APPENDIX

CLASSES OF LOCAL SYSTEMS OF HERMITIAN VECTOR SPACES

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For a local system $V$ on a topological manifold $S$ associated to a representation
$$\rho : \pi_1(S, s) \to GL(n, \mathbb{C})$$
of the fundamental group we denote by
$$\hat{c}_i(V) = \hat{c}_i(\rho) = \beta_i + \gamma_i \in H^{2i-1}(S, \mathbb{C}/\mathbb{Z})$$
the class defined in [4], [1] :
$$\beta_i \in H^{2i-1}(S, \mathbb{R}) \quad ([1], (2.20))$$
$$\gamma_i \in H^{2i-1}(S, \mathbb{R}/\mathbb{Z}) \quad ([4], \S 4).$$
If $f : X \to S$ is a smooth proper morphism of $C^\infty$ manifolds with orientable fibers, the Riemann-Roch theorem ([1], Theorem (0.2), Theorem (3.11)) says
$$\hat{c}_i(\sum_{j=0}^{\dim(X/S)} (-1)^j R^j f_* \mathbb{C}) = 0$$
in $H^{2i-1}(S, \mathbb{C}/\mathbb{Q})$, for all $i \geq 1$.

The purpose of this short note is to show how to apply Reznikov’s ideas [12] to obtain vanishing of the single classes $\hat{c}_i(R^j f_* \mathbb{C})$ under some assumptions.

**Definition 0.1.** Let $A$ be a ring with $\mathbb{Z} \subset A \subset \mathbb{C}$. A local system of $A$ hermitian vector spaces is a local system associated to a representation $\rho$ whose image $\rho(\pi_1(S, s))$ lies in $GL_n(A) \subset GL_n(\mathbb{C})$ and $U(p, q) \subset GL_n(\mathbb{C})$, for some pair $(p, q)$ with $n = p + q$, where $U(p, q)$ is the unitary group with respect to a non degenerate hermitian form with $p$ positive, and $q$ negative eigenvalues.

**Theorem 0.2.** Let $S$ be a topological manifold and let $\rho : \pi_1(S, s) \to GL(n, F)$ be a representation of the fundamental group with values in a number field $F$. Assume that for all real and complex embeddings $\sigma : F \to \mathbb{R}(\subset \mathbb{C})$ and $\sigma : F \to \mathbb{C}$, $\sigma \circ \rho : \pi_1(S, s) \to GL(n, \mathbb{C})$ is a local system of $\sigma(F)$ hermitian vector spaces. Then $\hat{c}_i(\rho) = 0$ in $H^{2i-1}(S, \mathbb{C}/\mathbb{Q})$ for all $i \geq 1$.

Examples of local systems of $\mathbb{Q}$ hermitian vector spaces are provided by $\mathbb{Q}$ variations of Hodge structures [9] I.2, whose main instances are the Gauß-Manin local systems $R^j f_* \mathbb{C}$, where $f : X \to S$ is a smooth proper morphism of complex manifolds with Kähler fibres. So Theorem 0.2 implies

**Theorem 0.3.** Let $f : X \to S$ be a smooth proper morphism of complex manifolds with Kähler fibres. Then
$$\hat{c}_i(R^j f_* \mathbb{C}) = 0$$
in $H^{2i-1}(S, \mathbb{C}/\mathbb{Q})$ for all $i \geq 1$, $j \geq 0$. 

In the $C^\infty$ category, other examples are provided by Poincaré duality:

**Theorem 0.4.** Let $f : X \to S$ be a smooth proper morphism of $C^\infty$ manifolds with orientable fibres. Then

\[
\hat{c}_i(R^j f_* \mathbb{C} \oplus R^{(\dim(X)/2-j)} f_* \mathbb{C}) = 0
\]

in $H^{2i-1}(S, \mathbb{C}/\mathbb{Q})$ and

\[
\hat{c}_i(R^{\dim(X)/2} f_* \mathbb{C}) = 0
\]

in $H^{2i-1}(S, \mathbb{C}/\mathbb{Q})$ if $\dim(X/S)$ is even.

**Proof of Theorem 0.2.**

The $U(p,q)$ flat bundle being isomorphic to the conjugate of its dual, the formula (\cite{1} (2.21)) says that $\gamma_i = 0$. Thus we just have to consider $\gamma_i$.

We may first assume that $\Lambda^n \rho : \pi_1(S, s) \to \mathbb{C}^*$ is trivial. In fact, it is torsion as a unitary and rational representation, say of order $N$, and $V \oplus \ldots \oplus V$ ($N$ times) has trivial determinant. On the other hand

\[
\hat{c}_i(V \oplus \ldots \oplus V) = N\hat{c}_i(V)
\]

in $H^{2i-1}(S, \mathbb{C}/\mathbb{Q})$, as

\[
\hat{c}_i(V) \hat{c}_j(V) = 0
\]

for $i \geq 1, j \geq 1$, in $H^{2i+j-1}(S, \mathbb{C}/\mathbb{Q})$. (The multiplication is defined by Image ($\hat{c}_i(V)$ in $H^{2i}(S, \mathbb{Z})$). $\hat{c}_j(V)$ \cite{4} (1.11)).

Furthermore, by adding trivial factors to $V$, one may assume that $n$ is as large as one wants.

There is an open cover $S = \cup_a S_a$ trivializing $V$ with transition functions

\[
\lambda_{a\beta} \in \Gamma(S_{a\beta}, SL_n(F))
\]

such that

\[
\sigma \circ \lambda_{a\beta} \in \Gamma(S_{a\beta}, SL_n(\sigma(F)) \cap U(p,q)).
\]

One has the continuous maps

\[
\varphi : S^\bullet \xrightarrow{\lambda} BSL_n(F) \xrightarrow{\sigma} BSL_n(\sigma(F)) \xrightarrow{\tau} BSL_n(\mathbb{C}) \xrightarrow{\lambda} BSL_n(\mathbb{C}),
\]

and

\[
\psi : S^\bullet \xrightarrow{\sigma \circ \lambda} BSU(p,q) \xrightarrow{\mu} BSL_n(\mathbb{C}),
\]

where $S^\bullet$ is the simplicial classifying manifold associated to the open cover $S_a$, $BSL_n(F)$ and $BSL_n(\sigma(F))$ are the simplicial classifying sets, $BG$ is the simplicial classifying manifold for

\[
G = SL_n(\mathbb{C}), SU(p,q),
\]

$BSL_n(\mathbb{C})$ is the discrete simplicial classifying set. So $\varphi = \psi$.

By \cite{4} §8, there is a class $\gamma_i^{\text{univ}} \in H^{2i-1}(BSL_n(\mathbb{C})_\delta, \mathbb{R})$, whose image

\[
\overline{\gamma}_i^{\text{univ}} \in H^{2i-1}(BSL_n(\mathbb{C})_\delta, \mathbb{R}/\mathbb{Q}) = H^{2i-1}(BSL_n(\mathbb{C})_\delta, \mathbb{R})/H^{2i-1}(BSL_n(\mathbb{C})_\delta, \mathbb{Q})
\]

verifies

\[
\gamma_i = \lambda^* \sigma^* \tau^* \overline{\gamma}_i^{\text{univ}}.
\]
We now apply Reznikov’s idea to use Borel’s theorem. By [2], (7.5), (11.3) and [3] (6.4) iii, (6.5), for \( n \) sufficiently large compared to \( i \), \( H^{2i-1}(BSL_n(F)), \mathbb{R} \) is generated by 

\[
\bigotimes_{\sigma} \sigma^* \tau^* t^* H^\bullet(BSL_n(\mathbb{C}), \mathbb{R})^{(2i-1)}
\]

where \( (2i - 1) \) denotes the part of the tensor product of degree \( (2i - 1) \). Thus \( \sigma^* \tau^* \gamma_i^{univ} \in H^{2i-1}(BSL_n(F)), \mathbb{R} \) is a sum of elements of the shape \( \otimes_{\sigma} \sigma^* \tau^* t^* x_{\sigma} \), where at least one \( x_{\sigma} \in H^{2j-1}_\text{cont}(SL_n(\mathbb{C}), \mathbb{R}) \), for some \( j \leq i \). This implies that 

\[
\gamma_i = \sum \otimes_{\sigma} (\sigma \circ \lambda)^* \mu^* x_{\sigma},
\]

and for each summand, there is at least one 

\[
\mu^* x_{\sigma} \in H^{2j-1}_\text{cont}(SU(p, q), \mathbb{R}).
\]

It remains to observe that 

\[
\text{for } p + q = n \text{ large } H^{2i-1}_\text{cont}(SU(p, q), \mathbb{R}) = 0.
\]

In fact, if \( p = q \), this is part of [2] 10.6. In general, the continuous cohomology of the \( \mathbb{R} \) valued points of the \( \mathbb{R} \) algebraic group \( SU(p, q) \) is computed by 

\[
H^\bullet_\text{cont}(SU(p, q), \mathbb{R}) = H^\bullet(\text{Hom}_K(\Lambda^\bullet \mathfrak{G}/\mathfrak{R}), \mathbb{R})
\]

where \( K \) is the maximal compact subgroup \( SU(p, q) \cap (U(p) \times U(q)) \), \( \mathfrak{R} \) is its Lie algebra, \( \mathfrak{G} \) is the Lie algebra of \( SU(p, q) \). The right hand side equals 

\[
H^\bullet(\text{Hom}_K(\Lambda^\bullet \mathfrak{G}_c/\mathfrak{R}), \mathbb{R}),
\]

where \( \mathfrak{G}_c \) is the Lie algebra of the compact form \( SU(p + q) \) of \( SU(p, q) \). This group is the de Rham cohomology of the manifold \( SU(p + q)/SU(p + q) \cap (U(p) \times U(q)) \), a Grassmann manifold without odd cohomology.

**Remark 0.5.** To a representation \( \rho \), one may also associate the classes 

\[
c_i(\rho) \in H^{2i-1}(S_\bullet/\mathbb{Z}(i))
\]

defined by \( \lambda^* c_i^{univ} \), where \( \lambda : S_\bullet \to BGL_n(\mathbb{C})_\delta \) is defined by locally constant transition functions of the local system, and 

\[
c_i^{univ} \in H^{2i-1}(BGL_n(\mathbb{C})_\delta, \mathbb{C}/\mathbb{Z}(i)) = H^D_B(BGL_n(\mathbb{C})_\delta, \mathbb{Z}(i)),
\]

where \( H_D \) is the Deligne-Beilinson cohomology, where \( c_i^{univ} \) are the restriction to \( BGL_n(\mathbb{C})_\delta \) of the Chern classes in the Deligne-Beilinson cohomology of the universal bundle on the simplicial algebraic manifold \( BGL_n \). One does not know in all generality that \( \lambda^* c_i^{univ} = c_i(V) \).

Again writing \( c_i^{univ} \) as \( b_i^{univ} + z_i^{univ} \), with 

\[
b_i^{univ} \in H^{2i-1}(BGL_n(\mathbb{C})_\delta, \mathbb{R}(i-1)),
\]

\[
z_i^{univ} \in H^{2i-1}(BGL_n(\mathbb{C})_\delta, \mathbb{R}(i)/\mathbb{Z}(i)),
\]

one knows that by definition \( b_i^{univ} \) lies in the image of the continuous cohomology of \( GL_n(\mathbb{C}) \): 

\[
H^{2i-1}_D(BGL_n(\mathbb{C})_\bullet, \mathbb{Z}(i)) \rightarrow H^{2i}_D(BGL_n(\mathbb{C})_\bullet, \mathbb{R}(i)) \rightarrow H^{2i-1}(BGL_n(\mathbb{C})_\bullet, S_{\mathbb{R}(i-1)}^\infty) \cong H^{2i-1}_\text{cont}(GL_n(\mathbb{C}), \mathbb{R}(i-1)) \rightarrow H^{2i-1}(BGL_n(\mathbb{C})_\delta, \mathbb{R}(i-1)),
\]

where \( S_{\mathbb{R}(i-1)}^\infty \) is the sheaf of \( \mathbb{R}(i-1) \) valued \( \mathcal{C}^\infty \) functions. (In fact Beilinson gave a precise identification of this class in terms of the Borel regulator. See [11] for details). Thus by the previous argument, \( \lambda^* b_i^{univ} = 0 \).
As before, we may assume that \( \rho \) has \( SU(p, q) \) values, since the multiplication
\[
c_i(\rho) \cdot c_j(\rho)
\]
factorizes through the Betti class in \( H^{2i}(S, \mathbb{Z}(i)) \) of \( \rho \) ([6] (3.4) proof). Furthermore, by definition, \( z^\text{univ}_i \) is a discrete cohomology class. Thus one can apply the same argument as in Theorem 0.2 to prove

**Theorem 0.6.** Let \( S \) be a topological manifold and let \( \rho: \pi_1(S, s) \to GL(n, F) \) be a representation of the fundamental group with values in a number field \( F \). Assume that for all real and complex embeddings \( \sigma: F \to \mathbb{R}(\subset \mathbb{C}) \) and \( \sigma: F \to \mathbb{C}, \sigma \circ \rho: \pi_1(S, s) \to GL(n, \mathbb{C}) \) is a local system of \( \sigma(F) \) hermitian vector spaces. Then \( c_i(\rho) = 0 \) in \( H^{2i-1}(S, \mathbb{C}/\mathbb{Q}) \) for all \( i \geq 1 \).

On the other hand, if \( S \) is an algebraic manifold, then the image of \( c_i(\rho) \) under the map
\[
H^{2i-1}(S, \mathbb{C}/\mathbb{Z}(i)) \to H^D_{2i}(S, \mathbb{Z}(i))
\]
is the Chern class \( c_i^D(E) \) of the underlying algebraic vector bundle \( E \) on \( V \otimes_{\mathbb{C}} \mathcal{O}_S \) [6], (3.5). So one has

**Corollary 0.7.** Let \( S \) be an algebraic manifold and let \( \rho: \pi_1(S, s) \to GL(n, F) \) be a representation of the fundamental group with values in a number field \( F \). Assume that for all real and complex embeddings \( \sigma: F \to \mathbb{R}(\subset \mathbb{C}) \) and \( \sigma: F \to \mathbb{C}, \sigma \circ \rho: \pi_1(S, s) \to GL(n, \mathbb{C}) \) is a local system of \( \sigma(F) \) hermitian vector spaces. Then the Chern classes of the underlying algebraic bundle in the Deligne cohomology are torsion.

**Remark 0.8.** Let \( f: X \to S \) be a proper equidimensional morphism of algebraic smooth complex proper varieties \( X \) and \( S \), such that \( f \) is smooth outside a normal crossing divisor \( \Sigma \), with \( D : = f^{-1}(\Sigma) \) a normal crossing divisor without multiplicities (that is \( f \) is “semi-stable” in codimension 1). Then the Gauß-Manin bundles
\[
\mathcal{H}^j = R^j f_* \Omega^\bullet_{X/S}(\log D)
\]
have an integrable holomorphic (in fact algebraic) connection with logarithmic poles along \( \Sigma \) whose residues are nilpotent (monodromy theorem, see eg [8], (3.1)). This implies [7], appendix B, that the de Rham classes of \( \mathcal{H}^j \) are zero. Therefore
\[
c_i^j(\mathcal{H}^j) \in H^{2i-1}(S, \mathbb{C}/\mathbb{Z}(i))/F^i \subset H^D_{2i-1}(S, \mathbb{Z}(i))
\]
that is, modulo torsion, \( c_i^j(\mathcal{H}^j) \) lies in the intermediate Jacobian, and \( c_i^j(\mathcal{H}^j|_{S-\Sigma}) \) is torsion(Corollary 0.7, Theorem 0.3). It would be interesting to understand those classes, in particular as one knows that there are only finitely many such classes for \( \mathcal{H}^j \) of a given rank, as there are, according to Deligne [5], finitely many \( \mathbb{Z} \) variations of Hodge structures of a given rank on \( S - \Sigma \), and \( \mathcal{H}^j \) is the canonical extension of \( R^j f|_{S-\Sigma}.\mathbb{C} \).

In fact, if \( f \) has relative (complex) dimension 1, even the Chern classes of \( \mathcal{H}^j \) in the Chow groups of \( S \) are torsion, as a consequence of Grothendieck-Riemann-Roch theorem [10] (5.2).

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REFERENCES


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