APPENDIX
CLASSES OF LOCAL SYSTEMS OF HERMITIAN VECTOR SPACES

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For a local system V on a topological manifold S associated to a representation

$$\rho : \pi_1(S, s) \to GL(n, \mathbb{C})$$

of the fundamental group we denote by

$$\hat{c}_i(V) = \hat{c}_i(\rho) = \beta_i + \gamma_i \in H^{2i-1}(S, \mathbb{C}/\mathbb{Z})$$

the class defined in [4], [1] :

$$\beta_i \in H^{2i-1}(S, \mathbb{R}) \quad ([1]\ , (2.20))$$

$$\gamma_i \in H^{2i-1}(S, \mathbb{R}/\mathbb{Z}) \quad ([4]\ , \S4).$$

If $f : X \to S$ is a smooth proper morphism of $\mathcal{C}^\infty$ manifolds with orientable fibers, the Riemann-Roch theorem ([1], Theorem (0.2), Theorem (3.11)) says

$$\hat{c}_i\Big(\sum_{j=0}^{\dim(X/S)} (-1)^j R^j f_*\mathbb{C}\Big) = 0$$

in $H^{2i-1}(S, \mathbb{C}/\mathbb{Q})$, for all $i \geq 1$.

The purpose of this short note is to show how to apply Reznikov's ideas [12] to obtain vanishing of the single classes $\hat{c}_i(R^j f_*\mathbb{C})$ under some assumptions.

Definition 0.1. Let A be a ring with $\mathbb{Z} \subset A \subset \mathbb{C}$. *A local system of* A *hermitian vector spaces* is a local system associated to a representation ρ whose image $\rho(\pi_1(S, s))$ lies in $GL_n(A) \subset GL_n(\mathbb{C})$ and $U(p, q) \subset GL_n(\mathbb{C})$, for some pair (p, q) with $n = p + q$, where $U(p, q)$ is the unitary group with respect to a non degenerate hermitian form with p positive, and q negative eigenvalues.

Theorem 0.2. *Let S be a topological manifold and let $\rho : \pi_1(S, s) \to GL(n, F)$ be a representation of the fundamental group with values in a number field F. Assume that for all real and complex embeddings $\sigma : F \to \mathbb{R}(\subset \mathbb{C})$ and $\sigma : F \to \mathbb{C}$, $\sigma \circ \rho : \pi_1(S, s) \to GL(n, \mathbb{C})$ is a local system of $\sigma(F)$ hermitian vector spaces. Then $\hat{c}_i(\rho) = 0$ in $H^{2i-1}(S, \mathbb{C}/\mathbb{Q})$ for all $i \geq 1$.*

Examples of local systems of $\mathbb{Q}$ hermitian vector spaces are provided by $\mathbb{Q}$ variations of Hodge structures [9] I.2, whose main instances are the Gauß-Manin local systems $R^j f_*\mathbb{C}$, where $f : X \to S$ is a smooth proper morphism of complex manifolds with Kähler fibres. So Theorem 0.2 implies

Theorem 0.3. *Let $f : X \to S$ be a smooth proper morphism of complex manifolds with Kähler fibres. Then*

$$\hat{c}_i(R^j f_*\mathbb{C}) = 0$$

in $H^{2i-1}(S, \mathbb{C}/\mathbb{Q})$ for all $i \geq 1$, $j \geq 0$.

In the $\mathcal{C}^\infty$ category, other examples are provided by Poincaré duality:

Theorem 0.4. *Let $f : X \to S$ be a smooth proper morphism of C^∞ manifolds with orientable fibres. Then*

$$\hat{c}_i(R^j f_*\mathbb{C} \oplus R^{(\dim(X/S)-j)} f_*\mathbb{C}) = 0$$

in $H^{2i-1}(S, \mathbb{C}/\mathbb{Q})$ and

$$\hat{c}_i(R^{\dim(X/S)/2} f_*\mathbb{C}) = 0$$

in $H^{2i-1}(S, \mathbb{C}/\mathbb{Q})$ if $\dim(X/S)$ is even.

Proof of Theorem 0.2.
The $U(p, q)$ flat bundle being isomorphic to the conjugate of its dual, the formula ([1] (2.21)) says that $\beta_i = 0$. Thus we just have to consider γ_i.

We may first assume that $\Lambda^n\rho : \pi_1(S, s) \to \mathbb{C}^*$ is trivial. In fact, it is torsion as a unitary and rational representation, say of order N, and $V \oplus \ldots \oplus V$ (N times) has trivial determinant. On the other hand

$$\hat{c}_i(V \oplus \ldots \oplus V) = N\hat{c}_i(V)$$

in $H^{2i-1}(S, \mathbb{C}/\mathbb{Q})$, as

$$\hat{c}_i(V).\hat{c}_j(V) = 0$$

for $i \geq 1, j \geq 1$, in $H^{2(i+j)-1}(S, \mathbb{C}/\mathbb{Q})$. (The multiplication is defined by Image ($\hat{c}_i(V)$ in $H^{2i}(S, \mathbb{Z})$). $\hat{c}_j(V)$ [4] (1.11)).

Furthermore, by adding trivial factors to V, one may assume that n is as large as one wants.

There is an open cover $S = \cup_\alpha S_\alpha$ trivializing V with transition functions

$$\lambda_{\alpha\beta} \in \Gamma(S_{\alpha\beta}, SL_n(F))$$

such that

$$\sigma \circ \lambda_{\alpha\beta} \in \Gamma(S_{\alpha\beta}, SL_n(\sigma(F)) \cap U(p, q)).$$

One has the continuous maps

$$\varphi : S_\bullet \xrightarrow{\lambda} BSL_n(F) \xrightarrow{\sigma} BSL_n(\sigma(F)) \xrightarrow{\tau} BSL_n(\mathbb{C})_\delta \xrightarrow{\iota} BSL_n(\mathbb{C}),$$

and

$$\psi : S_\bullet \xrightarrow{\sigma\circ\lambda} BSU(p, q) \xrightarrow{\mu} BSL_n(\mathbb{C}),$$

where $S_\bullet$ is the simplicial classifying manifold associated to the open cover S_α, $BSL_n(F)$ and $BSL_n(\sigma(F))$ are the simplicial classifying sets, BG is the simplicial ($\mathcal{C}^\infty$) classifying manifold for

$$G = SL_n(\mathbb{C}), SU(p, q),$$

$BSL_n(\mathbb{C})_\delta$ is the discrete simplicial classifying set. So $\varphi = \psi$.

By [4] §8 , there is a class $\gamma_i^{\text{univ}} \in H^{2i-1}(BSL_n(\mathbb{C})_\delta, \mathbb{R})$, whose image

$$\overline{\gamma}_i^{\text{univ}} \in H^{2i-1}(BSL_n(\mathbb{C})_\delta, \mathbb{R}/\mathbb{Q}) = H^{2i-1}(BSL_n(\mathbb{C})_\delta, \mathbb{R})/H^{2i-1}(BSL_n(\mathbb{C})_\delta, \mathbb{Q})$$

verifies

$$\gamma_i = \lambda^*\sigma^*\tau^*\overline{\gamma}_i^{\text{univ}}.$$

We now apply Reznikov's idea to use Borel's theorem. By [2], (7.5), (11.3) and [3] (6.4) iii, (6.5), for n sufficiently large compared to i, $H^{2i-1}(BSL_n(F)),\mathbb{R})$ is generated by

$$(\bigotimes_{\sigma}\sigma^*\tau^*\iota^*H^\bullet(BSL_n(\mathbb{C}),\mathbb{R}))^{(2i-1)}$$

where $(2i-1)$ denotes the part of the tensor product of degree $(2i-1)$. Thus $\sigma^*\tau^*\overline{\gamma_i}^{\rm univ}\in H^{2i-1}(BSL_n(F),\mathbb{R})$ is a sum of elements of the shape $\otimes_\sigma\sigma^*\tau^*\iota^*x_\sigma$, where at least one $x_\sigma\in H^{2j-1}_{cont}(SL_n(\mathbb{C}),\mathbb{R})$, for some $j\le i$. This implies that

$$\gamma_i=\sum\otimes_\sigma(\sigma\circ\lambda)^*\mu^*x_\sigma,$$

and for each summand, there is at least one

$$\mu^*x_\sigma\in H^{2j-1}_{cont}(SU(p,q),\mathbb{R}).$$

It remains to observe that

$$\text{for}\quad p+q=n\quad\text{large}\quad H^{2i-1}_{\rm cont}(SU(p,q),\mathbb{R})=0.$$

In fact, if $p=q$, this is part of [2] 10.6. In general, the continuous cohomology of the $\mathbb{R}$ valued points of the $\mathbb{R}$ algebraic group $SU(p,q)$ is computed by

$$H^\bullet_{\rm cont}(SU(p,q),\mathbb{R})=H^\bullet(Hom_K(\Lambda^\bullet\mathfrak{G}/\mathfrak{K}),\mathbb{R})$$

where K is the maximal compact subgroup $SU(p,q)\cap(U(p)\times U(q))$, $\mathfrak{K}$ is its Lie algebra, $\mathfrak{G}$ is the Lie algebra of $SU(p,q)$. The right hand side equals

$$H^\bullet(Hom_K(\Lambda^\bullet\mathfrak{G}_c/\mathfrak{K}),\mathbb{R}),$$

where $\mathfrak{G}_c$ is the Lie algebra of the compact form $SU(p+q)$ of $SU(p,q)$. This group is the de Rham cohomology of the manifold $SU(p+q)/SU(p+q)\cap(U(p)\times U(q))$, a Grassmann manifold without odd cohomology.

Remark 0.5. To a representation ρ, one may also associate the classes

$$c_i(\rho)\in H^{2i-1}(S,\mathbb{C}/\mathbb{Z}(i))$$

defined by $\lambda^*c_i^{\rm univ}$, where $\lambda: S_\bullet\to BGL_n(\mathbb{C})_\delta$ is defined by locally constant transition functions of the local system, and

$$c_i^{\rm univ}\in H^{2i-1}(BGL_n(\mathbb{C})_\delta,\mathbb{C}/\mathbb{Z}(i))=H^{2i}_{\mathcal{D}}(BGL_n(\mathbb{C})_\delta,\mathbb{Z}(i)),$$

where $H_{\mathcal{D}}$ is the Deligne-Beilinson cohomology, where $c_i^{\rm univ}$ are the restriction to $BGL_n(\mathbb{C})_\delta$ of the Chern classes in the Deligne-Beilinson cohomology of the universal bundle on the simplicial algebraic manifold BGL_n. One does not know in all generality that $\lambda^*c_i^{\rm univ}=\hat{c}_i(V)$.

Again writing $c_i^{\rm univ}$ as $b_i^{\rm univ}+z_i^{\rm univ}$, with

$$b_i^{\rm univ}\in H^{2i-1}(BGL_n(\mathbb{C})_\delta,\mathbb{R}(i-1)),$$
$$z_i^{\rm univ}\in H^{2i-1}(BGL_n(\mathbb{C})_\delta,\mathbb{R}(i)/\mathbb{Z}(i)),$$

one knows that by definition $b_i^{\rm univ}$ lies in the image of the continuous cohomology of $GL_n(\mathbb{C})$:

$$H^{2i-1}_{\mathcal{D}}(BGL_n(\mathbb{C})_\bullet,\mathbb{Z}(i))\longrightarrow H^{2i}_{\mathcal{D}}(BGL_n(\mathbb{C})_\bullet,\mathbb{R}(i))\longrightarrow$$
$$H^{2i-1}(BGL_n(\mathbb{C})_\bullet,\mathcal{S}^\infty_{\mathbb{R}(i-1)})\cong H^{2i-1}_{cont}(GL_n(\mathbb{C}),\mathbb{R}(i-1))\longrightarrow H^{2i-1}(BGL_n(\mathbb{C})_\delta,\mathbb{R}(i-1)),$$

where $\mathcal{S}^\infty_{\mathbb{R}(i-1)}$ is the sheaf of $\mathbb{R}(i-1)$ valued $\mathcal{C}^\infty$ functions. (In fact Beilinson gave a precise identification of this class in terms of the Borel regulator. See [11] for details). Thus by the previous argument, $\lambda^*b_i^{\rm univ}=0$.

As before, we may assume that ρ has $SU(p,q)$ values, since the multiplication

$$c_i(\rho)\cdot c_j(\rho)$$

factorizes through the Betti class in $H^{2i}(S,\mathbb{Z}(i))$ of ρ ([6] (3.4) proof). Furthermore, by definition, z_i^{univ} is a discrete cohomology class. Thus one can apply the same argument as in Theorem 0.2 to prove

Theorem 0.6. *Let S be a topological manifold and let $\rho:\pi_1(S,s)\to GL(n,F)$ be a representation of the fundamental group with values in a number field F. Assume that for all real and complex embeddings $\sigma:F\to\mathbb{R}(\subset\mathbb{C})$ and $\sigma:F\to\mathbb{C}$, $\sigma\circ\rho:\pi_1(S,s)\to GL(n,\mathbb{C})$ is a local system of $\sigma(F)$ hermitian vector spaces. Then $c_i(\rho)=0$ in $H^{2i-1}(S,\mathbb{C}/\mathbb{Q})$ for all $i\geq 1$.*

On the other hand, if S is an algebraic manifold, then the image of $c_i(\rho)$ under the map

$$H^{2i-1}(S,\mathbb{C}/\mathbb{Z}(i))\longrightarrow H^{2i}_{\mathcal{D}}(S,\mathbb{Z}(i))$$

is the Chern class $c_i^{\mathcal{D}}(E)$ of the underlying algebraic vector bundle E on $V\otimes_{\mathbb{C}}\mathcal{O}_{San}$ [6], (3.5). So one has

Corollary 0.7. *Let S be an algebraic manifold and let $\rho:\pi_1(S,s)\to GL(n,F)$ be a representation of the fundamental group with values in a number field F. Assume that for all real and complex embeddings $\sigma:F\to\mathbb{R}(\subset\mathbb{C})$ and $\sigma:F\to\mathbb{C}$, $\sigma\circ\rho:\pi_1(S,s)\to GL(n,\mathbb{C})$ is a local system of $\sigma(F)$ hermitian vector spaces. Then the Chern classes of the underlying algebraic bundle in the Deligne cohomology are torsion.*

Remark 0.8. Let $f:X\to S$ be a proper equidimensional morphism of algebraic smooth complex proper varieties X and S, such that f is smooth outside a normal crossing divisor Σ, with $D:=f^{-1}(\Sigma)$ a normal crossing divisor without multiplicities (that is f is "semi-stable" in codimension 1). Then the Gauß-Manin bundles

$$\mathcal{H}^j=R^jf_*\Omega^\bullet_{X/S}(\log D)$$

have an integrable holomorphic (in fact algebraic) connection with logarithmic poles along Σ whose residues are nilpotent (monodromy theorem, see eg [8], (3.1)). This implies [7], appendix B, that the de Rham classes of $\mathcal{H}^j$ are zero. Therefore

$$c_i^{\mathcal{D}}(\mathcal{H}^j)\in H^{2i-1}(S,\mathbb{C}/\mathbb{Z}(i))/F^i\subset H^{2i-1}_{\mathcal{D}}(S,\mathbb{Z}(i))$$

that is, modulo torsion, $c_i^{\mathcal{D}}(\mathcal{H}^j)$ lies in the intermediate Jacobian, and $c_i^{\mathcal{D}}(\mathcal{H}^j|_{S-\Sigma})$ is torsion(Corollary 0.7, Theorem 0.3). It would be interesting to understand those classes, in particular as one knows that there are only finitely many such classes for $\mathcal{H}^j$ of a given rank, as there are, according to Deligne [5], finitely many $\mathbb{Z}$ variations of Hodge structures of a given rank on $S-\Sigma$, and $\mathcal{H}^j$ is the canonical extension of $R^jf|_{S-\Sigma^*}\mathbb{C}$.

In fact, if f has relative (complex) dimension 1, even the Chern classes of $\mathcal{H}^j$ in the Chow groups of S are torsion, as a consequence of Grothendieck-Riemann-Roch theorem [10] (5.2).

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