K-Theory **6**: 45–56, 1992.

Characteristic Classes of Flat Bundles, II

HÉLÈNE ESNAULT
Universität Essen, Fachbereich 6, Mathematik, Universitätsstraße 3, D-4300 Essen 1, Germany

(Received February 1992)

Abstract. On a smooth variety X defined over a field K of characteristic zero, one defines characteristic classes of bundles with an integrable K-connection in a group lifting the Chow group, which map, when K is the field of complex numbers and X is proper, to Cheeger–Simons' secondary analytic invariants, compatibly with the cycle map in the Deligne cohomology.

Key words. Flat bundles, characteristic classes, Deligne cohomology, Chow groups.

Let $f = X \to S$ be a smooth morphism of smooth varieties defined over a field k of characteristic zero. Let $\Sigma \subset S$ be a divisor with normal crossings whose inverse image $Y := f^{-1}(\Sigma)$ is also a divisor with normal crossings. We consider a vector bundle E on X, together with a relative connection

$$\nabla : E \to \Omega^1_{X/S}(\log Y) \bigotimes_{\mathcal{O}_X} E.$$

In this note, we construct Chern classes $c_i(E, \nabla)$ (1.7) in a group $C^i(X)$ (1.4) mapping to the kernel of the map from the Chow group $CH^i(X)$ to the relative cohomology $H^i(X, \Omega^i_{X/S}(\log Y))$. If ∇ is integrable, then we construct classes in a group $C^i_{\mathrm{int}}(X)$ mapping to the kernel of the map from the Chow group $CH^i(X)$ to the cohomology $\mathbb{H}^{2i}(X, \Omega^{\geq i}_{X/S}(\log Y))$ of relative forms. Those classes are functorial and additive.

If $S = \operatorname{Spec} K$, where K is a field containing k, (and, of course, $\Sigma = \phi$), then following S. Bloch, one may just define $C^i_{\mathrm{int}}(X)$ as the group of cycles with an integrable K connection (2.6).

If $S = \operatorname{Spec} \mathbb{C}$, then the (algebraic) group $C^i_{\mathrm{int}}(X)$ maps to the (analytic) cohomology $H^{2i-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i))$, compatibly with the cycle map from the Chow group to the Deligne–Beilinson cohomology (1.5). More precisely, $c_i(E, \nabla)$ maps to the class

$$c_i^{\mathrm{an}}(E, \nabla) \in H^{2i-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i))$$

constructed in [3], identified with the Cheeger–Simons secondary invariant [1, 2] (at least when X is proper and ∇ is unitary) (1.7).

It defines a similar invariant

$$c_i^{\mathrm{an}}(E, \nabla) \in H^{2i-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i))/F^{i+1}$$

if ∇ is not integrable (1.9).

If $k = \mathbb{C}$ and f is proper, Griffiths [8] has defined an infinitesimal invariant of a normal function in $H^0(S_{\mathrm{an}}, \mathscr{H}^1)$ (2.1), where $\mathscr{H}^1$ is the first cohomology sheaf of the complex of analytic sheaves

$$\mathscr{F}^i \to \Omega_S^1(\log \Sigma) \otimes \mathscr{F}^{i-1} \to \Omega_S^2(\log \Sigma) \otimes \mathscr{F}^{i-2},$$

and where the $\mathscr{F}^j$ are the Hodge bundles. The groups $C_{\mathrm{int}}^i(X)$ and $C^i(X)$ map in fact naturally to some lifting of $H^0(S_{\mathrm{an}}, \mathscr{H}^1)$ (see (2.3), (2.5)). This fact partly explains the rigidity of the higher classes of flat bundles in the Deligne cohomology (see [5]).

1. Characteristic Classes

1.1. Recall that on X there is a map

$$\mathscr{K}_i \xrightarrow{D} \Omega^i_{X/\mathbb{Z}}$$

from the (Zariski) sheaf of higher K-theory to the (Zariski) sheaf of absolute i-forms, which restricts to the higher exterior power of the map

$$d \log \colon \mathscr{K}_1 \to \Omega^1_{X/\mathbb{Z}}$$

on the Milnor K-theory sheaves $\mathscr{K}_i^M$ (see [7]).

(a) For our construction of $C^i(X)$ and $C_{\mathrm{int}}^i(X)$, we need that this map extends to a complex

$$\mathscr{K}_i \xrightarrow{D} \Omega^i_{X/S} \to \Omega^{i+1}_{X/S} \to \cdots$$

(b) For the compatibility with the analytic classes in $H^{2i-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i))$ when $S = \operatorname{Spec} \mathbb{C}$, we need that the periods of a section $D(s)$, for $s \in \mathscr{K}_i$, are lying in $\mathbb{Z}(i)$, that is $D\mathscr{K}_i \subset \Omega^i_{\mathbb{Z}(i)}$, where

$$\Omega^a_{\mathbb{Z}(i)} := \ker \alpha_* \Omega^a_{X_{\mathrm{an}}, d \text{ closed}} \to \mathscr{H}^a(\mathbb{C}/\mathbb{Z}(i)).$$

Here $\mathbb{Z}(i)$ denotes $\mathbb{Z} \cdot (2\pi\sqrt{-1})^i$, $\alpha \colon X_{\mathrm{an}} \to X_{\mathrm{zar}}$ is the identity, and $\mathscr{H}^a(\mathbb{C}/\mathbb{Z}(i))$ denotes the sheaf associated to the (Zariski) presheaf

$$U \to H^a(U, \mathbb{C}/\mathbb{Z}(i)).$$

(c) Finally for the study of the Griffiths' invariant, we need that D extends to a complex

$$\mathscr{K}_i \xrightarrow{D} \Omega^i_{X/k} \to \Omega^{i+1}_{X/k} \to \cdots$$

Of course, the properties (a), (b), (c) are trivially fulfilled on $\mathscr{K}_i^M$, in particular also on $\mathscr{K}_i$ for $i \leqslant 2$. In this article, we replace $\mathscr{K}_i$ by the sheaf $\mathscr{K}_i^m$ of modified Milnor K-theory as introduced by O. Gabber [6] and M. Rost, whose definition we are recalling now. We show that this sheaf fulfills the conditions (a), (b) and (c).

1.2. They define $\mathscr{K}_i^m$ as the kernel of the map

$$K_i^M(k(X)) \xrightarrow{\partial} \bigoplus_{x \in X^{(1)}} K_{i-1}^M(k(X)),$$

where ∂ is the residue map from K_i^M from the function field of X to the function field of codimension 1 points. Of course

$$\mathscr{K}_i^m = \mathscr{K}_i$$

for $i \leqslant 2$. They prove:

(a) $CH^i(X) = H^i(X, \mathscr{K}_i^m)$

(b) The natural map

$$\mathscr{K}_i^M \to K_i^M(k(X))$$

is surjective onto $\mathscr{K}_i^m$ and has its kernel killed by $(i-1)!$.

(c) The cohomology of $\mathscr{K}_i^m$ satisfies the projective bundle formula: If E is a vector bundle on X, and $\mathbb{P}(E) \xrightarrow{q} X$ is the associated projective bundle, then one has

$$H^j(\mathbb{P}(E), \mathscr{K}_i^m) = \bigoplus q^* H^{j-s}(X, \mathscr{K}_{i-s}^m) \cup \mathscr{O}(1)^s$$

where $\mathscr{O}(1) \in H^1(\mathbb{P}(E), \mathscr{K}_1)$ is the class of the tautological bundle.

So by (b), the $d\log$ map

$$\mathscr{K}_i^M \to \Omega_{X/\mathbb{Z}}^i$$

factorizes through

$$\mathscr{K}_i^m \to \Omega_{X/\mathbb{Z}}^i$$

and extends to a complex

$$\mathscr{K}_i^m \to \Omega_{X/\mathbb{Z}}^i \to \Omega_{X/\mathbb{Z}}^{i+1} \to \cdots,$$

so that (1.1) (a) and (c) are fulfilled, and (1.1) (b) is fulfilled as well, as it is true for $i = 1$.

1.3. From now on, $f: X \to S$ is a smooth morphism of smooth varieties over a field k of characteristic zero, and E is a bundle on X with a relative connection

$$\nabla: E \to \Omega_{X/S}^1 \bigotimes_{\mathscr{O}_X} E$$

where $\Sigma \subset S$ and $Y = f^{-1}(\Sigma) \subset X$ are normal crossing divisors.

1.4. One defines

$$D^i(X) = \mathbb{H}^i(X, \mathcal{K}_i^m \to \Omega^i_{X/S}(\log Y)),$$
$$D^i_{\mathrm{int}}(X) = \mathbb{H}^i(X, \mathcal{K}_i^m \to \Omega^i_{X/S}(\log Y) \to \Omega^{i+1}_{X/S}(\log Y) \to \cdots).$$

Then $C^i(X)$ (resp. $C^i_{\mathrm{int}}(X)$) is the image of

$$\mathbb{H}^i(X, \mathcal{K}_i^m \to \Omega^i_{X/k}(\log Y)/f^*\Omega^i_{S/k}(\log \Sigma))$$

(resp. $\mathbb{H}^i(X, \mathcal{K}_i^m \to \Omega^i_{X/k}(\log Y)/f^*\Omega^i_{S/k}(\log \Sigma) \to \Omega^{i+1}_{X/S}(\log Y) \to \cdots)$) in $D^i(X)$ (resp. $D^i_{\mathrm{int}}(X)$).

Here $\Omega^i_{X/k}(\log Y)$ denotes as usual the sheaf of regular i-forms on X, relative to k, with logarithmic poles along Y. One has obvious maps

$$D^i_{\mathrm{int}}(X) \to D^i(X) \to CH^i(X) = H^i(X, \mathcal{K}_i^m).$$

1.5. Lemma. *If X is proper over $S = \operatorname{Spec} \mathbb{C}$, one has a commutative diagram*

$$\begin{array}{ccccc} D^i_{\mathrm{int}}(X) & \longrightarrow & D^i(X) & \longrightarrow & CH^i(X) \\ \phi_{\mathrm{int}}\downarrow & & \phi\downarrow & & \downarrow\psi \\ H^{2i-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i)) & \longrightarrow & H^{2i-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i))/F^{i+1} & \longrightarrow & H^{2i}_{\mathcal{D}}(X, i) \end{array}$$

where ψ is the cycle map in the Deligne cohomology.

Proof. To simplify the notations, we drop the subscript S in $\Omega^l_{X/S}$ which becomes simply Ω^l_X, the sheaf of regular l-forms over X relative to $\mathbb{C}$.

The complex

$$\mathcal{K}_i^m \to \Omega^i_X \to \Omega^{i+1}_X \to \cdots$$

maps to the complex

$$\Omega^i_{\mathbb{Z}(i)} \to \alpha_*\Omega^i_{X_{\mathrm{an}}} \to \alpha_*\Omega^{i+1}_{X_{\mathrm{an}}} \to \cdots$$

where the notations are as in (1.1) (b), and the complex

$$\mathcal{K}_i^m \to \Omega^i_X$$

maps to

$$\Omega^i_{\mathbb{Z}(i)} \to \alpha_*\Omega^i_{X_{\mathrm{an}}}.$$

One has an exact sequence

$$\begin{array}{c} 0 \\ \downarrow \\ \mathscr{H}^i(\mathbb{C})/\mathscr{H}^i(\mathbb{Z}(i))[-1] \\ \downarrow \\ (\Omega^i_{\mathbb{Z}(i)} \to \alpha_*\Omega^i_{X_{\mathrm{an}}} \to \alpha_*\Omega^{i+1}_{X_{\mathrm{an}}} \to \cdots) \\ \downarrow \\ \alpha_*\Omega^\cdot_{X_{\mathrm{an}}}/\tau_{\leqslant i}\alpha_*\Omega^\cdot_{X_{\mathrm{an}}}[i-1] \\ \downarrow \\ 0 \end{array}$$

The last complex of the exact sequence is quasi-isomorphic to

$$R\alpha_*\mathbb{C}/\tau_{\leqslant i}R\alpha_*\mathbb{C}[i-1],$$

and therefore via the map $\mathbb{C} \to \mathbb{C}/\mathbb{Z}(i)$, the complex

$$\Omega^i_{\mathbb{Z}(i)} \to \alpha_*\Omega^i_{X_{\mathrm{an}}} \to \alpha_*\Omega^{i+1}_{X_{\mathrm{an}}} \to \cdots$$

maps to

$$R\alpha_*\mathbb{C}/\mathbb{Z}(i)/\tau_{\leqslant(i-1)}R\alpha_*\mathbb{C}/\mathbb{Z}(i)[i-1],$$

which is an extension of

$$R\alpha_*\mathbb{C}/\mathbb{Z}(i)/\tau_{\leqslant i}R\alpha_*\mathbb{C}/\mathbb{Z}(i)[i-1]$$

by $\mathscr{H}^i(\mathbb{C}/\mathbb{Z}(i))[-1]$. As

$$H^i(X_{\mathrm{zar}}, R\alpha_*\mathbb{C}/\mathbb{Z}(i)/\tau_{\leqslant(i-1)}R\alpha_*\mathbb{C}/\mathbb{Z}(i)[i-1]) = H^{2i-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i)),$$

one obtains the left vertical arrow ϕ_{int}.

As for the definition of the middle vertical arrow ϕ, one similarly writes the complex

$$\Omega^i_{\mathbb{Z}(i)} \to \alpha_*\Omega^i_{X_{\mathrm{an}}}$$

as an extension of

$$R\alpha_*\mathbb{C}/\tau_{\leqslant i}R\alpha_*\mathbb{C}[i-1] + R\alpha_*\Omega^{\geqslant i+1}_{X_{\mathrm{an}}}$$

by $\mathscr{H}^i(\mathbb{C})/\mathscr{H}^i(\mathbb{Z}(i))[-1]$, and one argues as above.

Altogether this defines a commutative diagram as in (1.5) where ψ is replaced by the map

$$CH^i(X) = H^i(\mathscr{K}^m_i) \to H^i(\Omega^i_{\mathbb{Z}(i)}),$$

which is shown in [4] to factorize the cycle map. (In fact there, we wrote 'projective' in (1.3) (2), but proved the property for 'proper' in (1.5).) □

1.6. *Remark.* We see in fact that the image of $D^i_{\text{int}}(X)$ in

$$H^{2i-1}(X_{\text{zar}}, R\alpha_* \mathbb{C}/\mathbb{Z}(i)/\tau_{\leqslant i} R\alpha_* \mathbb{C}/\mathbb{Z}(i))$$

(resp. of $D^i(X)$ in

$$H^{2i-1}(X_{\text{zar}}, R\alpha_* \mathbb{C}/\mathbb{Z}(i)/\tau_{\leqslant i} R\alpha_* \mathbb{C}/\mathbb{Z}(i))/F^{i+1})$$

lifts naturally to

$$H^{2i-1}(X_{\text{zar}}, R\alpha_* \mathbb{C}/\tau_{\leqslant i} R\alpha_* \mathbb{C})$$

(resp.

$$H^{2i-1}(X_{\text{zar}}, R\alpha_* \mathbb{C}/\tau_{\leqslant i} R\alpha_* \mathbb{C})/F^{i+1}).$$

In particular, (1.7) will imply that the Betti class of a complex bundle E which carries a connection lies in the image of $H^{i-1}(X_{\text{zar}}, \mathscr{H}^i(\mathbb{C})/\mathscr{H}^i(\mathbb{Z}(i)))$ in $H^{2i}(X_{\text{an}}, \mathbb{Z}(i))$ (which embeds into the image of $H^{i-1}(X_{\text{zar}}, \mathscr{H}^i(\mathbb{C}/\mathbb{Z}(i))$ in $H^{2i}(X_{\text{an}}, \mathbb{Z}(i)))$.

1.7. THEOREM. *Let (E, ∇) be as in (1.3). Then there are Chern classes $c_i(E, \nabla) \in C^i(X)$ lifting the classes $c_i^{CH}(E) \in CH^i(X)$. They are fonctorial for any morphism*

$$\begin{array}{ccc} X' & \xrightarrow{\sigma'} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{\sigma} & S \end{array}$$

such that $\sigma^{-1}(\Sigma)$ and $\sigma'^{-1}(Y)$ are normal crossing divisors. They are additive in exact sequences of bundles with connection.

If ∇ is integrable, then there are Chern classes $c_i(E, \nabla) \in C^i_{\text{int}}(X)$ with the same properties.

If $S = \operatorname{Spec} \mathbb{C}$, then ϕ_{int} maps $c_i(E, \nabla)$ to the secondary analytic class $c_i^{\text{an}}(E, \nabla)$ defined in [3].

Proof. (a) For $i = 1$ one has $C^1(X) = D^1(X)$, $C^1_{\text{int}}(X) = D^1_{\text{int}}(X)$, and $C^1(X)$ is the group of bundles with connection (E, ∇) of rank 1 modulo isomorphisms, and contains $C^1_{\text{int}}(X)$ as the subgroup of those (E, ∇) for which $\nabla^2 = 0$.

Let $g: G \to X$ be the flag bundle of E, such that g^*E has a filtration by subbundles E_i whose successive quotients L_i have rank 1. In [3] we showed that ∇ defines a splitting

$$\tau: \Omega^1_{G/S}(\log Z) \to g^*\Omega^1_{X/S}(\log Y),$$

where $Z = g^{-1}(Y)$, such that $\tau\nabla$ stabilizes the subbundles E_i, and defines thereby $(L_i, \tau\nabla)$ as a class in

$$\mathbb{H}^1(G, \mathscr{K}_1 \to g^*\Omega^1_{X/S}(\log Y)).$$

If $\nabla^2 = 0$, then τ defines a splitting

$$\tau\colon \Omega^{\bullet}_{G/S}(\log Z) \to g^*\Omega^{\bullet}_{X/S}(\log Y)$$

of the de Rham complex, where the differential in $g^*\Omega^{\Phi}_{X/S}(\log Y)$ is the composite map

$$g^*\Omega^i_{X/S}(\log Y) \to \Omega^i_{G/S}(\log Z) \xrightarrow{d} \Omega^{i+1}_{G/S}(\log Z) \xrightarrow{\tau} g^*\Omega^{i+1}_{X/S}(\log Y).$$

Thereby $\tau\nabla$ is integrable and, therefore, $(L_i, \tau\nabla)$ are classes in the hypercohomology group

$$\mathbb{H}^1(G, \mathscr{K}_1 \xrightarrow{\tau \circ d \log} g^*\Omega^1_{X/S}(\log Y) \to g^*\Omega^2_{X/S}(\log Y) \to \cdots).$$

Write

$$(\xi^i_{\alpha\beta}, \omega^i_\alpha) \in (\mathscr{C}^1(\mathscr{K}_1) \times \mathscr{C}^0(g^*\Omega^1_{X/S}(\log Y)))_{\tau d - \delta}$$

for a Čech cocycle of $(L_i, \tau\nabla)$:

$$\tau\frac{d\xi^i_{\alpha\beta}}{\xi^i_{\alpha\beta}} - \delta\omega^i_\alpha = 0 \quad \text{and} \quad \tau d\omega^i_\alpha = 0 \quad \text{if } \nabla^2 = 0.$$

Then one defines

$$c_i(g^*(E, \nabla)) \in \mathbb{H}^i(G, \mathscr{K}^m_i \to g^*\Omega^i_{X/S}(\log Y))$$

(or in

$$\mathbb{H}^i(G, \mathscr{K}^m_i \to g^*\Omega^i_{X/S}(\log Y) \to \Omega^{i+1}_{X/S}(\log Y) \to \cdots)$$

if $\nabla^2 = 0$), as the class of the Čech cocycle

$$c_i = (c^i, c^{i-1}) \in (\mathscr{C}^i(\mathscr{K}^m_i) \times \mathscr{C}^{i-1}(g^*\Omega^i_{X/S}(\log Y)))_{\tau d + (-1)^i \delta}$$

by

$$c^i = \sum_{l_1 < \cdots < l_i} \xi^{l_1}_{\alpha_0\alpha_1} \cup \cdots \cup \xi^{l_r}_{\alpha_{i-1}\alpha_i},$$

$$c^{i-1} = (-1)^{i-1} \sum_{l_1 < \cdots < l_i} \omega^{l_1}_{\alpha_0} \wedge (\delta\omega)^{l_2}_{\alpha_0\alpha_1} \wedge \cdots \wedge (\delta\omega)^{l_i}_{\alpha_{i-2}\alpha_{i-1}},$$

$$\tau d c^{i-1} = 0 \quad \text{if } \nabla^2 = 0.$$

By definition, $c_i(g^*(E, \nabla))$ maps to $c^{CH}_i(g^*E) = g^*c^{CH}_i(E)$ in

$$g^*CH^i(X) \subset CH^i(G).$$

From (1.2) (c), one obtains that

$$H^{i-1}(G, \mathscr{K}^m_i) = g^*H^{i-1}(X, \mathscr{K}^m_i) \oplus \text{Rest},$$

where

$$\text{Rest} \subset \sum H^{i-2}(G, \mathscr{K}^m_{i-1}) \cup (L_l),$$

and where (L_l) is the class of L_l in $H^1(G, \mathscr{K}_1)$.

As (L_l) maps to zero in $\mathbb{H}^1(G, g^*\Omega^1_{X/S}(\log Y))$ (resp.

$$\mathbb{H}^1(G, g^*\Omega^1_{X/S}(\log Y) \to g^*\Omega^2_{X/S}(\log Y) \to \cdots)$$

if $\nabla^2 = 0$), the image I of $H^{i-1}(G, \mathscr{K}^m_i)$ in

$$H^{i-1}(G, g^*\Omega^i_{X/S}(\log Y)) = H^{i-1}(X, \Omega^i_{X/S}(\log Y))$$

(resp. in

$$\begin{aligned}&\mathbb{H}^{i-1}(G, g^*\Omega^i_{X/S}(\log Y) \to g^*\Omega^{i+1}_{X/S}(\log Y) \to \cdots)\\ &\quad = \mathbb{H}^{i-1}(X, \Omega^i_{X/S}(\log Y) \to \Omega^{i+1}_{X/S}(\log Y) \to \cdots))\end{aligned}$$

is the same as the image of $H^{i-1}(X, \mathscr{K}^m_i)$ in it. This shows, via the exact sequences

$$0 \to H^{i-1}(X, \Omega^i_{X/S}(\log Y))/I \to \mathbb{H}^i(G, \mathscr{K}^m_i \to g^*\Omega^i_{X/S}(\log Y)) \to H^i(G, \mathscr{K}^m_i)$$

(resp.

$$\begin{aligned}&0 \to \mathbb{H}^{i-1}(X, \Omega^i_{X/S}(\log Y) \to \Omega^{i+1}_{X/S}(\log Y) \to \cdots)/I \to\\ &\quad \to \mathbb{H}^i(G, \mathscr{K}^m_i \to g^*\Omega^i_{X/S}(\log Y) \to g^*\Omega^{i+1}_{X/S}(\log Y) \to \cdots) \to H^i(G, \mathscr{K}^m_i))\end{aligned}$$

and

$$0 \to H^{i-1}(X, \Omega^i_{X/S}(\log Y))/I \to \mathbb{H}^i(X, \mathscr{K}^m_i \to \Omega^i_{X/S}(\log Y)) \to H^i(X, \mathscr{K}^m_i)$$

(resp.

$$\begin{aligned}&0 \to \mathbb{H}^{i-1}(X, \Omega^i_{X/S}(\log Y) \to \Omega^{i+1}_{X/S}(\log Y) \to \cdots)/I \to\\ &\quad \to \mathbb{H}^i(X, \mathscr{K}^m_i \to \Omega^i_{X/S}(\log Y) \to \Omega^{i+1}_{X/S}(\log Y) \to \cdots) \to H^i(X, \mathscr{K}^m_i))\end{aligned}$$

that

$$\mathbb{H}^i(X, \mathscr{K}^m_i \to \Omega^i_{X/S}(\log Y))$$

(resp.

$$\mathbb{H}^i(X, \mathscr{K}^m_i \to \Omega^i_{X/S}(\log Y) \to \Omega^{i+1}_{X/S}(\log Y) \to \cdots))$$

injects into

$$\mathbb{H}^i(G, \mathscr{K}^m_i \to g^*\Omega^i_{X/S}(\log Y))$$

(resp.

$$\mathbb{H}^i(G, \mathscr{K}^m_i \to g^*\Omega^i_{X/S}(\log Y) \to g^*\Omega^{i+1}_{X/S}(\log Y) \to \cdots)),$$

with cokernel lying in

$$H^i(G, \mathscr{K}^m_i)/g^*H^i(X, \mathscr{K}^m_i) = CH^i(G)/g^*CH^i(X).$$

This proves the existence of the classes in $D^i(X)$ (resp. $D^i_{\mathrm{int}}(X)$).

(b) Consider the Atiyah class

$$\mathrm{At}_X(E) \in H^1\left(X, \Omega^1_{X/k}(\log Y) \underset{\mathcal{O}_X}{\otimes} \mathrm{End}\, E\right)$$

of E. Then its image in $H^1(X, \Omega^1_{X/S}(\log Y) \otimes_{\mathcal{O}_X} \mathrm{End}\, E)$ is vanishing, which implies that $\mathrm{At}_X(E)$ lies in fact in the image of $H^1(X, f^*\Omega^1_{S/k}(\log \Sigma) \otimes_{\mathcal{O}_X} \mathrm{End}\, E)$ in $H^1(X, \Omega^1_{X/k}(\log Y) \otimes_{\mathcal{O}_X} \mathrm{End}\, E)$ and therefore the exterior power $\wedge^i \mathrm{At}_X E$ lies in the image of $H^i(X, f^*\Omega^i_{S/k}(\log \Sigma) \otimes_{\mathcal{O}_X} \mathrm{End}\, E)$ in

$$H^i(X, \Omega^i_{X/k}(\log Y) \otimes_{\mathcal{O}_X} \mathrm{End}\, E).$$

In other words $\wedge^i \mathrm{At}_X(E)$ is vanishing in

$$H^i(X, \Omega^i_{X/k}(\log Y)/f^*\Omega^i_{S/k}(\log \Sigma) \otimes \mathrm{End}\, E),$$

as well as its trace in $H^i(X, \Omega^i_{X/k}(\log Y)/f^*\Omega^i_{S/k}(\log \Sigma))$. As the Chern class of E in $H^i(X, \Omega^i_{X/k}(\log Y))$ is a linear combination with $\mathbb{Q}$-coefficients of the traces of $\wedge^j \mathrm{At}_X E$, for $j \leqslant i$, one obtains that the class lies in $C^i(X)$, or in $C^i_{\mathrm{int}}(X)$ if $\nabla^2 = 0$.

(c) Additivity and functoriality are proven as in [3].

(d) We now compare with $c_i^{\mathrm{an}}(E, \nabla)$ if ∇ is integrable and defined over

$$S = \mathrm{Spec}\, \mathbb{C}.$$

By [3], (2.24), one has just to see that

$$(L_i, \tau\nabla) \in \mathbb{H}^1(G, \mathcal{K}_1 \to g^*\Omega^1_{X/\mathbb{C}} \to g^*\Omega^2_{X/\mathbb{C}} \to \cdots)$$

maps to the class of

$$(L_i, \tau\nabla) \in \mathbb{H}^1(G_{\mathrm{an}}, \mathcal{O}^*_{X_{\mathrm{an}}} \to g^*\Omega^1_{X_{\mathrm{an}}} \to g^*\Omega^2_{X_{\mathrm{an}}} \to \cdots).$$

This is just by definition. □

1.8. *Remark.* Lemma (1.5) together with Theorem (1.7) define functorial and additive secondary analytic classes

$$c_i^{\mathrm{an}}(E, \nabla) \in H^{2i-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i))/F^{i+1}$$

for a bundle E with a connection ∇ on X proper smooth over $\mathbb{C}$.

2. Griffiths' Invariant

2.1. Let f be as in (1.3). We assume in the sequel that f is proper.

We drop the subscript k in the differential forms, and we define as usual the Hodge bundles

$$\mathcal{F}^j := R^{2i-1} f_* \Omega^{\geqslant j}_{X/S}(\log Y).$$

We recall now the definition of Griffiths' infinitesimal invariant.

Let $\xi \in CH^i(X)$ be a codimension i cycle on X which is homologically torsion on the fibers $f^{-1}(s), s \in S - \Sigma$, by which we mean that its Hodge class in $\mathbb{H}^{2i}(X, \Omega_X^{\geq i}(\log Y))$ vanishes in

$$H^0(S - \Sigma, R^{2i}f_*\Omega_{X/S}^{\geq i}(\log Y)).$$

In fact, as the sheaf $R^{2i}f_*\Omega_{X/S}^{\geq i}(\log Y)$ is torsion free, the Hodge class vanishes in

$$H^0(S, R^{2i}f_*\Omega_{X/S}^{\geq i}(\log Y))$$

as well. Therefore, the class

$$v'(\xi) \in H^0(S, R^{2i}f_*\Omega_X^{\geq i}(\log Y))$$

induces a class

$$v(\xi) \in H^0(S, \Omega_S^1(\log \Sigma) \otimes \mathscr{F}^{i-1}/\mathscr{F}^i)$$

via the exact sequence

$$0 \to \Omega_S^1(\log \Sigma) \otimes \mathscr{F}^{i-1}/\mathscr{F}^i \to R^{2i}f_*(\Omega_X^{\geq i}(\log Y)/\langle \Omega_S^2(\log \Sigma)\rangle) \to R^{2i}f_*\Omega_{X/S}^{\geq i}(\log Y),$$

where $\langle \Omega_S^2(\log \Sigma)\rangle$ denotes the subcomplex of $\Omega_{X/S}^{\geq i}(\log Y)$ whose degree j sheaf is $f^*\Omega_S^2(\log \Sigma) \wedge \Omega_X^{j-2}(\log Y)$.

As $v(\xi)$ comes from $v'(\xi)$, it is Gauss–Manin flat and, therefore,

$$v(\xi) \in H^0(S, \mathscr{H}^1),$$

where $\mathscr{H}^1$ is the first homology sheaf of the complex

$$\mathscr{F}^i \to \Omega_S^1(\log \Sigma) \otimes \mathscr{F}^{i-1} \to \Omega_S^2(\log \Sigma) \otimes \mathscr{F}^{i-2}.$$

In fact, Griffiths' invariant is the image of $v(\xi)$ in $H^0((S - \Sigma)_{\text{an}}, \mathscr{H}^1_{\text{an}})$, if $k = \mathbb{C}$.

Griffiths defines it more generally for a normal function on $(S - \Sigma)$.

2.2. We assume that the connection ∇ is integrable. Then

$$\xi = c_i^{CH}(E) \in \mathrm{CH}^i(X)$$

fulfills the conditions of (2.1). This defines

$$v(c_i^{CH}(E)) \in H^0(S, \mathscr{H}^1).$$

2.3. PROPOSITION. *Griffiths' invariant*

$$v(c_i^{CH}(E)) \in H^0(S, \mathscr{H}^1)$$

lifts to a well defined functorial class

$$\gamma(c_i(E, \nabla)) \in H^0(S, \Omega_S^1(\log \Sigma) \otimes \mathscr{F}^i \to \Omega_S^2(\log \Sigma) \otimes \mathscr{F}^{i-1}).$$

Proof. We consider the complex

$$\mathscr{K}_i^m \to \Omega_X^i(\log Y)/\langle \Omega_S^2(\log \Sigma)\rangle \to \Omega_X^{i+1}(\log Y)/\langle \Omega_S^2(\log \Sigma)\rangle \to \cdots$$

as a Gauss–Manin like extension of the complex

$$\mathscr{K}_i^m \to \Omega^i_{X/S}(\log Y) \to \Omega^{i+1}_{X/S}(\log Y) \to \cdots$$

by

$$(f^*\Omega^1_S(\log \Sigma) \otimes \Omega^{i-1}_{X/S}(\log Y) \to f^*\Omega^1_S(\log \Sigma) \otimes \Omega^i_{X/S}(\log Y) \to \cdots)[-1].$$

This defines a class in

$$H^0(S, \Omega^1_S(\log \Sigma) \otimes \mathscr{F}^{i-1})$$

from which one knows that it is Gauss–Manin closed. As this class vanishes in $H^0(S, \Omega^1_S(\log \Sigma) \otimes R^i f_* \Omega^{i-1}_{X/S}(\log Y))$ by definition of $C^i_{\text{int}}(X)$, it is lying in

$$H^0(S, \Omega^1_S(\log \Sigma) \otimes \mathscr{F}^i \to \Omega^2_S(\log \Sigma) \otimes \mathscr{F}^{i-1}). \qquad \square$$

2.4. We assume now that the connection is not necessarily integrable, that $k = \mathbb{C}$, and that S is proper and one-dimensional.

2.5. PROPOSITION. *Griffiths' invariant $v(c_i^{CH}(E)) \in H^0(S, \mathscr{H}^1)$ lifts to a well defined functorial class in the image of $H^0(S, \Omega^1_S(\log \Sigma) \otimes \mathscr{F}^i)$ in*

$$\mathbb{H}^1(S, \mathscr{F}^i \to \Omega^1_S(\log \Sigma) \otimes \mathscr{F}^{i-1}).$$

Proof. The Betti Chern class of $\xi = c_i^{CH}(E)$ lies in $H^1(S_{\text{an}}, j_* R^{2i-1} f_* \mathbb{Q}(i))$, where $j: S - \Sigma \to S$ is the inclusion. In

$$H^1(S_{\text{an}}, j_* R^{2i-1} f_* \mathbb{C}) = \mathbb{H}^1(S_{\text{an}}, j_*(\Omega^{\bullet}_{S-\Sigma} \otimes R^{2i-1} f_* \Omega^{\bullet}_{(X-Y)/(S-\Sigma)})),$$

it lies in the subgroup

$$\mathbb{H}^1(S_{\text{an}}, \mathscr{F}^i \to \Omega^1_S(\log \Sigma) \otimes \mathscr{F}^{i-1})$$

which equals

$$\mathbb{H}^1(S, \mathscr{F}^i \to \Omega^1_S(\log \Sigma) \otimes \mathscr{F}^{i-1})$$

by the GAGA theorems ([9]).

Again considering $\mathscr{K}_i^m \to \Omega^i_X(\log Y)/\langle \Omega^2_S(\log \Sigma) \rangle$ as a Gauss–Manin-like extension of $\mathscr{K}_i^m \to \Omega^i_{X/S}(\log Y)$ by

$$(f^*\Omega^1_S(\log \Sigma) \otimes \Omega^{i-1}_{X/S}(\log Y))[-1],$$

one sees that $D^i(X)$ maps to

$$H^0(S, \Omega^1_S(\log \Sigma) \otimes (\mathscr{F}^{i-1}/\mathscr{F}^i))$$

which itself maps to

$$\mathbb{H}^1(S, \mathscr{F}^i \to \Omega^1_S(\log \Sigma) \otimes (\mathscr{F}^{i-1}/\mathscr{F}^i)).$$

In particular, the class in $\mathbb{H}^1(S, \mathscr{F}^i \to \Omega^1_S(\log \Sigma) \otimes \mathscr{F}^{i-1})$ maps to zero in $H^1(S, \mathscr{F}^i)$ and, therefore, lies in

$$H^0(S, \Omega^1_S(\log \Sigma) \otimes \mathscr{F}^{i-1})/H^0(S, \mathscr{F}^i).$$

As it comes from $C^i(X)$, it vanishes in $H^0(S, \Omega^1_S(\log \Sigma) \otimes (\mathscr{F}^{i-1}/\mathscr{F}^i))$. □

2.6. *Remark.* The rigidity property (2.2) could invite us – following S. Bloch – to define the group $C^i_{\mathrm{int}}(X)$ as the group of cycles with an integrable K connection, if $S = \operatorname{Spec} K$, where K is a field of characteristic zero. In this case,

$$\mathbb{H}^1(S, \mathscr{F}^i \to \Omega^1_S \otimes \mathscr{F}^{i-1} \to \Omega^2_S \otimes \mathscr{F}^{i-2}) = (\Omega^1_K \otimes \mathscr{F}^{i-1})^\nabla / \nabla \mathscr{F}^i$$

and the class is well defined in $(\Omega^1_K \otimes \mathscr{F}^i)^\nabla$.

2.7. *Remark.* The existence of the lifting of Griffiths' invariant in (2.5) will be used by H. Dunio to prove that the Deligne–Beilinson classes of (E, ∇) are locally constant, generalizing the result of [5] to the case where the morphism f is not constant.

References

1. Cheeger, J. and Simons, J.: Differential characters and geometric invariants, in J. Alexander, J. Harer (eds), *Geometry and Topology, Proceedings of the Special Year, University of Maryland 1983–84*, Lectures Notes in Math. 1167, Springer, Berlin, Heidelberg, New York (1985), pp. 50–80.
2. Dupont, J., Hain, R. and Zucker, S.: Regulators and characteristic classes of flat bundles, preprint 1992, Aarhus University 4.
3. Esnault, H.: Characteristic classes of flat bundles, *Topology* **27** (1988), 323–352.
4. Esnault, H.: A note on the cycle map, *J. Reine Angew. Math.* **411** (1990), 51–65.
5. Esnault, H. and Srinivas, V.: Chern classes of vector bundles with holomorphic connections on a complete smooth variety, *J. Differential Geom.*, to appear.
6. Gabber, O.: talk at the University Paris 7, Feb. 1992.
7. Gersten, S. M.: Some exact sequences in the higher K-theory of rings, *Algebraic* K-*Theory I*, Lecture Notes in Math. **341**, Springer, Berlin, Heidelberg, New York (1972), pp. 211–244.
8. Griffiths, P.: Infinitesimal variations of Hodge structure (III): Determinantal varieties and the infinitesimal invariant of normal functions, *Compositio Math.* **50** (1983), 267–325.
9. Zucker, S.: Hodge theory with degenerating coefficients: L_2 cohomology in the Poincaré metric, *Ann. of Math.* **109** (1979), 415–476.