Decomposability of Chow groups implies decomposability of cohomology.

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Let X be an n-dimensional complete irreducible smooth variety defined over the field $\mathbb{C}$ of complex numbers. For any Zariski open subset V of X, we have the following graded rings.

(i) $\bigoplus_{i=0}^{n} CH^i(V)_{\mathbb{Q}}$, where $CH^i(V)_{\mathbb{Q}}$ is the Chow group of algebraic cycles of codimension i on V with rational coefficients, modulo rational equivalence (see [F], Chapter 8, Prop. 8.3).

(ii) $\bigoplus_{i=0}^{n} H^i(V)/N^1H^i(V)$, where $H^i(V) = H^i(V_{an}, \mathbb{Q})$ is the singular cohomology of the underlying complex manifold V_{an}, and

$$N^aH^i(V) = \varinjlim_{\operatorname{codim} Z \geq a} \ker\Big(H^i(V) \longrightarrow H^i(V-Z)\Big)$$

defines Grothendieck's coniveau filtration (here Z runs over the Zariski closed subsets of V of codimension $\geq a$).

(iii) $\bigoplus_{i=0}^{n} H^0(V, \mathcal{H}^i_V)$, where $\mathcal{H}^i_V$ is the sheaf for the Zariski topology associated to the presheaf

$$U \longmapsto H^i(U) = H^i(U_{an}, \mathbb{Q}).$$

(iv) We also have a graded ring associated to X: $\bigoplus_{i=0}^{n} H^i(\mathbb{C}(X))$, where

$$\begin{aligned} H^i(\mathbb{C}(X)) :=& \varinjlim_{V\subset X} H^i(V) = \varinjlim_{V\subset X} H^i(V)/N^1H^i(V) \\ =& \varinjlim_{V\subset X} H^0(V, \mathcal{H}^i_V) \end{aligned}$$

Here the direct limits are over the non-empty Zariski open sets V in X, and $\mathbb{C}(X)$ denotes the function field of X. The first equality defines the cohomology of the function field; the right side of the equality is clearly a birational invariant of X.

In (ii), (iii), (iv) above, we consider only cohomology in degrees upto n, since the singular cohomology of an affine variety of dimension n vanishes in degrees larger than n, by the weak Lefschetz theorem (this implies that for any variety V of dimension n, we have $H^i(V) = N^1H^i(V)$ for $i > n$).

Theorem 1 *Let X be a smooth complete variety of dimension n over $\mathbb{C}$. Suppose there exists a non empty Zariski open subset $V \subset X$, and positive integers $n_1, \ldots, n_r$ with $\sum_i n_i = n$, such that one of the following product maps is surjective:*

(i) $CH^{n_1}(V)_{\mathbb{Q}} \otimes \cdots \otimes CH^{n_r}(V)_{\mathbb{Q}} \longrightarrow CH^n(V)_{\mathbb{Q}}$

(ii) $H^{n_1}(V)/N^1H^{n_1}(V) \otimes \cdots \otimes H^{n_r}(V)/N^1H^{n_r}(V) \longrightarrow H^n(V)/N^1H^n(V)$

(iii) $H^0(V, \mathcal{H}_V^{n_1}) \otimes \cdots \otimes H^0(V, \mathcal{H}_V^{n_r}) \longrightarrow H^0(V, \mathcal{H}_V^n)$

(iv) $H^{n_1}(\mathbb{C}(X)) \otimes \cdots \otimes H^{n_r}(\mathbb{C}(X)) \longrightarrow H^n(\mathbb{C}(X))$

Then the cup product map for the coherent cohomology

$$H^{n_1}(X, \mathcal{O}_X) \otimes H^{n_2}(X, \mathcal{O}_X) \otimes \cdots \otimes H^{n_r}(X, \mathcal{O}_X) \longrightarrow H^n(X, \mathcal{O}_X) \qquad (*)$$

is surjective.

The proof of (i) is motivated by Bloch's proof [B] of Mumford's theorem that for surfaces X with $H^2(X, \mathcal{O}_X) \neq 0$, the Chow group of 0-cycles $CH^2(X)$ is not 'finite dimensional' (see also the 'metaconjecture' in Chapter 1 of [B2]). Many other variants of Bloch's method have been considered by several authors. The method involves the action of correspondences on the cohomology. At the referee's suggestion, we try to make this argument with some care, though this type of reasoning is well known to experts.

The proofs of (ii), (iii) and (iv) are a consequence of the mixed Hodge structure on the cohomology of the open sets V (see [D]). For $V = X$, the surjectivity of the map (ii) trivially implies that $(*)$ is surjective, using the Hodge decomposition on cohomology, since the ring $\oplus H^i(X, \mathcal{O}_X)$ is a graded quotient of $\oplus(H^i(X)/N^1H^i(X)) \otimes \mathbb{C}$.

The proof of the theorem

We first discuss (i). Let $C = X - V$, and let $k \subset \mathbb{C}$ be a countable algebraically closed field of definition of X, C and V. Let X_0, C_0, V_0 be the corresponding models over k, and for any extension L of k, let $X_L = X_0 \times_k L$, etc. We embed $k(X_0) \hookrightarrow \mathbb{C}$ as a k-subalgebra, and consider the generic point of X_0 as a closed point $\eta \in X_{k(X_0)}$, hence as an element of $CH^n(X_{k(X_0)})_{\mathbb{Q}}$. By assumption, its image under the composite

$$CH^n(X_{k(X_0)})_{\mathbb{Q}} \longrightarrow CH^n(X)_{\mathbb{Q}} \longrightarrow CH^n(V)_{\mathbb{Q}}$$

decomposes as

$$\sum_{\text{finite}} m_{n_1} \cdot \dots \cdot m_{n_r}$$

where $m_{n_i} \in CH^{n_i}(V)_{\mathbb{Q}}$. The m_{n_i} are defined over a subfield $L \subset \mathbb{C}$ which is finitely generated over $k(X_0)$, and (see [B2], Lecture 1, Appendix, Lemma 3) the natural map

$$CH^n(V_L)_{\mathbb{Q}} \longrightarrow CH^n(V)_{\mathbb{Q}}$$

is injective, so

$$\sum_{\text{finite}} m_{n_1} \cdot \dots \cdot m_{n_r} = [\eta] \tag{1}$$

holds in $CH^n(V_L)_{\mathbb{Q}}$.

Let F be the algebraic closure of $k(X_0)$ in L; since L is finitely generated over $k(X_0)$, F is a finite algebraic extension of $k(X_0)$. We can find a non-singular affine F-variety W with function field L. The graded ring

$$\bigoplus_{i \geq 0} CH^i(V_L)$$

is the direct limit of the graded rings

$$\bigoplus_{i \geq 0} CH^i(V_F \times_F W'),$$

where W' runs over the non-empty Zariski open sets in W (see [B2], Lecture 1, Appendix, Lemma 1). So after replacing W by a nonempty open subset, we may assume given classes $m_{n_i} \in CH^{n_i}(V_F \times_F W)$ such that (1) holds in

$$CH^n(V_F \times_F W)_{\mathbb{Q}},$$

where $[\eta]$ now denotes the image in $CH^n(V_F \times_F W)_{\mathbb{Q}}$ of the earlier class

$$[\eta] \in CH^n(V_{k(X_0)})_{\mathbb{Q}} \subset CH^n(V_F)_{\mathbb{Q}}.$$

Let $P \in W$ be a closed point. Then there is a homomorphism of rings

$$f^* : \bigoplus_{i \geq 0} CH^i(V_F \times_F W) \to \bigoplus_{i \geq 0} CH^i(V_F \times_F \operatorname{Spec} F(P)),$$

where $f : V_F \times_F \operatorname{Spec} F(P) \to V_F \times_F W$ is induced by the inclusion of P into W (f is a morphism of non-singular F-varieties, hence by [F], Prop. 8.3, such a homomorphism f^* exists). Then $f^*[\eta]$ is just $[\eta]$ considered as an element of $CH^n(V_{k(X_0)})_{\mathbb{Q}} \subset CH^n(V_{F(P)})_{\mathbb{Q}}$. Hence

$$\sum_{\text{finite}} f^*(m_{n_1}) \cdot \cdots \cdot f^*(m_{n_r}) = [\eta] \tag{2}$$

holds in $CH^n(V_{F(P)})_{\mathbb{Q}}$, where $f^*(m_{n_i}) \in CH^{n_i}(V_{F(P)})_{\mathbb{Q}}$.

Hence, we are reduced to the situation when (1) holds, where L is a finite algebraic extension of $k(X_0)$, and $m_{n_i} \in CH^i(V_L)_{\mathbb{Q}}$.

By resolution of singularities, we can find a projective non-singular k-variety Z_0, together with a k-morphism $\sigma_0 : Z_0 \to X_0$, such that the induced map on function fields is the given inclusion $k(X_0) \to L$. Since L is a finite extension of $k(X_0)$, the morphism σ_0 is generically finite.

The (flat) k-morphism $\operatorname{Spec} L \to Z_0$ given by the inclusion of the generic point gives rise to a natural surjective homomorphism of graded rings

$$Cl : \bigoplus_{i \geq 0} CH^i(X_0 \times_k Z_0)_{\mathbb{Q}} \to \bigoplus_{i \geq 0} CH^n(V_L)_{\mathbb{Q}},$$

such that if $[\Delta_{\sigma_0}] \in CH^n(X_0 \times_k Z_0)_{\mathbb{Q}}$ is the class of the transposed graph of σ_0, then $Cl([\Delta_{\sigma_0}])$ is just $[\eta] \in CH^n(V_L)_{\mathbb{Q}}$. The kernel of

$$CH^n(X_0 \times_k Z_0)_{\mathbb{Q}} \to CH^n(V_L)$$

consists of the subgroup generated by the classes supported on subsets of the form $(C_0 \times_k Z_0) \cup (X_0 \times_k D_0)$, as D_0 runs over all proper subvarieties of Z_0 (see [B2], Lecture 1, Appendix, Lemma 1, and [F], Prop. 1.8). Thus we have an equation

$$[\Delta_{\sigma_0}] - \sum M_{n_1} \cdot \cdots \cdot M_{n_r} = \gamma_0 + \delta_0$$

in $CH^n(X_0 \times_k Z_0)$, where for some divisor $D_0 \subset Z_0$, we have

$$\begin{array}{rll} M_{n_i} \in CH^{n_i}(X_0 \times_k Z_0)_{\mathbb{Q}}, & M_{n_i} \mapsto & m_{n_i} \in CH^{n_i}(V_L)_{\mathbb{Q}} \\ \gamma_0 \in CH^n(X_0 \times_k Z_0)_{\mathbb{Q}}, & \operatorname{supp} \gamma_0 \subset & C_0 \times_k Z_0 \\ \delta_0 \in CH^n(X_0 \times_k Z_0)_{\mathbb{Q}}, & \operatorname{supp} \delta_0 \subset & X_0 \times_k D_0 \end{array}$$

Thus if $Z = Z_0 \times_k \mathbb{C}$, $\sigma : Z \longrightarrow X$ the induced map, $M'_{n_i} = (M_{n_i})_{\mathbb{C}}$, $\gamma = (\gamma_0)_{\mathbb{C}}$, $\delta = (\delta_0)_{\mathbb{C}}$, $C = (C_0)_{\mathbb{C}}$, $D = (D_0)_{\mathbb{C}}$, then

$$[\Delta_\sigma] - \sum M'_{n_1} \cdot \cdots \cdot M'_{n_r} = \gamma + \delta \tag{3}$$

in $CH^n(X \times Z)_{\mathbb{Q}}$, where γ is supported on $C \times Z$, and δ is supported on $X \times D$ (in the rest of this proof, $\times$ denotes $\times_{\mathbb{C}}$).

Elements of $CH^n(X \times Z)_{\mathbb{Q}}$ act on $H^n(X)$ as follows. First, there is a cycle class homomorphism of graded rings

$$\bigoplus_{i \geq 0} CH^i(X \times Z) \to \bigoplus_{i \geq 0} H^{2i}(X \times Z)$$

(see [F], Chapter 19, Cor. 19.2(b)). By [F], Prop. 16.1.2 and Example 19.2.7, an element $\alpha \in CH^n(X \times Z)_{\mathbb{Q}}$ yields mappings

$$\alpha_* : CH^i(X)_{\mathbb{Q}} \to CH^i(Z)_{\mathbb{Q}}, \qquad \alpha^* : CH^i(Z)_{\mathbb{Q}} \to CH^i(X)_{\mathbb{Q}}$$

on Chow groups, and

$$\alpha_* : H^i(X) \to H^i(Z), \qquad \alpha^* : H^i(Z) \to H^i(X)$$

on cohomology, where if $p : X \times Z \to X$, and $q : X \times Z \to Z$ are the projections, then $\alpha_*(x) = q_*(p^*(x) \cup \alpha)$, and $\alpha^*(y) = p_*(q^*(y) \cup \alpha)$. Since X, Z are proper and smooth over $\mathbb{C}$, the required operations exists on cohomology as well as Chow groups. Further, if α is the class of the transposed graph of a morphism $f : Z \to X$, then $\alpha_* = f^*$, and $\alpha^* = f_*$, where f^* is the natural map on cohomology, and f_* is the Gysin map (see [F], Prop. 16.1.2 and Example 19.2.7).

On the level of cohomology, the Gysin (push forward) map

$$q_* : H^m(X \times Z) \to H^{m-2n}(Z)$$

is defined via Poincaré duality. As we see below, an equivalent (up to sign) description of q_* is as follows: one may use the Künneth isomorphism

$$H^m(X \times Z) \cong \bigoplus_{i+j=m} H^i(X) \otimes H^j(Z)$$

to project onto the summand $H^{2n}(X) \otimes H^{m-2n}(Z)$, and then use the canonical isomorphism $\deg_X : H^{2n}(X) \xrightarrow{\cong} \mathbb{Q}$ (for any non-singular projective variety T over $\mathbb{C}$ of dimension d, let $\deg_T : H^{2d}(T) \xrightarrow{\cong} \mathbb{Q}$ denote the natural isomorphism). The map p_* is defined similarly.

To see that the two procedures for defining q_* are equivalent upto sign, note that the natural isomorphism (induced by $\deg_{X \times Z}$)

$$H^{2n}(X) \otimes H^{2n}(Z) = H^{4n}(X \times Z) \xrightarrow{\cong} \mathbb{Q}$$

is the tensor product of the natural isomorphisms

$$H^{2n}(X) \xrightarrow{\cong} \mathbb{Q}, \qquad H^{2n}(Z) \xrightarrow{\cong} \mathbb{Q}$$

(this is because a similar assertion is valid for integral cohomology – now we are comparing two isomorphisms $\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$, which are equal because the natural orientation on $X \times Z$ is the product orientation of those on X and Z). Now if $x \in H^m(X \times Z)$, then $q_*(x)$ defined via Poincaré duality is the unique element of $H^{m-2n}(Z)$ such that for any $x' \in H^{4n-m}(Z)$, we have

$$\deg_{X\times Z}(x \cup q^*(x')) = \deg_Z(q_*(x) \cup x').$$

But $x \cup q^*(x')$ depends only on the Künneth component of x in

$$H^{2n}(X) \otimes H^{m-2n}(Z).$$

If this Künneth component of x is $\sum_j p^*x_j \cup q^*y_j$, then

$$x \cup q^*(x') = \pm \sum_j p^*(x_j) \cup q^*(y_j \cup x'),$$

so that

$$\deg_Z(q_*(x) \cup x') = \deg_{X\times Z}(x \cup q^*(x')) = \pm \sum_j \deg_X(x_j) \deg_Z(y_j \cup x').$$

On the other hand, the second procedure for defining $q_*(x)$ yields the element $\sum_j \deg_X(x_j) y_j$, whose cup product with x' is $\sum_j \deg_X(x_j)(y_j \cup x')$, which thus has the same image under $\deg_Z$ as $q_*(x) \cup x'$, upto sign.

The cup product on the cohomology of $X \times Z$ is compatible upto signs with the Künneth decomposition, and the cup products on the cohomology of X and Z respectively. This is because we may view the Künneth component

$$H^i(X) \otimes H^j(Z) \subset H^{i+j}(X \times Z)$$

as image of the mapping given by

$$x \otimes y \mapsto p^*x \cup q^*y.$$

Now our assertion follows because the cup product is functorial, associative, and commutative upto sign.

In particular, the action of $\alpha \in H^{2n}(X \times Z)$ on $H^n(X)$ (via α_*) or on $H^n(Z)$ (via α^*) is determined by the Künneth component of α in $H^n(X) \otimes H^n(Z)$. If $\alpha = \sum_i p^*x_i \otimes q^*y_i$, then

$$\alpha^*(z) = \pm \sum_i \deg_Z(y_i \cup z) x_i,$$

where $y_i \cup z \in H^{2n}(Z)$, and $\deg_Z : H^{2n}(Z) \to \mathbb{Q}$ is the natural isomorphism.

The Künneth decomposition, as well as the action of classes of elements of $CH^n(X \times Z)_{\mathbb{Q}}$ on cohomology, are compatible with the Hodge decompositions on the various cohomology groups. Hence for $\alpha \in CH^n(X \times Z)_{\mathbb{Q}}$, the map

$$\alpha^* : H^n(Z, \mathbb{C})/F^1 H^n(Z, \mathbb{C}) \to H^n(X, \mathbb{C})/F^1 H^n(X, \mathbb{C})$$

depends only on the image of the class of α under the composite

$$CH^n(X \times Z)_{\mathbb{Q}} \to H^{2n}(X \times Z, \mathbb{C}) \to H^n(X, \mathbb{C}) \otimes H^n(Z, \mathbb{C}) \to \\ H^n(X, \mathcal{O}_X) \otimes H^0(Z, \Omega^n_{Z/\mathbb{C}}).$$

Here the last map is a tensor product of projections onto appropriate summands of the Hodge decompositions. This is because if $y \in H^n(X, \mathbb{C})$ is of Hodge type (p, q), $z \in H^n(X, \mathcal{O}_X)$ (*i.e.*,is of type $(0, n)$), and $x \in H^n(X, \mathcal{O}_X)$, then $\deg_Z(y \cup z)x$ is 0, unless y has type $(n, 0)$. Let

$$\sum_j t_j \otimes x_j \in H^n(X) \otimes H^n(X)$$

be the Künneth component of type (n, n) of the diagonal of $X \times X$, whose inverse image

$$\sum_i t_i \otimes \sigma^*(x_i) \in H^n(X) \otimes H^n(Z)$$

is the Künneth component of type (n, n) of Δ_σ. Then

$$\sigma_* = [\Delta_\sigma]^* : H^n(Z, \mathbb{C}) \to H^n(X, \mathbb{C})$$

is given by

$$z \mapsto \sum_j \deg_Z(\sigma^* x_j \cup z) t_j.$$

On the other hand, if $\alpha_{n_i} \in H^{2n_i}(X \times Z)$ is the cohomology class of M'_{n_i}, then

$$[M'_{n_1} \cdot \cdots \cdot M'_{n_r}]^* : H^n(Z) \to H^n(X)$$

is determined by the (n, n)th Künneth component of the cohomology class $\alpha_{n_1} \cup \cdots \cup \alpha_{n_r}$. Further, the map on the Hodge components of type $(0, n)$

$$\xi = [M'_{n_1} \cdot \cdots \cdot M'_{n_r}]^* : H^n(Z, \mathcal{O}_Z) \to H^n(X, \mathcal{O}_X)$$

depends only on the Hodge component of type $(0, n) \otimes (n, 0)$ in

$$H^n(X, \mathbb{C}) \otimes H^n(Z, \mathbb{C})$$

of $\alpha_{n_1} \cup \cdots \cup \alpha_{n_r}$. This is just $\alpha'_{n_1} \cup \cdots \cup \alpha'_{n_r}$, where α'_{n_i} is the Hodge component of α_{n_i} in $H^{n_i}(X, \mathcal{O}_X) \otimes H^0(Z, \Omega^{n_i}_{Z/\mathbb{C}})$. Hence ξ is expressible in the form

$$\xi(z) = \sum_{\text{finite}} \deg_Z(y_{n_1} \cup \cdots \cup y_{n_r} \cup z) z_{n_1} \cup \cdots \cup z_{n_r},$$

for suitable $y_{n_j} \in H^0(Z, \Omega^{n_j}_{Z/\mathbb{C}})$, and $z_{n_j} \in H^{n_j}(X, \mathcal{O}_X)$. In particular,

$$\text{image}\, \xi \subset \text{image}\ (H^{n_1}(X, \mathcal{O}_X) \otimes \cdots \otimes H^{n_r}(X, \mathcal{O}_X) \longrightarrow H^n(X, \mathcal{O}_X))\,.$$

The correspondence γ_* maps $H^n(Z)$ into $N^a H^n(X)$, where a is the codimension of C in X, whereas δ_* maps $H^n(Z)$ into $N^1 H^n(X)$ (see [B], and [J], proof of (10.1)).

Hence on the Hodge components of type $(0, n)$, the map

$$\sigma_* : H^n(Z, \mathbb{C}) \to H^n(X, \mathbb{C})$$

maps $H^n(Z, \mathcal{O}_Z)$ into

$$\text{image}\ (H^{n_1}(X, \mathcal{O}_X) \otimes \cdots \otimes H^{n_r}(X, \mathcal{O}_X) \longrightarrow H^n(X, \mathcal{O}_X))\,.$$

Finally, we note that $\sigma_* \circ \sigma^* : H^n(X, \mathbb{C}) \to H^n(X, \mathbb{C})$ is multiplication by the degree of σ; hence it is an isomorphism. Hence σ_* is surjective, *i.e.*,

$$H^{n_1}(X, \mathcal{O}_X) \otimes \cdots \otimes H^{n_r}(X, \mathcal{O}_X) \longrightarrow H^n(X, \mathcal{O}_X)$$

is surjective.

This proves that if the map (i) is surjective, so is the map $(*)$. Hence to complete the proof of the theorem, it suffices to show that if any of the maps (ii), (iii) or (iv) is surjective, so is $(*)$. From Hodge theory (see [D]), there is a surjection

$$\theta_V : H^i(V) \otimes \mathbb{C} \longrightarrow H^i(X, \mathcal{O}_X)$$

for any non empty Zariski open set $V \subset X$, which is compatible with cup products; this is just the quotient modulo the subspace $F^1(H^i(V) \otimes \mathbb{C})$, where $F^j(H^i(V) \otimes \mathbb{C})$ is the Hodge filtration for Deligne's mixed Hodge structure on $H^i(V)$. Further, for any inclusion of Zariski open sets $W \subset V \subset X$, the triangle

$$\begin{tikzcd} H^i(V) \otimes \mathbb{C} \arrow[rr] \arrow[dr, "\theta_V"'] & & H^i(W) \otimes \mathbb{C} \arrow[dl, "\theta_W"] \\ & H^n(X, \mathcal{O}_X) & \end{tikzcd}$$

commutes, by functoriality of the mixed Hodge structure.

Hence there is a commutative diagram of graded rings

$$\begin{tikzcd}
\bigoplus_{i=0}^{n}(H^i(V)/N^1H^i(V))\otimes\mathbb{C} \arrow[r] \arrow[dr, "\alpha"'] & \bigoplus_{i=0}^{n} H^0(V,\mathcal{H}^i_V)\otimes\mathbb{C} \arrow[r] \arrow[d, "\beta"] & \bigoplus_{i=0}^{n} H^i(\mathbb{C}(X))\otimes\mathbb{C} \arrow[dl, "\gamma"] \\
 & \bigoplus_{i=0}^{n} H^i(X,\mathcal{O}_X) &
\end{tikzcd}$$

where α, β and γ are induced by the θ_W for all open $W \subset V$, and are all surjective (incidentally the horizontal maps are known to be injective by [BO]). The surjections α, β and γ immediately imply that if the maps in (ii), (iii) or (iv) respectively are surjective, then so is $(*)$. $\square$

From the formulation of the proof, it appears that (ii), (iii) and (iv) are directly related to $(*)$ via the maps α, β and γ, while the relation between (i) and $(*)$ is indirect. It is possible to give another proof (which is really more or less a reformulation of the old one) which looks more like the proof in the other three cases, as follows.

We make use of the existence of a cycle class homomorphism

$$\bigoplus_{i=0}^{n} CH^i(X)_{\mathbb{Q}} \longrightarrow \bigoplus_{i=0}^{n} H^i(X,\Omega^i_{X/\mathbb{Z}}),$$

where $\Omega^i_{X/\mathbb{Z}}$ is the sheaf of absolute Kähler i-forms (see [S], for example; the proof below is motivated by the proof in [S] of the infinite dimensionality theorem for zero cycles). If k, X_0 are as in the proof above, this induces a ring homomorphism

$$\bigoplus_{i=0}^{n} CH^i(X)_{\mathbb{Q}} \longrightarrow \bigoplus_{i=0}^{n} H^i(X,\Omega^i_{X/X_0}) = H^i(X_0,\mathcal{O}_{X_0})\otimes_k \Omega^i_{\mathbb{C}/k}.$$

Suppose $l : H^n(X_0,\mathcal{O}_{X_0}) \longrightarrow k$ is a non-zero linear functional such that the composite

$$H^{n_1}(X_0,\mathcal{O}_{X_0})\otimes\cdots\otimes H^{n_r}(X_0,\mathcal{O}_{X_0}) \longrightarrow H^n(X_0,\mathcal{O}_{X_0}) \xrightarrow{l} k$$

is zero. Then the induced homomorphism

$$\begin{tikzcd}
CH^{n_1}(X)_{\mathbb{Q}}\otimes\cdots\otimes CH^{n_r}(X)_{\mathbb{Q}} \arrow[r] & CH^n(X)_{\mathbb{Q}} \arrow[r] \arrow[dr, "\mu"'] & H^n(X_0,\mathcal{O}_{X_0})\otimes_k\Omega^n_{\mathbb{C}/k} \arrow[d, "l\otimes 1"] \\
 & & \Omega^n_{\mathbb{C}/k}
\end{tikzcd}$$

clearly vanishes. We claim that

(a) for any $P \in C$, $[P] \in CH^n(X)_{\mathbb{Q}}$ lies in the kernel of the map (defined above using the functional l)

$$\mu : CH^n(X)_{\mathbb{Q}} \longrightarrow \Omega^n_{\mathbb{C}/k}$$

(b) if $k(X_0) \hookrightarrow \mathbb{C}$ yields the point $\eta \in X$, corresponding to the generic point of X_0, then $\mu([\eta]) \neq 0$.

These properties follow from certain properties of the cycle map

$$CH^n(X)_{\mathbb{Q}} \longrightarrow H^n(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^n_{\mathbb{C}/k},$$

discussed below. If $P \in X$ has ideal sheaf I, then there is an exact sequence

$$I/I^2 \xrightarrow{\psi} \Omega^1_{X/X_0} \otimes \mathcal{O}_P \longrightarrow \Omega^1_{P/X_0} \longrightarrow 0. \tag{4}$$

The image of $\bigwedge^n \psi$ under the composite

$$\mathrm{Hom}(\textstyle\bigwedge^n I/I^2, \Omega^n_{X/X_0} \otimes \mathcal{O}_P) \cong \mathrm{Ext}^n_X(\mathcal{O}_P, \Omega^n_{X/X_0}) \longrightarrow H^n_P(X, \Omega^n_{X/X_0}) \longrightarrow$$
$$\longrightarrow H^n(X, \Omega^n_{X/X_0}) = H^n(X_0, \mathcal{O}_{X_0}) \otimes \Omega^n_{\mathbb{C}/k}$$

is the cycle class of P (this follows from the definition of the cycle class given in [S]). If $Q \in X_0$ is the image of P (note that Q need not be a closed point), then the sequence (4) may be rewritten as

$$\mathbb{C}^n \xrightarrow{\psi} \Omega^1_{\mathbb{C}/k} \xrightarrow{\chi} \Omega^1_{\mathbb{C}/k(Q)} \longrightarrow 0$$

where χ is the natural surjection. Thus

$$\mathrm{rank}\ \psi = \mathrm{tr.deg.}\,(k(Q)/k).$$

Hence if $P \in C$, so that $Q \in C_0 \subset X_0$ but $C_0 \neq X_0$, then rank $\psi < n$, and $\bigwedge^n \psi = 0$. This proves (a).

Secondly the linear functional

$$l : H^n(X_0, \mathcal{O}_{X_0}) \longrightarrow k$$

is determined, via Serre duality, by a unique $\omega \in H^0(X_0, \Omega^n_{X_0/k})$. The embedding $k(X_0) \hookrightarrow \mathbb{C}$ used to determine $\eta \in X$ also yields an embedding

$$H^0(X_0, \Omega^n_{X_0/k}) \hookrightarrow \Omega^n_{k(X_0)/k} \hookrightarrow \Omega^n_{\mathbb{C}/k},$$

and it is shown in [S] that $\mu(\eta)$ is the image of ω under this map. In particular it is non-zero.

Further remarks

1. The theorem has been stated in the present form, as urged by the referee. However, possible applications would seem to be in the direction that if the cup product on coherent cohomology is *not* surjective, then none of the other products (i)-(iv) is surjective. This is because it is presumably easier to directly compute the cup product on coherent cohomology than to compute any of the products (i)-(iv), in most situations.

2. One might hope (this is consistent with the philosophy outlined in [B2], Chapter 1) that if

$$H^{n_1}(X)/N^1H^{n_1}(X) \otimes \cdots \otimes H^{n_r}(X)/N^1H^{n_r}(X) \longrightarrow H^n(X)/N^1H^n(X)$$

is not surjective, then for any non empty open set $V \subset X$,

$$CH^{n_1}(V)_{\mathbb{Q}} \otimes \cdots \otimes CH^{n_r}(V)_{\mathbb{Q}} \longrightarrow CH^n(V)_{\mathbb{Q}}$$

is not surjective. In an earlier version of the paper, the authors had claimed to prove this, but the argument was found to be incomplete. This statement is purely algebraic, and suggests an analogous theorem in arbitrary characteristics, if we interpret $H^i(X)$ as a suitable l-adic cohomology group, equipped with Grothendieck's coniveau filtration, defined as before.

However, note that if $(*)$ is surjective, then the image of

$$H^{n_1}(X) \otimes \cdots \otimes H^{n_r}(X) \to H^n(X)$$

is a $\mathbb{Q}$-Hodge substructure, which after tensoring with $\mathbb{C}$, maps onto $H^{(0,n)}(X)$. Hence this image maps onto the smallest quotient Hodge structure with the same space $H^{(0,n)}$. According to Grothendieck's generalized Hodge conjecture, this smallest quotient is just $H^n(X)/N^1H^n(X)$. Thus the surjectivity of the map in (ii) for $V = X$ is conjecturally equivalent to that of $(*)$.

3. Of course, it would be very interesting to have information in the converse direction to the theorem. For example, for $n = 2$ and surfaces of general type for which $H^2(X, \mathcal{O}_X) = 0$, one also knows that $H^1(X, \mathcal{O}_X) = 0$, so that $CH^1(X) = \mathrm{Pic}\,(X)$ is a finitely generated abelian group. Now the implication $(i) \Longrightarrow (*)$ is equivalent to Bloch's conjecture that $CH^2(X) = \mathbb{Z}$. (Here and below, by '(i)' or '$(*)$' we mean the surjectivity of the corresponding map, for some choices of $n_1, \ldots, n_r$; these choices will be fixed in each discussion.) This is because the subgroup of $CH^2(X)$ of

cycles of degree 0 is a divisible group ([B2], Lemma 1.3), so if it is finitely generated, it must be 0. Note that $(*)$ and (ii) are equivalent for surfaces; a generalisation of Bloch's conjecture is the assertion that $(ii) \Longrightarrow (i)$.

However, $(*) \Longrightarrow (iii)$ is false in general. If X is the Jacobian of a general curve of genus 3, then the natural map

$$H^i(X, \mathbb{Q}) \longrightarrow H^0(X, \mathcal{H}^i_X)$$

is surjective for $i \leq 2$, while the cokernel for $i = 3$ is the Griffiths group of codimension 2 cycles (with rational coefficients) homologous to 0 modulo algebraic equivalence, by results of [BO]. But Ceresa [C] has shown that this Griffiths group is a non-zero $\mathbb{Q}$-vector space. Hence the map (iii) is not surjective, while $(*)$ (and even (i)) is always surjective on an abelian variety (see [B3]).

We do not know an example where the map (iv) is known to be surjective.

4. In contrast to the situation in (iv), Bloch (see [B2], 5.12) wonders whether the graded ring

$$\bigoplus_{i=0}^{n} H^i(\mathbb{C}(X), \mathbb{Z}/m\mathbb{Z}) = \varinjlim_{V \subset X} \bigoplus_{i=0}^{n} H^i(V_{an}, \mathbb{Z}/m\mathbb{Z})$$

is generated by $H^1(\mathbb{C}(X), \mathbb{Z}/m\mathbb{Z})$ as a $\mathbb{Z}/m\mathbb{Z}$-algebra. If $n = 2$, this is known, from the Merkurjev-Suslin theorem, and Bloch (*loc. cit.*) states that

$$H^1(\mathbb{C}(X), \mathbb{Z}/m\mathbb{Z})^{\otimes n} \longrightarrow H^n(\mathbb{C}(X), \mathbb{Z}/m\mathbb{Z})$$

is always surjective. More generally, Kato has conjectured that for any field K containing a primitive l^{th} root of unity, the Galois cohomology ring with $\mathbb{Z}/l\mathbb{Z}$ coefficients, $l \neq \operatorname{char} K$, is generated by $H^1(K, \mathbb{Z}/l\mathbb{Z})$.

One may be tempted to argue using inverse limits that in view of the above conjectures, one should expect that

$$\bigoplus_{i=0}^{n} H^i(\mathbb{C}(X), \mathbb{Q}_l) = \bigoplus_{i=0}^{n} H^i(\mathbb{C}(X), \mathbb{Q}) \otimes \mathbb{Q}_l$$

is generated by $H^1(\mathbb{C}(X), \mathbb{Q}_l)$ as a $\mathbb{Q}_l$-algebra. However, the inverse systems

$$\{H^i(\mathbb{C}(X), \mathbb{Z}/l^m\mathbb{Z})\}_{m \geq 1}$$

do not satisfy the Mittag-Leffler condition, so the surjectivity of multiplication maps need not be preserved under taking inverse limits.

5. If $R = \bigoplus_{i=0}^{n} R_i$ is a graded $\mathbb{Q}$-algebra, define $x \in R_n$ to be *r-decomposable* if there is an expression
$$x = \sum_{i=1}^{r} x_i y_i$$
where the $x_i, y_i \in R$ are homogeneous of degree > 0. If x is not r-decomposable, we say that x is *r-indecomposable*.

Nori [N] has shown that if X is a proper smooth variety of dimension n over $\mathbb{C}$ with $H^n(X, \mathcal{O}_X) \neq 0$, then for any non-empty open subset $V \subset X$ and any $r > 0$, $CH^n(V)_{\mathbb{Q}}$ contains elements which are r-indecomposable in $\bigoplus_i CH^i(V)_{\mathbb{Q}}$. Nori's proof involves an argument analogous to the second proof of $(i) \Rightarrow (*)$ using the cycle class.

In a similar vein, suppose X is a smooth proper variety of dimension n over a universal domain Ω, such that $H^n_{\text{ét}}(X, \mathbb{Q}_l) \neq N^1 H^n_{\text{ét}}(X, \mathbb{Q}_l)$. Then one may raise the following questions.

(1) For any non-empty open set $V \subset X$ and any $r > 0$, does $CH^n(V)_{\mathbb{Q}}$ contain r-indecomposable elements?

(2) Does $H^n_{\text{ét}}(\Omega(X), \mathbb{Q}_l)$ contain elements which are r-indecomposable in
$$\bigoplus_i H^i_{\text{ét}}(\Omega(X), \mathbb{Q}_l),$$
for each $r > 0$? Is this true at least when $\Omega = \mathbb{C}$ and $H^n(X, \mathcal{O}_X) \neq 0$?

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