CHOW GROUPS OF PROJECTIVE VARIETIES OF VERY SMALL DEGREE

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Let k be a field. For a closed subset X of $\mathbb{P}_k^n$, defined by r equations of degree $d_1 \geqslant \cdots \geqslant d_r$, one has the numerical invariant

$$\kappa = \left[\frac{n - \sum_{i=2}^r d_i}{d_1} \right],$$

where $[\alpha]$ denotes the integral part of a rational number α. If k is the finite field $\mathbb{F}_q$, the number of k-rational points verifies the congruence

$$\# \mathbb{P}^n(\mathbb{F}_q) \equiv \# X(\mathbb{F}_q) \bmod q^\kappa,$$

while, if k is the field of complex numbers $\mathbb{C}$, one has the Hodge-type relation

$$F^\kappa H_c^i(\mathbb{P}_\mathbb{C}^n - X) = H_c^i(\mathbb{P}_\mathbb{C}^n - X) \quad \text{for all } i$$

(see [12], [5] and the references given there). These facts, together with various conjectures on the cohomology and Chow groups of algebraic varieties, suggest that the Chow groups of X might satisfy

$$\mathrm{CH}_l(X) \otimes \mathbb{Q} = \mathrm{CH}_l(\mathbb{P}_k^n) \otimes \mathbb{Q} = \mathbb{Q} \qquad (*)$$

for $l \leqslant \kappa - 1$ (compare with Remark 5.6 and Corollary 5.7).

This is explicitly formulated by V. Srinivas and K. Paranjape in [16, Conjecture 1.8]; the chain of reasoning goes roughly as follows. Suppose X is smooth. One expects a good filtration

$$0 = F^{j+1} \subset F^j \subset \cdots \subset F^0 = \mathrm{CH}^j(X \times X) \otimes \mathbb{Q},$$

whose graded pieces F^l/F^{l+1} are controlled by $H^{2j-l}(X \times X)$ (see [10]). According to Grothendieck's generalized conjecture [8], the groups $H^i(X)$ should be generated by the image under the Gysin morphism of the homology of a codimension-κ subset, together with the classes coming from $\mathbb{P}^n$. Applying this to the diagonal in $X \times X$ should then force the triviality of the Chow groups in the desired range.

For zero-cycles, the conjecture $(*)$ follows from Roitman's theorem (see [17] and [18]):

$$\mathrm{CH}_0(X) = \mathbb{Z} \quad \text{if} \quad \sum_{i=1}^{r} d_i \leqslant n. \tag{$**$}$$

In [16], K. Paranjape proves a version of $(*)$, showing that there is a finite bound $N = N(d_1, \ldots, d_r; l)$, such that, for $n \geqslant N$, one has $\mathrm{CH}_{l'}(X) = \mathbb{Z}$ for $0 \leqslant l' \leqslant l$. The bound $N(d_1, \ldots, d_r; l)$ grows quite rapidly as a function of the degrees; for example, if $l = r = 1$, one has the inductive inequality

$$N(d, 1) \geqslant \frac{1}{d} \binom{d + N(d-1, 1)}{d-1} + N(d-1, 1),$$

and $N(2, 1)$ is at least five.

In this article, we give the following improved bound (see Theorem 4.5). Suppose $d_1 \geqslant \ldots \geqslant d_r \geqslant 2$, and either $d_1 \geqslant 3$ or $r \geqslant l+1$. If

$$\sum_{i=1}^{r} \binom{l + d_i}{l + 1} \leqslant n,$$

then

$$\mathrm{CH}_{l'}(X) \otimes \mathbb{Q} = \mathbb{Q} \quad \text{for } 0 \leqslant l' \leqslant l.$$

If $d_1 = \cdots = d_r = 2$ and $r \leqslant l$, we have the same conclusion, assuming the modified inequality

$$\sum_{i=1}^{r} \binom{l + d_i}{l + 1} = r(l+2) \leqslant n - l + r - 1.$$

As an application, if we assume in addition to the above inequalities that X is smooth, we show in Section 5 that the primitive cohomology of X is generated by the image of the homology of a codimension-$(l+1)$ subset, in accordance with Grothendieck's conjecture, and we show that $\#\mathbb{P}^n(\mathbb{F}_q) \equiv \#X(\mathbb{F}_q) \bmod q^{l+1}$ for almost all primes p and X defined in characteristic zero.

The method of proof of the improved bound is a generalization of Roitman's technique, coupled with a generalization of Roitman's theorem $(**)$ to closed subsets of Grassmannians defined by the vanishing of sections of Sym^d of the tautological quotient bundle. This latter result is an elementary consequence of the theorem due to Kollár-Miyaoka-Mori [14] and Campana [3] that Fano varieties are rationally connected. The first part of the argument, the application of Roitman's technique to cycles of higher dimension, is completely geometric.

As an illustration, consider the case of surfaces on a sufficiently general hypersurface X of degree $d \geqslant 3$ in $\mathbb{P}^n$. Roitman shows that, if $d \leqslant n$ and p is a general point of a general X, there is a line L in $\mathbb{P}^n$ such that $L \cdot X = dp$.

Now take a surface Y on X, in a general position. Applying Roitman's construction to the general point y of Y, and specializing y over Y, we construct a three-dimensional cycle S in $\mathbb{P}^n$ with the property that

$$S \cdot X = NY + \sum_j n_j Y_j,$$

where N is some positive integer and the Y_j are ruled surfaces in X. If $\binom{d+1}{2} \leqslant n - 1$, we can find for each general line L on X a plane Π in $\mathbb{P}^n$ such that $\Pi \cdot X = dL$.

Assuming that the general line in each Y_j is in general position, we construct a three-dimensional cycle S' in $\mathbb{P}^n$ such that

$$S' \cdot X = N' \sum_j n_j Y_j + \sum_i m_i \Pi_i,$$

where N' is a positive integer, and the Π_i are two-planes in X. From this (ignoring the general position assumptions) it follows that all two-dimensional cycles on X are rationally equivalent to a sum of two-planes in X. We may then apply our result on zero-cycles of subsets of Grassmannians, which in this case implies that all the two-planes in X are rationally equivalent, assuming $\binom{d+2}{3} \leqslant n$. Putting this together gives
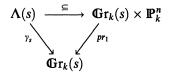
$$\mathrm{CH}_2(X) \otimes \mathbb{Q} = \mathbb{Q} \quad \text{if } d \geqslant 3 \text{ and } \binom{d+2}{3} \leqslant n.$$

One needs to refine this argument to treat cases of special position, as well as larger l and r. For the reader's convenience, we first give the argument in the case of hypersurfaces before giving the proof in general; the argument in the general case does not rely on that for hypersurfaces.

Throughout this article, we assume that k is algebraically closed, as the kernel of $\mathrm{CH}_l(X_k) \to \mathrm{CH}_l(X_{\bar{k}})$ is torsion (see [1]).

1. Flag and incidence varieties.

Let $X \subseteq \mathbb{P}^n_k$ be a closed reduced subscheme. For $0 \leqslant s \leqslant n - 1$, let $\mathbb{Gr}_k(s) = \mathbb{Gr}_k(s; n)$ denote the Grassmann variety of s-planes

in $\mathbb{P}_k^n$, and let

$$\Lambda(s) \xrightarrow{\subseteq} \mathbb{Gr}_k(s) \times \mathbb{P}_k^n$$

with γ_s and pr_1 projecting to $\mathbb{Gr}_k(s)$

be the universal family. We write $\mathbb{Gr}_k(s; X)$ for the closed subscheme of $\mathbb{Gr}_k(s)$ parametrizing s-planes in $\mathbb{P}_k^n$ which are contained in X. Correspondingly $\gamma: \Lambda(s; X) \to \mathbb{Gr}_k(s; X)$ denotes the restriction of γ_s to $\Lambda(s; X) = \gamma_s^{-1}(\mathbb{Gr}_k(s; X))$. In $\mathbb{Gr}_k(s; X) \times \mathbb{Gr}_k(s+1)$, we consider the flag manifold $\mathbb{F}(s, s+1; X)$ consisting of pairs $[H, H']$ with

$$\mathbb{P}_k^s \simeq H \subseteq X \quad \text{and} \quad H \subset H' \simeq \mathbb{P}_k^{s+1} \subseteq \mathbb{P}_k^n.$$

The projection $\mathbb{Gr}_k(s; X) \times \mathbb{Gr}_k(s+1) \to \mathbb{Gr}_k(s; X)$ induces a morphism

$$\mathbb{F}(s, s+1; X) \to \mathbb{Gr}_k(s+1)$$

and a surjection

$$\varphi: \mathbb{F}(s, s+1; X) \to \mathbb{Gr}_k(s; X).$$

By abuse of notation, we write $\Lambda(s; X)$ and $\Lambda(s+1)$ for the pullback of the universal families to $\mathbb{F}(s, s+1; X)$ and

$$\mathbb{F}(s, s+1; X) \times X \xleftarrow[\supseteq]{} \Lambda(s; X) \xrightarrow[\subseteq]{\tau} \Lambda(s+1) \xrightarrow{\subseteq} \mathbb{F}(s, s+1; X) \times \mathbb{P}_k^n$$

with pr_1, γ', η, pr_1 mapping to

$$\mathbb{F}(s, s+1; X) \xrightarrow{=} \mathbb{F}(s, s+1; X)$$

for the induced morphisms.

Assume that $X \subseteq \mathbb{P}_k^n$ is a hypersurface of degree d. Hence X is the zero set of $f(x_0, x_1, \ldots, x_n) \in k[x_0, \ldots, x_n]_d$. We consider the incidence varieties

$$\mathbb{H}' = \mathbb{H}'(s, s+1; X) = \mathbb{Gr}_k(s; X) \times \mathbb{Gr}_k(s+1; X) \cap \mathbb{F}(s, s+1; X) \quad \text{and}$$

$$\mathbb{H} = \mathbb{H}(s, s+1; X) = \{[H, H'] \in \mathbb{F}(s, s+1; X); H' \subseteq X \text{ or } H' \cap X = H\}.$$

Here "$H' \cap X$" denotes the set-theoretic intersection. "$H' \cap X = H$" implies that the zero-cycle of $f|_{H'}$ is H with multiplicity d. We will see in the proof of the following lemma that $\mathbb{H} \subseteq \mathbb{F}(s, s+1; X)$ is a closed subscheme.

By definition, one has $\mathbb{H}' \subseteq \mathbb{H}$. It might happen that for all $[H, H'] \in \mathbb{H}$ the $(s+1)$-plane H' is contained in X, or in different terms, that $\mathbb{H}' = \mathbb{H}$, but for a general hypersurface X, both are different. Generalizing Roitman's construction for $s = 0$ in [17], one obtains the following.

LEMMA 1.1. *Let* $X \subseteq \mathbb{P}^n_k$ *be a hypersurface of degree d, and let*

$$\pi_1 \colon \mathbb{H} \to \mathbb{G}\mathrm{r}_k(s; X) \qquad and \qquad \pi_1' \colon \mathbb{H}' \to \mathbb{G}\mathrm{r}_k(s; X)$$

be the restrictions of the projection

$$pr_1 \colon \mathbb{G}\mathrm{r}_k(s; X) \times \mathbb{G}\mathrm{r}_k(s+1) \to \mathbb{G}\mathrm{r}_k(s; X).$$

Then for all $[H] \in \mathbb{G}\mathrm{r}_k(s; X)$, *the fibres of* π_1 *(or* π_1'*) are subschemes of* $\mathbb{P}^{n-s-1}_k$ *defined by*

$$\binom{s+d}{s+1} - 1 \quad \left(or \ \binom{s+d}{s+1} \right)$$

equations. In particular, π_1 *(or* π_1'*) is surjective if*

$$\binom{s+d}{s+1} \leqslant n - s \quad \left(or \ \binom{s+d}{s+1} \leqslant n - s - 1 \right).$$

Proof. The first projection $p_1 \colon \mathbb{F} = \mathbb{F}(s, s+1; X) \to \mathbb{G}\mathrm{r}_k(s; X)$ is a $\mathbb{P}^{n-s-1}_k$ bundle (see, for example, [9, 11.40]). In fact, for $\mathrm{Spec}(A) \subseteq \mathbb{G}\mathrm{r}_k(s; X)$, let us fix coordinates in $\mathbb{P}^n_A$ such that $\Lambda_A = \gamma^{-1}(\mathrm{Spec}(A)) \subset \mathbb{P}^n_A$ is the linear subspace defined by

$$x_{s+1} = x_{s+2} = \cdots = x_n = 0.$$

Let Γ be the $(n - s)$-plane given by

$$x_1 = \cdots = x_s = 0.$$

An $(s+1)$-plane Λ_A' containing Λ_A is uniquely determined by the line $\Lambda_A' \cap \Gamma \subseteq \Gamma$, and each line in Γ which contains $(1 : 0 : \cdots : 0)$ determines some Λ_A'. In other terms, there is a $\mathrm{Spec}(A)$-isomorphism

$$\sigma \colon \mathbb{P}^{n-s-1}_A \xrightarrow{\cong} p_1^{-1}(\mathrm{Spec}(A)) \subseteq \mathbb{F},$$

given by $\sigma((a_0 : \cdots : a_{n-s-1})) = [\Lambda_A, \Lambda_A']$, where Λ_A' is spanned by Λ_A and by

$$(1 : 0 : \cdots : 0 : a_0 : \cdots : a_{n-s-1}).$$

An isomorphism $\mathbb{P}_A^{s+1} \xrightarrow{\cong} \Lambda_A'$ is given by

$$(\xi_0 : \cdots : \xi_{s+1}) \mapsto (\xi_0 : \cdots : \xi_s : a_0\xi_{s+1} : \cdots : a_{n-s-1}\xi_{s+1}).$$

Under this isomorphism, the intersection $X \times \mathrm{Spec}(A) \cap \Lambda_A' \subseteq \Lambda_A'$ is the zero set of

$$f(t_0, \ldots, t_s, a_0 t_{s+1}, \ldots, a_{n-s-1} t_{s+1}) \in A[t_0, \ldots, t_{s+1}].$$

Since $\Lambda_A \subset \Lambda_A'$ is the zero set of t_{s+1} and since $\Lambda_A \subset X \times \mathrm{Spec}(A)$, one can write this equation as

$$t_{s+1} g(t_0, \ldots, t_{s+1}),$$

where g is homogeneous of degree $d - 1$. The point $\sigma((a_0 : \cdots : a_{n-s-1})) = [\Lambda_A, \Lambda_A']$ is an A-valued point of $\mathbb{H}'$ if and only if $g(t_0, \ldots, t_{s+1}) = 0$, and it is an A-valued point of $\mathbb{H} - \mathbb{H}'$ if and only if

$$g(t_0, \ldots, t_{s+1}) = \varepsilon t_{s+1}^{d-1},$$

for $\varepsilon \in A^*$. Hence, writing

$$g(t_0, \ldots, t_{s+1}) = \sum_{i_0 + \cdots + i_{s+1} = d-1} g_{i_0, i_1, \ldots, i_{s+1}} t_0^{i_0} \cdots t_{s+1}^{i_{s+1}},$$

one obtains for $\mathbb{H}'$ the equations $g_{i_0, i_1, \ldots, i_{s+1}} = 0$ for all tuples $(i_0, \ldots, i_{s+1})$ with

$$\sum_{v=0}^{s+1} i_v = d - 1.$$

For $\mathbb{H}$, one obtains the same equations, except the one for $(0, \ldots, 0, d - 1)$. $\square$

Remark 1.2. Keeping the notation from the proof of 1.1, one can bound the degree of $\sigma^{-1}(\mathbb{H}) \subseteq \mathbb{P}_A^{n-s-1}$. In fact, writing $A[y_0, \ldots, y_{n-s-1}]$ for the coordinate ring and $g_{i_0, \ldots, i_{s+1}}$ for the equations for $\sigma^{-1}(\mathbb{H})$, then $g_{i_0, \ldots, i_{s+1}}$ is homogeneous of degree $i_{s+1} + 1$ in $y_0, \ldots, y_{n-s-1}$.

2. Hypersurfaces of small degree. For a closed reduced subscheme $X \subseteq \mathbb{P}_k^N$, we will write $\mathrm{CH}_l(X)$ for the Chow group of l-dimensional cycles.

Definition 2.1. (a) An l-dimensional closed subvariety Y of X will be called a *subvariety spanned by s-planes* if there exists an $(l - s)$-dimensional subvariety

$Z \subset \mathbb{Gr}_k(s, X)$ such that for the restriction of the universal family

$$\Lambda_Z = \gamma^{-1}(Z) \xrightarrow{\subseteq} Z \times X \subseteq Z \times \mathbb{P}_k^n,$$

the image of the composite $\Lambda_Z \xrightarrow{\subseteq} Z \times X \xrightarrow{pr_2} X$ is Y.

(b) $\mathrm{CH}_l^{(s)}(X)$ denotes the subgroup of $\mathrm{CH}_l(X)$ which is generated by l-dimensional subvarieties of X which are spanned by s-planes.

(c) We write $\mathrm{CH}_l(X)_{\mathbb{Q}} = \mathrm{CH}_l(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathrm{CH}_l^{(s)}(X)_{\mathbb{Q}} = \mathrm{CH}_l^{(s)}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

If Y is spanned by s-planes, it is spanned by $(s - 1)$-planes as well. Hence one has $\mathrm{CH}_l^{(s)}(X) \subseteq \mathrm{CH}_l^{(s-1)}(X)$. For $s > l$ one has $\mathrm{CH}_l^{(s)}(X) = \mathrm{CH}_l^{(s)}(X)_{\mathbb{Q}} = \{0\}$. The same holds true if X does not contain any s-plane. For $s = 0$, one obtains by definition $\mathrm{CH}_l^{(0)}(X) = \mathrm{CH}_l(X)$ and $\mathrm{CH}_l^{(0)}(X)_{\mathbb{Q}} = \mathrm{CH}_l(X)_{\mathbb{Q}}$.

Let $\Gamma \subset \mathbb{P}_k^n$ be an $(l + 1)$-dimensional closed subvariety or, more generally, an $(l + 1)$-cycle in $\mathrm{CH}_{l+1}(\mathbb{P}_k^n)$. By [6, 8.1], the intersection product $\Gamma \cdot X$ is a cycle in $\mathrm{CH}_l(|\Gamma| \cap X)$. By abuse of notation, we will write $\Gamma \cdot X$ for its image in $\mathrm{CH}_l(X)$ or $\mathrm{CH}_l(X)_{\mathbb{Q}}$, as well.

PROPOSITION 2.2. *Let $X \subseteq \mathbb{P}_k^n$ be an irreducible hypersurface of degree d, and let Y be an l-dimensional subvariety of X, spanned by s-planes but not by $(s + 1)$-planes. If*

$$\binom{s + d}{s + 1} \leqslant n - s,$$

then there exist an $(l + 1)$-dimensional subvariety $\Gamma \subset \mathbb{P}_k^n$ and a positive integer α with

$$\Gamma \cdot X \equiv \alpha Y \mod \mathrm{CH}_l^{(s+1)}(X).$$

Before giving the proof of Proposition 2.2, let us state the consequence we are mainly interested in.

COROLLARY 2.3. *Let $X \subseteq \mathbb{P}_k^n$ be an irreducible hypersurface of degree d. If*

$$\binom{l + d}{l + 1} \leqslant n - l, \tag{1}$$

then $\mathrm{CH}_l(X)_{\mathbb{Q}} = \mathbb{Q}$.

Proof. For $d \geqslant 2$, we have the identity

$$\binom{s + 1 + d}{s + 2} = \frac{s + 1 + d}{s + 2} \binom{s + d}{s + 1}.$$

Hence the inequality (1) implies that for all $s \leqslant l$

$$\binom{s+d}{s+1} \leqslant \binom{l+d}{l+1} \leqslant n - l \leqslant n - s. \qquad (2)$$

Let l' be the largest integer for which there exists an l-dimensional subvariety Y' of X, which is spanned by l'-planes. One has $l' \leqslant l$. In fact, as we will see in 4.2, the inequality (1) implies that $l' = l$, but this is not needed here.

By Proposition 2.2 the inequality (2) implies that some positive multiple of Y' is obtained as the intersection of X with some subvariety Γ' of $\mathbb{P}_k^n$. If Y is any other l-dimensional subvariety of X, spanned by s-planes, for $0 \leqslant s \leqslant l'$, then by inequality (2) and by Proposition 2.2, one finds some $\Gamma \in \mathrm{CH}_{l+1}(\mathbb{P}_k^n)_{\mathbb{Q}}$ such that $\Gamma \cdot X$ is rationally equivalent to Y modulo $\mathrm{CH}_l^{(s+1)}(X)_{\mathbb{Q}}$. Since $\mathrm{CH}_{l+1}(\mathbb{P}_k^n)_{\mathbb{Q}} = \mathbb{Q}$, the cycle Y is rationally equivalent to some rational multiple of Y' modulo $\mathrm{CH}_l^{(s+1)}(X)_{\mathbb{Q}}$.

For $s = l'$, this implies that $\mathrm{CH}_l^{(l')}(X)_{\mathbb{Q}} = \mathbb{Q}$. For $s < l'$, one obtains that

$$\mathrm{CH}_l^{(s)}(X)_{\mathbb{Q}} = \mathrm{CH}_l^{(s+1)}(X)_{\mathbb{Q}} = \cdots = \mathrm{CH}_l^{(l')}(X)_{\mathbb{Q}}. \qquad \square$$

Proof of 2.2. By definition, there exists an $(l - s)$-dimensional subvariety $Z' \subset \mathbb{G}\mathrm{r}_k(s, X)$ such that the image of the restriction $\Lambda_{Z'}$ of the universal family of s-planes to Z' maps surjectively to Y. In Section 1, we considered the morphism $\pi_1 \colon \mathbb{H} \to \mathbb{G}\mathrm{r}_k(s, X)$. By Lemma 1.1, the assumption made in 2.2 implies that π_1 is surjective. Hence there exists some variety Z, proper and generically finite over Z', such that the inclusion $Z' \subset \mathbb{G}\mathrm{r}_k(s, X)$ lifts to a morphism $Z \to \mathbb{H}$. The pullback $\Lambda_Z = \Lambda_{Z'} \times_{Z'} Z$ still dominates Y.

Replacing Z by a desingularization (if char $k = 0$) or by some variety generically finite over Z (if char $k \neq 0$; see [11]), we may assume that Z is nonsingular.

Let $\Lambda_Z' = \Lambda(s+1) \times_{\mathbb{F}(s,s+1;X)} Z$ be the pullback of the universal family of $(s+1)$-planes to Z. Putting this together, we have morphisms

$$
\begin{array}{ccccc}
Z \times X & \xleftarrow{\;\supseteq\;} & \Lambda_Z & \xrightarrow{\;\subseteq\;} & \Lambda_Z' & \xrightarrow{\;\subseteq\;} & Z \times \mathbb{P}_k^n \\
& & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \\
& & Z & \xrightarrow[=]{} & Z. & &
\end{array}
$$

Assume that $\Lambda_Z' \not\subset Z \times X$. Since $Z \to \mathbb{F}(s, s+1; X)$ factors through $\mathbb{H}$, for a general point $z \in Z$ the intersection satisfies $\psi^{-1}(z) \cdot X = d\varphi^{-1}(z)$. The codimension of $\Lambda_Z' \cap (Z \times X)$ in Λ_Z' is therefore one, and

$$\Lambda_Z' \cdot (Z \times X) = d\Lambda_Z + \sum \alpha_i \psi^{-1}(D_i) \in \mathrm{CH}_l(\Lambda_Z' \cap (Z \times X))$$

for some prime divisors D_i in Z.

If Λ'_Z is contained in $Z \times X$, one obtains the same equality in $CH_l(\Lambda'_Z)$. In fact, since Λ'_Z is a $\mathbb{P}^{s+1}_k$-bundle over the $(l-s)$-dimensional manifold Z, the group $CH_l(\Lambda'_Z)$ is generated by Λ_Z and by the pullback of divisors in Z. Since a general fibre of $\Lambda'_Z \to Z$ must intersect X in some cycle of degree d, the coefficient of Λ_Z in $\Lambda'_Z \cdot (Z \times X)$ must be d.

Since Y is not spanned by $(s+1)$-planes, Λ'_Z is generically finite over its image in $\mathbb{P}^n_k$ and the cycle $\Gamma = pr_{2*}(\Lambda'_Z)$ is nonzero. Let us write pr'_2 for the restriction of pr_2 to $\Lambda'_Z \cap (Z \times X)$. The projection formula (see [6, 8.1.7]), applied to $pr_2: Z \times \mathbb{P}^n_k \to \mathbb{P}^n_k$, implies that

$$\Gamma \cdot X = pr_{2*}(\Lambda'_Z) \cdot X = pr'_{2*}(\Lambda'_Z \cdot (Z \times X)) = dpr'_{2*}(\Lambda_Z) + \sum \alpha_i pr'_{2*}(\psi^{-1}(D_i))$$

in $CH_l(X \cap pr_2(\Lambda'_Z))$. Since Λ_Z is generically finite over the subvariety Y, and since $pr'_{2*}(\psi^{-1}(D_i))$ lies in $CH_l^{(s+1)}(X)$, one obtains for some positive multiple α of d the relation

$$\Gamma \cdot X \equiv \alpha Y \bmod CH_l^{(s+1)}(X). \qquad \square$$

3. Complete intersections of small degree. As for zero-cycles (see [18]), it is easy to generalize Proposition 2.2 and Corollary 2.3 to components of subschemes of $\mathbb{P}^n_k$ defined by equations of small degree.

PROPOSITION 3.1. *Let $X_1, \ldots, X_r$ be hypersurfaces in $\mathbb{P}^n_k$ of degrees $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_r \geqslant 2$, respectively, and let X be a union of irreducible components of $X_1 \cap X_2 \cap \cdots \cap X_r$, equidimensional of codimension r. If Y is an l-dimensional subvariety of X, spanned by s-planes, and if*

$$\sum_{i=1}^r \binom{s+d_i}{s+1} \leqslant n - s, \tag{3}$$

then there exists an effective cycle $\Gamma \in CH_{l+r}(\mathbb{P}^n_k)$ and a positive integer α with

$$\Gamma \cdot X \equiv \alpha Y \bmod CH_l^{(s+1)}(X).$$

In Section 4 we will need that, under more restrictive conditions, the cycle Γ is not zero.

Addendum 3.2. Assume in Proposition 3.1 that

$$\sum_{i=1}^r \binom{s+d_i}{s+1} \leqslant n - l. \tag{4}$$

Then one may choose Γ to be an $(l+r)$-dimensional subvariety of $\mathbb{P}^n_k$.

Proof of 3.1 and 3.2. For some $(l - s)$-dimensional subvariety $Z' \subset \mathbb{G}\mathrm{r}_k(s; X)$ the restriction $\Lambda_{Z'}$ of the universal family of s-planes maps surjectively to Y. Let us fix a subfamily of $(s - 1)$-planes Δ in $\Lambda_{Z'}$ over some open subscheme of Z'. The pullback of Δ to some blowing up $Z'' \to Z'$ extends to a projective subbundle

$$\Delta_{Z''} \text{ of } \Lambda_{Z''} = \Lambda_{Z'} \times_{Z'} Z''.$$

For $i = 1, \ldots, r$ let $\mathbb{H}_i$ be the subscheme of $\mathbb{F}(s, s + 1; X)$ parametrizing pairs $[H, H']$ with $H' \subseteq X_i$ or with $H' \cap X_i = H$. By Lemma 1.1, the fibres of the morphism $\mathbb{H}_i \to \mathbb{G}\mathrm{r}(s; X)$ are subvarieties of $\mathbb{P}_k^{n-s-1}$ defined by
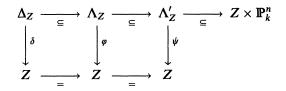
$$\binom{s + d_i}{s + 1} - 1$$

equations. By inequality (3), the dimension of the fibres of

$$\bigcap_{i=1}^{r} \mathbb{H}_i \to \mathbb{G}\mathrm{r}_k(s; X) \quad \text{and hence of } \mathbb{H}_{Z'} = \bigcap_{i=1}^{r} \mathbb{H}_i \times_{\mathbb{G}\mathrm{r}_k(s;X)} Z' \to Z'$$

is at least

$$n - s - 1 - \sum_{i=1}^{r} \left[\binom{s + d_i}{s + 1} - 1 \right] = n + r - s - 1 - \sum_{i=1}^{r} \binom{s + d_i}{s + 1} \geqslant r - 1. \quad (5)$$

We find an $(l + r - s - 1)$-dimensional subvariety Z of $\mathbb{H}_{Z'}$ which dominates Z'. Replacing Z by some blowing up, we may assume that Z dominates Z''. In characteristic zero, we can desingularize Z. In characteristic $p > 0$, we replace Z by the nonsingular generically finite cover constructed in [11]. Since the morphism $Z \to Z'$ factors over $\mathbb{H}_{Z'}$, we have the pullback families

$$
\begin{array}{ccccccc}
\Delta_Z & \xrightarrow{\subseteq} & \Lambda_Z & \xrightarrow{\subseteq} & \Lambda'_Z & \xrightarrow{\subseteq} & Z \times \mathbb{P}_k^n \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} & & \\
Z & \xrightarrow{=} & Z & \xrightarrow{=} & Z & &
\end{array}
$$

of $(s - 1)$, s, and $(s + 1)$-planes, respectively. By construction, the image of Δ_Z under the projection pr_2 to $\mathbb{P}_k^n$ is a divisor in Y, and the image of Λ_Z is Y. Let us consider the morphisms

$$Z \times \mathbb{P}_k^n \xrightarrow{\zeta} Z' \times \mathbb{P}_k^n \xrightarrow{p_2} \mathbb{P}_k^n,$$

their restrictions $\zeta' = \zeta|_{\Lambda'_Z \cap (Z \times X)}$ and $p'_2 = p_2|_{\zeta(\Lambda'_Z) \cap (Z' \times X)}$, and the induced maps

$$\mathrm{CH}_l(\Lambda'_Z \cap (Z \times X)) \xrightarrow{\zeta'_*} \mathrm{CH}_l(\zeta(\Lambda'_Z) \cap (Z' \times X)) \xrightarrow{p'_{2*}} \mathrm{CH}_l(X \cap pr_2(\Lambda'_Z)).$$

One has $\dim(\Lambda'_Z) = \dim(\zeta(\Lambda'_Z)) = l + r$ and the cycle $\Pi_Z = \zeta_*(\Lambda'_Z)$ is nonzero. For $\Gamma = pr_{2*}(\Lambda'_Z) = p_{2*}(\Pi_Z)$, the projection formula [6, 8.1.7] implies that

$$\Pi_Z \cdot (Z' \times X) = \zeta'_*(\Lambda'_Z \cdot (Z \times X)) \in \mathrm{CH}_l(\zeta(\Lambda'_Z) \cap (Z' \times X)) \quad \text{and that}$$

$$\Gamma \cdot X = p'_{2*}(\Pi_Z \cdot (Z' \times X)) = pr'_{2*}(\Lambda'_Z \cdot (Z \times X)) \in \mathrm{CH}_l(X \cap pr_2(\Lambda'_Z)).$$

In general, it might happen that $\Lambda'_Z \to pr_2(\Lambda'_Z)$ is not generically finite and, correspondingly, that the cycle Γ is zero. For the proof of 3.2 we need the next claim.

Claim 3.3. Under the additional assumption made in 3.2, we may choose Z such that the cycle $\Gamma = pr_{2*}(\Lambda'_Z)$ is nonzero and hence represented in $\mathrm{CH}_{l+r}(\mathbb{P}^n_k)$ by an $(l + r)$-dimensional subvariety.

Proof. Let $\Lambda' \subset \mathbb{H}_{Z'} \times \mathbb{P}^n_k$ denote the pullback of the universal family of $(s + 1)$-planes to $\mathbb{H}_{Z'}$, and let Π be the image of Λ' in $Z' \times \mathbb{P}^n_k$. The inequality (4) implies that the left-hand side in the inequality (5) is larger than or equal to $r - 1 + l - s$. Hence $\dim(\mathbb{H}_{Z'}) \geqslant r - 1 + 2l - 2s$ and $\dim(\Lambda') = \dim(\Pi) \geqslant r + 2l - s$. Since the fibres of $p_2|_{\Pi}$ are contained in Z', their dimension is at most $l - s$, and one finds $\dim(pr_2(\Lambda')) \geqslant l + r$. Choosing for Z a sufficiently general $(l + r - s - 1)$-dimensional subvariety of $\mathbb{H}_{Z'}$, one obtains $\dim(pr_2(\Lambda'_Z)) = l + r$. $\square$

To evaluate the intersection cycle $\Lambda'_Z \cdot (Z \times X)$, we distinguish three types of cycles.

Claim 3.4. The Chow group $\mathrm{CH}_l(\Lambda'_Z \cap (Z \times X))$ is generated by subvarieties

$$c_1, \ldots, c_\nu, \quad \psi^{-1} b_1, \ldots, \psi^{-1} b_\mu, \quad \text{and} \quad \varphi^{-1} a_1, \ldots, \varphi^{-1} a_\eta$$

with:
  (1) c_ι is an l-dimensional subvariety of $\Lambda'_Z \cap (Z \times X)$, with $\dim(\zeta(c_\iota)) < l$, for $\iota = 1, \ldots, \nu$.
  (2) b_j is an $(l - s - 1)$-dimensional subvariety of Z for which $pr_2(\psi^{-1}(b_j))$ is spanned by $(s + 1)$-planes, for $j = 1, \ldots, \mu$.
  (3) a_i is an $(l - s)$-dimensional subvariety of Z, surjective over Z', and hence $pr_2(\varphi^{-1}(a_i)) = Y$ for $i = 1, \ldots, \eta$.

Proof. Assume that $\Lambda'_Z \subset Z \times X$. Then $\Lambda'_Z = \Lambda'_Z \cap (Z \times X)$ is a $\mathbb{P}^{s+1}_k$-bundle over Z and

$$\mathrm{CH}_l(\Lambda'_Z) = \mathrm{CH}_l(\Delta_Z) + \varphi^* \mathrm{CH}_{l-s}(Z) + \psi^* \mathrm{CH}_{l-s-1}(Z).$$

For a prime cycle $c_0 \in \mathrm{CH}_l(\Delta_Z)$, one has

$$\zeta(c_0) \subseteq \zeta(\Delta_Z) \quad \text{and} \quad \dim(\zeta(c_0)) \leqslant \dim(Z'') + s - 1 = l - 1.$$

Similarly, for a prime cycle $c_0 \in \mathrm{CH}_{l-s}(Z)$ that does not dominate Z', $\zeta(\varphi^{-1}(c_0))$ is a family of s-planes over a proper subvariety of Z', hence of dimension strictly smaller than l. Choosing the "a_i" among the other cycles in $\varphi^* \mathrm{CH}_{l-s}(Z)$ and the "b_j" in $\psi^* \mathrm{CH}_{l-s-1}(Z)$, one obtains the generators of $\mathrm{CH}_l(\Lambda'_Z) = \mathrm{CH}_l(\Lambda'_Z \cap (Z \times X))$ asked for in 3.4.

If, on the other hand, $\Lambda'_Z \not\subset Z \times X$, then there is a proper subscheme A of Z with

$$\Lambda'_Z \cap (Z \times X) = \Lambda_Z \cup \psi^{-1}(A).$$

In fact, if for some $z \in Z$ the fibre $\psi^{-1}(z)$ is not contained in $Z \times X$, then $\psi^{-1}(z)$ is not contained in $Z \times X_i$ for one of the hyperplanes X_i cutting out X. Since $z \in \mathbb{H}_i$, one has

$$\psi^{-1}(z) \cap (Z \times X) \subset \psi^{-1}(z) \cap (Z \times X_i) = \varphi^{-1}(z) \subseteq \Lambda_Z.$$

As before, one can decompose the Chow group as

$$\mathrm{CH}_l(\Lambda'_Z \cap (Z \times X)) = \mathrm{CH}_l(\Delta_Z) + \varphi^* \mathrm{CH}_{l-s}(Z) + \psi^* \mathrm{CH}_{l-s-1}(A),$$

and again one obtains the generators asked for in 3.4. $\qquad\square$

By 3.4 we find integers $\gamma_1, \ldots, \gamma_\nu, \beta_1, \ldots, \beta_\mu, \alpha_1, \ldots, \alpha_\eta$ with

$$\Lambda'_Z \cdot (Z \times X) = \sum_{\iota=1}^{\nu} \gamma_\iota c_\iota + \sum_{j=1}^{\mu} \beta_j \psi^{-1}(b_j) + \sum_{i=1}^{\eta} \alpha_i \varphi^{-1}(a_i). \tag{6}$$

Since $\dim(pr_2(c_\iota)) < l$, for all ι, one obtains in $\mathrm{CH}_l(X \cap pr_2(\Lambda'_Z))$, and thereby in $\mathrm{CH}_l(X)$, the equation

$$pr'_{2*}(\Lambda'_Z \cdot (Z \times X)) = \sum_{j=1}^{\mu} \beta_j pr'_{2*}(\psi^{-1}(b_j)) + \sum_{i=1}^{\eta} \alpha_i pr'_{2*}(\varphi^{-1}(a_i)). \tag{7}$$

As stated in 3.4 (2), the first expression on the right-hand side of (7) is contained in $\mathrm{CH}_l^{(s+1)}(X)$. Let δ_i denote the degree of a_i over Z' or, equivalently, of $\varphi^{-1}(a_i)$ over $\Lambda_{Z'}$, and let ρ denote the degree of $\Lambda_{Z'}$ over Y. The second expression in (7) is nothing but $p_{2*}(\alpha' \Lambda_{Z'}) = \rho \alpha' Y$ for $\alpha' = \sum \alpha_i \delta_i$. One finds the equation

$$\Gamma \cdot X \equiv \rho \alpha' Y \mod \mathrm{CH}_l^{(s+1)}(X).$$

For $z' \in Z'$, let $H_{z'}$ denote the fibre of $\Lambda_{Z'} \to Z'$ over z'. If z' is chosen in sufficiently general position, the fibre $F \subseteq Z$ over z' meets the subvariety a_i of Z transversely in δ_i points, and it does not meet the cycles $b_1, \ldots, b_\mu$.

Let Λ_F and Λ'_F be the restrictions of Λ_Z and Λ'_Z to F. One has $\Lambda'_F \cap \psi^{-1}(b_j) = \varnothing$ for $j = 1, \ldots, \mu$ and $\zeta(\Lambda'_F) = \zeta(\Lambda'_Z) \cap (\{z'\} \times X))$. Since $\dim(\zeta(c_i)) < l$, one obtains from (6) the equation

$$\zeta_*((F \times \mathbb{P}^n_k) \cdot \Lambda'_Z \cdot (Z \times X)) = \sum_{j=1}^{\mu} \beta_j \zeta_*((F \times \mathbb{P}^n_k) \cdot \psi^{-1}(b_j))$$

$$+ \sum_{i=1}^{\eta} \alpha_i \zeta_*((F \times \mathbb{P}^n_k) \cdot \varphi^{-1}(a_i)).$$

The first term of the right-hand side is zero, and by the projection formula, one has

$$\zeta_*((F \times \mathbb{P}^n_k) \cdot \Lambda'_Z \cdot (Z \times X)) = \zeta_*(\Lambda'_F \cdot (Z \times X)) = \zeta_*(\Lambda'_F) \cdot Z' \times X.$$

Thus

$$\zeta_*(\Lambda'_F) \cdot Z' \times X = \sum_{i=1}^{\eta} \alpha_i \zeta_*(\varphi^{-1}(a_i)) \cdot (\{z'\} \times X) = \alpha' \Lambda_{Z'} \cdot (\{z'\} \times X)$$

$$= \alpha'(\{z'\} \times H_{z'}),$$

and $pr_{2*}(\Lambda'_F) \cdot X = \alpha' H_{z'}$ in $\mathrm{CH}_s(\mathbb{P}^n_k)$. Hence α', as the degree of the intersection of X with a nontrivial effective cycle, must be positive as well as $\alpha = \rho \alpha'$. $\square$

COROLLARY 3.5. *Let $X \subseteq \mathbb{P}^n_k$ be the union of some of the irreducible components of the intersection of r hyperplanes of degrees $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_r \geqslant 2$. If*

$$\sum_{i=1}^{r} \binom{l+d_i}{l+1} < n - l, \tag{8}$$

then $\mathrm{CH}_l(X)_{\mathbb{Q}} = \mathbb{Q}$.

In Lemma 4.2 in the next section, we will see that the inequality (8) implies that X contains an l-dimensional linear subspace H. Hence one may choose H as a generator of $\mathrm{CH}_l(X)_{\mathbb{Q}}$.

Proof. If Z is one of the irreducible components, say, of codimension t, then we can choose t of the equations in such a way that Z is a component of their zero locus. Using Proposition 2.2 and Addendum 3.2 instead of 2.2, the proof of the Corollary 2.3 carries over to prove that $\mathrm{CH}_l(Z)_{\mathbb{Q}} = \mathbb{Q}$.

Inequality (8) implies that $2r \leqslant n - l$ and hence $\dim(Z \cap Z') \geqslant l$ for two components Z and Z' of X. Since one may choose as generator for $\mathrm{CH}_l(Z)_{\mathbb{Q}}$ and $\mathrm{CH}_l(Z')_{\mathbb{Q}}$ the same cycle in the intersection, one obtains $\mathrm{CH}_l(Z \cup Z')_{\mathbb{Q}} = \mathbb{Q}$. $\quad\square$

4. An improved bound. It turns out that a slight modification of the methods of the previous sections enables us to improve our bound in Corollary 3.5 (with slightly different hypotheses), to

$$\sum_{i=1}^{r} \binom{d_i + l}{l + 1} \leqslant n.$$

Although this is, of course, a numerically insignificant improvement, it is really the appropriate bound given our methods, as explained in Remark 4.7 below.

Fix an algebraically closed base field k, and integers $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_r \geqslant 2$. As above, for a closed subset X of $\mathbb{P}_k^n$, we let $\mathrm{Gr}_k(l; X)$ denote the closed subset of $\mathrm{Gr}_k(l; n)$ consisting of the l-planes contained in X. We let $N_{n,d} = \binom{n+d}{d}$, and let

$$V(n; d_1, \ldots, d_r)_k := \mathbb{A}_k^{N_{n,d_1}} \times \cdots \times \mathbb{A}_k^{N_{n,d_r}}$$

parametrize r-tuples $(f_1, \ldots, f_r)$ of homogeneous equations of degrees $d_1, \ldots, d_r$ in variables $x_0, \ldots, x_n$. For

$$v = (f_1, \ldots, f_r) \in V(n; d_1, \ldots, d_r)_k,$$

we let X_v denote the closed subset of $\mathbb{P}_k^n$ defined by the equations $f_1 = \cdots = f_r = 0$.

We include the proofs of the following elementary results on $\mathrm{Gr}_k(l, X_v)$ for the convenience of the reader.

LEMMA 4.1. *Suppose* $\mathrm{char}(k) = 0$. *Let* $l \geqslant 0$ *be an integer such that*

$$\sum_{i=1}^{r} \binom{d_i + l}{l} < (l + 1)(n - l). \tag{9}$$

There is a nonempty Zariski-open subset $U_l = U_l(n; d_1, \ldots, d_r)_k$ *of* $V(n; d_1, \ldots, d_r)_k$ *such that for all* $v \in U_l$, *either* $\mathrm{Gr}_k(l; X_v)$ *is empty or*
  (i) $\mathrm{Gr}_k(l; X_v)$ *is smooth and has codimension* $\sum_{i=1}^{r} \binom{d_i + l}{l}$ *in* $\mathrm{Gr}_k(l; n)$;
  (ii) *if* $\sum_{i=1}^{r} \binom{d_i + l}{l+1} < n + 1$, *then the anticanonical bundle on* $\mathrm{Gr}_k(l; X_v)$ *is very ample;*
  (iii) *if* $\sum_{i=1}^{r} \binom{d_i + l}{l+1} = n + 1$, *then the canonical bundle on* $\mathrm{Gr}_k(l; X_v)$ *is trivial;*

(iv) *if $\sum_{i=1}^{r} \binom{d_i+l}{l+1} > n+1$, then the canonical bundle on $\mathbb{G}r_k(l; X_v)$ is very ample.*

In particular, if $\sum_{i=1}^{r} \binom{d_i+l}{l+1} \geqslant n+1$ and v is in U_l, then $\mathbb{G}r_k(l; X_v)$ has $p_g > 0$. Finally, we have

(v) *if the inequality (9) is not satisfied, then $\mathbb{G}r_k(l; X_v)$ is either empty, or has dimension zero, for all v in an open subset of $V(n; d_1, \ldots, d_r)_k$.*

Proof. We denote $\mathbb{G}r_k(l; n)$ by $\mathbb{G}r$. Let $V = k^{n+1}$, let $S \to \mathbb{G}r$ be the tautological rank-$(l+1)$ subsheaf of $\mathcal{O}_{\mathbb{G}r} \otimes_k V$, and let S^* be the dual of S, V^* the dual of V. Each $f \in \mathrm{Sym}^d(V^*)$ canonically determines a section f^S of $\mathrm{Sym}^d(S^*)$ over $\mathbb{G}r$. By the Bott theorem [4], sending f to f^S gives an isomorphism

$$\mathrm{Sym}^d(V^*) \to H^0(\mathbb{G}r, \mathrm{Sym}^d(S^*)).$$

As S^* is generated by global sections, so is $\mathrm{Sym}^d(S^*)$. Thus, there is a Zariski-open

$$U \subset \prod_{i=1}^{r} \mathrm{Sym}^{d_i}(V^*) = V(n; d_1, \ldots, d_r)_k$$

such that, for $(f_1, \ldots, f_r) \in U$, the subscheme $Y_{(f_1, \ldots, f_r)}$ of $\mathbb{G}r$ determined by the vanishing of the section $(f_1^S, \ldots, f_r^S)$ of $\bigoplus_{i=1}^{r} \mathrm{Sym}^{d_i}(S^*)$ is smooth, and has codimension equal to

$$\sum_{i=1}^{r} \mathrm{rank}(\mathrm{Sym}^{d_i}(S^*)) = \sum_{i=1}^{r} \binom{d_i + l}{l},$$

or is empty. In addition, we have $Y_{(f_1, \ldots, f_r)} = \mathbb{G}r_k(l; X_{(f_1, \ldots, f_r)})$. Taking U_l to be U proves (i), as well as (v).

We now compute the canonical sheaf $K_{\mathbb{G}r_k(l; X_v)}$ of $\mathbb{G}r_k(l; X_v)$ for $v \in U_l$, assuming that $\mathbb{G}r_k(l; X_v)$ is nonempty. The invertible sheaf $\Lambda^{\mathrm{top}} S^*$ is the very ample sheaf $\mathcal{O}_{\mathbb{G}r}(1)$ whose sections give the Plücker embedding of $\mathbb{G}r$. The tangent sheaf $T_{\mathbb{G}r}$ of $\mathbb{G}r$ fits into the standard exact sequence of sheaves on $\mathbb{G}r$

$$0 \to S \otimes S^* \to V \otimes_k S^* \to T_{\mathbb{G}r} \to 0,$$

giving the isomorphism

$$K_{\mathbb{G}r} \cong \mathcal{O}_{\mathbb{G}r}(-n-1). \tag{10}$$

For $v \in U_l$, we have the isomorphism $N_{\mathbb{G}r_k(l; X_v)/\mathbb{G}r} \cong \bigoplus_{i=1}^{r} \mathrm{Sym}^{d_i}(S^*) \otimes \mathcal{O}_{\mathbb{G}r_k(l; X_v)}$;

hence, we have the isomorphism

$$\Lambda^{\text{top}} N_{\mathbb{G}r_k(l;X_v)/\mathbb{G}r} \cong \bigotimes_{i=1}^{r} \Lambda^{\text{top}} \text{Sym}^{d_i}(S^*) \otimes \mathcal{O}_{\mathbb{G}r_k(l;X_v)} . \tag{11}$$

An elementary computation using the splitting principle gives

$$\Lambda^{\text{top}} \text{Sym}^{d_i}(S^*) \cong (\Lambda^{\text{top}} S^*)^{\otimes \binom{d_i+l}{l+1}} \cong \mathcal{O}_{\mathbb{G}r}\left(\binom{d_i+l}{l+1}\right) . \tag{12}$$

The exact sequence

$$0 \to T_{\mathbb{G}r_k(l;X_v)} \to T_{\mathbb{G}r} \to N_{\mathbb{G}r_k(l;X_v)/\mathbb{G}r} \to 0$$

gives the isomorphism

$$K_{\mathbb{G}r_k(l;X_v)} \cong K_{\mathbb{G}r} \otimes \Lambda^{\text{top}} N_{\mathbb{G}r_k(l;X_v)/\mathbb{G}r} \otimes \mathcal{O}_{\mathbb{G}r_k(l;X_v)} .$$

Combining this with (10)–(12) gives the isomorphism

$$K_{\mathbb{G}r_k(l;X_v)} \cong \mathcal{O}_{\mathbb{G}r}\left(\sum_{i=1}^{r} \binom{d_i+l}{l+1} - n - 1\right) \otimes \mathcal{O}_{\mathbb{G}r_k(l;X_v)} .$$

As $\mathcal{O}_{\mathbb{G}r}(1)$ is very ample, this proves (ii)–(iv). $\square$

LEMMA 4.2.   *Suppose that either* $d_1 \geqslant 3$, *or that* $r \geqslant l+1$. *If*

$$\sum_{i=1}^{r} \binom{d_i+l}{l+1} \leqslant n, \tag{13}$$

then one has the following.
    (a) *For each* $v \in V(n; d_1, \ldots, d_r)_k$ *and for each point* $x \in X_v$, *there exists an l-plane* $H \in \mathbb{G}r_k(l; X_v)$ *that contains* x. *In particular,* $\mathbb{G}r_k(l; X_v)$ *is nonempty.*
    (b) *If* $\text{char}(k)$ *is zero,* $\mathbb{G}r_k(l; X_v)$ *is irreducible for each* $v \in U_l(n; d_1, \ldots, d_r)$.
In addition, the inequality (13) implies the inequality (9).
    If $d_1 = 2$ *and* $1 \leqslant r \leqslant l$, *the same conclusions hold if* $r(l+2) \leqslant n+r-l-1$.

    Proof.   As we have seen in the proof of 2.3, the expression

$$\sum_{i=1}^{r} \binom{d_i+l}{l+1}$$

is an increasing function of l. So we may assume by induction on l that for all

points $v \in V(n; d_1, \ldots, d_r)_k$ and for all $x \in X_v$, there exists some $H \in \mathfrak{Gr}_k(l-1; X_v)$, passing through x. Moreover, if $\mathrm{char}(k) = 0$, we may assume that $\mathfrak{Gr}_k(l-1; X_v)$ is irreducible for all v in $U_{l-1}(n; d_1, \ldots, d_r)_k$.

Let $\mathfrak{Gr}_k(l; X_v)_H$ be the subset of $\mathfrak{Gr}_k(l; X_v)$ consisting of l-planes H' containing H. Returning to the notation introduced in Lemma 1.1, $\mathfrak{Gr}_k(l; X_v)_H$ is the intersection of the fibres of the morphisms $\pi'_1 : \mathbb{H}' \to \mathfrak{Gr}_k(l-1; Z_i)$ for the different hyperplanes Z_i, cutting out X_v. By 1.1, with l replaced by $l-1$, we find $\mathfrak{Gr}_k(l; X_v)_H$ to be a subscheme of $\mathbb{P}_k^{n-l}$, defined by

$$\sum_{i=1}^{r} \binom{d_i + l - 1}{l}$$

equations. So $\mathfrak{Gr}_k(l; X_v)_H$ has positive dimension if

$$\sum_{i=1}^{r} \binom{d_i + l - 1}{l} < n - l. \tag{14}$$

Using the standard identity for binomial coefficients

$$\binom{m+1}{n+1} = \binom{m}{n} + \binom{m}{n+1},$$

our inequality (13) implies (14) if

$$\sum_{i=1}^{r} \binom{d_i + l - 1}{l+1} \geqslant l + 1. \tag{15}$$

Since the left-hand side of (15) is an increasing function of the d_i, and since

$$\binom{l+2}{l+1} = l + 2 > l + 1 \quad \text{and} \quad \binom{l+1}{l+1} = 1,$$

the inequality (15) is satisfied if $d_1 \geqslant 3$, or if $r \geqslant l + 1$. In particular, it follows that $\dim(\mathfrak{Gr}_k(l; X_v)) > 0$ for general v. By Lemma 4.1 (v), this implies the inequality (9).

If $d_1 = 2$, then by assumption $r(l+1) \leqslant n - l - 1$, which easily implies the inequalities (14) and (9). In particular, $\mathfrak{Gr}_k(l; X_v)_H$ is nonempty, and we have obtained 4.2 (a).

In characteristic zero, it remains to show that $\mathfrak{Gr}_k(X_v)$ is connected for $v \in U_l$. Our inductive assumption, saying that $\mathfrak{Gr}_k(l-1; X_v)$ is irreducible for

$$v \in U_{l-1} = U_{l-1}(n; d_1, \ldots, d_r)_k,$$

reduces us to showing that $\mathfrak{Gr}_k(l; X_v)_H$ is connected for each $H \in \mathfrak{Gr}_k(l-1; X_v)$

and for v in some nonempty Zariski-open subset W of $U_l \cap U_{l-1}$. In fact, for those v the scheme $\mathbb{G}r_k(l; X_v)$ is connected, and by Lemma 4.1 (i) and by Zariski's connectedness theorem, one obtains the same for all $v \in U(n; d_1, \ldots, d_r)_k$.

By inequality (14), $\mathbb{G}r_k(X_v)$ is a subscheme of $\mathbb{P}_k^{n-l}$, defined by less than $n - l$ equations. The Fulton-Lazarsfeld connectedness theorem [7] shows the existence of W.   □

For a proper k-scheme X, we let $A_0(X)$ denote the subgroup of $CH_0(X)$ consisting of degree-zero zero-cycles.

A smooth projective variety Y is called *rationally connected* if, given a pair of general points y, y', there is an irreducible rational curve containing y and y'.

We call a projective k-scheme Y *weakly rationally connected* if for any pair of points y, y' of Y, there is a connected finite union C of rational curves on Y with y and y' in C. It is immediate that a weakly rationally connected Y has $A_0(Y) = 0$, even if Y is reducible.

LEMMA 4.3. *Let $\pi\colon Y \to Z$ be a flat, projective morphism of reduced schemes, with Z irreducible and normal, and with the geometric generic fibre $Y \times_Z \mathrm{Spec}(\overline{k(Z)})$ weakly rationally connected. Then, for each point z of Z, the geometric fibre $Y_z := Y \times_Z \mathrm{Spec}(\overline{k(z)})$ is weakly rationally connected. If, moreover, $E \subset Z$ is a subscheme with $A_0(E)_\mathbb{Q} = 0$, then $A_0(\pi^{-1}(E))_\mathbb{Q} = 0$.*

Proof. The first part follows directly from the following elementary fact (for a proof, see, e.g., Mumford [15]):

Let $p\colon C \to T$ be a projective morphism of reduced schemes of finite type over a Noetherian ring, with T normal and irreducible. Suppose that the reduced geometric fibre $(C \times_T \overline{k(T)})_{\mathrm{red}}$ is a connected union of rational curves. Let t be a closed point of T such that the fibre C_t over t has pure dimension one. Then $(C \times_T \overline{k(t)})_{\mathrm{red}}$ is a connected union of rational curves.

Since π is flat (of relative dimension, say, d), we have a well-defined pullback map

$$\pi^*\colon CH_0(E)_\mathbb{Q} \to CH_d(\pi^{-1}(E))_\mathbb{Q}.$$

Let z and z' be closed points of E. By assumption $z = z'$ in $CH_0(E)_\mathbb{Q}$, hence $\pi^*(z) = \pi^*(z')$ in $CH_d(\pi^{-1}(E))_\mathbb{Q}$. Intersecting $\pi^{-1}(E) \subset \mathbb{P}_k^N$ with a sufficiently general codimension-d linear subspace, this implies a relation in $CH_0(\pi^{-1}(E))_\mathbb{Q}$ of the form $a_z = a_{z'}$, where a_z is zero-cycle on $\pi^{-1}(z)$ and $a_{z'}$ is zero-cycle on $\pi^{-1}(z)$, both of positive degree. By the first part of Lemma 4.3, the fibres of π are weakly rationally connected; hence, a_z is a generator of $CH_0(Y_z)_\mathbb{Q}$ and $a_{z'}$ is a generator of $CH_0(Y_{z'})_\mathbb{Q}$. Since z and z' were arbitrary, this implies that a_z generates $CH_0(\pi^{-1}(E))_\mathbb{Q}$; hence, $A_0(\pi^{-1}(E))_\mathbb{Q} = 0$.   □

PROPOSITION 4.4. *Let v be in $V(n; d_1, \ldots, d_r)_k$. Suppose that either $d_1 \geqslant 3$, or*

that $r \geqslant l + 1$. *If*

$$\sum_{i=1}^{r} \binom{d_i + l}{l + 1} \leqslant n,$$

then $A_0(\mathbb{G}r_k(l; X_v))_{\mathbb{Q}} = 0$.

   If $d_1 = 2$, $1 \leqslant r \leqslant l$, *and* $r(l + 2) \leqslant n + r - l - 1$, *then* $A_0(\mathbb{G}r_k(l; X_v))_{\mathbb{Q}} = 0$.

   Proof. Denote $V(n; d_1, \ldots, d_r)_k$ by V_k. We have the correspondence $I_k \subset V_k \times \mathbb{G}r_k(l; n)$ consisting of pairs (v, Π) with $\Pi \subset X_v$. Let $p_1: I_k \rightarrow V_k$ and $p_2: I_k \rightarrow \mathbb{G}r_k(l; n)$ denote the restriction of the two projections. By an argument similar to the proof of Lemma 1.1, $p_2: I_k \rightarrow \mathbb{G}r_k(l; n)$ is a vector bundle and the restriction of p_1 to the fibres of p_2 gives isomorphisms with linear subspaces of V. In particular, I_k is smooth and irreducible.

   First, suppose that $\operatorname{char}(k) = 0$. By Lemma 4.1 and Lemma 4.2, $\mathbb{G}r_k(l; X_u)$ is a smooth, projective variety with very ample anticanonical bundle for all points $u \in U_k = U_l(n; d_1, \ldots, d_r)_k$. It follows from results of [3] or [14] that $\mathbb{G}r_k(l; X_u)$ is rationally connected. If $\operatorname{char}(k) = p > 0$, let R be a discrete valuation ring with residue field k and quotient field K having characteristic zero. Let V_R, I_R and $\mathbb{G}r_R(l; n)$ be the obvious R-schemes with fibre V_K, I_K, and $\mathbb{G}r_K(l; n)$ over K and V_k, I_k, and $\mathbb{G}r_k(l; n)$ over k. Since I_R and V_R are smooth, $I_R \rightarrow V_R$ is flat over the complement U_R of a closed subscheme of V_R of codimension at least two. Lemma 4.3 implies that $\mathbb{G}r_k(l; X_u)$ is rationally connected for all $u \in U_k = U_R \times_R k$.

   Of course, for all fields k the open subscheme $U_k \subset V_k$ is invariant under the action of $\mathbb{P}\mathrm{Gl}(n + 1, k)$ and for all $t \in I_k$ the image of $p_2(p_2^{-1}(t))$ will meet U_k.

   It remains to consider $\mathbb{G}r_k(l; X_v)$ for points $v \in V_k - U_k$ and to show that for two points t_1 and t_2 in $\mathbb{G}r_k(l; X_v)$ some multiple of the cycle $t_1 - t_2$ is rationally equivalent to zero. Let us choose for $i = 1, 2$ lines $G_i \subset p_2^{-1}(p_2(t_i))$ with $t_i \in G_i$ and with $p_1(G_i) \cap U_k \neq 0$. The lines $p_1(G_1)$ and $p_1(G_2)$ intersect in the point v and hence they span a two-dimensional linear subspace S of V_k, meeting U_k.

   The induced morphism from $S \cap U_k$ to the Hilbert scheme of subschemes of $\mathbb{G}r_k(l; n)$ extends to $\tilde{S}$ for some nonsingular blowup $\sigma: \tilde{S} \rightarrow V_k$. In other words, for the union J of all irreducible components of $I_k \times_{V_k} \tilde{S}$, which are dominant over $\tilde{S}$, the induced morphism $\pi: J \rightarrow \tilde{S}$ is flat. The reduced exceptional fibre $E = \sigma^{-1}(v)$ is the union of rational curves, and Lemma 4.3 implies that $A_0(\pi^{-1}(E))_{\mathbb{Q}} = 0$.

   Let $\tilde{\sigma}: J \rightarrow I_k$ be the induced morphism. By construction, the general points of the lines G_1 and G_2 lie in $\tilde{\sigma}(J)$. Since π is proper, one obtains $G_1, G_2 \subset \tilde{\sigma}(J)$ and hence the points t_1 and t_2 are contained in the image $\tilde{\sigma}(\pi^{-1}(E))$. $\qquad\square$

   LEMMA 4.5. *Let* $d_1, \ldots, d_r, l$ *be positive integers. If* $d_1 \geqslant 3$, *or if* $r \geqslant l - 1$, *then the inequality*

$$\sum_{i=1}^{r} \binom{d_i + l}{l + 1} \leqslant n$$

implies the inequality

$$\sum_{i=1}^{r} \binom{d_i + s}{s + 1} < n - s \tag{16}$$

for all $s < l$.

If $d_1 = 2$ and $1 \leqslant r \leqslant l$, then the inequality $r(l + 2) \leqslant n + r - l - 1$ implies the inequality (16) for all $s < l$.

Proof. As we saw in the proof of 2.3, the function of k

$$\sum_{i=1}^{r} \binom{d_i + k}{k + 1}$$

is increasing. Thus, we need only show (16) for $s = l - 1$. For $d \geqslant 3$ we have

$$\binom{d + l}{l + 1} = \binom{d + l - 1}{l} + \binom{d + l - 1}{l + 1} \geqslant \binom{d + l - 1}{l} + \binom{l + 2}{l + 1}$$

$$= \binom{d + l - 1}{l} + l + 2,$$

which verifies (16) for $s = l - 1$, completing the proof in case $d_1 \geqslant 3$.

If $d_i = 2$ for all i, we have

$$\sum_{i=1}^{r} \binom{d_i + s}{s + 1} = r(s + 2).$$

If $r \geqslant l - 1$, then $r(l + 2) \leqslant n \Rightarrow r(s + 2) < n - s$ for all $s < l$. If $1 \leqslant r \leqslant l$, then $r(l + 2) \leqslant n + r - l - 1 \Rightarrow r(s + 2) < n - s$ for all $s < l$. $\qquad\square$

THEOREM 4.6. *Let v be in $V(n; d_1, \ldots, d_r)_k$. Suppose that $d_1 \geqslant 3$, or that $r \geqslant l + 1$. If*

$$\sum_{i=1}^{r} \binom{d_i + l}{l + 1} \leqslant n,$$

then X_v contains a linear space of dimension l, and $\mathrm{CH}_s(X_v)_{\mathbb{Q}} = \mathbb{Q}$ for all $s \leqslant l$, with generator a linear space of dimension s. If $d_1 = 2$, $1 \leqslant r \leqslant l$, and

$$r(l + 2) = \sum_{i=1}^{r} \binom{d_i + l}{l + 1} \leqslant n + r - l - 1,$$

the same conclusion holds.

Proof. By Lemma 4.5, for all $0 \leqslant s < l$, the inequality

$$\sum_{i=1}^{r} \binom{d_i + s}{s + 1} \leqslant n - s \leqslant n \tag{17}$$

is satisfied. So it is sufficient to consider $\mathrm{CH}_l(X_v)_{\mathbb{Q}}$ in 4.6. Let Z be an irreducible component of X_v of codimension t in $\mathbb{P}_k^n$. By Lemma 4.2 (a), Z contains an l-plane H. Leaving out some of the equations f_i and correspondingly replacing the sum in (17) by a smaller one, we may apply Proposition 3.1 and Addendum 3.2.

The l-plane H is spanned by $(l-1)$-planes, and for $s = l - 1$, the equation (16) in 4.5 allows us to apply Addendum 3.2. Hence, there exists a $(t + l)$-dimensional subvariety Γ in $\mathbb{P}_k^n$ and a positive integer α with

$$\Gamma \cdot Z \equiv \alpha H \bmod \mathrm{CH}_l^{(l)},$$

and hence $\Gamma \cdot Z$ lies in $\mathrm{CH}_l^{(l)}$.

If Y is a dimension-l subvariety of Z spanned by σ-planes, for $0 \leqslant \sigma < l$, then by Proposition 3.1 there exists an effective cycle $\Gamma' \in \mathrm{CH}_{t+l}(\mathbb{P}_k^n)$ such that

$$\Gamma' \cdot Z = \alpha' Y + \sum_i \alpha_i Y_i$$

for some positive integer α', for $\alpha_i \in \mathbb{Z}$ and for l-dimensional subvarieties Y_i, spanned by $(\sigma + 1)$-planes in Z. Since $\mathrm{CH}_{t+l}(\mathbb{P}_k^n) = \mathbb{Z}$, the cycle Γ' is rationally equivalent to $\beta \Gamma$ for some rational number β. Hence $\beta \alpha H$ and $\alpha' Y$ are rationally equivalent modulo $\mathrm{CH}_l^{(\sigma+1)}(Z) \subseteq \mathrm{CH}_l^{(l)}(Z)$.

Thus, after finitely many steps, one obtains $\mathrm{CH}_l(Z)_{\mathbb{Q}} = \mathrm{CH}_l^{(l)}(Z)_{\mathbb{Q}}$, for all irreducible components Z of X_v. Of course, this implies that $\mathrm{CH}_l(X_v)_{\mathbb{Q}}$ is generated by the classes of l-planes contained in X_v.

On the other hand, by Proposition 4.4, $A_0(\mathbb{G}\mathrm{r}_k(l; X_v))_{\mathbb{Q}} = 0$; hence all the l-planes in X_v have the same class in $\mathrm{CH}_l(X_v)_{\mathbb{Q}}$. □

Remark 4.7. From the point of view of Hodge theory, or number theory, we have the essentially linear bound mentioned in the introduction

$$l + 1 \leqslant \left\lceil \frac{n - \sum_{i=2}^r d_i}{d_1} \right\rceil \tag{18}$$

rather than the degree-$(l + 1)$ bound

$$\sum_{i=1}^{r} \binom{d_i + l}{l + 1} \leqslant n \tag{19}$$

of Theorem 4.6. The statement

$$\sum_{i=1}^{r} \binom{d_i + l}{l + 1} \geqslant n + 1 \Rightarrow p_g(\mathbb{G}\mathrm{r}_k(l; X_v)) > 0$$

of Lemma 4.1 shows that one cannot hope to prove $\mathrm{CH}_l(X_v)_{\mathbb{Q}} = \mathbb{Q}$ for such $d_1, \ldots, d_r$ satisfying (18) but not (19) by only considering rational equivalences of l-planes among l-planes. Indeed, Roitman's theorem on the infinite dimensionality of zero-cycles [17] shows that the variety of l-planes has nontrivial zero-cycles once the inequality (19) fails. Hence, if it is indeed true that all l-planes in X_v are rationally equivalent (with $\mathbb{Q}$-coefficients), one must use rational equivalences which involve subvarieties of higher degree. The first interesting case is the question of whether $\mathrm{CH}_1(X)_{\mathbb{Q}} = \mathbb{Q}$ for X, a quartic hypersurface in $\mathbb{P}^8$.

Remark 4.8. In the case of irreducible quadric hypersurfaces $Q \subset \mathbb{P}^n$, Theorem 4.6 and Corollary 2.3 give the same bound; for a smooth quadric, this bound is sharp. Indeed, we have $\mathrm{CH}_s(Q) = \mathbb{Z}$ for all $s \leqslant l$ if and only if $n \geqslant 2l + 2$, which is exactly the bound of Corollary 2.3. (It is well known that the Chow groups of a smooth quadric are torsion-free.) This is also the bound given by the Hodge-theoretic considerations mentioned in Remark 4.7.

5. Decomposition of the diagonal. As pointed out by Bloch-Srinivas [2], results on triviality of Chow groups of a projective variety X give rise to a special structure on the diagonal in $X \times X$; this in turn leads to a decomposition of the motive of X and to the triviality of primitive cohomology. Variants of this have appeared in many works; we give here a brief account of this technique.

LEMMA 5.1. *Let X be a closed subset of $\mathbb{P}_k^n$ of pure dimension t. Suppose X contains a linear space $L \cong \mathbb{P}^l$ such that, for all algebraically closed fields $K \supset k$, $\mathrm{CH}_s(X_K)_{\mathbb{Q}}$ is generated by the class of a dimension-s linear space $L_s \subset L$. Let Y be a k-variety of dimension d with $t - l \leqslant d \leqslant t$. Then for each $\delta \in \mathrm{CH}_t(X \times Y)_{\mathbb{Q}}$, there is a proper closed subset $D \subset Y$, a cycle $\gamma \in \mathrm{CH}_t(X \times Y)_{\mathbb{Q}}$, supported in $X \times D$, and a rational number r, such that*

$$\delta = r[L_{t-d} \times Y] + \gamma,$$

where $[-]$ denotes the class in $\mathrm{CH}_t(X \times Y)_{\mathbb{Q}}$.

Proof. For a field extension F of $k(Y)$, we denote $X \times_k F$ by X_F, and let δ_F be the pullback of δ to X_F via the canonical map $X_F \to X \times Y$. We let $[-]_F$ denote the class of a cycle in $\mathrm{CH}_*(X_F)_{\mathbb{Q}}$.

Let K be the algebraic closure of the function field $k(Y)$. As δ_K is an element of $\mathrm{CH}_{t-d}(X_K)_{\mathbb{Q}}$, and $0 \leqslant t - d \leqslant l$, we have the identity $\delta_K = r[L_{t-d}]_K$ for some rational number r. This gives the identity $\delta_F = r[L_{t-d}]_F$ for some finite extension F of $k(Y)$. We may push forward by the map $X_F \to X_{k(Y)}$, giving the identity

$\delta_{k(Y)} = r[L_{t-d}]_{k(Y)}$. Thus there is a Zariski-open subset $j: U \to Y$ of Y such that we have the identity

$$(\mathrm{id}_X \times j)^*(\delta) = [L_{t-d} \times U]$$

in $\mathrm{CH}_t(X \times U)_{\mathbb{Q}}$. Let $D = Y \backslash U$ with inclusion $i: D \to Y$; the exact localization sequence

$$\mathrm{CH}_t(X \times D)_{\mathbb{Q}} \xrightarrow{(\mathrm{id}_X \times i)_*} \mathrm{CH}_t(X \times Y)_{\mathbb{Q}} \xrightarrow{(\mathrm{id}_X \times j)^*} \mathrm{CH}_t(X \times U)_{\mathbb{Q}} \to 0$$

together with the identity $[L_{t-d} \times U] = (\mathrm{id}_X \times j)^*([L_{t-d} \times Y])$ completes the proof. $\qquad\qquad\square$

THEOREM 5.2. *Let X be a closed subset of $\mathbb{P}_k^n$ of pure dimension t. Suppose that X is the intersection of hypersurfaces of degrees $d_1 \geqslant \cdots \geqslant d_r \geqslant 2$; suppose further either that*
(i) *$d_1 \geqslant 3$ or $r \geqslant l + 1$, and that*

$$\sum_{i=1}^r \binom{d_i + l}{l+1} \leqslant n,$$

or that
(ii) *$d_1 = 2$, $1 \leqslant r \leqslant l$ and that $r(l+2) \leqslant n + r - l - 1$.*
Then X contains a flag of linear spaces $L_0 \subset L_1 \subset \cdots \subset L_l$, with $\dim(L_j) = j$, and we may write the class of the diagonal Δ_X in $\mathrm{CH}_t(X \times X)_{\mathbb{Q}}$ as

$$[\Delta_X] = [L_0 \times X] + [L_1 \times A_1] + \cdots + [L_l \times A_l] + \gamma,$$

with $A_i \in \mathrm{CH}_{t-i}(X)_{\mathbb{Q}}$ and with γ supported in $X \times W$ for some pure codimension-$(l+1)$ closed subset W of X.
   In addition, if X is smooth, let h denote the class in $\mathrm{CH}^1(X)$ of a hyperplane section of X, and let $h^{(i)}$ denote the i-fold self-intersection. Then $[A_i] = h^{(i)}$ for $i = 1, \ldots, l$, in $\mathrm{CH}^i(X)_{\mathbb{Q}}$.

Proof. By Theorem 4.6, X contains a flag of linear spaces $L_0 \subset L_1 \subset \cdots \subset L_l$, such that, for each algebraically closed field $K \supset k$, $\mathrm{CH}_s(X_K)_{\mathbb{Q}}$ is generated by the class of L_s for $0 \leqslant s \leqslant l$.
   Suppose X has components $X_1, \ldots, X_p$; let Δ_X^i denote the image of Δ_{X_i} in $X \times X_i$. We now apply Lemma 5.1 to the cycle $[\Delta_X^i]$. This gives us a proper closed subset D_1^i of X_i, a rational number r_i, and a cycle γ_1^i in $\mathrm{CH}_t(X \times X_i)_{\mathbb{Q}}$, supported in $X \times D_1^i$ such that

$$[\Delta_X^i] = r_i[L_0 \times X_i] + \gamma_1^i \tag{20}$$

in $CH_t(X \times X_i)_{\mathbb{Q}}$. Applying the projection p_{2*} gives $r_i = 1$. Since D_1^i is a proper closed subset of X_i, we may suppose that D_1^i has pure codimension one on X_i.

Let $q_i : X_i \to X$ be the inclusion. Let $D_1 = \bigcup_{i=1}^p D_1^i$ and let

$$\gamma_1 = \sum_{i=1}^p (\mathrm{id}_X \times q_i)_* (\gamma_1^i).$$

Since

$$[\Delta_X] = \sum_{i=1}^p (\mathrm{id}_X \times q_i)_* ([\Delta_X^i])$$

$$[L_0 \times X] = \sum_{i=1}^p (\mathrm{id}_X \times q_i)_* ([L_0 \times X_i]),$$

applying $(\mathrm{id}_X \times q_i)_*$ to the relation (20) and summing gives the identity

$$[\Delta_X] = [L_0 \times X] + \gamma_1$$

in $CH_t(X \times X)_{\mathbb{Q}}$, with $\gamma_1 \in CH_t(X \times X)_{\mathbb{Q}}$ supported in $X \times D_1$.

The result then follows by induction: suppose we have an integer s, with $1 \le s \le l$, and the identity in $CH_t(X \times X)_{\mathbb{Q}}$:

$$[\Delta_X] = [L_0 \times X] + [L_1 \times A_1] + \cdots + [L_{s-1} \times A_{s-1}] + \gamma_s,$$

with $A_i \in CH_{t-i}(X)_{\mathbb{Q}}$ and with γ_s supported in $X \times D_s$ for some pure codimension-s closed subset D_s of X. If D_s has irreducible components $D_s^1, \ldots, D_s^p$, we may write γ_s as a sum

$$\gamma_s = \sum_{i=1}^p \gamma_s^i,$$

with γ_s^i supported in $X \times D_s^i$. As D_s^i has dimension $t - s$, we may apply Lemma 5.1, giving a proper closed subset D_{s+1}^i of D_s^i, a rational number r_i, and a cycle $\gamma_{s+1}^i \in CH_t(X \times D_s^i)$, supported in $X \times D_{s+1}^i$, such that the identity

$$\gamma_s^i = r_i [L_s \times D_s^i] + \gamma_{s+1}^i$$

holds in $CH_t(X \times D_s^i)$. We may suppose that D_{s+1}^i is a pure codimension-one subset of D_s^i.

Taking $A_s = \sum_{i=1}^p r_i [D_s^i]$, $\gamma_{s+1} = \sum_{i=1}^p \gamma_{s+1}^i$, and $D_{s+1} = \bigcup_{i=1}^p D_{s+1}^i$ gives the desired identity

$$[\Delta_X] = [L_0 \times X] + [L_1 \times A_1] + \cdots + [L_s \times A_s] + \gamma_{s+1},$$

verifying the induction.

It remains to show that $[A_j] = h^{(j)}$ for $j = 1, \ldots, l$ in case X is smooth. For cycles $\alpha \in \mathrm{CH}^*(X \times X)_{\mathbb{Q}}$ and $\beta \in \mathrm{CH}^*(X)_{\mathbb{Q}}$, let $\alpha_*(\beta) = p_{2*}(p_1^*\beta \cup \alpha)$. We note the identities

$$[\Delta_X]_*(h^{(j)}) = h^{(j)}$$

$$[L_i \times A_i]_*(h^{(j)}) = \delta_{ij}[A_i]$$

$$\gamma_*(h^{(j)}) = 0 \quad \text{for } 0 \leqslant j \leqslant l,$$

from which the identity $[A_i] = h^{(i)}$, $i = 1, \ldots, l$ follows immediately. $\qquad\square$

We recall the category of effective Chow motives over k, $\mathscr{M}_k^+$ (see [13] for details). The objects of $\mathscr{M}_k^+$ are pairs (X, γ), where X is a smooth projective k-variety, and $\gamma \in \mathrm{CH}^{\dim_k(X)}(X \times_k X)_{\mathbb{Q}}$ is an idempotent correspondence; the object (X, Δ_X) is denoted $m(X)$, and is called the *motive of X*. Morphisms from (X, γ) to (Y, δ) are given by correspondences. $\mathscr{M}_k^+$ is a tensor category, with tensor product induced by the operation of product over k. We have the Lefschetz motive $\mathscr{L}$, defined as the object $(\mathbb{P}^1, \mathbb{P}^1 \times 0)$; inverting the operation $(-) \otimes \mathscr{L}$ defines the category of Chow motives over k, $\mathscr{M}_k$.

COROLLARY 5.3. *Let X be as in Theorem 5.2, and assume in addition that X is smooth. Then the decomposition of the diagonal in Theorem 5.2 is a decomposition of $[\Delta_X]$ into mutually orthogonal idempotent correspondences, giving the decomposition of the motive $m(X)$ in $\mathscr{M}_k$ as*

$$m(X) = \bigoplus_{i=0}^{l} \mathscr{L}^i \bigoplus (X, \gamma).$$

Proof. One directly computes that the classes $[L_i] \times h^{(i)}$ are mutually orthogonal idempotents; as these classes clearly commute with $[\Delta_X]$, the decomposition of $[\Delta_X]$ in 5.2 is a decomposition into mutually orthogonal idempotents, as claimed. It remains to check that there is an isomorphism of $(X, [L_i] \times h^{(i)})$ with $\mathscr{L}^i$.

For this, we note that $\mathscr{L}^i$ is isomorphic to the motive defined by $(L_i, [L_i \times 0])$. Let $\iota_i \colon L_i \to X$ be the inclusion, giving the morphism

$$\iota_i^* \colon (X, [L_i] \times h^{(i)}) \to (L_i, [L_i \times 0]).$$

It is then easy to check that ι_i^* is an isomorphism, with its inverse given by the correspondence $L_i \times h^{(i)} \subseteq L_i \times X$. $\qquad\square$

One can use the decomposition of the diagonal in Theorem 5.2 to recover a part of the results of Ax and Katz on the congruence $\#\mathbb{P}^n(\mathbb{F}_q) \equiv \#X(\mathbb{F}_q)$, once

we make certain integrality assumptions on the decomposition and assume a weak form of resolution of singularities. This gives a proof of a weak version of the Ax-Katz result by essentially algebro-geometric means, without resorting to the use of zeta functions. To see this, we first note the following result.

LEMMA 5.4. *Let Y be a smooth variety over $\mathbb{F}_q$, and let Z be an irreducible closed subset of codimension s. Let Π_Y and Π_Z denote the graphs of the Frobenius endomorphisms Frob_Y and Frob_Z, respectively. Then*
  (i) *Π_Y intersects $Y \times_{\mathbb{F}_q} Z$ properly in $Y \times_{\mathbb{F}_q} Y$, and*

$$\Pi_Y \cdot_{Y \times_{\mathbb{F}_q} Y} Y \times_{\mathbb{F}_q} Z = q^s \Pi_Z,$$

  where we consider Π_Z as a cycle on $Y \times_{\mathbb{F}_q} Z$.
  (ii) *Π_Y intersects $Z \times_{\mathbb{F}_q} Y$ properly in $Y \times_{\mathbb{F}_q} Y$, and*

$$\Pi_Y \cdot_{Y \times_{\mathbb{F}_q} Y} Z \times_{\mathbb{F}_q} Y = \Pi_Z,$$

  where we consider Π_Z as a cycle on $Z \times_{\mathbb{F}_q} Y$.

Proof. We give the proof of (i); the proof of (ii) is similar and is left to the reader.

Since Π_Y is the locus of points (y, y^q), and similarly for Π_Z, it follows that

$$\Pi_Y \cap (Y \times_{\mathbb{F}_q} Z) = \Pi_Z$$

set-theoretically, which shows that the intersection $\Pi_Y \cap (Y \times_{\mathbb{F}_q} Z)$ is proper. Since Π_Z is irreducible, we have as well

$$\Pi_Y \cdot_{Y \times_{\mathbb{F}_q} Y} (Y \times_{\mathbb{F}_q} Z) = \mu \Pi_Z \tag{1}$$

for some positive integer μ. Since the intersection multiplicity is determined at the generic point of Π_Z, we may replace Y with any open subset which intersects Z; thus, we may assume that Y is affine, and, as $\mathbb{F}_q$ is perfect, that Z is smooth over $\mathbb{F}_q$.

Suppose we have an étale map of pairs $f : (Y, Z) \to (T, W)$, with (T, W) satisfying the hypotheses of the lemma. Then f induces étale maps

$$Y \times_{\mathbb{F}_q} Y \to T \times_{\mathbb{F}_q} Y, \qquad Y \times_{\mathbb{F}_q} Z \to T \times_{\mathbb{F}_q} W$$

$$\Pi_Y \to \Pi_T, \qquad \Pi_Z \to \Pi_W;$$

hence, we have

$$\Pi_T \cdot_{T \times_{\mathbb{F}_q} T} (T \times_{\mathbb{F}_q} W) = \mu \Pi_W,$$

with the same integer μ as in (1). Thus, it suffices to prove (i) for some (T, W).

Shrinking Y again if necessary, we may find an étale map of pairs

$$(Y, Z) \to (\mathbb{A}^{n+s}, \mathbb{A}^n)$$

where $\mathbb{A}^n$ is the subvariety of $\mathbb{A}^{n+s}$ defined by $x_{n+1} = \cdots = x_{n+s} = 0$, for global coordinates $x_1, \ldots, x_{n+s}$ on $\mathbb{A}^{n+s}$. Using coordinates x_i and y_j on $\mathbb{A}^{n+s} \times \mathbb{A}^{n+s}$, with the x_i being coordinates on the first factor and the y_j coordinates on the second factor, $\Pi_{\mathbb{A}^{n+s}}$ is defined by the equations

$$y_j - x_j^q = 0; \qquad j = 1, \ldots, n + s.$$

As $\mathbb{A}^{n+s} \times \mathbb{A}^n$ is defined by the equations

$$y_i = 0; \qquad i = n + 1, \ldots, n + s,$$

the identity

$$\Pi_{\mathbb{A}^{n+s}} \cdot_{\mathbb{A}^{n+s} \times \mathbb{A}^{n+s}} (\mathbb{A}^{n+s} \times \mathbb{A}^n) = q^s \Pi_{\mathbb{A}^n}$$

follows by a direct computation. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

PROPOSITION 5.5. *Let $X \subset \mathbb{P}^n_{\mathbb{F}_q}$ be a smooth projective variety over $\mathbb{F}_q$ containing a flag of linear spaces $L_0 \subset L_1 \subset \cdots \subset L_l$. Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at p, and suppose we have in $\mathrm{CH}^*(X \times_{\mathbb{F}_q} X) \otimes \mathbb{Z}_{(p)}$ the identity*

$$[\Delta_X] = [L_0 \times X] + [L_1 \times A_1] + \cdots + [L_l \times A_l] + [\gamma],$$

with A_i a codimension-i cycle (with $\mathbb{Z}_{(p)}$-coefficients) on X, and γ supported on $X \times W$ for some closed subset W of X. Suppose in addition that each irreducible component of γ dominates an irreducible component of W, that each irreducible component of W has codimension at least $l + 1$ on X, and that W admits a resolution of singularities over $\mathbb{F}_q$. Then

$$\# \mathbb{P}^n(\mathbb{F}_q) \equiv \# X(\mathbb{F}_q) \bmod q^{l+1}.$$

Proof. It follows as in the proof of the last statement of 5.2 that we have $[A_i] = h^{(i)}$ in $\mathrm{CH}^i(X) \otimes \mathbb{Z}_{(p)}$, with $h^{(i)}$ the class of the intersection of X with a dimension-$(n - i)$ linear subspace L'_{n-i} of $\mathbb{P}^n$. We may assume that L_i and L'_{n-i} intersect transversely in $\mathbb{P}^n$ at a single point zero. By repeated applications of the projection formula, together with 5.4, we have

$$\deg(\Pi_X \cdot_{X \times_{\mathbb{F}_q} X} (L_i \times h^{(i)})) = \deg(\Pi_X \cdot_{X \times_{\mathbb{F}_q} \mathbb{P}^n} (L_i \times_{\mathbb{F}_q} L_{n-i}))$$

$$= \deg(\Pi_{\mathbb{P}^n} \cdot_{\mathbb{P}^n \times_{\mathbb{F}_q} \mathbb{P}^n} (L_i \times_{\mathbb{F}_q} L_{n-i}))$$

$$= \deg(q^i \Pi_{L_{n-i}} \cdot {}_{\mathbb{P}^n \times_{\mathbb{F}_q} L_{n-i}} (L_i \times_{\mathbb{F}_q} L_{n-i}))$$

$$= q^i \deg(\Pi_{L_{n-i}} \cdot {}_{L_{n-i} \times_{\mathbb{F}_q} L_{n-i}} (0 \times_{\mathbb{F}_q} L_{n-i}))$$

$$= q^i \deg(\Pi_0 \cdot {}_{0 \times_{\mathbb{F}_q} L_{n-i}} (0 \times_{\mathbb{F}_q} L_{n-i})) = q^i.$$

Similarly, let γ_i be an irreducible component of γ, dominating an irreducible component W_i of W. By our assumption on γ and W, there is a resolution of singularities $\pi: \tilde{W}_i \to W_i$, and subvariety $\tilde{\gamma}_i$ of $X \times_{\mathbb{F}_q} \tilde{W}_i$ with $(\mathrm{id}_X \times \pi)_*(\tilde{\gamma}_i) = \gamma_i$. Let

$$p: X \times_{\mathbb{F}_q} \tilde{W}_i \to X \times_{\mathbb{F}_q} X$$

be the evident morphism. By applying the projection formula and 5.4, we have

$$\deg(\Pi_X \cdot {}_{X \times_{\mathbb{F}_q} X} \gamma_i) = \deg(\Pi_X \cdot {}_{X \times_{\mathbb{F}_q} X} p_*(\tilde{\gamma}_i))$$

$$= \deg(p^*(\Pi_X) \cdot {}_{X \times_{\mathbb{F}_q} \tilde{W}_i} \tilde{\gamma}_i).$$

On the other hand, since the projection $\Pi_{W_i} \to W_i$ is finite and surjective, it follows that the pullback $(\mathrm{id}_X \times \pi)^{-1}(\Pi_{W_i})$ is irreducible. This, together with 5.4, gives the identity

$$p^*(\Pi_X) = q^s \Pi_{\tilde{W}_i}$$

with $s = \mathrm{codim}_X(W_i) \geq l + 1$. Thus, we have

$$\Pi_X \cdot {}_{X \times_{\mathbb{F}_q} X} \gamma \equiv 0 \bmod q^{l+1}.$$

The identities $\#X(\mathbb{F}_q) = \deg(\Pi_X \cdot {}_{X \times_{\mathbb{F}_q} X} \Delta_X)$ and $\#\mathbb{P}^n(\mathbb{F}_q) = 1 + q + \cdots + q^n$ complete the proof. $\square$

Remark 5.6. If we have a smooth variety X over an algebraically closed field k of characteristic zero, for which our result 5.2 applies, we may consider the various specializations of X obtained by choosing a smooth projective model $\mathcal{X} \to \mathrm{Spec}(R)$ of X over a ring R, finitely generated over $\mathbb{Z}$, and taking the fibres of X over $\mathbb{F}_q$-points of R. The decomposition of Δ_X in 5.2 involves only finitely many denominators, and implies an analogous decomposition of $\Delta_{\mathcal{X}}$, after shrinking $\mathrm{Spec}(R)$ if necessary. Since we may assume that W has a resolution of singularities $\tilde{W}$, smooth and projective over R, it follows that 5.2 implies the Ax-Katz congruence on $\mathcal{X}(\mathbb{F}_q)$ for all but finitely many characteristics and with κ replaced by the smaller number $l + 1$.

For X defined over $\mathbb{C}$, one considers on the primitive Betti cohomology

groups

$$H^b(X)_{\mathrm{prim}} := H^b(X, \mathbb{Q})/H^b(\mathbb{P}^n, \mathbb{Q}),$$

the descending *coniveau filtration*

$$N^a H^b(X)_{\mathrm{prim}} := \{\sigma \in H^b(X)_{\mathrm{prim}}, \text{ there exists a closed subset } Z$$
$$\text{of codimension} \geqslant a \text{ such that } \sigma|_{X-Z} = 0\}.$$

The next corollary implies that the Hodge-type relation $F^l H_c^i(\mathbb{P}_{\mathbb{C}}^n - X) = H_c^i(\mathbb{P}_{\mathbb{C}}^n - X)$ holds true for all i.

COROLLARY 5.7. *Let X be as in Theorem 5.2. Assume moreover that X is smooth. Then for all b*

$$N^{l+1} H^b(X)_{\mathrm{prim}} = H^b(X)_{\mathrm{prim}}.$$

Proof. As usual, one applies the correspondence

$$[\Delta_X]_*(\sigma) = p_{2*}([\Delta_X] \cdot p_1^*(\sigma))$$

on $H^b(X)_{\mathrm{prim}}$, where $[\Delta_X]$ is the cohomology class of Δ_X in $H^{2\dim X}(X \times X)$. This is the identity. The correspondence with γ sends $H^b(X)$ into the image of the cohomology of W via the Gysin morphism, whereas the correspondence with $L_i \times h^{(i)}$ kills $H^b(X)$ for $b \neq 2i$, while for $b = 2i$, it sends $H^{2i}(X)$ into some multiple of the cohomology class of $h^{(i)}$.

On the other hand, for $\sigma \in H^{2i}(X)$, we have

$$\sigma = [\Delta_X]_*(\sigma) \equiv [L_i \times h^{(i)}]_*(\sigma) \bmod N^{l+1} H^{2i}(X) \equiv r[h^{(i)}] \bmod N^{l+1} H^{2i}(X)$$

for some $r \in \mathbb{Q}$. ∎

Note added in proof. For curves on hypersurfaces, i.e., for $r = l = 1$, J. Kollár obtained Theorem 4.6 independently in his book *Rational Curves on Algebraic Varieties*, Ergeb. Math. Grenzgeb. (3) **32**, Springer-Verlag, Berlin, 1996, Theorems IV.3.13 and V.4.2. His argument is in the same two stages as ours: first show that $\mathrm{CH}_1(X)_{\mathbb{Q}}$ is generated by lines, and then show that the lines are all equivalent. In characteristic zero, Kollár's proof of the second step is similar to the one given in Section 4. In characteristic $p > 0$ he shows the smoothness of the variety of lines, which we avoid by a degeneration argument.

For the first step, however, his arguments are different from the ones used here.

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