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Higher Kodaira-Spencer classes

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Contents

Introduction

In [1] Beilinson and Ginzburg announce a canonical description of the sheaf of differential operators on the moduli scheme (or moduli stack) M of principal G-bundles $\mathcal{E}$ on a smooth complex projective curve X of genus $g \geqq 2$, where G is a connected complex semisimple Lie group.

To be a little bit more precise, let $\mathcal{D}_M^n$ be the sheaf of regular differential operators on the (smooth) moduli space M of order smaller than or equal to n. The sheaf $\mathcal{O}_M$ is a subsheaf of $\mathcal{D}_M^n$. Then, starting from the bundle $\mathcal{E}_p$ corresponding to the point $p \in M$, Beilinson and Ginzburg consider a Σ_n equivariant blowing up $\hat{X}^n$ of the n-fold product $X \times \ldots \times X$ and they propose the definition of a Σ_n-sheaf $\mathcal{G}_n$ such that one has a canonical isomorphism

$$\phi_n : (\mathcal{D}_M^n/\mathcal{O}_M)_p^* \leftarrow H^0(\hat{X}^n, \mathcal{G}_n)^{-\Sigma_n}$$

where $(\ )_p$ denotes the geometric fibre and $(\ )^{-\Sigma_n}$ the subspace of anti invariants, i.e. of elements s with $\tau(s) = \text{sign}(\tau) \cdot s$ for $\tau \in \Sigma_n$. One has an exact sequence

$$0 \to \mathcal{D}_M^{n-1}/\mathcal{O}_M \to \mathcal{D}_M^n/\mathcal{O}_M \to S^n(T_M) \to 0$$

where T_M is the tangent sheaf. Therefore one should have a natural exact sequence

$$0 \to S^n H^0(X, \mathcal{G}_1) \to H^0(\hat{X}^n, \mathcal{G}_n)^{-\Sigma_n} \to H^0(\hat{X}^{n-1}, \mathcal{G}_{n-1})^{-\Sigma_{n-1}} \to 0$$

and a commutative diagram

$$\begin{array}{ccccc} S^n H^0(X,\mathcal{G}_1) & \longrightarrow & H^0(\hat{X}^n,\mathcal{G}_n)^{-\Sigma_n} & \longrightarrow & H^0(\hat{X}^{n-1},\mathcal{G}_{n-1})^{-\Sigma_{n-1}} \\ S^n(\phi_1) \downarrow \cong & & \phi_n \downarrow \cong & & \phi_{n-1} \downarrow \cong \\ S^n(T_M)^*_p & \longrightarrow & (\mathcal{D}^n_M/\mathcal{O}_M)^*_p & \longrightarrow & (\mathcal{D}^{n-1}_M/\mathcal{O}_M)^*_p \end{array}$$

where $\phi_1 : H^0(X, \mathcal{G}_1) \to T^*_{M,p}$ is the dual of the Kodaira-Spencer map.

This compatibility, also announced in [1], allows to consider ϕ_n as the dual of a lifting of the Kodaira-Spencer map to higher differential operators. In fact, if S is a non-singular variety, if $\mathcal{F}$ is a family of principal G-bundles on $X \times S$, and $\varphi : S \to M$ is the map to the moduli space induced by S, then for $s \in \varphi^{-1}(p)$ one has a map

$$\varphi_n : (\mathcal{D}^n_S/\mathcal{O}_S)_s \to (\mathcal{D}^n_M/\mathcal{O}_M)_p \to (H^0(\hat{X}^n, \mathcal{G}_n)^{-\Sigma_n})^* .$$

In this paper we want to present a different construction of φ_n for $G = Sl(r, \mathbb{C})$, which works as well for other deformation or moduli problems in any dimension. (It will turn out that in the announcement [1] one has to replace $H^0(\hat{X}^n, \mathcal{G}_n)^{-\Sigma_n}$ by a subspace $H^0(\hat{X}^n, \mathcal{G}_n)^{-\Sigma_n, -\Sigma_{n-1}, \ldots, -\Sigma_2}$, whose definition should be dual to the one used in (6.1)).

Let $f : X \to S$ be a projective flat family of smooth varieties defined over an algebraically closed field k of characteristic zero or a compact flat family of complex smooth analytic varieties (in which case we still write $k = \mathbb{C}$). We will consider a $f^{-1}\mathcal{O}_S$ Lie Algebra $\mathcal{A}$ on X with the following two properties:

a) *There is an extension*

$$0 \to \mathcal{A} \to \tilde{\mathcal{A}} \to f^{-1}T_S \to 0 .$$

The induced edge morphism

$$\varphi_S : T_S \to R^1 f_* \mathcal{A}$$

will be called the ***Kodaira-Spencer map****.*

b) *The $f^{-1}\mathcal{O}_S$ linear Lie bracket*

$$\mathcal{A} \otimes_{f^{-1}\mathcal{O}_S} \mathcal{A} \to \mathcal{A}$$

is induced by a k linear Lie bracket $\mathcal{A} \otimes_k \mathcal{A} \to \mathcal{A}$ which lifts to

$$\mathcal{A} \otimes_k \tilde{\mathcal{A}} \to \mathcal{A} .$$

The easiest example of such a Lie algebra $\mathcal{A}$ is the relative tangent sheaf. One has the exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{X/S} & \longrightarrow & T_X & \longrightarrow & f^*T_S & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & T_{X/S} & \longrightarrow & \tilde{T} & \longrightarrow & f^{-1}T_S & \longrightarrow & 0 \end{array}$$

where T_X, T_S and $T_{X/S}$ are the tangent sheaves and where the second sequence is induced by the first one. The usual Kodaira-Spencer map is the edge morphism

$$\varphi_S : T_S \to R^1 f_* T_{X/S}$$

of the first, or equivalently of the second exact sequence. The latter has been considered by Beilinson and Schechtman in [2], and they used already that $\tilde{T}$ allows a lifting of the Lie algebra structure on $T_{X/S}$. In fact, the $f^{-1}\mathcal{O}_S$ linear Lie bracket

$$T_{X/S} \otimes_{f^{-1}\mathcal{O}_S} T_{X/S} \xrightarrow{[,]} T_{X/S}$$

is defined by

$$[x,y](\lambda) = x(y(\lambda)) - y(x(\lambda)) ,$$

for $x,y \in T_{X/S}$ and $\lambda \in \mathcal{O}_X$. The k linear Lie bracket on T_X

$$T_X \otimes_k T_X \xrightarrow{[,]} T_X$$

restricts to

$$T_{X/S} \otimes_k \tilde{T} \xrightarrow{[,]} T_{X/S} .$$

In general, starting with any $f^{-1}\mathcal{O}_S$ Lie algebra satisfying a) and b) we construct in the first two sections of this paper a complex of sheaves $\mathcal{A}^\bullet(n)$ on the n-fold product

$$X \times_S \ldots X \times_S X \xrightarrow{f} S .$$

Choosing one diagonal embedding

$$X \times_S \ldots \times_S X((n-1) - \text{times}) \xrightarrow{\Delta} X \times_S \ldots \times_S X(n - \text{times})$$

we construct a map

$$R^{n-1} f_* \mathcal{A}^\bullet(n-1) \to R^n f_* \mathcal{A}^\bullet(n) .$$

The symmetric group Σ_n will act on $\mathcal{A}^\bullet(n)$. Let us write again $(\ )^{-G}$ for the anti invariants under a subgroup $G \subset \Sigma^n$. Since the fixgroup of Δ is a subgroup of Σ_n isomorphic to Σ_{n-2}, Δ induces a map

$$(R^{n-1} f_* \mathcal{A}^\bullet(n-1))^{-\Sigma_{n-2}} \to (R^n f_* \mathcal{A}^\bullet(n))^{-\Sigma_n}$$

whose cokernel will turn out to lie in $(R^n f_* \mathcal{A}^0(n))^{-\Sigma_n}$. We will define in Sect. 6 a quotient complex

$$(R^n f_* \mathcal{A}^\bullet(n))^{-\Sigma_n, \ldots, -\Sigma_2}$$

of $(R^n f_* \mathcal{A}^\bullet(n))^{-\Sigma_n}$ inductively by push forward in the diagram

$$\begin{array}{ccccc}
(R^{n-1} f_* \mathcal{A}^\bullet(n-1))^{-\Sigma_{n-1}, \ldots, -\Sigma_2} & \longrightarrow & (R^n f_* \mathcal{A}^\bullet(n))^{-\Sigma_n, \ldots, -\Sigma_2} & \longrightarrow & (R^n f_* \mathcal{A}^0(n))^{-\Sigma_n} \\
\text{surj.}\uparrow & & \text{surj.}\uparrow & & =\uparrow \\
(R^{n-1} f_* \mathcal{A}^\bullet(n-1))^{-\Sigma_{n-2}} & \longrightarrow & (R^n f_* \mathcal{A}^\bullet(n))^{-\Sigma_n} & \longrightarrow & (R^n f_* \mathcal{A}^0(n))^{-\Sigma_n} .
\end{array}$$

On the left hand side of the diagram we regard $(R^{n-1} f_* \mathcal{A}^\bullet(n-1))^{-\Sigma_{n-1}}$ as a quotient of $(R^{n-1} f_* \mathcal{A}^\bullet(n-1))^{-\Sigma_{n-2}}$ by means of the trace map (see Sect. 6).

Using this notation the main result of this paper says:

Assume that the $f^{-1}\mathcal{O}_S$ *Lie algebra* $\mathcal{A}$ *satisfies* a) *and* b) *and either one of the following two assumptions:*
c.1) $R^p f_* \mathcal{A} = 0$ *for* $p > 1$.
c.2) $f_* \mathcal{A} = 0$.

Then $(R^n f_* \mathcal{A}^0(n))^{-\Sigma_n} = S^n(R^1 f_* \mathcal{A})$ *and there exists a natural morphism of left* $\mathcal{O}_S$*-modules*

$$\varphi_{n,S} : \mathcal{D}^n_S/\mathcal{O}_S \to (R^n f_* \mathcal{A}^\bullet(n))^{-\Sigma_n,\ldots,-\Sigma_2}$$

such that the following diagram commutes

$$\begin{array}{ccc} 0 & & \\ \uparrow & & \\ S^n(T_S) & \xrightarrow{S^n(\varphi_S)} & S^n(R^1 f_* \mathcal{A}) \\ \uparrow & & \uparrow \\ \mathcal{D}^n_S/\mathcal{O}_S & \xrightarrow{\varphi_{n,S}} & (R^n f_* \mathcal{A}^\bullet(n))^{-\Sigma_n,\ldots,-\Sigma_2} \\ \uparrow & & \uparrow \\ \mathcal{D}^{n-1}_S/\mathcal{O}_S & \xrightarrow{\varphi_{n-1,S}} & (R^{n-1} f_* \mathcal{A}^\bullet(n-1))^{-\Sigma_{n-1},\ldots,-\Sigma_2} \\ \uparrow & & \\ 0\,. & & \end{array}$$

In particular, if $\varphi_S : T_S \to R^1 f_* \mathcal{A}$ is surjective the morphisms $\varphi_{n,S}$ are surjective as well. Under the assumption c.2) the morphism

$$(R^{n-1} f_* \mathcal{A}^\bullet(n-1))^{-\Sigma_{n-1},\ldots,-\Sigma_2} \to (R^n f_* \mathcal{A}^\bullet(n))^{-\Sigma_n,\ldots,-\Sigma_2}$$

is injective and hence the injectivity of

$$\varphi_S : T_S \to R^1 f_* \mathcal{A}$$

implies the injectivity of all the $\varphi_{n,S}$. Under the assumption c.1) a slightly different argument shows that $\varphi_{n,S}$ is an isomorphism if φ_S is an isomorphism.

As for the Kodaira-Spencer class φ_S itself, $\varphi_{n,S}$ comes from the edge morphism $\tilde{\varphi}_n$ of a n extension of $f^{-1}(T_S \otimes_k \ldots \otimes_k T_S)$ (n-times) by $\mathcal{A}^\bullet(n)$ on the n fold product $X \times_S \ldots \times_S X$ which we construct in the second half of Sect. 3. This n extension defines

$$\tilde{\varphi}_n : T_S \otimes_k T_S \otimes_k \ldots \otimes_k T_S \to R^n f_* \mathcal{A}^\bullet(n)\,.$$

In Sect. 4 we calculate in the language of Čech-cohomology the difference

$$\tilde{\varphi}_n(x_1 \otimes x_2 \otimes x_3 \otimes \ldots \otimes x_n) - \tilde{\varphi}_n(x_2 \otimes x_1 \otimes x_3 \otimes \ldots \otimes x_n)\,.$$

In Sect. 5 we discuss the properties of the sheaf of differential operators on a smooth variety. Finally, in Sect. 6 we define the sheaves $(R^n f_* \mathcal{A}^\bullet(n))^{-\Sigma_n,\ldots,-\Sigma_2}$ together with the natural maps occuring in our main result. We use the relation, obtained in Sect. 4, to show that $\tilde{\varphi}_n$ induces the maps $\varphi_{n,S}$.

Let us return to the example considered above, i.e. to the case $\mathcal{A} = T_{X/S}$. The condition c.2) is satisfied if the fibres of f have no infinitesimal automorphisms and our main result says that the Kodaira-Spencer map for such a family can be extended to the whole sheaf of differential operators. If S happens to be the nonsingular locus of a fine moduli space for certain manifolds without infinitesimal automorphisms, then this implies that $\mathcal{D}^n_S/\mathcal{O}_S$ is isomorphic to $(R^n f_* \mathcal{A}^\bullet(n))^{-\Sigma_n,\ldots,-\Sigma_2}$.

Other examples of families for which a $f^{-1}\mathcal{O}_S$ Lie algebra $\mathcal{A}$ exists with properties a) and b) and for which, of course, the Kodaira-Spencer map defined in a) is the usual one will be discussed in the first half of Sect. 3. They will include families of vectorbundles on curves (one example where the condition c.1) holds true), and families of stable vectorbundles over higher dimensional manifolds.

This paper grew out of an attempt of Schechtman and us to understand the results announced in [1]. The influence of the ideas of Beilinson and Ginzburg on our work is obvious.

1 The sheaves $\mathcal{A}^l(n)$ and the Σ_n action

Let $f : X \to S$ be a flat morphism of schemes over an algebraically closed field k of characteristic 0 or a flat morphism of analytic spaces. Let $\mathcal{A}$ be a locally, free $\mathcal{O}_X$-module. If $X \times_S \ldots \times_S X$ is the n-fold product we denote the structure map again by

$$f : X \times_S \ldots \times_S X \to S ,$$

and the projection to the i-th factor by

$$\mathrm{pr}_i : X \times_S \ldots \times_S X \to X .$$

In Sect. 2 we assume that we have a bracket

$$[\,,\,] : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

and we will construct a complex $\mathcal{A}^\bullet(n)$, concentrated in degrees $0, \ldots, n-1$ together with an action of the symmetric group Σ_n. In this section we start by defining the sheaves $\mathcal{A}^l(n)$, supported on the diagonals of codimension $l \cdot \dim(X)$; then we define the Σ_n action and consider the antiinvariants under this group.

1.1 Notations. For $i = 1, \ldots, r$, let $I_i \subseteq \{1, \ldots, n\}$ be subsets with

$$\bigcup_{i=1}^{r} I_i = \{1, \ldots, n\} \quad \text{and} \quad I_i \cap I_j = \emptyset \quad \text{for } i \neq j .$$

We write $\Delta_{I_1,\ldots,I_r}$ for the diagonal given by local equations

$$\{\underline{t}_\mu = \underline{t}_\nu;\ \nu, \mu \in I_i,\ i = 1, \ldots, r\}$$

where $\underline{t} = t^{(1)}, \ldots, t^{(d)}$ are coordinates of X and $\underline{t}_\nu = \mathrm{pr}_\nu^*(\underline{t})$. We will regard

$$\underline{I} = (I_1, \ldots, I_r)$$

as an ordered tupel of subsets of $\{1, \ldots, n\}$ and write

$$\Delta_{\underline{I}} = \Delta_{I_1,\ldots,I_r} .$$

Obviously one has

1.2 *Properties.* a) $\Delta_{\{1\},\ldots,\{n\}} = X \times_S \ldots \times_S X$.

b) For $\pi \in \Sigma_r$ we have $\Delta_{I_1,\ldots,I_r} = \Delta_{I_{\pi(1)},\ldots,I_{\pi(r)}}$.

c) $\mathrm{codim}(\Delta_{I_1,\ldots,I_r} \subset X \times_S \ldots \times_S X) = (n-r) \cdot \dim X$.

d) $\Delta_{I_1,\ldots,I_r} \simeq X \times_S \ldots \times_S X$ (r-times).

The group Σ_n acts on $X \times_S \ldots \times_S X$ by permuting the coordinate functions on the different factors.

1.3 Lemma and definition. a) *For* $\underline{I} = (I_1, \ldots, I_r)$ *as in* (1.1) *let*

$$F(\underline{I}) = \{\sigma \in \Sigma_n; \sigma(\Delta_{\underline{I}}) = \Delta_{\underline{I}}\} .$$

Then

$$F(\underline{I}) = \{\sigma \in \Sigma_n; \text{ for some } \pi \in \Sigma_r \text{ one has } \sigma(I_i) = I_{\pi(i)} \text{ for all } i\} .$$

b) *Let us call* $\underline{I}$ ***ordered***, *if*

$$\mathrm{Min}\{\nu \in I_i\} < \mathrm{Min}\{\mu \in I_{i+1}\}$$

for $i = 1, \ldots, r-1$.

c) *If* $\underline{I}$ *is any tuple as in* (1.1), *then there is a unique element* $\pi \in \Sigma_r$ *such that* $(I_{\pi(1)}, \ldots, I_{\pi(r)})$ *is ordered.*

1.4 Notation. Let $\mathrm{pr}_{I_i} : \Delta_{\underline{I}} \to X$ be the projection to the i-th factor (using (1.2.d). Hence $\mathrm{pr}_{I_i} = \mathrm{pr}_\nu|_{\Delta_{\underline{I}}}$ for all $\nu \in I_i$. Let $\mathcal{R}$ be a sheaf of rings on S such that the sheaf $\mathcal{A}$ is a sheaf of $f^{-1}\mathcal{R}$-modules, and such that the Lie product

$$[\,,] : \mathcal{A} \otimes_{f^{-1}\mathcal{R}} \mathcal{A} \to \mathcal{A}$$

is defined over $f^{-1}\mathcal{R}$. Then we define

$$\mathcal{A}_{\underline{I}} = \mathcal{A}_{I_1} \boxtimes \mathcal{A}_{I_2} \boxtimes \ldots \boxtimes \mathcal{A}_{I_r} = \boxtimes_{i=1}^r \mathcal{A}_{I_i}$$

to be the sheaf

$$\bigotimes_{i=1}^{r} f^{-1}(\mathcal{R}) \mathrm{pr}_{I_i}^{-1} \mathcal{A} .$$

1.5 *Remarks.* a) We will consider the cases $\mathcal{R} = \mathcal{O}_S$ and $\mathcal{R} = k$, the constant sheaf on S, where k is the field of definition for S, or $k = \mathbb{C}$ in case that X and S are analytic manifolds.

b) As long as the bracket $[\,,] : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ in (2.1) is $\mathcal{O}_X$-linear, everything said in this section for $\mathcal{A}_{\underline{I}}$ remains true for

$$\bigotimes_{\mathcal{O}_{X \times_S \cdots \times_S X}} \mathrm{pr}_{I_i}^* \mathcal{A} .$$

c) Let us write

$$F_0(\underline{I}) = \{\sigma \in \Sigma_n; \sigma(I_i) = I_i\} .$$

Then $F_0(\underline{I}) \subset F(\underline{I})$ is a subgroup and for $\sigma \in F_0(\underline{I})$ one has $\mathcal{A}_{\underline{I}} = \mathcal{A}_{\sigma(\underline{I})}$.

1.6 Notation. Let us write for $l = n - r$

$$\mathcal{A}^l(n) = \mathcal{A}_r(n) = \bigoplus \mathcal{A}_{\underline{I}} ,$$

where the direct sum is taken over all ordered tuples $\underline{I} = (I_1, \ldots, I_r)$. In particular, if $\underline{I} = (I_1, \ldots, I_r)$ is any tuple as in (1.1), then $\mathcal{A}^l(n)$ has exactly one summand with support $\Delta_{\underline{I}}$. If $\pi \in \Sigma_r$ is chosen such that $(I_{\pi(1)}, \ldots, I_{\pi(r)})$ is ordered, then this summand is $\mathcal{A}_{I_{\pi(1)}, \ldots, I_{\pi(r)}}$.

1.7 *The action of Σ_n.* The Σ_n action on $X \times_S \ldots \times_S X$ extends to a Σ_n action on $\mathcal{A}^l(n)$:

For $\tau \in \Sigma_n$ and $\underline{I}$ ordered we can find some $\pi \in \Sigma_r$ such that for $\underline{I}' = \tau(\underline{I})$ the tupel $(I'_{\pi(1)}, \ldots, I'_{\pi(r)})$ is ordered. Then an action of Σ_n on $\mathcal{A}^l(n)$ – which is not yet compatible with the differential defined in Sect. 2 – is given by

$$\tau^{-1}(a_{I_1} \boxtimes \ldots \boxtimes a_{I_r}) = b_{I'_{\pi(1)}} \boxtimes \ldots \boxtimes b_{I'_{\pi(r)}}$$

where $b_{I'_{\pi(i)}} = a_{I_i}$. Here τ is acting on the elements of $\{1, \ldots, n\}$ and π on the indices $\{1, \ldots, r\}$

$$(I_1, \ldots, I_r) \xrightarrow{\tau} (I'_1, \ldots, I'_r) \xrightarrow{\pi} (I'_{\pi(1)}, \ldots, I'_{\pi(r)}) .$$

In other terms, π is the element of Σ_r with

$$\mathrm{Min}\{\nu \in I'_{\pi(i)}\} < \mathrm{Min}\{\nu \in I'_{\pi(i+1)}\}$$

for $i = 1, \ldots, r-1$.

1.8 Definition. *We consider $\mathcal{A}^l(n)$ as a Σ_n sheaf with the action*

$$\tau^\# = \mathrm{sign}(\tau) \cdot \mathrm{sign}(\pi) \cdot \tau^{-1}$$

on $\mathcal{A}_{\underline{I}}$.

1.9 *Examples.* Assume that $\underline{I}$ is ordered.

a) If $\underline{I} = (\{1\}, \{2\}, \ldots, \{n\})$ then $\pi = \tau^{-1}$ and $\tau^\# = \tau^{-1}$.

b) If $\underline{I} = (\{1,2,3\}, \{4,5\})$ and $\tau = (1,5) \in \Sigma_5$, then $\tau(\underline{I}) = (\{5,2,3\}, \{4,1\})$ and $\pi = (1,2) \in \Sigma_2$.

c) For $\sigma \in F_0(\underline{I})$ (see (1.5)) the map $\tau^\# = \mathcal{A}_{\underline{I}} \to \mathcal{A}_{\underline{I}}$ is the multiplication by $\mathrm{sign}(\tau)$. In fact, the corresponding π is the identity.

d) Let $\tau = (\nu, \mu)$ be a two cycle in Σ_n and assume that $\tau \notin F_0(\underline{I})$ or, equivalently, that $\nu \in I_i$ and $\mu \in I_j$ for some $i \neq j$. Then $\tau^\# = \tau^{-1}$ if $\nu = \mathrm{Min}\, I_i$ and $\mu = \mathrm{Min}\, I_j$, but $\tau^\# = -\tau^{-1}$ if $\nu > \mathrm{Min}\, I_i$ and $\mu > \mathrm{Min}\, I_j$.

1.10 Notations. Let $S^n(X)$ be the quotient $(X \times_S \ldots \times_S X)/\Sigma_n$. If X is a projective (or an analytic) variety, then $S^n(X)$ exists as a projective (or analytic) variety. Let

$$\begin{array}{ccc} X \times_S \ldots \times_S X & \xrightarrow{\delta} & S^n(X) \\ f\downarrow & & g\downarrow \\ S & \xrightarrow{=} & S \end{array}$$

denote the induced morphisms and let $G \subset \Sigma_n$ be a subgroup. If $\mathcal{F}$ is a G-sheaf on $X \times_S \ldots \times_S X$, then we write $(\delta_* \mathcal{F})^{-G}$ for the sheaf of antiinvariants, i.e: $(\delta_* \mathcal{F})^{-G}$ is the subsheaf of $\delta_* \mathcal{F}$ on which $\tau \in G$ acts by multiplication with $\mathrm{sign}(\tau)$.

1.11 *Examples.* Assume that $\underline{I}$ is ordered.

a) $(\delta_* \mathcal{A}_{\underline{I}})^{-F_0(\underline{I})} = \delta_* \mathcal{A}_{\underline{I}}$ (here π is the identity).

b) If $F_1 \subset F(\underline{I})$ is a subgroup such that $F_0(\underline{I})$ and F_1 generate $F(\underline{I})$. Then

$$(\delta_* \mathcal{A}_{\underline{I}})^{-F(\underline{I})} = (\delta_* \mathcal{A}_{\underline{I}})^{-F_1} .$$

c) Let $G \subseteq \Sigma_n$ be a subgroup such that $F_0(\underline{I})$ and $G \cap F(\underline{I}) = F_1$ generate $F(\underline{I})$. Let $\underline{I} = \underline{I}^{(1)}, \underline{I}^{(2)}, \ldots, \underline{I}^{(s)}$ be ordered tuples such that $\{\Delta_{\underline{I}^{(1)}}, \ldots, \Delta_{\underline{I}^{(s)}}\}$ is the G-orbit of $\Delta_{\underline{I}}$. Then

$$\left(\delta_* \bigoplus_{j=1}^{s} \mathcal{A}_{\underline{I}(j)}\right)^{-G} = (\delta_* \mathcal{A}_{\underline{I}})^{-F(\underline{I})} .$$

Proof. a) is a special case of b) and both follow from (1.9.c).

If $\underline{a} = \bigoplus_{j=1}^{s} a_{\underline{I}(j)}$ is G antiinvariant, then $a_{\underline{I}(1)}$ is F_1-antiinvariant. Moreover, $\underline{a}$ is determined by the component $a_{\underline{I}(1)}$. Hence, for a given F_1-antiinvariant $a_{\underline{I}(1)}$, we just have to show that we find some $a_{\underline{I}(i)}$ such that the local section

$$\underline{a} = \bigoplus_{i=1}^{s} a_{\underline{I}(i)} \in \delta_* \bigoplus_{j=1}^{s} \mathcal{A}_{\underline{I}(i)}$$

is G-antiinvariant.

Let $\sigma_1 = \mathrm{id}, \sigma_2, \ldots, \sigma_r$ represent the different cosets of G/F_1 and let $\pi_j \in \Sigma_r$ be chosen such that

$$(\sigma_j(I_{\pi_j^{-1}(1)}), \ldots, \sigma_j(I_{\pi_j^{-1}(r)}))$$

is ordered. We can number the σ_j such that

$$\underline{I}^{(j)} = (\sigma_j(I_{\pi_j^{-1}(1)}), \ldots, \sigma_j(I_{\pi_j^{-1}(r)})) .$$

Then we define

$$a_{\underline{I}(j)} = \mathrm{sign}(\pi_j) \cdot \sigma_j^{-1}(a_{\underline{I}}) .$$

This definition is independent of the choice of σ_j. In fact, for $\gamma \in F_1$ we have

$$\gamma^{\#} a_{\underline{I}} = \mathrm{sign}(\gamma) \cdot a_{\underline{I}} ,$$

as $a_{\underline{I}}$ is F_1-antiinvariant. On the other hand, if $\tau \in \Sigma_r$ with

$$\underline{I} = (\gamma(I_{\tau^{-1}(1)}), \ldots, \gamma(I_{\tau^{-1}(r)}))$$

then

$$\gamma^{\#} a_{\underline{I}} = \mathrm{sign}(\gamma) \cdot \mathrm{sign}(\tau) \cdot \gamma^{-1}(a_{\underline{I}})$$

and we obtain $\mathrm{sign}(\tau) \cdot \gamma^{-1}(a_{\underline{I}}) = a_{\underline{I}}$. Since replacing σ_j by $\sigma_j \circ \gamma$ forces us to replace π_j by $\pi_j \circ \tau$, we obtain the independence. By definition we have for $\gamma \in F_1$

$$(\sigma_j \circ \gamma)^{\#}(a_{\underline{I}}) = \mathrm{sign}(\sigma_j \cdot \gamma) \cdot a_{\underline{I}(i)} .$$

On the other hand, since $F_j = G \cap F(\underline{I}^{(j)})$ is conjugate to F_1, the element $a_{\underline{I}(j)}$ is F_j-antiinvariant, and repeating this argument, we have for $\sigma \in G$ with $\sigma(\Delta_{\underline{I}(j)}) = \Delta_{\underline{I}(j)}$

$$\sigma^{\#}(a_{\underline{I}(j)}) = \mathrm{sign}(\delta) \cdot a_{\underline{I}(j)} .$$

Hence $\underline{a}$ is G-antiinvariant.

1.12 Notations. Let $r = n - l$ and let $\mathcal{D}_2^l(n)$ be the subsheaf of $\mathcal{A}^l(n)$ given by

$$\mathcal{D}_2^l(n) = \bigoplus \mathcal{A}_{\underline{I}} ,$$

where the direct sum is taken over all ordered tupels $\underline{I} = (I_1, \ldots, I_r)$ with

$$\{1,2\} \subset I_1 .$$

Then Σ_{n-2}, as the group of permutations of $\{3, \ldots, n\}$, and Σ_2, as the group of permutations of $\{1,2\}$ act on $\mathcal{D}_2^l(n)$.

1.13 Lemma. *One has, for* $l \geqq 1$,

$$(\delta_* \mathcal{A}^l(n))^{-\Sigma_n} = (\delta_* \mathcal{D}_2^l(n))^{-\Sigma_2 \times \Sigma_{n-2}} = (\delta_* \mathcal{D}_2^l(n))^{-\Sigma_{n-2}} .$$

Proof. Since the 2-cycle (1, 2) acts by multiplication with (-1) on each $\mathcal{A}_{\underline{I}}$ with

$$\{1,2\} \subset I_1$$

the second equality is obvious. By (1.11.c)

$$\left(\delta_* \mathcal{A}^l(n)\right)^{-\Sigma_n} = \bigoplus_{\nu=1}^{\eta} (\mathcal{A}_{\underline{J}^{(\nu)}})^{-F(\underline{J}^{(\nu)})}$$

where $\underline{J}^{(1)}, \ldots, \underline{J}^{(\eta)}$ are ordered tuples representing the different Σ_n orbits

$$\{\Delta_{\underline{I}^{(1)}}, \ldots, \Delta_{\underline{I}^{(s)}}\} .$$

Of course we can choose $\underline{J}^{(\nu)} = (J_1^{(\nu)}, \ldots, J_r^{(\nu)})$ such that $\{1,2\} \subseteq J_1^{(\nu)}$. Then

$$(1,2) \in F(\underline{J}^{(\nu)})$$

and by (1.11.c) again, applied to $G = \Sigma_{n-2}$, one obtains

$$\left(\delta_* \mathcal{D}_2^l(n)\right)^{-\Sigma_{n-2}} = \bigoplus_{\nu=1}^{\eta} (\mathcal{A}_{\underline{J}^{(\nu)}})^{-F(\underline{J}^{(\nu)})} .$$

2 The $\mathcal{A}^\bullet$ (n) complex

2.1 Notations. Using the notations from Sect. 1, we assume that we have an $f^{-1}\mathcal{R}$ linear pairing

$$[\,,] : \mathcal{A} \otimes_{f^{-1}\mathcal{R}} \mathcal{A} \to \mathcal{A}$$

with

$$a \otimes b \mapsto [a,b]$$

satisfying:

a) $[a,b] = -[b,a]$

b) (Jacobi-identity) $[[a,b],c] + [[b,c],a] + [[c,a],b] = 0.$

The next aim is to use [,] to define a Σ_n invariant differential

$$d : \mathcal{A}^l(n) \to \mathcal{A}^{l+1}(n)$$

such that $(\mathcal{A}^\bullet(n), d)$ is a complex of $f^{-1}\mathcal{R}$ sheaves with a $f^{-1}\mathcal{R}$ linear differential.

2.2 Notation. Let $\underline{J} = (J_1, \ldots, J_{r-1})$ and $\underline{I} = (I_1, \ldots, I_r)$ be ordered tuples as defined in (1.1) and (1.3). We write $\underline{J} \prec \underline{I}$ if there exist numbers v and μ with $1 \leqq \mu < v \leqq r$ such that

$$J_i = \begin{cases} I_i & \text{for} \quad i < \mu \text{ and } \mu < i < v \\ I_{i+1} & \text{for} \quad i \geqq v \\ I_v \cup I_\mu & \text{for} \quad i = \mu \,. \end{cases}$$

For $\underline{I}$, μ and v given we can construct some $\underline{J} \prec \underline{I}$ by choosing $J_i = I_i$, for $i < \mu$ and $\mu < i < v$, $J_i = I_{i+1}$ for $i \geqq v$ and by defining $J_\mu = I_v \cup I_\mu$. Of course the tuple $\underline{J}$ obtained is ordered.

2.3 Definition. *For $\underline{J} \prec \underline{I}$ and $\mu < v$ as above we define*

$$d_{\underline{J}\,\underline{I}} : \mathcal{A}_{\underline{I}} \to \mathcal{A}_{\underline{J}}$$

by

$$d_{\underline{J}\,\underline{I}}(a_{I_1} \boxtimes \ldots \boxtimes a_{I_r}) = (-1)^{n+v} \cdot b_{J_1} \boxtimes \ldots \boxtimes b_{J_{r-1}}$$

where

$$b_{J_i} = \begin{cases} a_{I_i} & \text{for} \quad i < \mu \quad \text{and } \mu < i < v \\ a_{I_{i+1}} & \text{for} \quad i \geqq v \\ [a_{I_\mu}, a_{I_v}] & \text{for} \quad i = \mu \,. \end{cases}$$

2.4 *Remark.* Up to the $(-1)^n$ factor, which we added to get compatibility of the sign rules for different n, this corresponds to the usual conventions. If one drops the condition of "ordered" for the tuples $\underline{J}$ one could choose

$$d(a_{I_1} \boxtimes \ldots \boxtimes a_{I_r})_{\underline{J}} = (-1)^{n+v+\mu}[a_{I_\mu}, a_{I_v}] \boxtimes \ldots \boxtimes \widehat{a_{I_\mu}} \boxtimes \ldots \boxtimes \widehat{a_{I_v}} \boxtimes \ldots \boxtimes a_{I_r} \,.$$

Then, rearranging the tuple forces us to introduce an additional sign $(-1)^\mu$.

2.5 Definition. *For $l = n - r$ we define*

$$d : \mathcal{A}^l(n) \to \mathcal{A}^{l+1}(n)$$

by

$$d|_{\mathcal{A}_{\underline{I}}} = \sum_{\underline{J} \prec \underline{I}} d_{\underline{J}\,\underline{I}} \,.$$

2.6 *Example.* Let us consider the case $n = 3$. We have maps

$$\mathcal{A}^0(3) \xrightarrow{d} \mathcal{A}^1(3) \xrightarrow{d} \mathcal{A}^2(3)$$

given by:

$$\mathcal{A}_{\{1\}\{2\}\{3\}} \xrightarrow{-} \mathcal{A}_{\{1,2\}\{3\}} \xrightarrow{-} \mathcal{A}_{\{1,2,3\}}$$

$$\mathcal{A}_{\{1\}\{2\}\{3\}} \xrightarrow{+} \mathcal{A}_{\{1,3\}\{2\}} \xrightarrow{-} \mathcal{A}_{\{1,2,3\}}$$

$$\mathcal{A}_{\{1\}\{2\}\{3\}} \xrightarrow{+} \mathcal{A}_{\{1\}\{2,3\}} \xrightarrow{-} \mathcal{A}_{\{1,2,3\}}$$

where we just give the signs, the maps themselves being the obvious ones, for example

$$\begin{aligned} \mathcal{A}_{\{1\}\{2\}\{3\}} &\xrightarrow{+} \mathcal{A}_{\{1,3\}\{2\}} \\ a \otimes b \otimes c &\mapsto +[a,c] \otimes b \end{aligned}$$

or

$$\begin{aligned} \mathcal{A}_{\{1,3\}\{2\}} &\xrightarrow{-} \mathcal{A}_{\{1,2,3\}} \\ a \otimes b &\mapsto -[a,b] \,. \end{aligned}$$

We obtain

$$\begin{aligned} d^2(a \otimes b \otimes c) &= [[a,b],c] - [[a,c],b] - [a,[b,c]] \\ &= [[a,b],c] + [[c,a],b] + [[b,c],a] = 0 \,. \end{aligned}$$

Before we show that $d^2 = 0$ for all n, let us give an inductive description for d.

2.7. We can write $\mathcal{A}^l(n)$ as a direct sum of $\mathcal{B}^l(n)$ and $\mathcal{D}^l(n)$, where we take for

$$\mathcal{B}^l(n) = \bigoplus \mathcal{A}_{\underline{I}}$$

the direct sum over all ordered tuples $\underline{I} = (\{1\}, I_2, \ldots, I_r)$, and for

$$\mathcal{D}^l(n) = \bigoplus \mathcal{A}_{\underline{I}}$$

we take the direct sum over ordered tuples $\underline{I} = (I_1, I_2, \ldots, I_r)$ with $\sharp I_1 > 1$. Of course, since $\underline{I}$ is supposed to be ordered, $1 \in I_1$. We have

$$d(\mathcal{D}^l(n)) \subseteq \mathcal{D}^{l+1}(n) \,.$$

In fact, we can go one step further. If for $\eta \in \{2, \ldots n\}$ we write (as in (1.12))

$$\mathcal{D}^l_\eta(n) = \bigoplus_{\{1,\eta\} \subseteq I_1,\, \underline{I} \text{ ordered}} \mathcal{A}_{\underline{I}} \,,$$

then $d(\mathcal{D}^l_\eta(n)) \subseteq \mathcal{D}^{l+1}_\eta(n)$. Identifying

$$\Delta_{\{1,\eta\},\{2\},\ldots,\widehat{\{\eta\}},\ldots,\{n\}}$$

with the $(n-1)$ fold product $X \times_S \ldots \times_S X$, let $\mathcal{A}^l(n-1)$ be the sheaf constructed for the index set

$$\{1, \ldots, \eta - 1, \widehat{\eta}, \eta + 1, \ldots n\} \,.$$

Then "leaving out η" gives an isomorphism

$$\mathcal{D}^l_\eta(n) \xleftarrow[\iota^l]{\cong} \mathcal{A}^{l-1}(n-1)$$

and, due to the $(-1)^n$ in the definition of d, the differential changes signs. Hence we have a commutative diagram:

2.8.

$$\begin{array}{ccccccc} \mathcal{A}^{l-1}(n-1) & \xrightarrow{(-1)^l\cdot\iota^l} & \mathcal{D}^l_\eta(n) & \longrightarrow & \mathcal{D}^l(n) & \longrightarrow & \mathcal{A}^l(n) \\ \downarrow d & & \downarrow d & & \downarrow d & & \downarrow d \\ \mathcal{A}^l(n-1) & \xrightarrow[(-1)^{l+1}\cdot\iota^{l+1}]{} & \mathcal{D}^{l+1}_\mu(n) & \longrightarrow & \mathcal{D}^{l+1}(n) & \longrightarrow & \mathcal{A}^{l+1}(n)\,. \end{array}$$

On the other hand, $\mathcal{B}^l(n) = \mathrm{pr}_1^{-1}\mathcal{A}\boxtimes\mathcal{A}^l(n-1)$, where now $\{2,\ldots,n\}$ is the index set for $\mathcal{A}^l(n-1)$ on the $(n-1)$-fold product $X\times_S\ldots\times_S X$, and where we identify $\Delta_{\{1\},I_2,\ldots,I_r}$ with $X\times_S\Delta_{I_2,\ldots,I_r}$. We have a commutative diagram:

2.9.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D}^l(n) & \longrightarrow & \mathcal{A}^l(n) & \longrightarrow & \mathcal{B}^l(n) = \mathrm{pr}_1^{-1}\mathcal{A}\boxtimes\mathcal{A}^l(n-1) & \longrightarrow & 0 \\ & & \downarrow d & & \downarrow d & & \downarrow \mathrm{id}\boxtimes d & & \\ 0 & \longrightarrow & \mathcal{D}^{l+1}(n) & \longrightarrow & \mathcal{A}^{l+1}(n) & \longrightarrow & \mathcal{B}^{l+1}(n) = \mathrm{pr}_1^{-1}\mathcal{A}\boxtimes\mathcal{A}^{l+1}(n-1) & \longrightarrow & 0\,. \end{array}$$

If we consider $\mathcal{B}^l(n)$ as a subsheaf of $\mathcal{A}^l(n)$, then d does not respect $\mathcal{B}^\bullet(n)$.

2.10 Definition. *Let $x\in\mathcal{A}$ and $\underline{a}\in\mathcal{A}^l(n-1)$ be local sections on X and on the $(n-1)$-fold product $X\times_S\ldots\times_S X$, respectively, with*

$$\underline{a} = \bigoplus_{\underline{I}'} a_{I_1'}\boxtimes a_{I_2'}\boxtimes\ldots\boxtimes a_{I_{r-1}'} = \bigoplus_{\underline{I}'} a_{\underline{I}'}\,.$$

Define $[x,\underline{a}] = \sum_{\underline{I}'}[x,a_{\underline{I}'}]$ with $[x,a_{\underline{I}'}] = \bigoplus_{\underline{J}} b_{\underline{J}}$, where the sum is taken over ordered tuples $\underline{J} = (J_1,\ldots,J_{r-1})$ and

$$b_{\underline{J}} = \begin{cases} (-1)^{n+\nu+1} & [x,a_{I_\nu'}]\boxtimes a_{I_1'}\boxtimes\ldots\boxtimes\widehat{a_{I_\nu'}}\boxtimes\ldots\boxtimes a_{I_{r-1}'} \\ & \text{if } J_1 = \{1\}\cup I_\nu' \text{ and } (J_2,\ldots,J_{r-1}) = (I_1',\ldots,\widehat{I_\nu'},\ldots I_{r-1}') \\ 0 & \text{otherwise}\,. \end{cases}$$

2.11 Lemma. *For $x\in\mathcal{A}$ and $\underline{a}\in\mathcal{A}^l(n-1)$ we have*

$$d(x\boxtimes\underline{a}) = [x,\underline{a}] + x\boxtimes d(\underline{a})$$

where $d(x\boxtimes\underline{a})$ is the differential in $\mathcal{A}^l(n)$ and $d(\underline{a})$ the differential in $\mathcal{A}^l(n-1)$.

Proof. For $\underline{a} = a_{I_1'}\boxtimes\ldots\boxtimes a_{I_{r-1}'}$ we have $x\boxtimes\underline{a}\in\mathcal{A}_I$ for $\underline{I} = (\{1\},I_1',\ldots,I_{r-1}')$. We can verify (2.11) in each $\mathcal{A}_{\underline{J}}$ with $\underline{J}\prec\underline{I}$ separately.

Case a) $J_1 = I_1 = \{1\}$.

Hence we have $1<\mu<\nu\leqq n$ and

$$J_i = \begin{cases} I_i = I_{i-1}' & \text{for } 1<i<\mu \text{ and } \mu<i<\nu \\ I_{i+1} = I_i' & \text{for } i\geqq\nu \\ I_\nu\cup I_\mu = I_{\nu-1}'\cup I_{\mu-1}' & \text{for } i=\mu\,. \end{cases}$$

In $\mathcal{A}_{\underline{J}}$ only the second summand occurs and

$$\begin{aligned} d_{\underline{J}\,\underline{I}}(x\boxtimes\underline{a}) &= (-1)^{n+\nu}x\boxtimes a_{I_2}\boxtimes\ldots\boxtimes a_{I_{\mu-1}}\boxtimes[a_{I_\mu},a_{I_\nu}]\boxtimes\ldots\boxtimes\widehat{a_{I_\nu}}\boxtimes\ldots\boxtimes a_{I_r} \\ &= (-1)^{(n-1)+(\nu-1)}x\boxtimes a_{I'_1}\boxtimes\ldots\boxtimes a_{I'_{\mu-2}}\boxtimes[a_{I'_{\mu-1}},a_{I'_\nu-1}] \\ &\quad \boxtimes\ldots\boxtimes\widehat{a_{I'_{\nu-1}}}\boxtimes\ldots\boxtimes a_{I'_{r-1}} \\ &= x\boxtimes d_{\underline{J}'\underline{I}'}(\underline{a}) \end{aligned}$$

for

$$\underline{J}' = (J_2,\ldots,J_{r-1})\,.$$

Case b) $J_1 = \{1\}\cup I_\nu = \{1\}\cup I'_{\nu-1}$.

Only the first summand occurs and

$$\begin{aligned} d_{\underline{J}\,\underline{I}}(x\boxtimes\underline{a}) &=(-1)^{n+\nu}[x,a_{I_\nu}]\boxtimes a_{I_2}\boxtimes\ldots\boxtimes\widehat{a_{I_\nu}}\boxtimes\ldots\boxtimes a_{I_r} \\ &=(-1)^{n+\nu}[x,a_{I'_{\nu-1}}]\boxtimes a_{I'_1}\boxtimes\ldots\boxtimes\widehat{a_{I'_{\nu-1}}}\boxtimes\ldots\boxtimes a_{I'_{r-1}} \end{aligned}$$

is the element $b_{\underline{J}}$ of (2.10) for

$$\underline{J} = (J_1,I'_1,\ldots,\widehat{I'_{\nu-1}},\ldots I'_{r-1})\,.$$

□

2.12 Proposition. a) $(\mathcal{A}^\bullet(n),d)$ *is a complex of sheaves.*

b) *In the notation of* (2.7) $(\mathcal{D}^\bullet(n),d)$ *is a subcomplex of* $(\mathcal{A}^\bullet(n),d)$ *and*

$$0\to(\mathcal{D}^\bullet(n),d)\to(\mathcal{A}^\bullet(n),d)\to(\mathrm{pr}_1^{-1}\mathcal{A}\boxtimes\mathcal{A}^\bullet(n-1),\mathrm{id}\boxtimes d)\to 0$$

is an exact sequence of complexes.

c) *For* $\eta\in\{2,\ldots,n\}$, $(\mathcal{D}^\bullet_\eta(n),d)$ *is a subcomplex of* $(\mathcal{A}^\bullet(n),d)$ *and we have an isomorphism of complexes*

$$\mathcal{A}^{\bullet-1}(n-1)\xrightarrow{(-1)^\bullet\cdot\iota^\bullet}\mathcal{D}^\bullet_\eta(n)\,.$$

Proof. b) and c) follow from a) and (2.8) or (2.9). To prove a) we can assume that, by induction, $\mathcal{A}^\bullet(n-1)$ is a complex. Hence (2.9) implies that $\mathcal{D}^\bullet_\eta(n)$ is a complex for all $\eta\in\{2,\ldots,n\}$. Hence $\mathcal{D}^\bullet(n)$ is a complex as well.

It remains to verify that $d^2(x\boxtimes\underline{a})=0$ for $x\in\mathcal{A}$ and $\underline{a}\in\mathcal{A}^l(n-1)$. We can assume that $\underline{a}=a_{I'_1}\boxtimes\ldots\boxtimes a_{I'_{r-1}}$ for an ordered tupel $(I'_1,\ldots,I'_{r-1})$ for the index set $\{2,\ldots,n\}$. We have

$$d^2(x\boxtimes\underline{a}) = d[x,\underline{a}]+d(x\boxtimes d\underline{a}) = d[x,\underline{a}]+[x,d\underline{a}]+x\boxtimes d^2\underline{a} = d[x,\underline{a}]+[x,d\underline{a}]\,.$$

Let us write

$$\underline{I} = (\{1\},I'_1,\ldots,I'_{r-1}) = (I_1,I_2,\ldots,I_r)\,.$$

The component of $d^2(x\boxtimes\underline{a})$ in $\mathcal{A}_{\underline{K}}$ with $\underline{K}\prec\underline{J}\prec\underline{I}$ is zero if $K_1=\{1\}$. Assume that

$$K_1 = \{1\}\cup I_\eta = \{1\}\cup I'_{\eta-1}\,.$$

There are two possible tuples $\underline{J}^{(i)}$, for $i = 1,2$, with $\underline{K} \prec \underline{J}^{(i)} \prec \underline{I}$:

$$\underline{J}^{(1)} = (\{1\} \cup I_\eta, I_2, \ldots, \widehat{I_\eta}, \ldots, I_r)$$
$$\underline{J}^{(2)} = (\{1\}, I_2, \ldots, J_\mu, \ldots, \widehat{I_\nu}, \ldots, I_r)$$

with $J_\mu = K_{\mu'} = I_\mu \cup I_\nu$ for

$$\mu' = \begin{cases} \mu - 1 & \text{if } \eta < \mu \\ \mu & \text{if } \eta > \mu . \end{cases}$$

The signs in the definition of d are

$$\mathcal{A}_{\underline{I}} \overset{(-1)^{n+\eta}}{\longrightarrow} \mathcal{A}_{\underline{J}^{(1)}} \overset{(-1)^{n+\nu'}}{\longrightarrow} \mathcal{A}_{\underline{K}}$$

$$\mathcal{A}_{\underline{I}} \overset{(-1)^{\nu}}{\longrightarrow} \mathcal{A}_{\underline{J}^{(2)}} \overset{(-1)^{n+\eta'}}{\longrightarrow} \mathcal{A}_{\underline{K}}$$

where

$$\nu' = \begin{cases} \nu - 1 & \text{if } \eta < \nu \\ \nu & \text{if } \eta > \nu \end{cases}$$

and

$$\eta' = \begin{cases} \eta & \text{if } \eta < \nu \\ \eta - 1 & \text{if } \eta > \nu . \end{cases}$$

Hence $\nu + \nu' + \eta + \eta'$ is odd for all $\nu \neq \eta$. We have

$$\begin{aligned} d^2(x \boxtimes \underline{a})_{\underline{K}} &= d_{\underline{K}\,\underline{J}^{(2)}} \circ d_{\underline{J}^{(1)}\underline{I}}(x \boxtimes \underline{a}) + d_{\underline{K}\,\underline{J}^{(1)}} \circ d_{\underline{J}^{(2)}\underline{I}}(x \boxtimes \underline{a}) \\ &= (-1)^{\eta+\nu'}[x, a_{I_\eta}] \boxtimes a_{I_2} \boxtimes \ldots \boxtimes [a_{I_\mu}, a_{I_\nu}] \boxtimes a_{I_{\mu+1}} \boxtimes \ldots \boxtimes \widehat{a_{I_\nu}} \boxtimes \ldots \boxtimes a_{I_r} \\ &\quad + (-1)^{\nu+\eta'}[x, a_{I_\eta}] \boxtimes a_{I_2} \boxtimes \ldots \boxtimes [a_{I_\mu}, a_{I_\nu}] \boxtimes a_{I_{\mu+1}} \boxtimes \ldots \boxtimes \widehat{a_{I_\nu}} \boxtimes \ldots \boxtimes a_{I_r} \\ &= 0 . \end{aligned}$$

If $\underline{K}$ satisfies $K_1 = \{1\} \cup I_\mu \cup I_\nu$ with $\mu < \nu$, then there are three possible $\underline{J}$, as indicated in the following diagram:

$$\mathcal{A}_{\underline{I}} \overset{(-1)^{n+\nu}}{\longrightarrow} \mathcal{A}_{\{1\} \cup I_\nu, I_2, \ldots, \widehat{I_\nu}, \ldots, I_r} \overset{(-1)^{n+\mu}}{\longrightarrow} \mathcal{A}_{\underline{K}}$$

$$\mathcal{A}_{\underline{I}} \overset{(-1)^{n+\mu}}{\longrightarrow} \mathcal{A}_{\{1\} \cup I_\mu, I_2, \ldots, \widehat{I_\mu}, \ldots, I_r} \overset{(-1)^{n+\nu+1}}{\longrightarrow} \mathcal{A}_{\underline{K}}$$

$$\mathcal{A}_{\underline{I}} \overset{(-1)^{n+\nu}}{\longrightarrow} \mathcal{A}_{\{1\}, I_2, \ldots, I_{\mu-1}, I_\mu \cup I_\nu, \ldots, \widehat{I_\nu}, \ldots, I_r} \overset{(-1)^{n+\mu}}{\longrightarrow} \mathcal{A}_{\underline{K}} .$$

Hence

$$\begin{aligned} d^2(x \boxtimes \underline{a})_{\underline{K}} &= (-1)^{\nu+\mu}\{[[x, a_{I_\nu}], a_{I_\mu}] - [[x, a_{I_\mu}], a_{I_\nu}] \\ &\quad + [x, [a_{I_\nu}, a_{I_\mu}]]\} \boxtimes a_{I_2} \ldots \boxtimes \widehat{a_{I_\mu}} \boxtimes \ldots \boxtimes \widehat{a_{I_\nu}} \boxtimes \ldots \boxtimes a_{I_r} \\ &= (-1)^{\nu+\mu}\{[[x, a_{I_\nu}], a_{I_\mu}] + [[a_{I_\mu}, x], a_{I_\nu}] \\ &\quad + [[a_{I_\nu}, a_{I_\mu}], x]\} \boxtimes a_{I_2} \boxtimes \ldots \boxtimes \widehat{a_{I_\mu}} \boxtimes \ldots \boxtimes \widehat{a_{I_\nu}} \boxtimes \ldots \boxtimes a_{I_r} \end{aligned}$$

and $d^2(x \boxtimes \underline{a})_{\underline{K}} = 0$ by the Jacobi-identity.

In Sect. 1 we have defined an action of Σ_n on $\mathcal{A}^l(n)$. As already mentioned, the sign rules in the definition of this action were made to get the following statement.

2.13 Proposition. *Σ_n acts on the complex $\mathcal{A}^\bullet(n)$.*

Proof. Let $\underline{I}$ be ordered and $a_{\underline{I}} = a_{I_1} \boxtimes \ldots \boxtimes a_{I_r}$. In order to prove (2.13) we have to verify that

$$d(\sigma^\# a_{\underline{I}}) = \sigma^\#(da_{\underline{I}}) \quad \text{for all } \sigma \in \Sigma_n .$$

Of course it is sufficient to consider 2 cycles $(m, l) \in \Sigma_n$. Let us fix some $\underline{J} \prec \underline{I}$ and let $\mu < \nu$ be as in (2.2). Let us choose $\pi \in \Sigma_r$ and $\tau \in \Sigma_{r-1}$ such that:

$$\underline{I}' = (\sigma(I_{\pi^{-1}(1)}), \ldots, \sigma(I_{\pi^{-1}(r)}))$$

and

$$\underline{J}' = (\sigma(J_{\tau^{-1}(1)}), \ldots, \sigma(J_{\tau^{-1}(r-1)}))$$

are both ordered.

Of course, $\underline{J}' \prec \underline{I}'$. Let $\mu' < \nu'$ be chosen with $J'_{\mu'} = I'_{\mu'} \cup I'_{\nu'}$. The diagram

$$\begin{array}{ccc} a_{I_1} \boxtimes \ldots \boxtimes a_{I_r} & \xrightarrow[d]{} & (-1)^{n+\nu} a_{I_1} \boxtimes \ldots \boxtimes a_{I_{\mu-1}} \boxtimes [a_{I_\mu}, a_{I_\nu}] \boxtimes \ldots \boxtimes \widehat{a_{I_\nu}} \boxtimes \ldots \boxtimes a_{I_r} \\ \downarrow \sigma^{-1} & & \downarrow \sigma^{-1} \\ b_{I'_1} \boxtimes \ldots \boxtimes b_{I'_r} & \xrightarrow[d]{} & (-1)^{n+\nu'} b_{I'_1} \boxtimes \ldots \boxtimes b_{I'_{\mu'-1}} \boxtimes [b_{I'_{\mu'}}, b_{I'_{\nu'}}] \boxtimes \ldots \boxtimes \widehat{a_{I'_{\nu'}}} \boxtimes \ldots \boxtimes b_{I'_r} \end{array}$$

commutes up to sign, where, of course, $b_{I'_j} = a_{I_{\pi^{-1}(j)}}$. We have

$$\mu' = \mathrm{Min}\{\pi^{-1}(\mu), \pi^{-1}(\nu)\} \quad \text{and} \quad \nu' = \mathrm{Max}\{\pi^{-1}(\mu), \pi^{-1}(\nu)\} .$$

Write

$$\varepsilon = \begin{cases} 0 & \text{if} \quad \nu' = \pi^{-1}(\nu) \\ 1 & \text{if} \quad \nu' = \pi^{-1}(\mu) . \end{cases}$$

It remains to show that the diagram commutes, if we replace σ^{-1} on both sides by $\sigma^\#$. Hence we have to verify that for $\sigma = (l, m)$ one has

$$(-1)^{n+\nu} \cdot \mathrm{sign}(\sigma) \cdot \mathrm{sign}(\tau) = (-1)^{n+\nu'} \cdot \mathrm{sign}(\sigma) \cdot \mathrm{sign}(\pi) \cdot (-1)^\varepsilon$$

or equivalently (∗):

$$(-1)^{\nu+\nu'+\varepsilon} = \mathrm{sign}(\tau) \cdot \mathrm{sign}(\pi) .$$

Obviously, if $l, m \in I_i$ for some i, we have $\tau = \mathrm{id}, \pi = \mathrm{id}, \nu = \nu'$ and $\varepsilon = \mathrm{id}$ and both sides of (∗) are $+1$.

Hence it remains to consider $\sigma = (l, m)$ where for some $\varrho \neq \eta$ one has $l = \mathrm{Min}\, I_\varrho$ and $m = \mathrm{Min}\, I_\eta$. Of course, one has $\pi = (\varrho, \eta)$ in this situation.

Case 1. If $\varrho = \mu$ and $\eta = \nu$, then $\tau = \mathrm{id}, \nu' = \nu = \pi^{-1}(\mu)$ and $\varepsilon = 1$. Hence both sides of (∗) are -1.

Case 2. If $\varrho = \mu$ but $\eta < \nu$, then $\tau = (\mu, \eta)$, $\nu' = \nu = \pi^{-1}(\nu)$ and $\varepsilon = 0$. Both sides of (∗) are 1.

Case 3. If $\{\varrho, \eta\} \cap \{\mu, \nu\} = \varnothing$, then $\tau = (\eta', \varrho')$ for some $\eta' \neq \varrho'$. Since $\nu = \nu' = \pi^{-1}(\nu)$ we have $\varepsilon = 1$ and, again, both sides of (∗) are 1.

Case 4. $\varrho = \mu$ and $\nu < \eta$.

If $\nu > 2$, then there is some $\eta' < \nu$ with $\eta' \neq \mu = \varrho$. By case 3 we can interchange η' and η and by case 2 we obtain $(*)$. If $\nu = 2$, then $\mu = 1$ and, using case 3 again we can assume $\eta = 3$. We have $l = 1 \in J_1$ and $m \in J_2 = I_3$. However, since $1 < \operatorname{Min} I_2$, the ordered tuple $\underline{J}'$ is $(\sigma(J_2), \sigma(J_1), J_3, \ldots, J_{r-1})$ and $\tau = (1,2)$. We have $\nu' = 3 = \pi^{-1}(\mu)$ and $\varepsilon = 1$. Then the right hand side of $(*)$ is $(-1)^{2+3+1} = 1$, hence the same as the left hand side.

Case 5. $\varrho \neq \mu$ and $\eta = \nu$.

This case follows, using the cases 1 and 2 or 1 and 4.

The Proposition 2.13 allows to consider the complex

$$(\delta_* \mathcal{A}^\bullet(n), d)^{-\Sigma_n}$$

of antiinvariants, where again

$$\delta : X \times_S \ldots \times_S X \to S^n(X) = X \times_S \ldots \times_S X/\Sigma_n$$

is the natural quotient map.

Let $\mathcal{A}^{\bullet \geqq 1}(n)$ be the subcomplex of $\mathcal{A}^\bullet(n)$ with

$$(\mathcal{A}^{\bullet \geqq 1}(n))^l = \begin{cases} 0 & \text{for } l = 0 \\ \mathcal{A}^l(n) & \text{for } l > 0 . \end{cases}$$

Of course, the complex $\mathcal{D}_2^\bullet(n)$ considered in (2.7) and (2.8) lies in $\mathcal{A}^{\bullet \geqq 1}(n)$. We have

2.14 Corollary. *We have isomorphisms*

$$(\delta_* \mathcal{A}^{\bullet -1}(n-1))^{-\Sigma_{n-2}} \xrightarrow[(-1)^\bullet \cdot \iota^\bullet]{\simeq} (\delta_* \mathcal{D}_2^\bullet(n))^{-\Sigma_{n-2}} \xrightarrow{\simeq} (\delta_* \mathcal{A}^{\bullet \geqq 1}(n))^{-\Sigma_n}$$

and an exact sequence of complexes

$$0 \to (\delta_* \mathcal{A}^{\bullet \geqq 1}(n))^{-\Sigma_n} \to (\delta_* \mathcal{A}^\bullet(n))^{-\Sigma_n} \to (\delta_* \mathcal{A}^0(n))^{-\Sigma_n} \to 0 .$$

Proof. The exact sequence follows directly from the definition of $\mathcal{A}^{\bullet \geqq 1}(n)$. The first isomorphism follows since

$$(-1)^\bullet \cdot \iota^\bullet : \mathcal{A}^{\bullet -1}(n-1) \to \mathcal{D}_2^\bullet(n)$$

is an isomorphism and Σ_{n-2} as group of permutations of $\{3, \ldots, n\}$ is compatible with the morphism $\iota^\bullet$ given by "leaving out 2" in the index sets. The second isomorphism was shown in (1.13).

2.15 *Remark.* Let us remind that $\sigma \in \Sigma_n$ is acting on $\mathcal{A}^0(n)$ by $\sigma^\# = \sigma^{-1}$ (see (1.9.a)). Hence the local sections of $(\delta_* \mathcal{A}^0(n))^{-\Sigma_n}$ are given by local sections $\underline{a}$ of $\mathcal{A}^0(n)$ with $\sigma^{-1}(\underline{a}) = \operatorname{sign}(\sigma) \cdot \underline{a}$.

3 An extension by $\mathcal{A}^\bullet(n)$

As already mentioned in the introduction the sheaf $\mathcal{A}$ considered in Sects. 1 and 2 should be an $f^{-1}\mathcal{O}_S$ Lie algebra which controls a deformation problem. The corresponding Kodaira-Spencer map should be the edge morphism of an extension

$$0 \to \mathcal{A} \to \tilde{\mathcal{A}} \to f^{-1}T_S \to 0 .$$

Before we discuss some examples let us state in an axiomatic way the additional assumptions used in this and in the next chapters.

3.1 *Assumptions.* Let $f : X \to S$ be a flat morphism of schemes over an algebraically closed field k of characteristic zero or a flat morphism of analytic spaces. In addition to the locally free $\mathcal{O}_X$-module $\mathcal{A}$ we consider an $\mathcal{O}_S$-module T' and an $f^{-1}\mathcal{O}_S$-module T such that $f^{-1}T'$ maps to T. (As in [2] the example we have in mind is the sheaf $T' = T_S$ and $T = f^{-1}T_S$.)

Let $\tilde{\mathcal{A}}$ be a $f^{-1}\mathcal{O}_S$-module obtained as an extension of T by $\mathcal{A}$:

$$0 \to \mathcal{A} \to \tilde{\mathcal{A}} \to T \to 0 .$$

We assume that $\mathcal{A}$ is a $f^{-1}\mathcal{O}_S$ Lie algebra, i.e. that one has a $f^{-1}\mathcal{O}_S$-bilinear bracket

$$[\,,] : \mathcal{A} \otimes_{f^{-1}\mathcal{O}_S} \mathcal{A} \to \mathcal{A} ,$$

antisymmetric and satisfying the Jacobi-identity (2.1).

The Lie-bracket $[\,,]$ extends to a left $f^{-1}\mathcal{O}_S$-product

$$[\,,] : \tilde{\mathcal{A}} \otimes_k \mathcal{A} \to \mathcal{A} ,$$

i.e. a product, k-linear on the right and $f^{-1}\mathcal{O}_S$-linear on the left, satisfying the Jacobi identity

$$[[\tilde{a}, b], c] - [[\tilde{a}, c], b] - [\tilde{a}, [b, c]] = 0$$

for local sections $\tilde{a}$ of $\tilde{\mathcal{A}}$ and b, c of $\mathcal{A}$.

3.2 *Remarks.* a) If X and S are schemes over a field k, all sheaves considered are sheaves for the Zariski topology. For analytic spaces we take the analytic sheaves.

b) For some deformation problems (for example in (3.6)) the Lie-product is even defined on $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}$.

3.3. In (3.1) we have the following situation in mind. Let

$$Y \xrightarrow{\pi} X \xrightarrow{f} S \quad \text{and} \quad g = f \circ \pi$$

be the composite of smooth projective morphisms (or of flat morphisms of analytic varieties with compact manifolds as fibres). Then $T_{Y/S}$ has a $g^{-1}\mathcal{O}_S$ Lie algebra structure and $\pi_* T_{Y/S}$ has an induced $f^{-1}\mathcal{O}_S$ Lie algebra structure. Let $\mathcal{A} = \pi_* T_{Y/S}$ and call T_Y^{top} the inverse image of $g^{-1}T_S$ in T_Y via the natural map $T_Y \to g^* T_S$. One has an exact sequence

$$0 \to T_{Y/S} \to T_Y^{\text{top}} \to g^{-1}T_S \to 0$$

and the usual Lie bracket

$$[\,,]: T_{Y/S} \otimes_{g^{-1}\mathcal{O}_S} T_{Y/S} \to T_{Y/S} .$$

Locally $T_{Y/S}$ consists of those sections ∂ of T_Y^{top} which, applied to $g^{-1}\mathcal{O}_S$ are zero. In particular, for local sections $\tilde{\partial}$ of T_Y^{top} and ∂ of $T_{Y/S}$ one has

$$\tilde{\partial}(g^{-1}\mathcal{O}_S) \subset g^{-1}\mathcal{O}_S \quad \text{and} \quad \partial\tilde{\partial}(g^{-1}\mathcal{O}_S) = \tilde{\partial}\partial(g^{-1}\mathcal{O}_S) = 0 .$$

Hence the k Lie algebra structure

$$T_Y^{\mathrm{top}} \otimes_k T_Y^{\mathrm{top}} \to T_Y^{\mathrm{top}}$$

restricts to

$$T_Y^{\mathrm{top}} \otimes_k T_{Y/S} \to T_{Y/S}$$

and verifies $[\lambda\tilde{x},y] = \lambda[\tilde{x},y]$ for $\lambda \in \mathcal{O}_S$, $\tilde{x} \in T_Y^{\mathrm{top}}$ and $y \in T_{Y/S}$.

Let us assume that $R^1\pi_* T_{Y/S} = 0$ or, more generally, that

$$R^1\pi_* T_{Y/S} \to R^1\pi_* T_Y^{\mathrm{top}}$$

is injective. Then for $\tilde{\mathcal{A}} = \pi_* T_Y^{\mathrm{top}}$ one has an exact sequence

$$0 \to \mathcal{A} \to \tilde{\mathcal{A}} \to f^{-1}T_S \to 0 ,$$

and the Lie-bracket extends to a bracket

$$[\,,]: \tilde{\mathcal{A}} \otimes_k \mathcal{A} \to \pi_*(T_Y^{\mathrm{top}} \otimes_k T_{Y/S}) \to \mathcal{A} ,$$

$f^{-1}\mathcal{O}_S$-linear on the left. Altogether, the assumptions made in (3.1) are satisfied in this case.

3.4 *Example.* For $Y = X$ we obtain, as in the introduction, $\mathcal{A} = T_{X/S}$ and $\tilde{\mathcal{A}} = T_X^{\mathrm{top}}$.

Let us note for later use, that $f_*\mathcal{A} = 0$ if the fibres of $f: X \to S$ have no infinitesimal automorphisms. In particular this holds true if the Kodaira dimension of the fibres of f is maximal.

3.5 *Example.* Let $\mathcal{E}$ be a vector bundle on X and $Y = \mathbb{P}(\mathcal{E})$ the corresponding projective bundle over X. Then $R^1\pi_* T_{Y/X} = 0$ and $\pi_* T_{Y/X} = \mathcal{E}nd^0(\mathcal{E})$ is the sheaf of endomorphisms of trace zero. We have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{E}nd^0(\mathcal{E}) & \longrightarrow & \mathcal{A} & \longrightarrow & T_{X/S} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{E}nd^0(\mathcal{E}) & \longrightarrow & \tilde{\mathcal{A}} & \longrightarrow & T_X^{\mathrm{top}} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & f^{-1}T_S & \xrightarrow{=} & f^{-1}T_S & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where the right hand side sequence describes the deformations of the fibres $X_s = f^{-1}(s)$ of f, whereas the middle describes the deformations of both, X_s and $\mathbb{P}(\mathcal{E})|_{X_s}$.

If the fibres X_s have no infinitesimal automorphisms and if the bundles $\mathcal{E}|_{X_s}$ are stable, then both, $f_*\mathcal{E}nd^0(\mathcal{E})$ and $f_*T_{X/S}$ are zero and hence $f_*\mathcal{A} = 0$ as well.

3.6 *Example.* If in (3.5) one has $X = Z \times S$ and $f = \mathrm{pr}_2$, then the right hand vertical exact sequence in (3.5) splits and one obtains a diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \mathcal{E}nd^0(\mathcal{E}) & \longrightarrow & \mathcal{A} & \longrightarrow & T_{X/S} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & =\downarrow & & \\ 0 & \longrightarrow & \tilde{\mathcal{A}}_1 & \longrightarrow & \tilde{\mathcal{A}} & \longrightarrow & T_{X/S} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & f^{-1}T_S & \overset{=}{\longrightarrow} & f^{-1}T_S & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0\,. & & & & \end{array}$$

In this case the Kodaira-Spencer map is the edge morphism of the left hand vertical sequence and one should take $\mathcal{A}_1 = \mathcal{E}nd^0(\mathcal{E})$ in (3.1). The bracket

$$[\,,] : \mathcal{A}_1 \otimes_{f^{-1}\mathcal{O}_S} \mathcal{A}_1 \to \mathcal{A}_1\,,$$

is given by $[\varphi, \sigma] = \varphi \circ \sigma - \sigma \circ \varphi$. In particular it is $\mathcal{O}_X$-bilinear.

3.7. Let us return to the notations introduced in (3.1). Starting from the extension

$$(\varepsilon_1) \qquad 0 \to \mathcal{A} \to \tilde{\mathcal{A}} \to T \to 0$$

of $f^{-1}\mathcal{O}_S$-modules we want to construct n-extensions (ε_n) of the left $f^{-1}\mathcal{O}_S$-module

$$T_{\{1\}} \boxtimes_k T_{\{2\}} \boxtimes_k \ldots \boxtimes_k T_{\{n\}} = \mathrm{pr}_1^{-1} T \otimes_k \mathrm{pr}_2^{-1} T \otimes_k \ldots \otimes_k \mathrm{pr}_n^{-1} T$$

by $\mathcal{A}^\bullet(n)$. Here we consider $T_{\{1\}} \boxtimes_k \ldots \boxtimes_k T_{\{n\}}$ as a left $f^{-1}\mathcal{O}_S$-module by multiplication on the first factor and $\mathcal{A}^\bullet(n)$ is the complex constructed in Sect. 2 for $\mathcal{R} = \mathcal{O}_S$.

3.8. Let us denote by $\tilde{\mathcal{A}}^\bullet(n)$ the complex one obtains if one replaces in the definition of $\tilde{\mathcal{A}}^\bullet(n)$ the sheaves

$$\mathcal{A}_{\{1\}, I_2, \ldots I_r} = \mathrm{pr}_1^{-1} \mathcal{A} \otimes_{f^{-1}\mathcal{O}_S} (\mathcal{A}_{I_2, \ldots I_r})$$

by the sheaf

$$\tilde{\mathcal{A}}_{\{1\}, I_2, \ldots I_r} = \mathrm{pr}_1^{-1} \tilde{\mathcal{A}} \otimes_k (\mathcal{A}_{I_2, \ldots I_r})$$

whereas

$$\tilde{\mathcal{A}}_{I_1, \ldots I_r} = \mathcal{A}_{I_1, \ldots I_r}$$

remains unchanged for $\{1\} \subset I_1$ but $\{1\} \neq I_1$. Again, multiplication on the left gives $\tilde{\mathcal{A}}_{I_1, \ldots I_r}$ a $f^{-1}\mathcal{O}_S$-module structure.

Since $[\,,]: \tilde{\mathcal{A}} \otimes_k \mathcal{A} \to \tilde{\mathcal{A}}$ has its image in $\mathcal{A}$ and is left $f^{-1}\mathcal{O}_S$ linear, the map defined in (2.3) lifts to a left $f^{-1}\mathcal{O}_S$-linear map

$$\tilde{d}_{\underline{J}\,\underline{I}} : \tilde{\mathcal{A}}_{\underline{I}} \to \tilde{\mathcal{A}}_{\underline{J}} .$$

In fact, $\tilde{d}_{\underline{J}\,\underline{I}} = d_{\underline{J}\,\underline{I}}$ if $\{1\} \subset I_1$, but $\{1\} \neq I_1$, whereas $\tilde{d}_{\underline{J}\,\underline{I}} = \mathrm{id} \boxtimes (d_{\underline{J}'\,\underline{I}'})$ if $I_1 = J_1 = \{1\}$ and $\underline{J}' = (J_2, \ldots, J_{r-1}), \underline{I}' = (I_2, \ldots, I_r)$. Finally, for $I_1 = \{1\}$ and $J_1 = \{1\} \cup I_\nu$ one obtains

$$\tilde{d}_{\underline{J}\,\underline{I}}(\tilde{a}_{I_1} \boxtimes a_{I_2} \boxtimes \ldots \boxtimes a_{I_r}) = [\tilde{a}_{I_1}, a_{I_\nu}] \boxtimes a_{I_2} \boxtimes \ldots \boxtimes \widehat{a_{I_\nu}} \boxtimes \ldots \boxtimes a_{I_r} .$$

Since $[\tilde{a}_{I_1}, a_{I_r}]$ is a local section of $\mathcal{A}$, the map $\tilde{d}_{\underline{J}\,\underline{I}}$ is well defined. As in (2.5) we define a left $f^{-1}\mathcal{O}_S$-linear morphism

$$\tilde{d} : \tilde{\mathcal{A}}^l(n) \to \tilde{\mathcal{A}}^{l+1}(n)$$

by

$$\tilde{d}|_{\tilde{\mathcal{A}}_{\underline{I}}} = \sum_{\underline{J} \prec \underline{I}} \tilde{d}_{\underline{J}\,\underline{I}} .$$

The calculations made in (2.7)–(2.12) remain true since we only used the sign rules and the Jacobi identity, as it was stated in (3.1). In particular as in (2.12):

a) one finds $(\tilde{\mathcal{A}}^\bullet(n), \tilde{d})$ to be a complex of sheaves and, as in (2.12)

b) one finds an exact sequence of complexes

$$0 \to (\mathcal{D}^\bullet(n), d) \to (\tilde{\mathcal{A}}^\bullet(n), \tilde{d}) \to (\mathrm{pr}_1^{-1}\tilde{\mathcal{A}} \otimes_k (\mathcal{A}^\bullet(n)), \mathrm{id} \otimes d) \to 0$$

where $(\mathcal{D}^\bullet(n), d)$ is the complex defined in (2.7) for $\mathcal{R} = f^{-1}\mathcal{O}_S$.

3.9. The above exact sequence induces a surjective map of complexes

$$(\tilde{\mathcal{A}}^\bullet(n), \tilde{d}) \to (\mathrm{pr}_1^{-1} T \otimes_k (\mathcal{A}^\bullet(n-1), \mathrm{id} \otimes d) ,$$

by mapping $\mathrm{pr}_1^{-1}\tilde{\mathcal{A}}$ to $\mathrm{pr}_1^{-1}T$. The kernel of this map is the subcomplex $\mathcal{A}'^\bullet(n)$ of $\tilde{\mathcal{A}}^\bullet(n)$ given by

$$\mathcal{A}'_{\{1\}, I_2, \ldots, I_r} = \mathrm{pr}_1^{-1}\mathcal{A} \otimes_k (\mathcal{A}_{I_2, \ldots I_r})$$

and

$$\mathcal{A}'_{I_1, \ldots I_r} = \mathcal{A}_{I_1, \ldots, I_r}$$

for $\{1\} \subset I_1, \{1\} \neq I_1$.

The different complexes give a commutative diagram of exact sequences

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{D}^\bullet(n) & \longrightarrow & \mathcal{A}'^\bullet(n) & \longrightarrow & \mathrm{pr}_1^{-1}\mathcal{A} \otimes_k \mathcal{A}^\bullet(n-1) \longrightarrow 0 \\
 & & \downarrow = & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{D}^\bullet(n) & \longrightarrow & \tilde{\mathcal{A}}^\bullet(n) & \longrightarrow & \mathrm{pr}_1^{-1}\tilde{\mathcal{A}} \otimes_k \mathcal{A}^\bullet(n-1) \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \mathrm{pr}_1^{-1}T \otimes_k \mathcal{A}^\bullet(n-1) & \xrightarrow{=} & \mathrm{pr}_1^{-1}T \otimes_k \mathcal{A}^\bullet(n-1) \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0 .
\end{array}$$

3.10. One has a natural surjection

$$\mathcal{A}'^\bullet(n) \to \mathcal{A}^\bullet(n)$$

by replacing "$\otimes_k$" by "$\otimes f^{-1}\mathcal{O}_S$" and from the middle vertical exact sequence (ε') in (3.9) one obtains by push forward

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathcal{A}'^\bullet(n) & \longrightarrow & \tilde{\mathcal{A}}^\bullet(n) & \longrightarrow & \mathrm{pr}_1^{-1}T \otimes_k \mathcal{A}^\bullet(n-1) & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \mathcal{A}^\bullet(n) & \longrightarrow & \mathcal{A}^\bullet(n) \times \tilde{\mathcal{A}}^\bullet(n)/\mathcal{A}'^\bullet(n) & \longrightarrow & \mathrm{pr}_1^{-1}T \otimes_k \mathcal{A}^\bullet(n-1) & \longrightarrow 0\,. \end{array}$$

Let us denote the extension given by the bottom line by (ε).
Let us assume, by induction that we have constructed the $(n-1)$-extension

$$(\varepsilon_{n-1}) \text{ of } T_{\{2\}} \boxtimes_k T_{\{3\}} \boxtimes_k \ldots \boxtimes_k T_{\{n\}} \text{ by } \mathcal{A}^\bullet(n-1)\,.$$

Tensoring on the left by $T_{\{1\}} = \mathrm{pr}_1^{-1}T$ over k one obtains an extension

$$(1 \boxtimes_k \varepsilon_{n-1}) \text{ of } T_{\{1\}} \boxtimes_k T_{\{2\}} \boxtimes_k \ldots \boxtimes T_n \text{ by } \mathrm{pr}_1^{-1}T \otimes_k \mathcal{A}^\bullet(n-1)\,,$$

and composing with (ε) one obtains an extension

$$(\varepsilon_n) \text{ of } T_{\{1\}} \boxtimes_k T_{\{2\}} \boxtimes_k \ldots \boxtimes T_{\{n\}} \text{ by } \mathcal{A}^\bullet(n)\,.$$

Since $(1 \boxtimes_k \varepsilon_{n-1})$ and (ε) are extensions of left $f^{-1}\mathcal{O}_S$-modules the same holds true for (ε_n).

3.11. In particular, the extension (ε_n) gives rise to a left $\mathcal{O}_S$-linear morphism of sheaves on S

$$\tilde{\varphi}_n\colon T' \otimes_k \ldots \otimes_k T' \to f_*(T_{\{1\}} \boxtimes_k \ldots \boxtimes_k T_{\{n\}}) \to R^n f_* \mathcal{A}^\bullet(n)$$

where again

$$f\colon X \times_S \ldots \times_S X \to S$$

is the structure map.

3.12 *Remarks.* a) Of course, one can modify the construction of $\tilde{\varphi}_n$. For example, if $\mathcal{A}^\bullet(n)_k$ denotes the complex from Sect. 2 for $\mathcal{R} = k$, and if $\tilde{\mathcal{A}}^\bullet(n)_k$ denotes the complex with $\tilde{\mathcal{A}}$ instead of $\mathcal{A}$ at the first factor and with all tensor products over k, then the extension (ε')

$$0 \to \mathcal{A}^\bullet(n)_k \to \tilde{\mathcal{A}}^\bullet(n)_k \to T_{\{1\}} \boxtimes_k \mathcal{A}^\bullet(n-1)_k \to 0$$

gives rise to

$$\tilde{\varphi}'_n\colon T' \otimes_k \ldots \otimes_k T' \to R^n f_* \mathcal{A}^\bullet(n)_k\,.$$

Composing $\tilde{\varphi}'_n$ with the natural map

$$R^n f_* \mathcal{A}(n)_k \to R^n f_* \mathcal{A}^\bullet(n)$$

one obtains again $\tilde{\varphi}_n$.

b) If the bracket $\mathcal{A} \otimes_{f^{-1}\mathcal{O}_S} \mathcal{A} \longrightarrow \mathcal{A}$ in (3.1) is $\mathcal{O}_X$-linear, for example in (3.6), then one can replace $f^{-1}\mathcal{O}_S$ in the definition of $\mathcal{A}^\bullet(n)$ by $\mathcal{O}_X$. If $\mathcal{A}^\bullet(n)_{\mathcal{O}_X}$ is the corresponding complex, then the above constructions give a left $f^{-1}\mathcal{O}_S$ extension of $T_{\{1\}}\boxtimes_k \dots \boxtimes_k T_{\{n\}}$ by $\mathcal{A}^\bullet(n)_{\mathcal{O}_X}$, and again a left $\mathcal{O}_S$-linear morphism

$$T' \otimes_k \dots \otimes_k T' \to R^n f_* \mathcal{A}^\bullet(n)_{\mathcal{O}_X} .$$

However, this morphism can as well be obtained by composing $\tilde{\varphi}_n$ in (3.11) with

$$R^n f_* \mathcal{A}^\bullet(n) \to R^n f_* \mathcal{A}^\bullet(n)_{\mathcal{O}_X} .$$

3.13. Proposition (2.12), b) and c) allows to add up the maps $\tilde{\varphi}_n$ to obtain

$$\tilde{\Phi}_n : \bigoplus_{v=1}^{n} (T' \otimes_k \dots \otimes_k T'(v \text{ times})) \to R^n f_* \mathcal{A}^\bullet(n)$$

where $\tilde{\Phi}_n$ restricted to the last summand is

$$\tilde{\varphi}_n : T' \otimes_k \dots \otimes_k T'(n \text{ times}) \to R^n f_* \mathcal{A}^\bullet(n)$$

and $\tilde{\Phi}_n$ restricted to the first $(n-1)$ summands is the composition of $\tilde{\Phi}_{n-1}$ with

$$R^{n-1} f_* \mathcal{A}^\bullet(n-1) \overset{(-1)^\bullet \iota^\bullet}{\longrightarrow} R^n f_* \mathcal{D}_2^\bullet(n) \to R^n f_* \mathcal{A}^\bullet(n) .$$

This map, as we will see in the following three sections induces the lifting of the Kodaira-Spencer map.

4 Antisymmetrization of $\tilde{\varphi}_n$ and Čech cohomology

Our next aim is to study $\tilde{\varphi}_n$ in the language of Čech cohomology in order to show that $\tilde{\Phi}_n$ maps certain "relations" to the invariant part of $R^n f_* \mathcal{A}^\bullet(n)$ under the Σ_n action constructed in (2.13). Let us first construct certain covers of $X \times_S \dots \times_S X$ which are Σ_n invariant.

4.1 Lemma. *Let X be a projective variety or a complex compact analytic variety. Then there exists a cover $\{V_i\}_{i \in I}$ of X by open sets such that the higher cohomology groups of coherent sheaves on V_i are trivial and such that $\{U_i\}_{i \in I}$ is an open cover of $X \times_S \cdots \times_S X$ for*

$$U_i = \bigcap_{j=1}^{n} \mathrm{pr}_j^{-1} V_i \subset X \times_S \dots \times_S X .$$

Proof. We may assume that S is affine or a Stein space. For given V_i the open sets U_i cover $X \times_S \dots \times_S X$ if and only if for arbitrary points $x_1, x_2, \dots, x_n \in X$ there is an $i \in I$ for which $x_1, x_2, \dots, x_n$ are all in V_i. If X is projective, one can choose $\{V_i\}_{i \in I}$ to be the set of all affine subvarieties. In the analytic case one may choose for each

$$i = (x_1, \dots x_n) \in I = X \times_S \dots \times_S X$$

small Stein neighbourhoods $V(x_j)$ which are not meeting each other for different x_j and define V_i to be the union of the $V(x_j)$. Of course, in both cases we may replace the index set I by some finite subset. □

4.2 Notations. From now on let $\mathcal{U} = \{U_i\}$ be the Σ_n invariant cover of $X \times_S \dots \times_S X$ constructed in (4.1). We denote by $C^\bullet(\mathcal{A}^\bullet(n))$ the corresponding Čech complex. One has

$$C^N(\mathcal{A}^\bullet(n)) = \bigoplus_{l=0}^{N} C^{N-l}(\mathcal{A}^l(n))$$

and the differential is

$$d + (-1)^{N+1} \cdot \delta$$

where d is the differential in $\mathcal{A}^\bullet(n)$ and δ is the Čech differential.

4.3. Let $z \otimes x_2 \otimes \dots \otimes x_n$ be a local section of $T' \otimes_k T' \otimes_k \dots \otimes_k T'$. In order to compute $\tilde{\varphi}_n(z \otimes x_2 \otimes \dots \otimes x_n)$ let us return to the construction of the n extension (ε_n) in (3.10). There we considered the one extension (ε) obtained as push forward from the one extension

$$(\varepsilon') \qquad 0 \to \mathcal{A}'^\bullet(n) \to \tilde{\mathcal{A}}^\bullet(n) \to T_{\{1\}} \boxtimes_k \mathcal{A}^\bullet(n-1) \to 0 \,.$$

Let

$$B_\varepsilon : R^{n-1} f_*(T_{\{1\}} \boxtimes_k \mathcal{A}^\bullet(n-1)) \to R^n f_* \mathcal{A}'^\bullet(n)$$

be the induced edge morphism. As (ε_n) was the composition of $(1 \boxtimes \varepsilon_{n-1})$ and (ε) one finds

$$\tilde{\varphi}_n(z \otimes x_2 \otimes \dots \otimes x_n) = \tau(B_{\varepsilon'}(\mathrm{pr}^{-1} z \boxtimes_k \tilde{\varphi}_{n-1}(x_2 \otimes \dots \otimes x_n)))$$

where $\tau : R^n f_* \mathcal{A}'^\bullet(n) \to R^n f_* \mathcal{A}^\bullet(n)$ is the natural map. Abusing notations we will, until the end of this section, identify $B_{\varepsilon'}$ and $\tau \circ B_{\varepsilon'} = B_\varepsilon$ and suppress the map τ, and we will replace $\mathrm{pr}^{-1} z$ by z and $\boxtimes$ by $\otimes$.

By the choice of the cover $\{V_i\}$ of X in (4.1) z has on V_α a lifting $Z_\alpha \in \Gamma(V_\alpha, \tilde{A})$. We may assume, moreover, that $\tilde{\varphi}_{n-1}(x_2 \otimes \dots \otimes x_n)$ is represented by a Čech cocycle

$$(\underline{a}) = (a^{n-1}, \dots, a^1) \in C^{n-1}(\mathcal{A}^\bullet(n-1))_{d+(-1)^n \cdot \delta} \,,$$

for $a^i \in C^i(\mathcal{A}^{n-1-i}(n-1))$.

4.4 Lemma. *$\tilde{\varphi}_n(z \otimes x_2 \otimes \dots \otimes x_n)$ is represented by the cocycle $(\underline{b}) = (b^n, b^{n-1}, \dots, b^1)$ in $C^n(\mathcal{A}^\bullet(n))_{d+(-1)^{n+1} \cdot \delta} = (C^n(\mathcal{A}^0(n)) \oplus \dots \oplus C^1(\mathcal{A}^{n-1}(n))_{d+(-1)^{n+1} \cdot \delta}$ with*

$$b^j = d(Z \otimes a^j) + (-1)^n \delta(Z \otimes a^{j-1}) = [Z_\alpha, a^j] + (-1)^n (\delta Z)_{\alpha\beta} \otimes a^{j-1} \,.$$

Of course this equality for $b^j \in C^j(\mathcal{A}^{n-j}(n))$ means that on

$$U_{\alpha\beta\gamma\dots} = U_\alpha \cap U_\beta \cap U_\gamma \cap \dots \,,$$

one has

$$b^j_{\alpha\beta\gamma\dots} = d(Z_\alpha \otimes a^j_{\alpha\beta\gamma\dots}) + (-1)^n \delta(Z \otimes a^{j-1})_{\alpha\beta\gamma\dots} = [Z_\alpha, a^j_{\alpha\beta\gamma\dots}] + (-1)^n (\delta Z)_{\alpha\beta} \otimes a^{j-1}_{\beta\gamma\dots} \,.$$

Proof. The first equality is just the computation of the Bockstein $B_{\varepsilon'}$, whereas the second one comes from (2.11):

$$d(Z \otimes a^j) + (-1)^n \delta(Z \otimes a^{j-1})$$
$$= [Z, a^j] + Z \otimes da^j + (-1)^n (\delta Z) \otimes a^{j-1} + (-1)^n Z \otimes \delta a^{j-1} .$$

Since $(\underline{a})$ is a cocycle the latter is equal to $[Z, a^j] + (-1)^n (\delta Z) \otimes a^{j-1}$. □

We now consider $\tilde{\varphi}_{n+1}$ applied to a local section

$$y \otimes z \otimes x_3 \otimes \ldots \otimes x_{n+1}$$

of

$$T' \otimes_k T' \otimes_k \ldots \otimes_k T' .$$

Let again Y_α and Z_α be local liftings of y and z in $\Gamma(V_\alpha, \tilde{\mathcal{A}})$ and let

$$(\underline{a}) = (a^{n-1}, \ldots, a^1)$$

be a Čech cocycle of $\tilde{\varphi}_{n-1}(x_3 \otimes \ldots \otimes x_{n-1})$ in $C^{n-1}(\mathcal{A}^\bullet(n-1))_{d+(-1)^n \cdot \delta}$.

We denote by $(\underline{c}) = (c^{n+1}, \ldots, c^1)$ a Čech cocycle for

$$\tilde{\varphi}_{n+1}(y \otimes z \otimes x_3 \otimes \ldots \otimes x_{n+1}) .$$

4.5 Lemma. *One has in* $C^j(\mathcal{A}^{n+1-j}(n+1))$:

$$c^j = [Y_\alpha, [Z_\alpha, a^j]] + (-1)^n [Y_\alpha, (\delta Z)_{\alpha\beta} \otimes a^{j-1}]$$
$$+ (-1)^{n+1} (\delta Y)_{\alpha\beta} \otimes [Z_\beta, a^{j-1}] - (\delta Y)_{\alpha\beta} \otimes (\delta Z)_{\beta\gamma} \otimes a^{j-2} .$$

Proof. As explained in (4.3) one has

$$(\underline{c}) = B_{\varepsilon'}(y \otimes B_{\varepsilon'}(z \otimes (\underline{a}))) = B_{\varepsilon'}(y \otimes (\underline{b}))$$

where $(\underline{b})$ is the cocycle given in (4.4).

Hence taking into account that n is replaced by $n+1$, one has

$$c^j = [Y_\alpha, b^j] + (-1)^{n+1} (\delta Y)_{\alpha\beta} \otimes b^{j-1}$$

and, applying (4.4) a second time, one finds

$$c^j = [Y_\alpha, [Z_\alpha, a^j]] + (-1)^n [Y_\alpha, (\delta Z)_{\alpha\beta} \otimes a^{j-1}]$$
$$+ (-1)^{n+1} (\delta Y)_{\alpha\beta} \otimes [Z_\beta, a^{j-1}] + (-1)^{2n+1} (\delta Y)_{\alpha\beta} \otimes (\delta Z)_{\beta\gamma} \otimes a^{j-2} .$$ □

As mentioned in (3.13) one can compose the map

$$\tilde{\varphi}_n : T' \otimes_k \ldots \otimes T' (n \text{ times}) \to R^n f_* \mathcal{A}^\bullet(n)$$

with the map

$$\theta_{12} : R^n f_* \mathcal{A}^\bullet(n) \to R^{n+1} f_* \mathcal{D}_2^\bullet(n+1) \to R^{n+1} f_* \mathcal{A}^\bullet(n+1)$$

constructed in Sect. 2 (see (2.12)). Using this notation one has the following formula.

4.6 Proposition.

$$\tilde{\varphi}_{n+1}((y\otimes z - z\otimes y)\otimes x_3 \otimes \ldots \otimes x_{n+1}) - \theta_{12}(\tilde{\varphi}_n([y,z]\otimes x_3 \otimes \ldots x_{n+1})$$

lies in the sheaf $(R^{n+1}f_*\mathcal{A}^\bullet(n+1))^{(1,2)}$, *where* $(\ )^{(1,2)}$ *denotes the invariants under the element* $(1,2) \in \Sigma_{n+1}$.

4.7. If one writes correspondingly

$$\theta_{i,i+1}: R^n f_*\mathcal{A}^\bullet(n) \to R^{n+1}f_*\mathcal{A}^\bullet(n+1)$$

for the morphism induced by considering

$$\mathcal{A}^{\bullet-1}(n) \xrightarrow{\cong} \mathcal{D}^\bullet_{\{i,i+1\}}(n+1)$$

for the subcomplex $\mathcal{D}^\bullet_{\{i,i+1\}}(n+1)$ of $\mathcal{A}^\bullet(n+1)$, which is built up by those $\mathcal{A}_{I_1,\ldots,I_r}$ with $\{i,i+1\} \subset I_\nu$ for some ν, then, due to the inductive definition of the morphism $\tilde{\varphi}_\bullet$ one has as well that

$$\tilde{\varphi}_{n+1}(y_1\otimes\ldots\otimes y_{i-1}\otimes(x_i\otimes x_{i+1} - x_{i+1}\otimes x_i)\otimes y_{i+2}\otimes\ldots\otimes y_{n+1})$$
$$-\theta_{i,i+1}\tilde{\varphi}_n(y_1\otimes\ldots\otimes y_{i-1}\otimes[x_i,x_{i+1}]\otimes y_{i+2}\otimes\ldots\otimes y_{n+1})$$

lies in $R^{n+1}f_*\mathcal{A}^\bullet(n+1)))^{(i,i+1)}$.

Proof of (4.6). Let us drop θ_{12} keeping in mind that

$$\tilde{\varphi}_n: R^n f_*\mathcal{A}^\bullet(n) \to R^{n+1}f_*\mathcal{A}^\bullet(n+1)$$

comes from $\mathcal{D}^\bullet_2(n+1) \hookrightarrow \mathcal{A}^\bullet(n+1)$ and the identification $\mathcal{A}^{\bullet-1}(n) \xrightarrow{(-1)^\bullet} \mathcal{D}^\bullet_2(n+1)$. One obtains maps

$$\begin{array}{ccc} C^{n+1}(\mathcal{A}^{\bullet-1}(n)) & \longrightarrow & C^{n+1}(\mathcal{D}^\bullet_2(n+1)) \\ =\downarrow & & \downarrow= \\ (\oplus)_j C^j(\mathcal{A}^{(n-j)}(n)))_{d+(-1)^{n+2}\cdot\delta} & \longrightarrow & (\oplus_j C^j((\mathcal{D}_2^{n+1-j}(n+1))_{d+(-1)^{n+2}\cdot\delta} \end{array}$$

with $e^j \mapsto (-1)^{n-j+1}\cdot e^j$ for $e^j \in C^j(\mathcal{A}^{(n-j)}(n))$.

To describe $R^n f_*\mathcal{A}^\bullet(n) \to R^{n+1}f_*\mathcal{D}^\bullet(n+1)$ one has to compose this map with the shift operator

$$C^n(\mathcal{A}^\bullet(n))_{d+(-1)^{n+1}\cdot\delta} \to C^{n+1}(\mathcal{A}^{\bullet-1}(n))_{d+(-1)^{n+2}\cdot\delta}$$

given by $e^j \mapsto (-1)^j\cdot e^j$ for $e^j \in C^j(\mathcal{A}^{(n-j)}(n))$. Hence the map

$$R^n f_*\mathcal{A}^\bullet(n) \to R^{n+1}f_*\mathcal{D}^\bullet(n+1)$$

is induced by the multiplication of Čech cocycles e^j with $(-1)^{n+1}$.

By (4.5) the first term in (4.6) is given by the cocycle

$$\begin{aligned} \Delta^j := {} & [Y_\alpha,[Z_\alpha,a^j]] - [Z_\alpha,[Y_\alpha,a^j]] \\ & + (-1)^n([Y_\alpha,(\delta Z)_{\alpha\beta}\otimes a^{j-1}] - [Z_\alpha,(\delta Y)_{\alpha\beta}\otimes a^{j-1}]) \\ & + (-1)^{n+1}((\delta Y)_{\alpha\beta}\otimes[Z_\beta,a^{j-1}] - (\delta Z)_{\alpha\beta}\otimes[Y_\beta,a^{j-1}]) \\ & - ((\delta Y)_{\alpha\beta}\otimes(\delta Z)_{\beta\gamma} - (\delta Z)_{\alpha\beta}\otimes(\delta Y)_{\beta\gamma})\otimes a^{j-2} \end{aligned}$$

whereas the second one is given by

$$B^j = [[Y,Z]_\alpha, a^j] + (-1)^n (\delta[Y,Z])_{\alpha\beta} \otimes a^{j-1} .$$

Here $[Y,Z]_\alpha = [Y_\alpha, Z_\alpha]$ and $(\delta[Y,Z])_{\alpha\beta} = [Y_\beta, Z_\beta] - [Y_\alpha, Z_\alpha]$.
Considering B^j as an element of

$$\mathcal{C}^j(\mathcal{D}^{n+1-j}(n+1)) \hookrightarrow \mathcal{C}^j(\mathcal{A}^{n+1-j}(n+1))$$

one has to prove that the cocycle $\Delta^j - (-)^{n+1}B^j$ is, modulo a coboundary, symmetric with respect to $\sigma = (1,2) \in \Sigma_{n+1}$.

4.8 Claim. *One has* $[Y_\alpha, Z_\alpha \otimes a^j] = (-1)^{n+1}[Y_\alpha, Z_\alpha] \otimes a^j + \sigma^{-1}(Z_\alpha \otimes [Y_\alpha, a^j])$.

Proof. Returning to the notations used in Sect. 2, one can verify (4.8) for

$$a^j = a_{I_3} \boxtimes \ldots \boxtimes a_{I_r} ,$$

where $n+1-j = n+1-r$ or $j = r$. By definition of $[\,,]$ in (2.10), for $\underline{I} = (\{1\}, \{2\}, I_2, \ldots, I_r)$ and n replaced by $n+1$, one has

$$[Y, Z \otimes a^r] = \bigoplus_{v=2}^{r} b_{\underline{J}^{(v)}} ,$$

where for $v = 3, \ldots, r$ one writes

$$\underline{J}^{(v)} = (\{1\} \cup I_v, \{2\}, I_3, \ldots, \widehat{I_v}, \ldots, I_r)$$

and

$$b_{\underline{J}^{(v)}} = (-1)^{n+1+v}[Y, a_{I_v}] \boxtimes Z \boxtimes a_{I_3} \boxtimes \ldots \boxtimes \widehat{a_{I_v}} \boxtimes \ldots \boxtimes a_{I_r} .$$

The tuple $\underline{J}^{(2)}$ is $\underline{J}^{(2)} = (\{1\} \cup \{2\}, I_3, \ldots, I_r)$ and

$$b_{\underline{J}^{(2)}} = (-1)^{n+3}[Y,Z] \boxtimes a_{I_3} \boxtimes \boxtimes \ldots \boxtimes a_{I_r} = (-1)^{n+1}[Y,Z] \otimes a^r .$$

If we take the bracket for $\underline{I}' = (\{2\}, I_3, \ldots, I_r)$ and the index set $\{2,3,4,\ldots, n+1\}$, then (2.10) gives

$$\sigma^{-1}(Z \otimes [Y, a^j]) = \sigma^{-1}\left(Z \boxtimes \left(\bigoplus_{v=3}^{r} (-1)^{n+(v-1)} [Y, a_{I_v}] \boxtimes a_{I_3} \boxtimes \ldots \boxtimes \widehat{a_{I_v}} \boxtimes \ldots \boxtimes a_{I_r} \right)\right)$$
$$= \bigoplus_{v=3}^{r} b_{\underline{J}^{(v)}} .$$

4.9 Claim. *One has*

$$[(\delta Y)_{\alpha\beta}, (\delta Z)_{\alpha\beta}] = (\delta[Y,Z])_{\alpha\beta} - [Y_\alpha, (\delta Z)_{\alpha\beta}] + [Z_\alpha, (\delta Y)_{\alpha\beta}] .$$

Proof. The left hand side is

$$[Y_\beta - Y_\alpha, Z_\beta - Z_\alpha] = [Y_\beta, Z_\beta] + [Y_\alpha, Z_\alpha] - [Y_\beta, Z_\alpha] - [Y_\alpha, Z_\beta]$$

and the right hand side is

$$[Y_\beta, Z_\beta] - [Y_\alpha, Z_\alpha] - [Y_\alpha, Z_\beta - Z_\alpha] + [Z_\alpha, Y_\beta - Y_\alpha] .$$

□

4.10 Claim. *The Jacobi identity gives*

$$[Y_\alpha, [Z_\alpha, a^j]] - [Z_\alpha, [Y_\alpha, a^j]] - (-1)^{n+1}[[Y, Z]_\alpha, a^j] = 0 .$$

Proof. As in the proof of (4.8) we consider $a^r = a_{I_3} \boxtimes \dots \boxtimes a_{I_r}$. By definition, the expression considered in (4.10) just occurs for index tuples

$$\underline{K} = (\{1,2\} \cup I_\nu, I_3, \dots, \widehat{I_\nu}, \dots, I_r)$$

and the expression is

$$((-1)^{\nu+1}[Y_\alpha, [Z_\alpha, a_{I_\nu}]] - (-1)^{\nu+1}[Z_\alpha, [Y_\alpha, a_{I_\nu}]]$$
$$-(-1)^{n+1}(-1)^{n+1+\nu-1}[[Y_\alpha, Z_\alpha], a_{I_\nu}])\boxtimes$$
$$\boxtimes a_{I_1} \dots \boxtimes \widehat{a_{I_\nu}} \boxtimes a_{I_{\nu+1}} \boxtimes \dots a_{I_r} .$$

However the Jacobi identity (3.1) gives

$$[Y_\alpha, [Z_\alpha, a_{I_\nu}]] - [Z_\alpha, [Y_\alpha, a_{I_\nu}]] - [[Y_\alpha, Z_\alpha], a_{I_\nu}] = 0 .$$

Applying (4.8) to the second term in Δ^j and rearranging according to the a terms, one finds

$$\begin{aligned}
\Delta^j - (-)^{n+1}B^j = {} & [Y_\alpha, [Z_\alpha, a^j]] - [Z_\alpha, [Y_\alpha, a^j]] - (-1)^{n+1}[[Y, Z]_\alpha, a^j] \\
& + (-[Y_\alpha, (\delta Z)_{\alpha\beta}] + [Z_\alpha, (\delta Y)_{\alpha\beta}] + \delta[Y, Z]_{\alpha\beta}) \otimes a^{j-1} \\
& + (-1)^n(\sigma^{-1}(\delta Z)_{\alpha\beta} \otimes [Y_\alpha, a^{j-1}] - \sigma^{-1}(\delta Y)_{\alpha\beta} \otimes [Z_\alpha, a^{j-1}]) \\
& + (-1)^{n+1}((\delta Y)_{\alpha\beta} \otimes [Z_\beta, a^{j-1}] - (\delta Z)_{\alpha\beta} \otimes [Y_\beta, a^{j-1}]) \\
& - ((\delta Y)_{\alpha\beta} \otimes (\delta Z)_{\beta\gamma} - (\delta Z)_{\alpha\beta} \otimes (\delta Y)_{\beta\gamma}) \otimes a^{j-2} .
\end{aligned}$$

Applying (4.10) to the first three expressions and (4.9) to the fourth one, one finds (rearranging the other terms in a more complicated way):

$$\begin{aligned}
\Delta^j - (-1)^{n+1}B^j = {} & [(\delta Y)_{\alpha\beta}, (\delta Z)_{\alpha\beta}] \otimes a^{j-1} \\
& + (-1)^{n+1}(1 + \sigma)^{-1}((\delta Y)_{\alpha\beta} \otimes [Z_\alpha, a^{j-1}]) \\
& + (-1)^n(1 + \sigma)^{-1}((\delta Z)_{\alpha\beta} \otimes [Y_\alpha, a^{j-1}]) \\
& + (-1)^{n+1}(\delta Y)_{\alpha\beta} \otimes [(\delta Z)_{\alpha\beta}, a^{j-1}] \\
& + (-1)^n(\delta Z)_{\alpha\beta} \otimes [(\delta Y)_{\alpha\beta}, a^{j-1}] \\
& - ((\delta Y)_{\alpha\beta} \otimes (\delta Z)_{\beta\gamma} - (\delta Z)_{\alpha\beta} \otimes (\delta Y)_{\beta\gamma}) \otimes a^{j-2} .
\end{aligned}$$

Here $(1 + \sigma)^{-1}(\underline{c})$ stands for $\underline{c} + \sigma^{-1}(\underline{c})$.

Let us write, for all j,

$$\xi^j = (\delta Y)_{\alpha\beta} \otimes (\delta Z)_{\alpha\beta} \otimes a^{j-1} .$$

4.11 Claim. *For any sheaf $\mathcal{F}$ and $n, m \in C^1(\mathcal{F})$ with $\delta m = \delta n = 0$ one has*

$$\delta(m_{\alpha\beta} \otimes n_{\alpha\beta})_{\alpha\beta\gamma} = -m_{\alpha\beta} \otimes n_{\beta\gamma} - m_{\beta\gamma} \otimes n_{\alpha\beta} .$$

Proof. The left hand side is

$$\begin{aligned} &m_{\beta\gamma} \otimes n_{\beta\gamma} - m_{\alpha\gamma} \otimes n_{\alpha\gamma} + m_{\alpha\beta} \otimes n_{\alpha\beta} \\ &= m_{\beta\gamma} \otimes n_{\beta\gamma} - (m_{\beta\gamma} + m_{\alpha\beta}) \otimes n_{\alpha\gamma} + m_{\alpha\beta} \otimes n_{\alpha\beta} \\ &= m_{\beta\gamma} \otimes (n_{\beta\gamma} - n_{\alpha\gamma}) + m_{\alpha\beta} \otimes (n_{\alpha\beta} - n_{\alpha\gamma}) . \end{aligned}$$

Using (2.11) and (4.8) the coboundary of (ξ^j) has the expansion

$$\begin{aligned} d\xi^j + (-1)^{n+1}\delta\xi^{j-1} = &(-1)^{n+1}[(\delta Y)_{\alpha\beta}, (\delta Z)_{\alpha\beta}] \otimes a^{j-1} \\ &+ \sigma^{-1}((\delta Z)_{\alpha\beta} \otimes [(\delta Y)_{\alpha\beta}, a^{j-1}] \\ &+ (\delta Y)_{\alpha\beta} \otimes [(\delta Z)_{\alpha\beta}, a^{j-1}] + (\delta Y)_{\alpha\beta} \otimes (\delta Z)_{\alpha\beta} \otimes da^{j-1} \\ &+ (-1)^{n+1}\delta((\delta Y)_{\alpha\beta} \otimes (\delta Z)_{\alpha\beta}) \otimes a^{j-2} \\ &+ (-1)^{n+2}(\delta Y)_{\alpha\beta} \otimes (\delta Z)_{\alpha\beta} \otimes \delta a^{j-2} . \end{aligned}$$

$\underline{a}$ is a cocycle, and using (4.11), one obtains

$$\begin{aligned} (-1)^{n+1}d\xi^j + \delta\xi^{j-1} = &[(\delta Y)_{\alpha\beta}, (\delta Z)_{\alpha\beta}] \otimes a^{j-1} \\ &+ (-1)^{n+1}\sigma^{-1}((\delta Z)_{\alpha\beta} \otimes [(\delta Y)_{\alpha\beta}, a^{j-1}]) \\ &+ (-1)^{n+1}(\delta Y)_{\alpha\beta} \otimes [(\delta Z)_{\alpha\beta}, a^{j-1}]) \\ &- ((\delta Y)_{\alpha\beta} \otimes (\delta Z)_{\beta\gamma} + (\delta Y)_{\beta\gamma} \otimes (\delta Z)_{\alpha\beta}) \otimes a^{j-2} . \end{aligned}$$

Altogether one obtains that the cohomology class in (4.6) is represented by the cocycle

$$\begin{aligned} &\Delta^j - (-1)^{n+1}B^j - (-1)^{n+1}(d\xi^j + (-1)^{n+1}\delta\xi^{j-1}) \\ &= (-1)^{n+1}(1+\sigma)^{-1}((\delta Y)_{\alpha\beta} \otimes [Z_\alpha, a^{j-1}]) + (-1)^n(1+\sigma)^{-1}((\delta Z)_{\alpha\beta} \otimes [Y_\alpha, a^{j-1}]) \\ &\quad + (-1)^n(\delta Z)_{\alpha\beta} \otimes [(\delta Y)_{\alpha\beta}, a^{j-1}] - (-1)^{n+1}\sigma^{-1}((\delta Z)_{\alpha\beta} \otimes [(\delta Y)_{\alpha\beta}, a^{j-1}] \\ &\quad + ((\delta Z)_{\alpha\beta} \otimes (\delta Y)_{\beta\gamma} + (\delta Y)_{\beta\gamma} \otimes (\delta Z)_{\alpha\beta}) \otimes a^{j-2} \\ &= (1+\sigma)^{-1}\{(-1)^{n+1}((\delta Y)_{\alpha\beta} \otimes [Z_\alpha, a^{j-1}]) + (-1)^n((\delta Z)_{\alpha\beta} \otimes [Y_\alpha, a^{j-1}]) \\ &\quad + (-1)^n((\delta Z)_{\alpha\beta} \otimes [(\delta Y)_{\alpha\beta}, a^{j-1}]) + (\delta Z)_{\alpha\beta} \otimes (\delta Y)_{\beta\gamma} \otimes a^{j-2}\} . \end{aligned}$$

5 Differential Operators

In this section we recall a definition of the sheaf $\mathcal{D}_S$ of the ring of differential operators on a non singular variety S, and of the left $\mathcal{O}_S$-module $\mathcal{D}_S^n$.

5.1. The sheaf

$$\mathcal{T} := \bigoplus_{\nu \geq 0} T_S^{\oplus\nu} = \mathcal{O}_S \oplus T_S \oplus T_S \otimes_k T_S \oplus \ldots \oplus T_S \otimes_k \ldots \otimes_k T_S \oplus \ldots$$

is a sheaf of left $\mathcal{O}_S$-modules, where $\mathcal{O}_S$ acts on $T_S \otimes_k \ldots \otimes_k T_S$ by multiplication from the left on the left factor.

5.2. A right $\mathcal{O}_S$-module structure on $\mathcal{T}$ is given (inductively) by the product

$$T_S \times \mathcal{O}_S \to \mathcal{O}_S \oplus T_S$$
$$(x, \lambda) \mapsto x(\lambda) + \lambda \cdot x \,.$$

For example,

$$(x \otimes y) \cdot \lambda = x \otimes \lambda \cdot y + x \cdot y(\lambda) = x \otimes \lambda \cdot y + y(\lambda) \cdot x + x(y(\lambda)) \,.$$

We drop the $\cdot$ in the sequel.

5.3. One defines a ring structure on $\mathcal{T}$ by the tensor product

$$T_S^{\otimes v} \times T_S^{\otimes \mu} \to T_S^{\otimes v+\mu}$$

and by the left and the right $\mathcal{O}_S$-module structure $\mathcal{O}_S \times T_S^{\otimes v} \to T_S^{\otimes v}$ and

$$T_S^{\otimes v} \times \mathcal{O}_S \to \bigoplus_{i=0}^{v} T_S^{\otimes i} \,.$$

5.4. The ring $\mathcal{D}_S$ of differential operators on S can be defined as $\mathcal{D}_S = \mathcal{T}/I$, where

$$I = \langle x \otimes y - y \otimes x - [x,y]; x,y \in T_S \rangle_{\mathcal{T}}$$

is the two-sided ideal. In other terms, I is the $\mathcal{O}_S$-bisubmodule of $\mathcal{T}$ generated by expressions (*)

$$y_1 \otimes \ldots \otimes y_{i-1} \otimes (x_i \otimes x_{i+1} - x_{i+1} \otimes x_i - [x_i, x_{i+1}]) \otimes y_{i+2} \otimes \ldots \otimes y_n \,,$$

where $n \geqq 2$. One has for a local section λ of $\mathcal{O}_S$

$$\begin{aligned}(x \otimes y - y \otimes x - [x,y])\lambda = & - \lambda(x \otimes y - y \otimes x - [x,y]) \\ & + (x \otimes \lambda y - \lambda y \otimes x - [x, \lambda y]) \\ & + (\lambda x \otimes y - y \otimes \lambda x - [\lambda x, y])\end{aligned}$$

by the definition of the right module structure in (5.2). Hence, as an $\mathcal{O}_S$-left module I is as also generated by the expressions (*) for $y_1 \ldots y_{i-1}, y_{i+2}, \ldots y_n, x_i, x_{i+1} \in T_S$.

5.5. $\mathcal{T}$ is filtered by the $\mathcal{O}_S$ bimodules $\mathcal{T}^n = \bigoplus_{v=0}^{n} T_S^{\otimes v}$ and correspondingly $\mathcal{D}_S$ is filtered by the $\mathcal{O}_S$ bimodules

$$\mathcal{D}_S^n = \mathcal{T}^n / I \cap \mathcal{T}^n \,.$$

In general, since the relation (*) mixes up the degrees, one has no natural splitting $\mathcal{D}_S = \mathcal{D}_S^n \oplus \mathcal{D}_S/\mathcal{D}_S^n$. However, for $n = 0$, one has $\mathcal{D}_S^0 = \mathcal{O}_S$ and the inclusion $\mathcal{O}_S \hookrightarrow \mathcal{D}_S$ gives an isomorphism of left $\mathcal{O}_S$-modules

$$\mathcal{D}_S = \mathcal{O}_S \oplus \mathcal{D}_S/\mathcal{O}_S$$

and

$$\mathcal{D}_S^n = \mathcal{O}_S \oplus \mathcal{D}_S^n/\mathcal{O}_S \,,$$

since

$$I \cup \mathcal{O}_S = \{0\} \,.$$

5.6 Claim. *One has a commutative diagram of exact sequences of $\mathcal{O}_S$ bimodules*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_S & \xrightarrow{=} & \mathcal{O}_S & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{D}_S^{n-1} & \longrightarrow & \mathcal{D}_S^n & \longrightarrow & S^n_{\mathcal{O}_S}(T_S) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{D}_S^{n-1}/\mathcal{O}_S & \longrightarrow & \mathcal{D}_S^n/\mathcal{O}_S & \longrightarrow & S^n_{\mathcal{O}_S}(T_S) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0\,. & & & &
\end{array}$$

Proof. One just has to show that the middle horizontal exact sequence exists. Since the image of I in

$$T_S^{\otimes n} = \mathcal{T}^n/\mathcal{T}^{n-1}$$

is generated by expressions

$$y_1 \otimes \ldots \otimes y_{i-1} \otimes (x_i \otimes x_{i+1} - x_{i+1} \otimes x_i) \otimes y_{i+2} \otimes \ldots \otimes y_n$$

one has a surjection $\mathcal{D}_S^n \longrightarrow S_k^n(T_S)$ and hence a surjection $\mathcal{D}_S^n \longrightarrow S^n_{\mathcal{O}_S}(T_S)$. For $n \geqq 2$, the kernel is generated by $\mathcal{D}_S^{n-1}$ and by expression

$$r = y_1 \otimes .. \otimes y_{i-1} \otimes (x_i \otimes \lambda x_{i+1} - \lambda x_i \otimes x_{i+1}) \otimes y_{i+2} \otimes .. \otimes y_n$$

for $\lambda \in \mathcal{O}_S$. However, in $\mathcal{D}_S^n$ one has that $r \in \mathcal{D}_S^{n-1}$ as, using the relation (5.4), (*), one has $x_i \otimes \lambda x_{i+1} - \lambda x_i \otimes x_{i+1} = \lambda[x_{i+1}, x_i] + [x_i, \lambda x_{i+1}] \in \mathcal{D}_S^1$.

5.7 Corollary. *Assume that $\mathcal{B}(n)$, for $n \geqq 1$, are left $\mathcal{O}_S$-modules, with left $\mathcal{O}_S$-linear morphisms*

$$\mathcal{B}(1) \xrightarrow{\varrho_1} \mathcal{B}(2) \xrightarrow{\varrho_2} \mathcal{B}(3) \ldots \xrightarrow{\varrho_n} \mathcal{B}(n+1) \longrightarrow \ldots\,.$$

Assume moreover that one has left $\mathcal{O}_S$-linear morphisms

$$\varphi'_n : T_S^{\otimes n} = T_S \otimes_k \ldots \otimes_k T_S \to \mathcal{B}(n)$$

and a product

$$\tau : T_S \otimes \mathcal{B}(n-1) \to \mathcal{B}(n)$$

such that, for all n:

a) $\tau(\mathrm{id} \otimes \varphi'_{n-1}) = \varphi'_n$

b) $\varphi'_{n+1}((y \otimes z - z \otimes y) \otimes x_3 \otimes \ldots \otimes x_{n+1}) - \varrho_n(\varphi'_n([y,z] \otimes x_3 \otimes \ldots \otimes x_n)) = 0\,.$

Then the induced morphism

$$\Phi'_n = \sum_{\nu=1}^{n} \varphi'_\nu = \mathcal{T}^n/\mathcal{O}_S \to \mathcal{B}(n)$$

factors through

$$\Phi_n : \mathcal{D}_S^n/\mathcal{O}_S \to \mathcal{B}(n)\,.$$

Proof. By (5.4) one just has to show that $\Phi'_n(I) = 0$ or, that

$$\Phi'_n(y_1 \otimes \ldots \otimes y_{i-1} \otimes (x_i \otimes x_{i+1} - x_{i+1} \otimes x_i) \otimes y_{i+2} \otimes \ldots \otimes y_n)$$
$$-\varrho_{n-1}(\varphi'_{n-1}(y_1 \otimes \ldots \otimes y_{i-1} \otimes [x_i, x_{i+1}] \otimes y_{i+2} \otimes \ldots \otimes y_n)) = 0 .$$

By induction on n and using a) we have this property for $i \geqq 1$. For $i = 1$ it is just assumption b).

6 The definition of the higher order Kodaira-Spencer class

For $\mathcal{A}$ as in (3.1) we had constructed in (2.14) an exact sequence

$$0 \to (\delta_* \mathcal{A}^{\bullet -1}(n-1))^{-\Sigma_{n-2}} \to (\delta_* \mathcal{A}^{\bullet}(n))^{-\Sigma_n} \to (\delta_* \mathcal{A}^0(n))^{-\Sigma_n} \to 0$$

where

$$\delta: X \times_S \ldots \times_S X \to S^n(X) = X \times_S \ldots \times_S X/\Sigma_n$$

is the quotient map.

Unfortunately, due to the fact that the fixgroup of the diagonal

$$\Delta_{\{1,2\}\{3\},\ldots,\{n\}} = X \times_S \ldots \times_S X((n-1) \text{ times})$$

is not Σ_{n-1}, we have just the antiinvariants under Σ_{n-2} on the left hand side. This makes the definition in this section a little more unpleasant. From now on, we use the trace maps $\frac{1}{|G|}\sum \text{sign}(\tau) \cdot \tau$ to consider the sheaves of antiinvariants as quotient sheaves. We keep the notations from Sect. 3, in particular we write

$$f: X \times_S \ldots \times_S X \to S .$$

6.1 Definition. *We define a quotient sheaf*

$$\mathcal{B}(n) = (R^n f_* \mathcal{A}^{\bullet}(n))^{-\Sigma_n, -\Sigma_{n-1}, \ldots, -\Sigma_2}$$

of $(R^n f_* \mathcal{A}^{\bullet}(n))^{-\Sigma_n}$ *recursively by:*

a) $\mathcal{B}(1) = R^1 f_* \mathcal{A}^{\bullet}(1) = R^1 f_* \mathcal{A}.$

b) *Assume that we have defined* $\mathcal{B}(n-1)$. *Since* $(R^{n-1} f_* \mathcal{A}^{\bullet}(n-1))^{-\Sigma_{n-1}}$ *is a quotient of* $(R^{n-1} f_* \mathcal{A}^{\bullet}(n-1))^{-\Sigma_{n-2}}$ *the complex* $\mathcal{B}(n-1)$ *is a quotient of*

$$(R^{n-1} f_* \mathcal{A}^{\bullet}(n-1))^{-\Sigma_{n-2}} .$$

Let

$$0 \to \mathcal{K}(n-1) \to (R^{n-1} f_* \mathcal{A}^{\bullet}(n-1))^{-\Sigma_{n-2}} \to \mathcal{B}(n-1) \to 0$$

be the induced exact sequence.

By (2.14) *we have an exact sequence*

$$(R^{n-1} f_* \mathcal{A}^{\bullet}(n-1))^{-\Sigma_{n-2}} \underset{\tilde{\varrho}_{n-1}}{\to} (R^n f_* \mathcal{A}^{\bullet}(n))^{-\Sigma_n} \underset{\tilde{\gamma}_n}{\to} (R^n f_* \mathcal{A}^0(n))^{-\Sigma_n}$$

and we define

$$\mathcal{B}(n) = (R^n f_* \mathcal{A}^{\bullet}(n))^{-\Sigma_n} / \tilde{\varrho}_{n-1}(\mathcal{K}(n-1)) .$$

c) *In particular one has an exact sequence*

$$\mathcal{B}(n-1) \underset{\varrho_{n-1}}{\to} \mathcal{B}(n) \underset{\gamma_n}{\to} (R^n f_* \mathcal{A}^0(n))^{-\Sigma_n}\,.$$

6.2 *Remark.* (i) Obviously $\mathcal{B}(n)$ is again an $\mathcal{O}_S$ left module.
(ii) If $\tilde{\varrho}_{n-1}$ is injective, then ϱ_{n-1} is injective as well.
(iii) If $\tilde{\gamma}_n$ is surjective, γ_n is again surjective.

6.3. In (3.11), for $T_S = T'$, we had defined a left $\mathcal{O}_S$ linear morphism

$$\tilde{\varphi}_n : T_S \otimes_k \ldots \otimes_k T_S \to R^n f_* \mathcal{A}^\bullet(n)\,.$$

By composition with

$$R^n f_* \mathcal{A}^\bullet(n) \to (R^n f_* \mathcal{A}^\bullet(n))^{-\Sigma_n} \to \mathcal{B}(n)$$

one obtains morphisms

$$\varphi'_n : T_S \otimes_k \ldots \otimes_k T_S \to \mathcal{B}(n)\,.$$

Together with $\varrho_i : \mathcal{B}(i) \longrightarrow \mathcal{B}(i+1)$, as in (3.13), one gets

$$\Phi'_n : \mathcal{T}^n/\mathcal{O}_S = \bigoplus_{\nu=1}^{n} T_S^{\otimes_k \nu} \to \mathcal{B}(n)$$

with

$$\Phi'_n | T_S^{\otimes \nu} = \varrho_{n-1} \circ \varrho_{n-2} \ldots \circ \varrho_\nu \circ \varphi'_\nu\,.$$

6.4 Theorem. *The morphism*

$$\Phi'_n : \mathcal{T}^n/\mathcal{O}_S \to \mathcal{B}(n)$$

factors through

$$\Phi_n : \mathcal{D}^n/\mathcal{O}_S \to \mathcal{B}(n)\,.$$

We will call Φ_n the ***Kodaira-Spencer map of order*** *n.*

Proof. Let us remind that in (3.11) $\tilde{\varphi}_n$ was constructed as the edge morphism of an extension (ε_n) which was obtained composing $(1 \boxtimes_k \varepsilon_{n-1})$ with an extension (ε). Hence one has a commutative diagram

$$\begin{array}{ccc} T_S \otimes_k \ldots \otimes_k T_S & \xrightarrow{\tilde{\varphi}_n} & R^n f_* \mathcal{A}^\bullet(n) \\ {\scriptstyle 1 \otimes_k \tilde{\varphi}_{n-1}} \downarrow & & \uparrow {\scriptstyle \varphi_\varepsilon} \\ T_S \otimes_k R^{n-1} f_* \mathcal{A}^\bullet(n-1) & \underset{\tilde{\tau}}{\to} & R^{n-1} f_*(\mathrm{pr}_1^{-1} T_S \otimes_k \mathcal{A}^\bullet(n-1)) \end{array}$$

where φ_ε is the edge morphism of (ε) and $\tilde{\tau}$ the map induced by the Künneth decomposition. The induced map

$$T_S \otimes_k R^{n-1} f_* \mathcal{A}^\bullet(n-1) \underset{\varphi_\varepsilon \circ \tilde{\tau}}{\to} \mathcal{B}(n)$$

factors, by construction of $\mathcal{B}(n)$, through

$$\tau: T_S \otimes_k \mathcal{B}(n-1) \to \mathcal{B}(n)$$

and $\tau(1 \otimes \varphi'_{n-1}) = \varphi'_n$.

The assumption b) of (5.7) has been verified in (4.6) and hence (5.7) implies (6.4). □

Before we study some properties of Φ_n and of the sheaves $\mathcal{B}(n)$, let us recall a well known statement from cohomology theory.

6.5 Lemma. *Let Σ_n act on*

$$\mathcal{A}^0(n) = \mathrm{pr}_1^{-1}\mathcal{A} \otimes_{f^{-1}\mathcal{O}_S} \cdots \otimes_{f^{-1}\mathcal{O}_S} \mathrm{pr}_n^{-1}\mathcal{A}$$

and on

$$R^1f_*\mathcal{A} \otimes_{\mathcal{O}_S} \ldots \otimes_{\mathcal{O}_S} R^1f_*\mathcal{A} \ (n \ times)$$

by permuting the factors. Then the Künneth map

$$R^1f_*\mathcal{A} \otimes_{\mathcal{O}_S} \ldots \otimes_{\mathcal{O}_S} R^1f_*\mathcal{A} \to R^nf_*(\mathcal{A}^0(n))$$

sends the invariants $S^n_{\mathcal{O}_S}(R^1f_*\mathcal{A})$ of the left hand side to the antiinvariants

$$(R^nf_*(\mathcal{A}^0(n))^{-\Sigma_n}$$

on the right hand side.

Proof. It is sufficient to verify (6.5) for a two cycle and hence one may assume that $n = 2$.

Any Σ_2 invariant element in $R^1f_*\mathcal{A} \otimes_{\mathcal{O}_S} \otimes R^1f_*\mathcal{A}$ is the sum of elements $\xi = m \otimes n + n \otimes m$ for $m, n \in R^1f_*\mathcal{A}$. For some covering $\{V_\alpha\}$ of X let $\underline{n} = (n_{\alpha\beta})$ and $\underline{m} = (m_{\alpha\beta})$ be representatives. The image of ξ is represented by the 2 cocycle

$$p_{\alpha\beta\gamma} = m_{\alpha\beta} \otimes n_{\beta\gamma} + n_{\alpha\beta} \otimes m_{\beta\gamma} \,.$$

One has for $\sigma = (1,2) \in \Sigma_2$:

$$(1+\sigma^{-1})p_{\alpha\beta\gamma} = m_{\alpha\beta} \otimes n_{\beta\gamma} + n_{\alpha\beta} \otimes m_{\beta\gamma} + n_{\beta\gamma} \otimes m_{\alpha\beta} + m_{\beta\gamma} \otimes n_{\alpha\beta} \,.$$

By (4.11) one has

$$(1+\sigma^{-1})p_{\alpha\beta\gamma} = -\delta(m_{\alpha\beta} \otimes n_{\alpha\beta} + n_{\alpha\beta} \otimes m_{\alpha\beta})$$

and

$$\xi = -\sigma^{-1}\xi \,.$$ □

6.6 Corollary. *Using the notations from* (6.1.c) *and* (6.3), *the image of the morphism*

$$\mathcal{D}^n_S/\mathcal{O}_S \xrightarrow{\Phi_n} \mathcal{B}(n) \xrightarrow{\gamma_n} (R^nf_*\mathcal{A}^0(n))^{-\Sigma_n}$$

lies in $S^n_{\mathcal{O}_S}(R^1f_*\mathcal{A})$. Moreover, for

$$x_1 \otimes \ldots \otimes x_n \in T_S \otimes_k \ldots \otimes_k T_S$$

one has

$$\gamma_n \circ \varphi'_n(x_1 \otimes \ldots \otimes x_n) = \varphi'_1(x_1) \otimes \ldots \otimes \varphi'_1(x_n) \,.$$

Proof. By (6.5) the first statement follows from the second one. For the latter one just remember that the extension (ε_n) which gave rise to $\tilde{\varphi}_n$ is a lifting of the n extension $(\varepsilon_1) \cup (\varepsilon_1) \cup \ldots \cup (\varepsilon_1)$ of

$$T_{\{1\}} \boxtimes_k \ldots \boxtimes_k T_{\{n\}} = \mathrm{pr}_1^{-1} T \otimes_k \ldots \otimes_k \mathrm{pr}_n^{-1} T$$

by

$$\mathcal{A}^0(n) = \mathrm{pr}_1^{-1} \mathcal{A} \otimes_{f^{-1}\mathcal{O}_S} \ldots \otimes_{f^{-1}\mathcal{O}_S} \mathrm{pr}_n^{-1} \mathcal{A} .$$

(See (3.11)).

Putting everything together one has obtained up to now the first half of :

6.7 Theorem. *Under the assumptions of* (3.1) *let*

$$\varphi_1 : T_S \to R^1 f_* \mathcal{A} = \mathcal{B}(1)$$

be the edge morphism of the exact sequence

$$0 \to \mathcal{A} \to \tilde{\mathcal{A}} \to f^{-1} T_S \to 0 .$$

Then for Φ_n as in (6.4) *one has a commutative diagram of exact sequences*

$$\begin{array}{ccccccc} 0 \to & \mathcal{D}_S^{n-1}/\mathcal{O}_S & \to & \mathcal{D}_S^n/\mathcal{O}_S & \to & S^n_{\mathcal{O}_S}(T_S) & \to 0 \\ & \Phi_{n-1} \downarrow & & \Phi_n \downarrow & & \downarrow \Psi_n & \\ & \mathcal{B}(n-1) & \underset{\varrho_{n-1}}{\to} & \mathcal{B}(n) & \underset{\gamma_n}{\to} & (R^n f_* \mathcal{A}^0(n))^{-\Sigma_n} & \end{array}$$

such that

a) $\Phi_1 = \varphi_1$

b) *Ψ_n factorizes through*

$$S^n_{\mathcal{O}_S}(T_S) \xrightarrow{S^n(\varphi_1)} S^n_{\mathcal{O}_S}(R^1 f_* \mathcal{A}) \to (R^n f_* \mathcal{A}^0(n))^{-\Sigma_n}$$

where the second map is the Künneth map.

c) *If $f_* \mathcal{A} = 0$, then*

$$\varrho_n : \mathcal{B}(n-1) \to \mathcal{B}(n)$$

is injective,

$$S^n_{\mathcal{O}_S}(R^1 f_* \mathcal{A}) \cong (R^n f_* \mathcal{A}^0(n))^{-\Sigma_n}$$

and

$$\Psi_n = S^n(\varphi_1) .$$

d) *If $R^i f_* \mathcal{A} = 0$, for $i \geqq 2$, then again*

$$S^n_{\mathcal{O}_S}(R^1 f_* \mathcal{A}) \cong (R^n f_* \mathcal{A}^0(n))^{-\Sigma_n}$$

and

$$\Psi_n = S^n(\varphi_1) .$$

Proof. It just remains to show c) and d).

By the Künneth formula one has

$$R^m f_* \mathcal{A}^0(n) = \oplus R^{\nu_1} f_* \mathcal{A} \otimes_{\mathcal{O}_S} \otimes R^{\nu_2} f_* \mathcal{A} \otimes_{\mathcal{O}_S} \ldots \otimes_{\mathcal{O}_S} R^{\nu_n} f_* \mathcal{A} ,$$

where the sum is taken over all tuples $(\nu_1, \ldots, \nu_n)$ with $\sum_{i=1}^n \nu_i = m$.

Under the assumptions made in c) and d), for $m = n$ the only tuple is $(1, \dots, 1)$, and in c), for $m = n - 1$, there is no such tuple. □

6.8 Corollary. *Keeping the notations and assumptions from* (6.7) *assume that* $f_*\mathcal{A} = 0$. *Then one has*

a) *If* φ_1 *is surjective, then the sequence*

$$0 \to \mathcal{B}(n-1) \underset{\varrho_{n-1}}{\to} \mathcal{B}(n) \underset{\gamma_n}{\to} S^n_{\mathcal{O}_S}(R^1 f_* \mathcal{A}) \longrightarrow 0$$

is exact and Φ_n *is surjective for all* n.

b) *If* φ_1 *is injective, then* Φ_n *is injective for all* n.

Proof. By (6.7) ϱ_{n-1} is injective. One obtains a) and b) by induction on n, starting with (6.7.a). □

6.9 Corollary. *Under the assumptions of* (6.7.d), (*i.e.* $R^i f_* \mathcal{A} = 0$ *for* $i \geqq 2$) *one has*:

a) *If* φ_1 *is surjective, then the sequence*

$$0 \to \mathcal{B}(n-1) \underset{\varrho_{n-1}}{\to} \mathcal{B}(n) \underset{\gamma_n}{\to} S^n_{\mathcal{O}_S}(R^1 f_* \mathcal{A}) \to 0$$

is exact and Φ_n *is surjective for all* n.

b) *If* φ_1 *is an isomorphism, then* Φ_n *is an isomorphism for all* n.

Proof. The surjectivity of $S^n(\varphi_1)$ implies the surjectivity of γ_n. By induction one obtains the surjectivity of Φ_n. Moreover, b) follows from a). Hence, (6.9) follows from the following claim. □

6.10 Claim. *Under the assumptions made in* (6.9.a) *the morphisms*

$$R^i f_* \mathcal{A}^\bullet(n) \underset{\gamma_i}{\to} R^i f_* \mathcal{A}^0(n)$$

are surjective for all $i \leqq n$.

Proof. By induction on n we may assume that

$$R^{i-1} f_* \mathcal{A}(n-1) \to R^{i-1} f_* \mathcal{A}^0(n-1)$$

is surjective. In (3.10) we had constructed an extension (ε) of $T_{\{1\}} \boxtimes_k \mathcal{A}^\bullet(n-1)$ by $\mathcal{A}^\bullet(n)$. If we write $\tilde{\mathcal{A}}'^\bullet(n)$ for the corresponding sheaf, this extension is a lifting of $(\varepsilon_1 \boxtimes 1)$, i.e. one has a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow \mathcal{A}^\bullet(n) & \longrightarrow & \tilde{\mathcal{A}}'^\bullet(n) & \longrightarrow & T_{\{1\}} \boxtimes_k \mathcal{A}^\bullet(n-1) & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow \mathcal{A}^0(n) & \longrightarrow & \tilde{\mathcal{A}} \boxtimes_{f^{-1}\mathcal{O}_S} \mathcal{A}^0(n-1) & \longrightarrow & T_{\{1\}} \boxtimes_{f^{-1}\mathcal{O}_S} \mathcal{A}^0(n-1) & \longrightarrow 0 \end{array}$$

inducing

$$\begin{array}{ccc} R^{i-1}f_*(T_{\{1\}}\boxtimes_k\mathcal{A}^\bullet(n-1)) & \xrightarrow{\alpha} & R^if_*\mathcal{A}^\bullet(n) \\ \varrho\downarrow & & \downarrow\gamma_i \\ R^{i-1}f_*\{T_{\{1\}}\boxtimes_{f^{-1}\mathcal{O}_S}\mathcal{A}^0(n-1)\} & \underset{\beta}{\to} & R^if_*\mathcal{A}^0(n)\,. \end{array}$$

If

$$(R^{i-\nu-1}f_*T)\otimes_k(R^\nu f_*\mathcal{A}^\bullet(n-1))$$

is a Künneth component of $R^{i-1}f_*(T_{\{1\}}\boxtimes_k\mathcal{A}^\bullet(n-1))$, then its image under ϱ lies in

$$(R^{i-\nu-1}f_*T)\otimes_{\mathcal{O}_S}(R^\nu f_*\mathcal{A}^0(n-1))$$

and by induction ϱ is surjective.

By assumption $R^jf_*\mathcal{A}=0$ for $j\geqq 2$ and hence

$$R^{j-1}f_*T\to R^jf_*\mathcal{A}$$

is surjective for $j\geqq 2$. For $j=1$ this is nothing but φ_1, hence surjective by assumption. β decomposes into a direct sum of maps

$$\beta_\nu:(R^{i-\nu-1}f_*T)\otimes_{\mathcal{O}_S}(R^\nu f_*\mathcal{A}^0(n-1))\to(R^{i-\nu-1}f_*\mathcal{A})\otimes_{\mathcal{O}_S}(R^\nu f_*\mathcal{A}^0(n-1))\,,$$

which, as we just remarked, are surjective for all ν. Then β is surjective, as well as $\gamma_i\circ\alpha$ and γ_i. □

6.11 *Remarks.* a) The assumptions made in (6.7.c) are, for example, satisfied for families $f:X\to S$ of non singular projective varieties X_s without infinitesimal automorphisms (see (3.4)), or for families of stable projective bundles (see (3.5), (3.6)).

b) The assumptions made in (6.7.d) are satisfied for families of projective bundles over a curve (see (3.6)), or for families $g:Y\to S$ where $Y_s=g^{-1}(s)$ is a projective bundle over a curve.

c) Throughout this paper, except for the introduction, we avoided to talk about moduli. However, if M is a fine moduli scheme for a moduli problem controlled by a Lie algebra $\mathcal{A}$ satisfying (3.1) and the additional condition $f_*\mathcal{A}=0$ or $R^if_*\mathcal{A}=0$ for $i\geqq 2$, then (6.8) a) and b) or (6.9.b) imply that the sheaf of differential operators $\mathcal{D}^n_{M^0}/\mathcal{O}_{M^0}$ is equal to $\mathcal{B}(n)|_{M^0}$, where $M^0=M-\mathrm{Sing}(M)$ and $\mathcal{B}(n)$ is the sheaf constructed in (6.1), for the universal family over M.

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Schechtman's approach to "higher Kodaira-Spencer maps" is different. He wants to consider $Rf_*\mathcal{A}$ as a kind of differential graded Lie algebra and wants to extend this structure to anti-symmetric products of this complex. We hope to be able to compare his approach with ours in the near future. At this point we thank him for explaining to us his ideas and for contributing to this work hereby.

Finally, we thank P. Deligne for his interest and the explanation of his understanding of the problem [3]. Even, if we were not able to compute in his language of differential graded Lie algebras, his presentation helped us and we largely profited from his viewpoint.

After a first version of the paper was written we learned at the occasion of a talk on this subject during the "Journées de Géométrie Algébriques d'Orsay, 1992", that Z. Ran worked on similar problems and was about to announce results (see [4]), presumably overlapping with those

presented here. (In between his announcement appeared in: Int. Math. Res. Notices, Duke **4**, 93–106 (1993))

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