$$

Thus the connection

$$\pi_* D: \pi_*\mathcal{E} \to \Omega^1_{\mathbb{P}^1\times S}(\log\ D) \otimes \pi_*\mathcal{E}$$

restricts to

$$\pi_* D|_{\{t\}\times S}: \pi_*\mathcal{E}|_{\{t\}\times S} \to (\Omega^1_S \oplus \mathcal{O}_S) \otimes \pi_*\mathcal{E}|_{\{t\}\times S}$$

defining, by projection, the connection

$$\overline{\pi_* D}|_{\{t\}\times S}: \pi_*\mathcal{E}|_{\{t\}\times S} \to \Omega^1_S \otimes (\pi_*\mathcal{E})|_{\{t\}\times S}.$$

The integrability of $\pi_* D|_{\{t\}\times S}$ implies the integrability of the genuine connection $\overline{\pi_* D}|_{\{t\}\times S}$, and

$$c_n((\mathcal{E}, D)|\varphi^{-1}(t) \times S) = c_n(\pi_*\mathcal{E}|_{\{t\}\times S}, \overline{\pi_* D}|_{\{t\}\times S}).$$

Now by [7],

$$c_n(\pi_*(\mathcal{E}, D)) = p_2^* a + \sum_i p_2^* b_i \cup p_1^* c_i,$$

where $a \in \mathbb{H}^n(S, \Omega_S^\infty \mathcal{K}_n)$, and the preceding discussion shows that

$$c_n((\mathcal{E}, D)|_{\varphi^{-1}(t)\times S}) = a \in \mathbb{H}^n(S, \Omega_S^\infty \mathcal{K}_n)$$

whether $t \in B$ or $t \notin B$. $\square$

2. Proof of the Theorem

2.1. NOTATIONS

Let $H_0, \ldots, H_N$ be the restriction of the coordinate hyperplanes $H'_0, \ldots, H'_N$ in an embedding $Y \subset \mathbb{P}^N$. Since $\Omega^1_{\mathbb{P}^N}(\log H'_0 \cup \ldots H'_N) \cong \oplus_1^N \mathcal{O}_{\mathbb{P}^N}$, the sheaf $\Omega^1_Y(\log H_0 \cup \ldots \cup H_N)$ is globally generated. We denote by $H^{(\ell)}$ the normalization of the ℓ by ℓ intersections of the H_j, $H^{(0)} = X$, $H^{(\delta)} = \emptyset$ for $\delta > d$, by H the union of the H_j, by

$$\Omega^a_{H^{(\ell)}}(\log\ H^{(\ell+1)})$$

the sheaf of a forms on $H^{(\ell)}$ with logarithmic poles along the $(\ell+1)$ by $(\ell+1)$ intersections. One has the following resolution of the de Rham complex:

$$\begin{aligned}\Omega^\bullet_{X/S} &\to \Omega^\bullet_{X/S}(\log(H \times S)) \to \Omega^{\bullet-1}_{H^{(1)}\times S/S}(\log(H^{(2)} \times S)) \to \cdots \\ &\to \Omega^{\bullet-\ell}_{H^{(\ell)}\times S/S}(\log\ (H^{(\ell+1)} \times S)) \to \cdots \\ &\to \Omega^{\bullet-d}_{H^{(d)}\times S/S} \to 0. \end{aligned} \tag{2.1.1}$$

2.2. PROOF OF THEOREM 0.0.1

The resolution 2.1.1 is compatible with the Gauss–Manin connection, as

$$H^{(\ell)} \times S$$

is dominant over S.

Therefore, in the K group $K(S, \text{flat})$ of bundles on S with an integrable connection, one has

$$\begin{aligned}
&\sum_j (-1)^j R^j f_*(\Omega^\bullet_{X/S} \otimes E, \nabla) \\
&\quad = \sum (-1)^j R^j f_*(\Omega^\bullet_{X/S}(\log(H \times S)) \otimes E, \nabla) + \\
&\qquad + \sum (-1)^j R^j f_*(\Omega^\bullet_{H^{(1)} \times S/S}(\log(H^{(2)} \times S)) \otimes E, \nabla) + \cdots \\
&\qquad + \sum (-1)^j R^j f_*(\Omega^\bullet_{H^{(\ell)} \times S/S}(\log(H^{(\ell)} \times S)) \otimes E, \nabla) + \cdots \\
&\qquad + f_*(\Omega^\bullet_{H^{(d)} \times S/S} \otimes E, \nabla).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
c_d(\Omega^1_{X/S}) = \; & c_d(\Omega^1_{X/S}(\log(H \times S))) - \\
& -c_{d-1}(\Omega^1_{H^{(1)} \times S/S}(\log(H^{(2)} \times S)) + \cdots \\
& +(-1)^\ell c_{d-\ell}(\Omega^1_{H^{(\ell)} \times S/S}(\log(H^{(\ell+1)} \times S)) + \cdots \\
& +(-1)^d [H^{(d)} \times S],
\end{aligned}$$

where $[H^{(d)} \times S]$ means the codimension d cycle, image of $H^{(d)} \times S$ in $Y \times S$.

Therefore, one just has to prove the following formula:

$$\begin{aligned}
& c_n \left(\sum (-1)^j (R^j f_*(\Omega^\bullet_{X/S}(\log D \times S) \otimes E), \nabla), \nabla) \right) \\
& \quad = (-1)^d f_* c_n(E, \nabla)|_{c_d(\Omega^1_{X/S}(\log(H \times S))}.
\end{aligned}$$

We denote by $\tau\colon \Omega^1_X(\log(H \times S)) \to f^*\Omega^1_S$ the splitting of the one forms, which induces a splitting

$$\tau\colon \Omega^\bullet_X(\log(H \times S)) \to f^*\Omega^\bullet_S$$

of the de Rham complex, where the differential $f^*\Omega^i_S \to f^*\Omega^{i+1}_S$ is defined by $\tau d\iota$, $\iota\colon f^*\Omega^i_S \to \Omega^i_X(\log(H \times S))$ being the natural embedding. This defines a $f^*\Omega^1_S$ valued connection ∇_τ on $\Omega^i_{X/S}(\log(H \times S)) \otimes E$ by embedding $\Omega^i_{X/S}(\log(H \times S)) \otimes E$ into $\Omega^i_X(\log(H \times S)) \otimes E$ via the splitting, then taking ∇, then projecting onto the factor

$$f^*\Omega^1_S \otimes \Omega^1_{X/S}(\log(H \times S)) \otimes E$$

with the sign $(-1)^i$. The integrability condition $\nabla^2 = 0$ then implies that

$$\nabla_\tau \circ \nabla_{X/S} = \nabla_{X/S} \circ \nabla_\tau,$$

where

$$\nabla_{X/S}\colon \Omega^i_{X/S}(\log(H \times S)) \otimes E \to \Omega^{i+1}_{X/S}(\log(H \times S)) \otimes E.$$

Taking cohomology defines a flat connection, still denoted by ∇_τ

$$\nabla_\tau\colon R^j f_* \Omega^i_{X/S}(\log(H \times S)) \otimes E \to \Omega^1_S \otimes R^j f_* \Omega^i_{X/S}(\log(H \times S)) \otimes E$$

which is compatible with the Gauss–Manin connection. The integrability of ∇ implies the integrability of ∇_τ. Therefore, in $K(S, \text{flat})$ one has

$$\begin{aligned}&(R^j f_*(\Omega^\bullet_{X/S}(\log(H \times S)) \otimes E, \nabla), \nabla)\\ &\quad = \oplus_i ((R^{j-i} f_* \Omega^i_{X/S}(\log(H \times S)) \otimes E, \nabla_\tau), \nabla_\tau)\end{aligned}$$

and

$$\begin{aligned}&\sum_j (-1)^j ((R^j f_* \Omega^\bullet_{X/S}(\log\,(H \times S)) \otimes E, \nabla), \nabla)\\ &\quad = \sum_{i,j} (-1)^{i+j} ((R^j f_* \Omega^i_{X/S}(\log(H \times S)) \otimes E, \nabla_\tau), \nabla_\tau).\end{aligned}$$

Let $\alpha \in H^0(Y, \Omega^1_Y(\log H))$ be a nontrivial generic section. We still denote by α the corresponding form $p_1^* \alpha \in H^0(X, \Omega^1_{X/S}(\log(H \times S)))$. Then, it defines a morphism

$$\alpha_{X/S}\colon \Omega^i_{X/S}(\log(H \times S)) \otimes E \to \Omega^{i+1}_{X/S}(\log(H \times S)) \otimes E$$

by $\alpha_{X/S}(w \otimes e) = \alpha \wedge w \otimes e$. As $d\alpha = 0$, one has

$$\alpha_{X/S} \circ \nabla_\tau = \nabla_\tau \circ \alpha_{X/S}.$$

Thus in $K\ (S, \text{flat})$ one has

$$\begin{aligned}&\sum_{i,j} (-1)^{i+j} (R^j f_*(\Omega^i_{X/S}(\log(H \times S)) \otimes E, \nabla_\tau), \nabla_\tau)\\ &\quad = \sum_j (-1)^j (R^j f_*(\Omega^\bullet_{X/S}(\log(H \times S)) \otimes E, \alpha_{X/S}), \nabla_\tau).\end{aligned}$$

On the other hand, the complex

$$(\Omega^\bullet_{X/S}(\log(H \times S)), \alpha_{X/S})$$

is quasi-isomorphic to $\mathcal{O}_{\Sigma\times S}[-d]$, where Σ is the zero set of α, and one, furthermore, has a commutative diagram of complexes

$$\begin{array}{ccc} (\Omega^\bullet_{X/S}(\log(H\times S))\otimes E, \alpha_{X/S}) & \xrightarrow{\nabla_\tau} & f^*\Omega^1_S\otimes(\Omega^\bullet_{X/S}(\log(H\times S))\otimes E, \alpha_{X/S}) \\ \downarrow & & \downarrow \\ E|_{\Sigma\times S}[-d] & \xrightarrow{\nabla|_{\Sigma\times S}} & \Omega^1_{\Sigma\times S}\otimes E[-d] \end{array}$$

This shows the equality in $K\ (S, \text{flat})$

$$\begin{aligned} &\sum_j (-1)^j((R^j f_*(\Omega^\bullet_{X/S}(\log(H\times S))\otimes E, \alpha), \nabla_\tau) \\ &\quad = (-1)^d \oplus_{\sigma\in\Sigma} (E, \nabla)|_{\sigma\times S} \end{aligned}$$

and completes the proof.

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