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Coverings with odd ramification and Stiefel-Whitney classes

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J.-P. Serre considers in [23] a covering $\pi: Y \to X$ of degree n of Riemann surfaces whose ramification indices e_y for $y \in Y$ are odd. One obtains a natural relative theta characteristic

$$D_{Y/X} = \sum_{y \in Y} \frac{e_y - 1}{2} \cdot y,$$

and therefore, via duality theory, a unimodular quadratic bundle

$$E = \pi_* \mathcal{O}_Y(D_{Y/X}),$$

whose quadratic form q_E at the generic point $\mathbb{C}(X)$ of X is given by

$$\mathrm{Tr}_{\mathbb{C}(Y)/\mathbb{C}(X)}(x^2).$$

He proves a geometric formula ([23], Théorème 1, (6)) relating the second Stiefel-Whitney class

$$w_2(E, q_E) \in H^2_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$$

of (E, q_E) to a class $w_2^S(\pi) \in \mathbb{Z}/2\mathbb{Z}$, characterised by the following property:

Let G be the Galois group of the Galois hull $\pi^{\mathrm{gal}}: Y^{\mathrm{gal}} \to X$ of π and

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \tilde{G} \to G \to 0$$

be the extension obtained from the "pinor extension" [22]

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \tilde{\mathbf{O}}(n) \to \mathbf{O}(n) \to 0$$

as a pullback under the natural representation

$$(*) \qquad G \to \mathcal{S}_n \to \mathbf{O}(n).$$

Then $w_2^S(\pi) = 0$ if and only if $\tilde{G}$ occurs as the Galois group over X of an étale quadratic covering $\tilde{Y}^{\mathrm{gal}}$ of Y^{gal}. Of course, the formula may be interpreted as a geometric obstruction for the existence of $\tilde{Y}^{\mathrm{gal}}$.

To this aim Serre uses a formula ([23], Théorème 2, 9) relating $w_2(E, q_E)$ to the Atiyah-Mumford-Riemann invariant

$$h^0(X, \mathscr{L} \otimes E) \in \mathbb{Z}/2\mathbb{Z},$$

where $\mathscr{L}$ is any theta characteristic on X, a formula that he proves in the same spirit as Mumford's proof of the quadraticity of this invariant for $E = \mathcal{O}_X$ (see [2], [20]).

On the other hand, if L is a finite extension of degree n of a field K of characteristic $\neq 2$, Serre ([22], Théorème 1) had proved earlier an arithmetic formula relating

$$w_2\big(\mathrm{Tr}_{L/K}(x^2)\big) \in H^2_{\text{ét}}(K, \mathbb{Z}/2\mathbb{Z})$$

to the Stiefel Whitney class $w_2(\pi)$ of the Galois group G of the Galois hull L^{gal} of L.

$w_2(\pi)$ is obtained by pulling back through (*) the universal class

$$w_2 \in H^2\big(\mathbf{BO}(n), \mathbb{Z}/2\mathbb{Z}\big)$$

to $H^2(G, \mathbb{Z}/2\mathbb{Z})$ and sending it to $H^2_{\text{ét}}(K, \mathbb{Z}/2\mathbb{Z})$. This formula can be interpreted as an obstruction for the existence of a lifting of the representation

$$\mathrm{Gal}(K) \to G$$

to $\tilde{G}$.

In this article, we give a common generalisation of these two formulae, thereby answering a question raised by Serre ([23], p. 549).

More precisely, let X be a Dedekind scheme on which 2 is invertible and let $\pi: Y \to X$ be a tame finite flat covering of Dedekind schemes, whose ramification indices e_y are all odd.

Following Serre's own hint ([23], § 3, remarque) of using Grothendieck's theory of equivariant cohomology [11], we define invariants

$$w_i(\pi) \in H^i_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$$

by sending into $H^i_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$ those naturally defined in $H^i(G, \mathbb{Z}/2\mathbb{Z})$ as the pull-back of the universal Stiefel-Whitney classes in $H^i\big(\mathbf{BO}(n), \mathbb{Z}/2\mathbb{Z}\big)$ via (*) (see § 1). With this definition, we prove the formula:

(S) $$w_2(E, q_E) + \omega(Y, X) = w_2(\pi) + (2) \cup \big(w_1(E, q_E)\big),$$

where $(E, q_E) := (\pi_* \mathcal{O}_Y(D_{Y/X}), \mathrm{Tr}_{Y/X}(x^2))$, is a quadratic bundle defined as in the geometrical case via duality theory, where (2) is the class of $2 \in \Gamma(X, \mathbb{G}_m)$ in $H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$ via the Kummer sequence, $w_i(E, q_E) \in H^i_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$ are the Stiefel-Whitney classes of (E, q_E), and where

$$\omega(Y/X) := \sum_{\substack{x \text{ closed point} \\ \text{in } X}} \left(\sum_{\substack{y \text{ closed point} \\ \text{in } \pi^{-1}(x)}} \frac{e_y^2 - 1}{8} [k(y) : k(x)] \right) \cdot x \in \operatorname{Pic} X/2 \hookrightarrow H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$$

is a divisor on X modulo 2 (here $k(x)$ is residue field of the closed point x).

Localizing (S) at the generic point K of X on which $\omega(Y/S) = 0$, one recovers Serre's arithmetical formula

$$(\mathrm{S}_{\text{ét}}) \qquad w_2(E, q_E) = w_2(\pi) + (2) \cup (d_{(E, q_E)}) .$$

Going from k to $\mathbb{C}$ if X is a proper smooth curve defined over a field k of characteristic zero, (2) becomes zero and (S) gives

$$(\mathrm{S}_{\text{geom}}) \qquad w_2(E, q_E) + w(Y/X) = w_2(\pi) .$$

In particular this proves a posteriori that Serre's ad hoc geometrical invariant $w_2^S(\pi)$ coincides with the invariant $w_2(\pi)$ used here, as $(\mathrm{S}_{\text{geom}})$ is nothing but Serre's formula in the geometrical case. It also provides a proof of the geometric formula without using theta characteristics on X and the related invariant

$$h^0(X, \mathscr{L} \otimes E) \in \mathbb{Z}/2\mathbb{Z} .$$

We proceed as follows:

Thanks to the Grothendieck style definition of $w_2(\pi)$, the right hand side of (S) is functorial for base change $\varphi : Z \to X$. Constructing such a φ for which

$$\pi_Z : T := \text{normalization of } Z \times_X Y \to Z$$

is étale (see (4.6)), (S) is then a consequence of two special formulae: the first one is $(\mathrm{S}_{\text{ét}})$ when π is étale, the second one is

$$(\mathrm{S}_{\text{lc}}) \qquad w_2(\varphi^*(E, q_E)) + \varphi^* \omega(Y/X) = w_2(F, q_F)$$

where $(F, q_F) := (\pi_{Z_*} \mathcal{O}_T, \mathrm{Tr}_{T/Z}(x^2))$. Here lc stands for local contributions as one has to compute the difference between the 2nd Stiefel-Whitney classes of two quadratic bundles (E, q_E) and (F, q_F) which coincide at the generic point. We give in section 6 a general method to evaluate those local contributions, using the theory of split bundles.

The first section of this paper contains an overview of Stiefel-Whitney classes w_i as obtained from the classifying space of the orthogonal group [15], [16]. We reproduce Fröhlich's definition of the pinor group [7], appendix 1, and prove that w_2 is obtained by the connecting map associated to the pinor extension 1.14. Actually this section contains more details than really needed for the proof of the formula (S).

Section 2 proves ($S_{\text{ét}}$), section 3 defines the classes $w_i(\pi)$ mentioned above and section 4 contains a geometric construction needed for the reduction of (S) to ($S_{\text{ét}}$).

Section 5 makes explicit the splitting principle of [18] for Stiefel-Whitney classes and, in particular, gives detailed proofs of it.

In section 6 we prove (S_{lc}) and finally (S) is proved in section 7.

This article would not have existed without [23] and [22]. We thank J.-P. Serre for the interest he showed in our work and for his comments which helped us to improve it.

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§ 1. Stiefel-Whitney classes

In this section we recapitulate various definitions of Stiefel-Whitney classes for orthogonal vector bundles and equivariant such, and prove that they coincide.

Stiefel-Whitney classes for orthogonal vector bundles and for orthogonal representations of a finite group are special cases of more general Stiefel-Whitney classes associated to equivariant orthogonal vector bundles. These in turn are special cases of Stiefel-Whitney classes associated to a simplicial orthogonal vector bundle over a simplicial scheme. Even though simplicial schemes are rather wild objects for usual needs, they provide the right generality for constructions like the splitting principle to cover also the case of group representations. It seems most convenient to develop the theory in this context in view of future applications.

It would also be possible to attach Stiefel-Whitney classes to orthogonal vector bundles over a locally ringed topos in the vein of [13] (provided 2 is invertible on the topos). We do not do this because Jardine computed the étale cohomology of $\mathbf{BO}(n)$ in [14], [15] as a simplicial scheme, not as a topos. In any case, the étale cohomology of the simplicial scheme $\mathbf{BO}(n)/X$ and that of the topos $(\mathbf{BO}(n)/X)_{\text{ét}}$ coincide for any scheme X over Spec $\mathbb{Z}[\frac{1}{2}]$ by [8], th. 1.12.

For the étale cohomology of simplicial schemes (more generally, étale simplicial sheaves over a base scheme) we refer to [15], § 2. All simplicial schemes are over Spec $\mathbb{Z}[\frac{1}{2}]$. We recall once and for all that if $X_\bullet$ is a simplicial scheme, the étale sheaf μ_2 of square roots of unity is constant over $(X_\bullet)_{\text{ét}}$, canonically isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The same applies to all Tate twists $\mu_2^{\otimes i}$, and we shall switch between notations without mention, unless necessary for the understanding.

For simplicity, we drop the subscript "ét" from étale cohomology groups, as no other topology will be used here.

1.1. Notations. Let $X_\bullet$ be a simplicial scheme (henceforth abbreviated to *sscheme*). Recall ([9], ex. 1.1) that a *simplicial vector bundle* over $X_\bullet$ is by definition a morphism of sschemes

$$E_\bullet \to X_\bullet$$

such that every

$$E_n \to X_n$$

defines a vector bundle and, furthermore, face and degeneracy maps induce isomorphisms of vector bundles. An *orthogonal (or quadratic) vector bundle* over $X_\bullet$ is a simplicial vector bundle $E_\bullet$ over $X_\bullet$ provided with an unimodular symmetric bilinear form

$$b: E_\bullet \times E_\bullet \to \mathbf{1},$$

where $\mathbf{1}$ denotes the trivial vector bundle of rank 1. Equivalently, b corresponds to a symmetric isomorphism

$$\tilde{b}: E_\bullet \to \mathscr{H}om(E_\bullet, \mathbf{1}).$$

Orthogonal vector bundles of rank n over $X_\bullet$ are classified by two objects (compare [9], ex. 1.1, and [8], proof of lemma 4):

(i) The nonabelian cohomology set $H^1(X_\bullet, \mathbf{O}(n))$. In particular, orthogonal line bundles are classified by elements of

$$H^1(X_\bullet, \mathbf{O}(1)) = H^1(X_\bullet, \mu_2).$$

(ii) The set of homotopy classes of maps

$$\left[X_\bullet, \mathbf{BO}(n)/\operatorname{Spec}\left[\frac{1}{2}\right]\right].$$

From now on, we abbreviate the sscheme

$$\mathbf{BO}(n)/\operatorname{Spec}\mathbb{Z}\left[\frac{1}{2}\right] \quad \text{by} \quad \mathbf{BO}(n).$$

1.2. Remark. The natural map

$$[X, \mathbf{BO}(n)/X] \to \left[X, \mathbf{BO}(n)/\operatorname{Spec}\mathbb{Z}\left[\frac{1}{2}\right]\right] = [X, \mathbf{BO}(n)]$$

is bijective. Therefore we could work with $\mathbf{BO}(n)/X$ as well.

To an orthogonal vector bundle $E_\bullet$ we will associate in the sequel characteristic classes

$$w_i(E_\bullet) \in H^i(X_\bullet, \mathbb{Z}/2\mathbb{Z}),$$

called the *Stiefel-Whitney classes of* $E_\bullet$. By convention, $w_0(E_\bullet) = 1$. We denote by $w(E)$ the sum of all $w_i(E_\bullet)$ in $H^*(X_\bullet, \mathbb{Z}/2\mathbb{Z})$. Stiefel-Whitney classes have the following properties:

1.3. (a) *Functoriality*. Let $f: Y_\bullet \to X_\bullet$ be a morphism of sschemes, and $f^* E_\bullet$ the inverse image of $E_\bullet$ over $Y_\bullet$. Then

$$w_i(f^* E_\bullet) = f^* w_i(E_\bullet)$$

for all $i \geqq 0$.

(b) *Whitney formula*. Let $E_\bullet$ and $F_\bullet$ be two orthogonal vector bundles over $X_\bullet$. Then

$$w(E_\bullet \oplus F_\bullet) = w(E_\bullet) \cdot w(F_\bullet).$$

(c) *Normalisation*. Let $L_\bullet$ be a line bundle over $X_\bullet$. Then $w_i(L_\bullet) = 0$ for $i \geqq 2$, and $w_1(L_\bullet)$ is the class of $L_\bullet$ in $H^1(X_\bullet, \mathbb{Z}/2\mathbb{Z})$ (cf. 1.1 (i)).

(d) If $\operatorname{rank} E_\bullet \leqq n$, then $w_i(E_\bullet) = 0$ for $i > n$.

Properties (a), (b) and (c) characterize the Stiefel-Whitney classes w_i. They can be defined in one of the following ways:

- By means of the splitting principle.
- Via the cohomology of the simplicial scheme $\mathbf{BO}(n)$ [15], [16].

In this section we restrict ourselves to the second approach. In §5 we develop the splitting principle, for simplicity only for orthogonal vector bundles over ordinary schemes. However the simplicial case is the same, mutatis mutandis.

We proceed as follows. By 1.1 (ii), $E_\bullet$ defines a homotopy class of maps

$$[E_\bullet] \in [X_\bullet, \mathbf{BO}(n)].$$

By [16], $H^*(\mathbf{BO}(n), \mathbb{Z}/2\mathbb{Z})$ is a polynomial algebra over $H^*(\mathbb{Z}[\tfrac{1}{2}], \mathbb{Z}/2\mathbb{Z})$ on certain generators $HW_1, \ldots, HW_n$ in degree $1, \ldots, n$. We define $w_i(E_\bullet)$ as $[E_\bullet]^*(HW_i)$.

Note that $\mathbf{BO}(n)$ carries a canonical orthogonal vector bundle $\mathscr{E}_\bullet$ of rank n, corresponding to the class of the identity map in $[\mathbf{BO}(n), \mathbf{BO}(n)]$. This bundle is universal in the sense that any rank n orthogonal vector bundle is the pull-back of $\mathscr{E}_\bullet$ via its classifying map.

1.4. Proposition. (i) *The classes constructed via the splitting principle* (*see section* 5) *and those constructed via the cohomology of the simplicial scheme coincide.*

(ii) *Jardine's class* HW_i *coincides with* $w_i(\mathscr{E}_\bullet)$, *where* $\mathscr{E}_\bullet$ *is the universal orthogonal bundle on* $\mathbf{BO}(n)$.

Proof. (i) By the splitting principle, it is enough to treat the case where $E_\bullet$ is an orthogonal direct sum of line bundles $L_1, \dots, L_n$. In this case, each L_i defines a class in $[X_\bullet, \mathbf{BO}(1)]$, hence a class

$$c \in [X_\bullet, \mathbf{BO}(1)^n]$$

such that $\Delta \circ c = [E_\bullet]$, where

$$\Delta : \mathbf{BO}(1)^n \to \mathbf{BO}(n)$$

is induced by the natural inclusion

$$\mathbf{O}(1)^n \to \mathbf{O}(n).$$

Therefore,

$$[E_\bullet]^* HW_i = c^* \Delta^* HW_i.$$

But, by definition of the HW_i, the inverse image $\Delta^* HW_i$ is the i-th elementary symmetric function on the e_i, where

$$e_i = 1 \otimes \dots \otimes HW_1 \otimes \dots \otimes 1 \in H^i(\mathbf{BO}(1)^n, \mathbb{Z}/2\mathbb{Z}).$$

Here we have used the "cross product"

$$H^i(\mathbf{BO}(1), \mathbb{Z}/2\mathbb{Z})^{\otimes n} \to H^i(\mathbf{BO}(1)^n, \mathbb{Z}/2\mathbb{Z}).$$

On the other hand,

$$HW_1 \in H^1(\mathbf{BO}(1), \mathbb{Z}/2\mathbb{Z}) = H^1(\Gamma^* \mathbf{B}(\mu_2), \mathbb{Z}/2\mathbb{Z}) = H^1(\Gamma^* \mathbf{B}(\mu_2), \mu_2)$$

is by definition the image in $H^1(\Gamma^* \mathbf{B}(\mu_2), \mu_2)$ of the generator of $H^1(\mathbf{B}(\mu_2), \mu_2)$ via the natural map. But this generator gives the class of the identity map in $[\mathbf{B}(\mu_2), \mathbf{B}(\mu_2)]$ (simplicial sets), hence in $[\Gamma^* \mathbf{B}(\mu_2), \Gamma^* \mathbf{B}(\mu_2)]$ (constant simplicial sheaves over $\operatorname{Spec} \mathbb{Z}[\frac{1}{2}]$). Therefore, $[L_i]^* HW_1$ is by the normalisation 1.3 (c) nothing else than $w_1(L_i)$. It follows, for all i, that $c^* \Delta^* HW_i$ is the i-th elementary symmetric function in the $w_1(L_i)$, which is $w_i(E_\bullet)$ by the Whitney formula 1.3 (b).

(ii) is an obvious consequence of (i). □

1.5. Main example. Let X be an (ordinary) scheme and G a group scheme acting on X. An *equivariant orthogonal vector bundle* over X is an orthogonal vector bundle provided with an action of G which is compatible both with the action on X and the quadratic structure. An *equivariant étale sheaf* on (X, G) is an étale sheaf provided with a continuous action of G (in the sense that the stabiliser of any element is open in G). Equivariant étale cohomology of (X, G) is the collection of derived functors of the functor

$$\mathscr{F} \mapsto \Gamma(X, \mathscr{F})^G$$

defined on the category of abelian equivariant étale sheaves on (X, G) (cf. [13]); they are denoted by $H^*_{\text{ét}}(X, G; \mathscr{F})$.

Consider the simplicial scheme $GX_\bullet = \mathbf{E}G \times_G X_\bullet$. Any equivariant sheaf $\mathscr{F}$ on (X, G) gives rise to a simplicial sheaf

$$G\mathscr{F}_\bullet = \mathbf{E}G \times_G \mathscr{F} \quad \text{on} \quad GX_\bullet .$$

This applies to abelian étale sheaves as well as to orthogonal equivariant vector bundles.

1.6. Definition. Let E be an equivariant orthogonal vector bundle over (X, G). The i-th *Stiefel-Whitney class* $w_i(E)$ of E is $w_i(GE)$.

We can regard $w_i(E)$ as an element of $H^i(X, G; \mathbb{Z}/2\mathbb{Z})$ in view of the following

1.7. Proposition. *For any abelian group A, one has a canonical isomorphism*

$$H^*(X, G; A) \cong H^*(GX_\bullet, A) .$$

The proof is as in [8], th. 1.12.

1.8. Special cases. (a) $G = 1$. One recovers the Stiefel-Whitney classes of § 3.

(b) $X = \operatorname{Spec} F$, where F is a separably closed field (of characteristic $\neq 2$), and G discrete. An (X, G) orthogonal vector bundle is simply an orthogonal representation of G defined over F. Its Stiefel-Whitney classes are the pull backs of the universal ones in the cohomology of $\mathbf{BO}(n)$. However, the map

$$H^*(\mathbf{BO}(n), \mathbb{Z}/2\mathbb{Z}) \to H^*(G, \mathbb{Z}/2\mathbb{Z})$$

factors through

$$H^*(\mathbf{BO}(n)/\operatorname{Spec} F, \mathbb{Z}/2\mathbb{Z})$$

(remark 1.2). By [6], the cohomology of $\mathbf{BO}(n)/\operatorname{Spec} F$ canonically coincides with that of $\mathbf{BO}(n, \mathbb{C})$ (the topological classifying space of the Lie group $\mathbf{O}(n, \mathbb{C})$). Hence one recovers the usual Stiefel-Whitney classes of an orthogonal complex representation.

(c) (Fröhlich's setting).

$$G = \operatorname{Gal}(\pi'), \quad \text{where} \quad \pi' : Y' \to X$$

is an étale Galois covering. Let E be an (X, G)-orthogonal vector bundle. We denote by $\tilde{E} = (\pi_* \pi^* E)^G$, where G acts diagonally, its *Fröhlich twisted bundle*. This is an orthogonal vector bundle over X. Let H be a subgroup of G and

$$\pi : Y \to X$$

be the corresponding subcovering of π'. Let V be an (X, H)-orthogonal vector bundle. There is an (X, G) induced orthogonal vector bundle

$$E = \operatorname{Ind}_H^G V .$$

Then $\tilde{E} = (\pi_* \pi^* V)^{\sim}$ (cf. [7], th. 6). When $V = \mathcal{O}_X$ with trivial H-action, this gives back lemma 2.14. This is Fröhlich's approach to Serre's formula, that we don't develop here (although one should).

1.9. The Clifford algebra and the group $\tilde{\mathbf{O}}(n)$. In this section, we "recall" the construction of an algebraic group scheme $\tilde{\mathbf{O}}(n)$, following Fröhlich [7], appendix 1. Slightly more generally, we construct an algebraic group scheme $\tilde{\mathbf{O}}(q)$ (denoted by $\widetilde{\mathrm{Pin}}(q)$ in *loc. cit.*) for any quadratic bundle (E, q).

Let (E, q) be a quadratic bundle over a scheme X. Its *Clifford algebra* $\mathrm{Cl}(q)$ is the quotient of the tensor algebra $T(E)$ by the (two-sided) ideal generated by the

$$x \otimes x - q(x)1,$$

where x runs through E. It is a locally free $\mathcal{O}_X$-algebra. If E has constant rank n, then $\mathrm{Cl}(q)$ has constant rank 2^n. It inherits a μ_2-graduation: an element in $\mathrm{Cl}(q)$ is odd (even) if it is a sum of products of odd (even) numbers of elements of E (viewed as embedded in $\mathrm{Cl}(q)$).

The sheaf of algebras $\mathrm{Cl}(q)$ enjoys an involution (anti-automorphism with square 1)

$$x \mapsto x_t, \quad \text{characterised by } x_t = x \text{ for } x \in E.$$

The Clifford group $C^*(q)$ is the subgroup of homogeneous invertible elements x in $\mathrm{Cl}(q)$ satisfying $x \cdot v \cdot x^{-1} \in E$ for all $v \in E$. It is representable by an algebraic group scheme over X, and splits as

$$C^*(q) = C^*_+(q) \cup C^*_-(q).$$

The map $N: \mathrm{Cl}(q) \to \mathrm{Cl}(q)$ defined by $N(x) = x \cdot x_t$ restricts to a homomorphism

$$N: C^*(q) \to \mathbb{G}_m.$$

We define $\tilde{\mathbf{O}}(q)$ as the kernel of this homomorphism. It is a smooth algebraic group scheme over X.

Let an element x of $C^*_\varepsilon(q)$, for $\varepsilon = +1$ or $\varepsilon = -1$, act on E by

$$r(x)v = \varepsilon \cdot x \cdot v \cdot x^{-1}.$$

This defines a homomorphism $r: C^*(q) \to \mathbf{O}(q)$. The restriction of r to $\tilde{\mathbf{O}}(q)$ has kernel μ_2.

1.10. Lemma. *Assume that* $X = \operatorname{Spec} R$, *where* R *is a strictly henselian local ring. Then the map on rational points*

$$r: \tilde{\mathbf{O}}(q)(R) \to \mathbf{O}(q)(R)$$

is surjective.

Proof. It is known [14], cor. 4.3, that over a local ring, the group $\mathbf{O}(q)$ is generated by hyperplane reflections; therefore it is enough to see that every such reflection is in the image of r. This reduces us to the case $n = 1$. In this case, we can write $E = R \cdot t$ (for some basis element t) and $q(x \cdot t) = u \cdot x^2$ for some unit u. Since R is strictly henselian with residue characteristic $\neq 2$, the unit u is a square and we may even assume that $u = 1$. Then

$$\mathrm{Cl}(q) = R[T]/\langle T^2 - 1\rangle = R \oplus R \cdot t, \quad \text{for } t \in \tilde{\mathbf{O}}(q)(R) \text{ and } r(t) = -1 .$$

Lemma 1.10 shows that one has a short exact sequence of étale sheaves of groups (hence of smooth algebraic group schemes):

$$(*) \qquad 1 \to \mu_2 \to \tilde{\mathbf{O}}(n) \to \mathbf{O}(n) \to 1 .$$

Taking global sections on some sscheme $X_\bullet$ one obtains a long(er) exact sequence:

$$(**) \quad 1 \to \mu_2 \to \Gamma(X_\bullet, \tilde{\mathbf{O}}(n)) \to \Gamma(X_\bullet, \mathbf{O}(n)) \xrightarrow{\mathrm{sp}} H^1(X_\bullet, \mu_2) \to$$
$$\to H^1(X_\bullet, \tilde{\mathbf{O}}(n)) \to H^1(X_\bullet, \mathbf{O}(n)) \xrightarrow{\Delta} H^2(X_\bullet, \mu_2) .$$

The map sp generalises the spinor norm ([7], p. 118).

1.11. Lemma. *Let $n = 2$. The restriction of $(*)$ to the "maximal torus" of diagonal matrices $\mu_2 \times \mu_2$ is the dihedral extension whose class in*

$$H^2(\mu_2 \times \mu_2, \mu_2)$$

is $\alpha \cdot \beta$, where

$$\alpha \quad \text{and} \quad \beta \in H^1(\mu_2 \times \mu_2, \mu_2)$$

are the two "coordinate" characters of $\mu_2 \times \mu_2$.

Proof. As the cohomology $H^*(\mu_2 \times \mu_2, \mu_2)$ is a polynomial algebra on the two 'coordinate' characters, it is enough to check that further restriction of $(*)$ to each factor

$$\mu_2 \times 1 \quad \text{and} \quad 1 \times \mu_2$$

is trivial while its restriction to the diagonal subgroup is non trivial. For the first two we are reduced to the case $n = 1$. Then $\mathrm{Cl}(E)$ is commutative, and the generator v of E such that $q(v) = 1$ is in $\tilde{\mathbf{O}}(1)$ and satisfies $r(v) = -1$ (notation as in 1.9); furthermore $v^2 = 1$. For the last one, consider an orthonormal basis (v, w) of E. Then $v \cdot w \in \tilde{\mathbf{O}}(2)$. Using the relations

$$v^2 = w^2 = 1 \quad \text{and} \quad v \cdot w + w \cdot v = 0 ,$$

we compute:

$$v \cdot w \cdot v \cdot (v \cdot w)^{-1} = v \cdot w \cdot v \cdot w \cdot v = -v \cdot w \cdot v \cdot v \cdot w = -v \cdot w \cdot w = -v ;$$
$$v \cdot w \cdot w \cdot (v \cdot w)^{-1} = v \cdot w \cdot w \cdot w \cdot v = v \cdot w \cdot v = -v \cdot v \cdot w = -w ;$$

therefore $r(v \cdot w) = \operatorname{diag}(-1, -1) \in \mathbf{O}(2)$. On the other hand:

$$(v \cdot w)^2 = v \cdot w \cdot v \cdot w = -v \cdot v \cdot w \cdot w = -1\,,$$

so that the restriction of $(*)$ to the diagonal is not split. □

Low degrees.

1.12. Proposition. *Let $X_\bullet$ be a sscheme and $E_\bullet$ a quadratic bundle on $X_\bullet$ of rank n.*

Then $w_1(E_\bullet) = w_1(\Lambda^n E_\bullet)$.

Proof. Reduce by the splitting principle to the case where $E_\bullet$ is a sum of line bundles. □

1.13. Corollary. *With the notations of* 1.12, *let $[E_\bullet]$ be the class of $E_\bullet$ in $H^1(X_\bullet, \mathbf{O}(n))$ and* $\det : \mathbf{O}(n) \to \mu_2$ *be the determinant map.*

Then $\det_*[E_\bullet] = w_1(E_\bullet)$.

1.14. Theorem. *With the notations of* 1.12, *let*

$$\Delta : H^1(X_\bullet, \mathbf{O}(n)) \to H^2(X_\bullet, \mu_2)$$

be the map defined in 1.10, $(**)$. *Then* $\Delta[E_\bullet] = w_2(E_\bullet)$.

Proof. It suffices to prove the statements in the universal case for which

$$X_\bullet = \mathbf{BO}(n) \quad \text{and} \quad E_\bullet = \mathscr{E}_\bullet\,.$$

Consider the commutative ladder:

$$\begin{array}{ccc}
H^1(\mathbf{BO}(n), \mathcal{O}(n)) & \longrightarrow & H^2(\mathbf{BO}(n), \mu_2) \\
\downarrow & & \delta\downarrow \\
H^1(\mathbf{BO}(2), \mathcal{O}(n)) & \longrightarrow & H^2(\mathbf{BO}(2), \mu_2) \\
\uparrow & & =\uparrow \\
H^1(\mathbf{BO}(2), \mathcal{O}(2)) & \longrightarrow & H^2(\mathbf{BO}(2), \mu_2) \\
\downarrow & & \eta\downarrow \\
H^1(\Gamma^*\mathbf{B}(\mu_2)^2, \mathcal{O}(n)) & \longrightarrow & H^2(\Gamma^*\mathbf{B}(\mu_2)^2, \mu_2) \\
\uparrow & & =\uparrow \\
H^1(\Gamma^*\mathbf{B}(\mu_2)^2, (\mu_2)^2) & \longrightarrow & H^2(\Gamma^*\mathbf{B}(\mu_2)^2, \mu_2) \\
\uparrow & & \gamma\uparrow \\
H^1(\mathbf{B}(\mu_2)^2, (\mu_2)^2) & \longrightarrow & H^2(\mathbf{B}(\mu_2)^2, \mu_2)
\end{array}$$

where $\Gamma^*\mathbf{B}(\mu_2)^2$ denotes the sscheme $\mathbf{B}(\mu_2)^2/\operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$ while $\mathbf{B}(\mu_2)^2$ denotes the simplicial set which classifies the discrete group $(\pm 1)^2$, and the map

$$\Gamma^*\mathbf{B}(\mu_2)^2 \to \mathbf{BO}(2)$$

comes from the embedding of $(\mu_2)^2$ into $\mathbf{O}(2)$ as diagonal matrices. The morphism γ is obviously injective; δ is bijective and η is injective by [16], th. 2.8, and its proof (see end of proof in *loc. cit.*). By lemma 1.11, the class of the restriction of $(*)$ to $(\mu_2)^2$ and

$$w_2 \in H^2(\mathbf{BO}(2), \mu_2)$$

have the same image in $H^2(\Gamma^*\mathbf{B}(\mu_2)^2, \mu_2)$. Therefore, to prove the theorem, it is enough to prove

1.15. Lemma. *Let*

$$1 \to Z \to \tilde{G} \to G \to 1$$

be a central extension of finite groups, with class $e \in H^2(G, Z)$. *Let G act on itself by inner automorphisms and on* $\tilde{G}$ *by the action induced by inner automorphisms of* $\tilde{G}$. *Let*

$$\Delta_e : H^1(G, G) \to H^2(G, Z)$$

be the nonabelian boundary associated to this extension and $-\mathbf{1} \in H^1(G, G)$ *be the class of the cocycle* $g \mapsto g^{-1}$. *Then*

$$\Delta_e(-\mathbf{1}) = e^{-1}.$$

1.16. Remark. The map $g \mapsto g$ does not define a 1-cocycle, unless G is abelian!

Proof. Choose a section $s : G \to \tilde{G}$ of the projection; then $g \mapsto s(g)^{-1}$ is a cochain of G with values in $\tilde{G}$ lifting the cocycle of the lemma. By definition of Δ_e (see [24], annex to ch. VII), $\Delta_e(-\mathbf{1})$ is represented by the cocycle

$$c_{g,h} = s(g)^{-1} \cdot {}^g s(h)^{-1} \cdot s(gh).$$

But ${}^g s(h)^{-1} = s(g) \cdot s(h)^{-1} \cdot s(g)^{-1}$, hence

$$c_{g,h} = s(h)^{-1} \cdot s(g)^{-1} \cdot s(gh) = s(gh) \cdot s(h)^{-1} \cdot s(g)^{-1} = \left(s(g) \cdot s(h) \cdot s(gh)^{-1}\right)^{-1}$$

(we used that $c_{g,h}$ is central). The cohomology class of this cocycle is precisely e^{-1} (e.g. [24], ch. VII, § 3). □

1.17. Corollary (cf. [25], formula (4.6) when X is the spectrum of a field). *Let* (E, q) *be a quadratic bundle over X and* $[E]$ *be its class in* $H^1_{\text{ét}}(X, \mathbf{O}(n))$. *Let*

$$\partial : H^1_{\text{ét}}(X, \mathbf{O}(n)) \to H^2_{\text{ét}}(X, \mu_2)$$

be the boundary map associated to extension $(*)$.

Then, $\partial[E] = w_2(E)$.

1.18. Corollary. *Let Γ be a profinite group, let K be a separably closed field of characteristic $\neq 2$, let $\varrho : \Gamma \to \mathbf{O}(n, K)$ be a continuous orthogonal representation and $[\varrho]$ be its class in*

$$H^1(\Gamma, \mathbf{O}(n, K)) = \mathrm{Hom}(\Gamma, \mathbf{O}(n, K))/\sim$$

(here "$\sim$" denotes the conjugation by elements of $\mathbf{O}(n, K)$). Let

$$\delta : H^1(\Gamma, \mathbf{O}(n, K)) \to H^2(\Gamma, \mu_2)$$

be the boundary map associated to the K-points of extension $(*)$.

Then, $\delta[r] = w_2(\varrho)$.

1.19. Corollary. *Let X be a scheme, $\varrho : \pi_1(X) \to \mathbf{O}(n, K)$ be a continuous orthogonal representation of $\pi_1(X)$, and $[\varrho]$ be its class in $H^1(\pi_1(X), \mathbf{O}(n, K))$. We also write $[\varrho]$ for the image of $[\varrho]$ in $H^1_{\text{ét}}(X, \mathbf{O}(n, K))$. Let*

$$\delta : H^1_{\text{ét}}(X, \mathbf{O}(n, K)) \to H^2_{\text{ét}}(X, \mu_2)$$

be the boundary map associated to the K-points of extension $(*)$.

Then, $\delta[\varrho] = w_2(\varrho)$.

1.20. Remark. Theorem 1.14 can also be obtained as a consequence of the following more general functoriality principle.

Let G be a sheaf of groups over $X_{\text{Ét}}$, the big étale site over X. Let S be an $X_{\text{Ét}}$-simplicial sheaf and $E \to S$ be a principal homogeneous space with structural group G. As in 1.1 (i) and (ii) we can associate to E a class in two different sets:

- a class $[E]$ in the nonabelian cohomology set $H^1(S, G)$;

- a class $\{E\}$ in the set $[S, \mathbf{B}G/X]_X$ of homotopy classes of X-maps from S to the classifying simplicial sheaf $\mathbf{B}G/X$.

Let A be an abelian sheaf over $X_{\text{Ét}}$ and $\alpha \in H^2(\mathbf{B}G/X, A)$. We associate to it a class

$$\alpha(E) := \{E\}^* \alpha \in H^2(S, A).$$

On the other hand, α defines a central extension of sheaves of groups

$$1 \to A \to \tilde{G} \to G \to 1$$

with a boundary map:

$$\Delta_\alpha : H^1(S, G) \to H^2(S, A).$$

1.21. Theorem. $\Delta_\alpha([E]) = \alpha(E)$.

Proof. This follows from [10], VIII. 6.2.10 (ii), taking account of *op. cit.*, IV. 2.5.8, IV. 3.5.4 (ii), IV. 3.6.1 and VIII. 6.2.10 (i). Presumably there is a more elementary "cocycle" proof along the lines of the proof of lemma 1.15, 6.4. □

§ 2. Serre's formula in the étale case

In this section we consider a connected scheme X over $\mathbb{Z}[\frac{1}{2}]$ and an étale covering

$$\pi : Y \to X$$

of (constant) degree n. To π we associate elements

$$w_i(E) \quad \text{and} \quad w_i(\pi)$$

in $H^i_{\text{ét}}(X, \mu_2)$ in the following way.

2.1. (A) On the bundle $E = \pi_* \mathcal{O}_Y$ there is a unimodular symmetric bilinear pairing

$$E \otimes_{\mathcal{O}_X} E \to \mathcal{O}_X$$

defined over an affine open set $U = \operatorname{Spec} A \subset X$ by

$$(f, g) \to \operatorname{Tr}_{B/A}(fg),$$

where $B = \Gamma(\pi^{-1}(U), \mathcal{O}_Y)$ and $f, g \in B$. We consider the i-th Stiefel-Whitney class $w_i(E)$ of the quadratic bundle E (see section 1).

(B) Let $\pi_1(X)$ be the fundamental group of X (based at some geometric point). The covering π corresponds to a permutation representation of $\pi_1(X)$ of degree n, that is an action of $\pi_1(X)$ on a set with n elements. Let K be a separable closure of the residue field at some point x of X and

$$\varrho : \pi_1(X) \to S_n \subset \mathbf{O}(n, K)$$

be the orthogonal representation naturally associated to this action. We define $w_i(\pi)$ as the image of $w_i(\varrho) \in H^i(\pi_1(X), \mu_2)$ in $H^i_{\text{ét}}(X, \mu_2)$ (1.8, b)).

2.2. Remark. Actually the choice of K is irrelevant (see 1.8 (b)). We take the separable closure of the residue field at some point to make an argument below a little more obvious.

Serre's formula relates $w_2(E)$ and $w_2(\pi)$ as follows:

2.3. Theorem. $w_2(E) = w_2(\pi) + (2) \cdot d$, *where*

$$d = w_1(E) = w_1(\pi) \in H^1_{\text{ét}}(X, \mu_2)$$

is the "discriminant" of the covering π, (2) *is the image of* $2 \in \Gamma(X, \mathbb{G}_m)$ *in* $H^1_{\text{ét}}(X, \mu_2)$ *by the boundary map associated to the Kummer exact sequence of étale sheaves*

$$1 \to \mu_2 \to \mathbb{G}_m \xrightarrow{2} \mathbb{G}_m \to 1,$$

and the cup-product "$\cdot$" *is induced by the pairing*

$$\mu_2 \times \mu_2 \to \mu_2$$

obtained from the canonical identification of μ_2 *with* $\mathbb{Z}/2\mathbb{Z}$ (*i.e.* $\varepsilon \cdot \varepsilon' = -1$ *if* $\varepsilon = \varepsilon' = -1$, $\varepsilon \cdot \varepsilon' = +1$ *otherwise*).

2.4. Conjecture (cf. [22], p. 665, question). *For any* $i \geqq 1$,

$$w_i(E) = w_i(\pi) \quad \text{if } i \text{ is odd}, \quad w_i(E) = w_i(\pi) + (2) \cdot w_{i-1}(\pi) \quad \text{if } i \text{ is even}.$$

This conjecture is compatible with the Wu formulae. In particular it is true for $i = 3$.

2.5. Notations. In this section, we consider a sheaf of groups G and an abelian sheaf A over $X_{\text{ét}}$ provided with an action of G.

We denote by $\mathscr{H}^i(G, A)$ (resp. $H^i_X(G, A)$) the derived functors of the left exact functor $A \mapsto A^G$ (resp. $A \mapsto \Gamma(X, A^G)$), where A^G (resp. $\Gamma(X, A^G)$) is the sheaf $U \mapsto A(U)^{G(U)}$ (resp. its global sections). There is a Grothendieck (composite functor) spectral sequence:

$$H^p_{\text{ét}}(X, \mathscr{H}^i(G, A)) \Rightarrow H^{p+q}_X(G, A).$$

One knows that extensions

$$(\alpha) \qquad 1 \to A \to \tilde{G} \to G \to 1$$

are classified by $\alpha \in H^2_X(G, A)$.

From now on, we assume that G acts trivially on A. Then an extension (α) as above is *central*. We associate to it, as in [11], cor. à la prop. 3.4.2, a "boundary map":

$$\Delta_\alpha : H^1_{\text{ét}}(X, G) \to H^2_{\text{ét}}(X, A).$$

The map Δ_α depends functorially on α in the following sense: if

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & \tilde{G} & \longrightarrow & G & \longrightarrow & 1 \\ & & {\scriptstyle f}\downarrow & & \downarrow & & {\scriptstyle g}\downarrow & & \\ 1 & \longrightarrow & A' & \longrightarrow & \tilde{G}' & \longrightarrow & G' & \longrightarrow & 1 \end{array}$$

is a commutative diagram of extensions, where the lower extension is denoted by (α'), then the diagram

$$\begin{array}{ccc} H^1_{\text{ét}}(X,G) & \xrightarrow{\Delta_\alpha} & H^2_{\text{ét}}(X,A) \\ f_*\downarrow & & g_*\downarrow \\ H^1_{\text{ét}}(X,G') & \xrightarrow{\Delta_{\alpha'}} & H^2_{\text{ét}}(X,A') \end{array}$$

commutes.

2.6. Lemma. *Let α_1, α_2 be two elements of $H^2_X(G,A)$. Then*

$$\Delta_{\alpha_1+\alpha_2} = \Delta_{\alpha_1} + \Delta_{\alpha_2}.$$

Proof. Let

$$(\alpha_1) \qquad 1 \to A_1 \to \tilde{G}_1 \to G \to 1$$

and

$$(\alpha_2) \qquad 1 \to A_2 \to \tilde{G}_2 \to G \to 1$$

be two central extensions. We define their direct sum

$$(\alpha_1 + \alpha_2) \qquad 1 \to A_1 \oplus A_2 \to \tilde{G} \to G \to 1$$

by $\tilde{G} = \tilde{G}_1 \times_G \tilde{G}_2$. Then, functoriality implies that:

$$\Delta_{\alpha_1 \oplus \alpha_2} = (\Delta_{\alpha_1}, \Delta_{\alpha_2}).$$

Assume now that $A_1 = A_2 = A$. Let $\Sigma : A \oplus A \to A$ be the sum defined by $\Sigma(a,b) = a + b$. Then the extension corresponding to $\alpha_1 + \alpha_2$ is $\Sigma_*(\alpha_1 \oplus \alpha_2)$. Therefore one has:

$$\Delta_{\alpha_1+\alpha_2} = \Delta_{\Sigma_*(\alpha_1 \oplus \alpha_2)} = \Sigma_* \circ \Delta_{\alpha_1 \oplus \alpha_2} = \Sigma_* \circ (\Delta_{\alpha_1}, \Delta_{\alpha_2}) = \Delta_{\alpha_1} + \Delta_{\alpha_2}. \quad \square$$

From the spectral sequence 2.5, we get in particular a differential

$$d_2^{11} : H^1_{\text{ét}}(X, \mathscr{H}^1(G,A)) \to H^3_{\text{ét}}(X, \mathscr{H}^0(G,A))$$

and a surjective map

$$\operatorname{Ker}(H^2_X(G,A) \to H^0_{\text{ét}}(X, \mathscr{H}^2(G,A))) \to \operatorname{Ker} d_2^{11}.$$

On the other hand, consider the cup-product

$$H^1_{\text{ét}}(X, \mathscr{H}^1(G,A)) \times H^1_{\text{ét}}(X, G^{\text{ab}}) \to H^2_{\text{ét}}(X,A),$$

induced by the natural pairing

$$\mathscr{H}^1(G,A) \times G^{\text{ab}} \to A.$$

Observe that, by assumption, one has

$$\mathscr{H}^1(G, A) = \mathscr{H}^1(G^{ab}, A) = \mathscr{H}om(G^{ab}, A).$$

2.7. Lemma. *Let*

$$\alpha \in \operatorname{Ker}\left(H_X^2(G, A) \rightarrow H_{\text{ét}}^0(X, \mathscr{H}^2(G, A))\right),$$

and denote by f its image in $\operatorname{Ker} d_2^{11}$. *Then, for any $x \in H_{\text{ét}}^1(X, G)$, one has*

$$\Delta_\alpha(x) = f \cdot \det(x),$$

where $\det(x)$ *denotes the image of x in $H_{\text{ét}}^1(X, G^{ab})$.*

Proof. The hypothesis means that α is locally split. Since G acts trivially on A, the map $\mathscr{H}^1(G^{ab}, A) \rightarrow \mathscr{H}^1(G, A)$ is an isomorphism. Therefore α comes from a locally split extension:

$$(\alpha^{ab}) \qquad 1 \rightarrow A \rightarrow \tilde{G}' \rightarrow G^{ab} \rightarrow 1.$$

The fact that $\tilde{G}'$ is locally split implies in particular that it is *abelian*. By [17], prop. A 3.1 (for $i = 1$), $\Delta_{\alpha^{ab}}(y) = f \cdot y$ for any $y \in H_{\text{ét}}^1(X, G^{ab})$. The lemma follows by functoriality. □

2.8. Proposition. *Let α_1, α_2 be two locally isomorphic (central) extensions of G by A. Let $f \in H_{\text{ét}}^1(X, \mathscr{H}^1(G, A))$ be the element associated to $\alpha_1 - \alpha_2$ as in 2.7. Then for any $x \in H_{\text{ét}}^1(X, G)$, one has*

$$\Delta_{\alpha_1}(x) - \Delta_{\alpha_2}(x) = f \cdot \det(x),$$

where $\det(x)$ *denotes the image of x in $H_{\text{ét}}^1(X, G^{ab})$.*

This follows from lemmas 2.6 and 2.7.

2.9. Two extensions of the symmetric group S_n. In this section we consider extensions of the constant sheaf $G = S_n$ by the constant sheaf $A = \mu_2$, which are not necessarily constant.

Let $\mathbf{O}(n)$ be the standard orthogonal group of rank n. This is a group scheme over X (even over $\mathbb{Z}$). We associate to it two sheaves of groups:

a) the sheaf $\mathbf{O}(n): U \mapsto \mathbf{O}(n, U)$;

b) the constant sheaf with stalks $\mathbf{O}(n, K)$ everywhere, where $\operatorname{Spec} K$ is some fixed geometric point of X. We denote this sheaf by $\mathbf{O}(n, K)$.

Let Q be the standard quadratic form of rank n

$$Q(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2,$$

with orthogonal group $\mathbf{O}(n)$, and let $\mathrm{Cl}(Q)$ its Clifford algebra 1.9. One constructs from $\mathrm{Cl}(Q)$ a two-fold covering $\tilde{\mathbf{O}}(n)$ of $\mathbf{O}(n)$ (1.9):

$$(1) \qquad 1 \to \mu_2 \to \tilde{\mathbf{O}}(n) \to \mathbf{O}(n) \to 1\,,$$

which defines two exact sequences of étale sheaves:

$$(2) \qquad 1 \to \mu_2 \to \tilde{\mathbf{O}}(n) \to \mathbf{O}(n) \to 1\,,$$

and

$$(3) \qquad 1 \to \mu_2 \to \tilde{\mathbf{O}}(n, K) \to \mathbf{O}(n, K) \to 1\,.$$

Actually (3) is just the K-points of (2), considered as an exact sequence of constant sheaves over $X_{ét}$. We now consider the symmetric group S_n on n letters. It acts on $\mathbb{Z}^n$ by permuting its canonical basis. This defines an orthogonal representation $S_n \subset \mathbf{O}(n)$ where we view S_n as an algebraic group of dimension zero. At the sheaf level, this representation yields two sheaf homomorphisms,

$$S_n \to \mathbf{O}(n) \quad \text{and} \quad S_n \to \mathbf{O}(n, K)\,.$$

This defines two central extensions of S_n by μ_2, the pull-backs of (2) and (3):

$$(\alpha) \qquad 1 \to \mu_2 \to \tilde{S}'_n \to S_n \to 1\,,$$

$$(\beta) \qquad 1 \to \mu_2 \to \tilde{S}_n \to S_n \to 1\,.$$

The sheaf $\tilde{S}_n$ is constant, as the pull-back of a constant sheaf by another. It can be checked that its value is the group defined at [22], p. 654 (or [21], p. 355) (see [22], p. 662, remarque 2), but we shall not need this. We shall see in lemma 2.11 that the sheaf $\tilde{S}'_n$ is not constant in general. However:

2.10. Lemma. *The two extensions* (α) *and* (β) *are locally isomorphic.*

Proof. It is clear that (α) and (β) coincide at the geometric point $\mathrm{Spec}(K)$. But if k and K are two separably closed fields, a standard argument shows that the extensions $\tilde{S}'_n(k)$ and $S'_n(K)$ are isomorphic. Indeed, this is obvious if $k \subset K$, hence if k and K have the same characteristic. If $K = \mathbb{Q}_p$ and $k = \mathbb{F}_p$ for ($p \neq 2$), we get the same conclusion passing through the Witt vectors of k. More generally, if R is strict henselian with residue field k,

$$\tilde{S}'_n(R) \to \tilde{S}'_n(k)$$

is an isomorphism. Hence the conclusion holds in general. □

As in 2.7, to the class

$$\alpha - \beta \in H^2_X(S_n, \mu_2)$$

one associates an element

$$f \in H^1_{ét}(X, \mathscr{H}^1(S_n, \mu_2)) = H^1_{ét}(X, \mu_2)\,.$$

We now determine this element.

2.11. Lemma. a) *The image of* $S'_n(\mathbb{Z}[\frac{1}{2}])$ *in* S_n *equals the alternating group* A_n.

b) *The homomorphism* $\tilde{S}'_n(\mathbb{Z}[\frac{1}{2}, \sqrt{2}]) \to S_n$ *is surjective.*

Proof. We argue as in [22], p. 659. Denote by (e_i) the canonical basis of $V = \mathbb{Z}[\frac{1}{2}]^n$, and also that of

$$V[\sqrt{2}] = V \otimes_{\mathbb{Z}[\frac{1}{2}]} \mathbb{Z}\left[\frac{1}{2}, \sqrt{2}\right].$$

We provide V with the quadratic form Q, with standard orthonormal basis e_i. Then $V[\sqrt{2}]$ inherits by extension of the scalars the form $Q \otimes \mathbb{Z}[\sqrt{2}]$ with the same orthonormal basis. Let $i \neq j$. The elements

$$x_{ij} = \frac{1}{\sqrt{2}} \cdot (e_i - e_j)$$

belong to $\mathrm{Cl}(Q \otimes \mathbb{Z}[\sqrt{2}])$ and are of norm $N(x_{ij}) = 1$. Therefore they are elements of the group $\tilde{\mathbf{O}}(n)(\mathbb{Z}[\frac{1}{2}, \sqrt{2}])$ (see 1.9). More precisely, one checks that (with the notations of section 1.9):

$$r(x_{ij})e_k = e_k \quad \text{if} \quad k \neq i, j;$$

$$r(x_{ij})e_i = e_j; \quad \text{and} \quad r(x_{ij})e_j = e_i.$$

Hence $r(x_{ij}) = (ij) \in S_n$. As S_n is generated by transpositions (ij), this proves b).

To prove a), we first observe that for any i, j, k, l one has

$$x_{ij}\, x_{kl} \in \tilde{\mathbf{O}}(n)\left(\mathbb{Z}\left[\frac{1}{2}\right]\right).$$

Since A_n is generated by products of two transpositions, it just remains to prove that

$$(12) \notin \mathrm{im}\left(\tilde{S}'_n\left(\mathbb{Z}\left[\frac{1}{2}\right]\right) \to S_n\right).$$

Let (if possible) $x \in \tilde{S}'_n(\mathbb{Z}[\frac{1}{2}])$ such that $r(x) = (12)$. Then, in

$$\tilde{\mathbf{O}}\left(\mathbb{Z}\left[\frac{1}{2}, \sqrt{2}\right]\right),$$

one has

$$x^{-1} x_{12} \in \ker r = \mu_2.$$

This implies that $x_{12} \in \tilde{\mathbf{O}}(n)(\mathbb{Z}[\frac{1}{2}])$, a contradiction. □

2.12. Proposition. *For any scheme X over $\mathbb{Z}[\frac{1}{2}]$, the image of $\alpha - \beta$ in $H^1_{\text{ét}}(X, \mu_2)$ is equal to* (2).

Proof. By functoriality, it is enough to see this when $X = \operatorname{Spec} \mathbb{Z}[\frac{1}{2}]$. Let f be this image. By lemma 2.11, a) (resp. 2.11, b)), f is non trivial (resp.

$$f \in \ker\left(H^1\left(\mathbb{Z}\left[\frac{1}{2}\right], \mu_2\right) \to H^1\left(\mathbb{Z}\left[\frac{1}{2}, \sqrt{2}\right], \mu_2\right)\right).$$

The conclusion now follows from the fact that

$$\ker\left(H^1\left(\mathbb{Z}\left[\frac{1}{2}\right], \mu_2\right) \to H^1\left(\mathbb{Z}\left[\frac{1}{2}, \sqrt{2}\right], \mu_2\right)\right)$$

is generated by (2). □

Denote by ∂ (resp. by δ) the boundary map:

$$H^1_{\text{ét}}(X, S_n) \to H^2_{\text{ét}}(X, \mu_2)$$

associated to the extension (α) (resp. (β)). From propositions 2.8 and 2.12, we obtain:

2.13. Proposition. *Let $x \in H^1_{\text{ét}}(X, S_n)$; denote by $\det(x)$ its image in $H^1_{\text{ét}}(X, \mu_2)$ via the map induced by the signature*:

$$\partial(x) - \delta(x) = (2) \cdot \det(x).$$

Proof of Serre's formula 2.3. Let $e : \pi_1(X) \to S_n$ be the permutation representation associated to π; denote by $[e] \in H^1_{\text{ét}}(X, S_n)$ the image in étale cohomology of the class of e in $H^1(\pi_1(X), S_n)$ and by $[E]$ (resp. $[\pi]$) the class of E (resp. π) in $H^1_{\text{ét}}(X, \mathbf{O}(n))$ (resp. $H^1_{\text{ét}}(X, \mathbf{O}(n, K))$), cf. 2.1.

2.14. Lemma (cf. [22], 1.4). *With the notations introduced above, $[E]$ (resp. $[\pi]$) is the image of $[e]$ via the coefficient homomorphisms*

$$S_n \hookrightarrow \mathbf{O}(n) \quad \textit{and} \quad S_n \hookrightarrow \mathbf{O}(n, K).$$

In view of 2.14, 2.13, 1.19 and 1.21, to prove 2.3 it suffices to observe that both composites

$$S_n \to \mathbf{O}(n) \xrightarrow{\det} \mu_2 \quad \text{and} \quad S_n \to \mathbf{O}(n, K) \xrightarrow{\det} \mu_2$$

coincide with signature. Hence,

$$\det(e) = \det[E] = \det[\pi] = w_1(E) = w_1(\pi). \quad \square$$

2.15. A generalisation. The arguments of this section apply in fact to any subgroup G of $\Gamma(X, \mathbf{O}(n))$ instead of S_n, provided we know the difference $f_G \in H^1_{\text{ét}}(X, \operatorname{Hom}(G, \mu_2))$

between the classes of the corresponding extensions (α_G) and (β_G). By functoriality, it is enough to treat the 'maximal' case

$$G = \Gamma(X, \mathbf{O}(n)) =: \mathbf{O}(N, X).$$

This leads to Fröhlich's theory of the "spinor class" [7]. We shall not develop it here.

§ 3. Stiefel-Whitney classes attached to a tame covering with odd ramification

3.1. Equivariant cohomology. Let Y be a scheme and G a group acting on Y. Consider the category $\mathbf{A}$ of étale G-sheaves on Y. In [13], § 2, Grothendieck studies equivariant cohomology, i.e. the derived functors $H^*(Y, G; -)$ of the left exact functor

$$\mathscr{F} \mapsto \Gamma(Y, \mathscr{F})^G, \quad \text{for} \quad \mathscr{F} \in \mathbf{A}.$$

There are two spectral sequences converging to $H^*(Y, G; \mathscr{F})$:

$$\text{(a)} \qquad I_2^{pq} = H^p(G, H^q(Y, \mathscr{F}));$$

$$\text{(b)} \qquad II_2^{pq} = H^p(Y/G, \mathscr{H}^q(G, \mathscr{F})),$$

(cf. [13], pp. 244–245 and [11], formulae 5.2.5). The second one is defined when the quotient Y/G exists in an appropriate sense, e.g. when Y is covered by G-invariant open affine subschemes. By definition, $\mathscr{H}^q(G, -)$ is the q-th derived functor of $\mathscr{H}^0(G, -)$, where $\mathscr{H}^0(G, \mathscr{F})$ is the sheaf on Y/G associated to the presheaf

$$U \mapsto \mathscr{F}(U)^G.$$

These are Leray spectral sequences associated to appropriate morphisms of sites.

For $y \in Y$, let D_y denote the *decomposition group* of y, i.e. the stabiliser of y in G. Then D_y acts on the residue field k_y of y. Let I_y be the kernel of this action, the *inertia group of* y.

3.2. Proposition (cf. [11], cor. to th. 5.3.1). *Assume that for any point $y \in Y$, the inertia group I_y of y in G has finite order ε_y and that multiplication by ε_y on $\mathscr{F}$ is an isomorphism. Then*

$$\mathscr{H}^q(G, \mathscr{F}) = 0 \quad \text{for any } q > 0;$$

therefore the spectral sequence (b) *degenerates into isomorphisms*:

$$H^i(Y/G, \mathscr{H}^0(G, \mathscr{F})) \xrightarrow{\cong} H^i(Y, G; \mathscr{F}).$$

3.3. Corollary. *Assume that $\mathscr{F}$ is the constant sheaf $\mathbb{Z}/2\mathbb{Z}$ and that all the ε_y are odd. Then there are canonical isomorphisms*:

$$H^i(Y/G, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cong} H^i(Y, G; \mathbb{Z}/2\mathbb{Z}).$$

In particular this defines "edge homomorphisms":

$$H^i(G, \mathbb{Z}/2\mathbb{Z}) \to H^i(Y/G, \mathbb{Z}/2\mathbb{Z})$$

for all $i \geqq 0$.

This follows from 3.2 and the spectral sequence I_2^{pq}.

3.4. Functoriality. We keep the notations of 3.1, and assume G to be finite. Let X be a scheme provided with the trivial action of G and $\pi : Y \to X$ be an equivariant, finite and flat morphism. We assume that:

(a) π identifies X with the quotient Y/G.

(b) For any $y \in Y$, the inertia group I_y of y in G has odd order ε_y.

By corollary 3.3, one obtains homomorphisms:

$$\pi^* : H^i(G, \mathbb{Z}/2\mathbb{Z}) \to H^i(X, \mathbb{Z}/2\mathbb{Z}) .$$

The following lemmas 3.5 and 3.6 are obvious by functoriality.

3.5. Lemma. *Let Y' be another scheme on which G acts, $\pi' : Y' \to X$ an equivariant morphism satisfying condition* 3.4 (a). *Suppose that there exists an equivariant map $g : Y' \to Y$ such that $\pi' = \pi \circ g$. Then π' satisfies condition* 3.4 (b) *and* $\pi'^* = \pi^*$.

3.6. Lemma. *With the notations of* 3.4, *let $\phi : Z \to X$ be a morphism and $p : T' \to Z$ be the pull-back of π by ϕ. Then p satisfies conditions* 3.4 (a) *and* 3.4 (b). *Furthermore one has* $p^* = \phi^* \circ \pi^*$.

3.7. Proposition. *With the notations of* 3.4 *and* 3.6, *assume Z to be locally noetherian and normal. Assume further that π is separable (i.e. separable at all generic points of Z). Let T be the normalisation of T' and $\pi_Z : T \to Z$ be the composite*

$$T \to T' \to Z .$$

Then π_Z satisfies conditions 3.4 (a) *and* 3.4 (b), *and* $(\pi_Z)^* = \phi^* \circ \pi^*$.

Proof. By 3.6 and 3.2, we need only to show that π_Z satisfies condition 3.4 (a). For this, we may assume that Z is affine, integral and noetherian, say $Z = \operatorname{Spec} A$. Since π is finite and flat, so is π_Z, and $T = \operatorname{Spec} B$ with B a flat finitely generated A-module. We also set $T' = \operatorname{Spec} B'$. Let $A' = B'^G$. We have to show that $A' = A$.

First assume that A is a field. Then B is a finite dimensional A-algebra, with radical R, and $B' = B/R$. By assumption, $B = \prod B_i$ where the B_i are separable field extensions of A. Therefore the projection $B \to B'$ has a unique splitting. Let us still write B' for the image of B' in B via this splitting. By uniqueness of the splitting, G keeps B' invariant in B, therefore the injection

$$A = B^G \rightarrow B'^G = A'$$

has a section, hence is surjective.

In general, let L be the total quotient ring of B. Since $A = B^G$ and B is finite over A, we have $K = L^G$. Besides, the total quotient ring L' of B' is the integral closure of L. By the special case above, $A' = B'^G$ is identified to a subring of K containing A. On the other hand, since L' is a product of separable extensions of K, the algebra B' which is the integral closure of A in L' is finite over A (see [3], V, § 3, n° 7, cor. 1 to prop. 18). Since A is noetherian, A' is finite over A; since A is integrally closed, it follows that $A' = A$. □

Oddly ramified coverings.

3.8. Definition. A scheme X is a *Dedekind scheme* if it is noetherian, normal and one-dimensional.

3.9. Remarks. (i) We are mostly interested in two cases, in smooth curves over a field and in schemes of the form Spec $\mathcal{O}_k$, where $\mathcal{O}_k$ is the ring of integers of a number field k. Another example of possible interest is

$$X = \operatorname{Spec}(R) - \{x\},$$

where R is a normal two-dimensional local ring and x is its maximal ideal.

(ii) An open subscheme of a Dedekind scheme is Dedekind.

(iii) Assume that X is a Dedekind scheme. Then X is either a curve defined over a field or X is a quasi-affine scheme. In fact, if $A = H^0(X, \mathcal{O}_X)$ then the induced morphism $f: X \rightarrow \operatorname{Spec} A$ either maps X to a point, or it is quasi-finite. By [12], IV, 18.12.13, f is quasi-affine in the second case and by definition of a quasi-affine morphism X is a quasi-affine scheme.

3.10. Definition. (i) Let $\pi: Y \rightarrow X$ be a finite, flat morphism of Dedekind schemes. We say that π is a *ramified covering* if furthermore, for any generic point $\zeta = \operatorname{Spec} K$ of X and any generic point $\eta = \operatorname{Spec} L$ of Y above ζ, the extension L/K is separable. Then there is an open subset U of Y such that $\pi|_U$ is étale. The complement Z' of a maximal such U is called the *ramification locus* of π in Y; its image Z in X is called the *ramification locus* of π in X.

(ii) We define the *ramification index* e_y of a closed point $y \in Y$ as follows:

$\mathcal{O}_{Y,y}$ is a discrete valuation ring containing $\mathcal{O}_{X,x}$, where $x = \pi(y)$. Then e_y is the ramification index of the extension

$$\mathcal{O}_{Y,y} \quad \text{of} \quad \mathcal{O}_{X,x}.$$

So $e_y > 1$ if and only if $y \in Z'$.

(iii) We say that π is *tame* if for all y, the ramification index e_y is prime to the characteristic of the residue field of y and the residue extension at y is separable.

(iv) Let l be a prime number. We say that π has ramification prime to l if all the e_y are prime to l.

Assume that π is Galois and let $G = \mathrm{Gal}(\pi)$ be the Galois group. By definition G acts on Y and $X = Y/G$. For any $y \in Y$, the numbers e_y and ε_y coincide, where ε_y is as in 3.2.

3.11. Proposition. *Let*

$$\pi : Y \to X \quad \textit{and} \quad \pi' : Y' \to X$$

be two ramified coverings. Let $\pi'' : Y'' \to X$ denote the normalisation of

$$Y \times_X Y' \to X.$$

Then π'' is a ramified covering.

If π and π' are tame, *so is π''. If moreover π and π' have ramification prime to l, so does π''.*

Proof. The claim is local so we may assume that $X = \mathrm{Spec}\, R$, where R is a discrete valuation ring. Then $Y = \mathrm{Spec}\, S$ and $Y' = \mathrm{Spec}\, S'$, where S and S' are semilocal principal ideal rings. Let K (resp. L, L') be the field of fractions of R (resp. of S and S'). Then L/K and L'/K are finite extensions, S and S' are the integral closures of R in L and L', and $Y \times_X Y'$ corresponds to the integral closure of R in a compositum L'' of L and L'. Up to completing K, we may even assume that K, L, L' and L'' are complete; then S, S' and S'' are discrete valuation rings. Let k be the residue field of K. Since ramification indices do not change by unramified extensions, we may assume that k is separably closed (cf. [24], ch. III, th. 2). Then, by the structure of complete tame totally ramified extensions (e.g. [19], ch. II, prop. 12), L/K and L'/K are cyclic and the proposition becomes obvious.

3.12. Corollary. *Let X be an irreducible Dedekind scheme, with function field F. Let F_s be a separable closure of F and $\Gamma = \mathrm{Gal}(F_s/F)$.*

Given a prime number l, there exists a quotient $\pi_1(X)^{\mathrm{tr},l}$ of Γ with the following property:

> *Let $F^{\mathrm{tr},l}$ be the extension of F corresponding to $\pi_1(X)^{\mathrm{tr},l}$ and $\eta^{\mathrm{tr},l} = \mathrm{Spec}\, F^{\mathrm{tr},l}$. Then for any ramified covering $\pi : Y \to X$ such that Y is irreducible, π is tame with ramification prime to l if and only if the canonical morphism $\eta^{\mathrm{tr},l} \to X$ factors through π.*

There is an anti-equivalence of categories between the category of tame coverings of X with ramification prime to l and the category of finite sets provided with a continuous $\pi_1(X)^{\mathrm{tr},l}$-action. In particular, any tame covering $\pi : Y \to X$ with ramification prime to l, such that Y is irreducible, is covered by a Galois covering of this type.

By 3.3, there exist canonical homomorphisms

$$H^i(\pi_1(X)^{\mathrm{tr},2}, \mathbb{Z}/2\mathbb{Z}) \to H^i(X, \mathbb{Z}/2\mathbb{Z}).$$

3.13. Proposition. *Let Z be another Dedekind scheme and $\phi : Z \to X$ a morphism. Assume Z irreducible. Then ϕ defines a continuous homomorphism*

$$\phi : \pi_1(Z)^{\mathrm{tr},2} \to \pi_1(X)^{\mathrm{tr},2}.$$

Furthermore, the diagram

$$\begin{array}{ccc} H^i(\pi_1(X)^{\mathrm{tr},2}, \mathbb{Z}/2\mathbb{Z}) & \longrightarrow & H^i(X, \mathbb{Z}/2\mathbb{Z}) \\ \phi^* \downarrow & & \phi^* \downarrow \\ H^i(\pi_1(Z)^{\mathrm{tr},2}, \mathbb{Z}/2\mathbb{Z}) & \longrightarrow & H^i(Z, \mathbb{Z}/2\mathbb{Z}) \end{array}$$

is commutative.

Proof. This follows from 3.6 and 3.10. □

3.14. Stiefel-Whitney classes. Let $\pi : Y \to X$ be a tame covering of degree n, with odd ramification. By 3.12, π corresponds to a continuous permutation representation of degree n of $\pi_1(X)^{\mathrm{tr},2}$. This in turn corresponds to a continuous homomorphism $G \to \mathscr{S}_n$. By composition with the natural embedding

$$\mathscr{S}_n \hookrightarrow \mathbf{O}(n, \mathbb{C}),$$

we obtain a continuous orthogonal representation

$$\varrho : \pi_1(X)^{\mathrm{tr},2} \to \mathbf{O}(n, \mathbb{C}),$$

hence Stiefel-Whitney classes:

$$w_i(\varrho) \in H^i(\pi_1(X)^{\mathrm{tr},2}, \mathbb{Z}/2\mathbb{Z}),$$

as in 1.8 (b).

3.15. Definition. The i-th Stiefel-Whitney class of π is the image

$$w_i(\pi) \in H^i(X, \mathbb{Z}/2\mathbb{Z}) \quad \text{of} \quad w_i(\varrho)$$

by the homomorphism of 3.4. The total Stiefel-Whitney class of π is

$$w(\pi) = \sum w_i(\pi) \quad \text{where we define} \quad w_0 = 1.$$

3.16. Theorem. *Let π be as in* 3.10 *and Z, T, T', ϕ, p, π_Z be as in* 3.5 *and* 3.6, *with Z a Dedekind scheme. Then, $w(\pi_Z) = \phi^* w(\pi)$.*

Proof. This follows from 3.13. □

3.17. Remark. If π is étale, $w_i(\pi)$ coincides with the class of 2.1.

§ 4. Reduction to the étale case

4.1. Definition. In this section X and Z are *Dedekind schemes* and $\varphi : Z \to X$ is a ramified covering (see 3.8 and 3.10).

We say that the covering φ is a *Kummer covering* if the extension of the fields of rational functions K_Z/K is Kummer, that is, a field extension obtained by adding to K a N-th root of some element $f \in K$.

A closed point z is *totally ramified* if $e_z = [K_Z : K]$, the degree of the field extension. In this case one has $k(z) = k(\varphi(z))$.

4.2. Kummer coverings. We shall construct Kummer coverings as follows.

A natural number $N \geqq 2$ being given, let $\mathscr{A}$ be an invertible sheaf such that

$$\mathscr{A}^N \cong \mathcal{O}_X\left(\sum \nu_x \cdot x\right),$$

where $D = \sum \nu_x \cdot x$ is an effective divisor on X. Then the section s of $\mathscr{A}$ with zero set D or rather its inverse

$$\mathcal{O}_X\left(-\sum \nu_x \cdot x\right) \xrightarrow{s^{-1}} \mathcal{O}_X$$

gives the $\mathcal{O}_X$-module $\mathscr{R}' := \bigoplus_{0}^{N-1} \mathscr{A}^{-i}$ and $\mathcal{O}_X$-algebra structure.

Define $Z = \mathbf{Spec}_{\mathcal{O}_X} \mathscr{R}$ to be the normalization of $\mathbf{Spec}_{\mathcal{O}_X} \mathscr{R}'$.

For example, if $X = \operatorname{Spec} A$, we have $\mathscr{A} = \mathcal{O}_X$ and $f \in A$ with zero divisor

$$(f) = \sum \nu_x \cdot x \, .$$

Of course, $Z = \operatorname{Spec} R$ where R is the integral closure of the ring $R' = A[t]/\langle t^N - f\rangle$.

In general, each point has an affine neighbourhood U such that

$$\Gamma(U, \mathscr{R}') = A[t]/\langle t^N - f\rangle \, ,$$

where $A = \Gamma(U, \mathcal{O}_X)$ and f is the local equation of $\sum \nu_x \cdot x$ on U, that is of the shape $\tau^{\nu_x} \cdot u$ in the localization $\mathcal{O}_{X,x}$ of A in x, where $u \in \mathcal{O}_{X,x}^*$ is a unit in $\mathcal{O}_{X,x}$ and τ is a local parameter. Of course $\Gamma(\varphi^{-1}(U), \mathcal{O}_X)$ is again the integral closure of $\Gamma(U, \mathscr{R}')$.

We will assume, for simplicity, that for $\eta = \gcd\{\nu_x, \nu_x \geqq 1\}$ one has either $\eta = 1$ or otherwise the section s is not the η-th power of a section

$$s' \in \Gamma\left(X, \mathcal{O}_X\left(\sum \frac{\nu_x}{\eta} \cdot x\right)\right).$$

4.3. Proposition. *Under the assumptions made in* 4.2 *one has*:

(i) *Z is irreducible and $\varphi : Z \to X$ is a Kummer covering.*

(ii) *φ has ramification index* 1 *at the points x for which $v_x = 0$, and has ramification index*

$$\frac{N}{\gcd(N, v_{\varphi(z)})}$$

at a point $z \in Z$ such that $v_{\varphi(z)} \geqq 1$.

(iii) *One has*

$$\mathscr{R} = \bigoplus_0^{N-1} \mathscr{A}^{-i}\left(\sum_{x \in X}\left[\frac{v_x \cdot i}{N}\right] \cdot x\right).$$

Proof. This ought to be well-known, at least if X is a curve over a field k (see for example [5]). We give a proof for the reader's convenience. Of course we may assume that $N \geqq 2$. Locally at some point $x \in X$,

$$R' = \mathcal{O}_{X,x}[t]/\langle t^N - \tau^{v_x} \cdot u\rangle .$$

If $\eta = \gcd\{v_x, v_x \geqq 1\} = 1$ then $\tau^{v_x} \cdot u$ cannot be a power. For $\eta > 1$ we assumed that s and hence $\tau^{v_x} \cdot u$ is not a power. So Z is irreducible and this proves (i).

If $v_x = 0$, then R' is normal as φ is unramified at x. If $v_x = 1$, then R' is normal and φ is totally ramified.

If $0 \leqq i, j < N$ with $i + j < N$,

$$\mathscr{A}^{-(i+j)}\left(\sum\left(\left[\frac{v_x \cdot i}{N}\right] + \left[\frac{v_x \cdot j}{N}\right]\right) \cdot x\right)$$

injects into

$$\mathscr{A}^{-(i+j)}\left(\sum\left[\frac{v_x \cdot (i+j)}{N}\right] \cdot x\right).$$

Otherwise $i + j = k + N$ for some $0 \leqq k < N$, and

$$\mathscr{A}^{-(k+N)}\left(\sum\left(\left[\frac{v_x \cdot i}{N}\right] + \left[\frac{v_x \cdot (k-i)}{N}\right]\right) \cdot x\right)$$

injects into

$$\mathscr{A}^{-k}\left(\sum\left[\frac{v_x \cdot k}{N}\right] \cdot x\right).$$

Therefore

$$\mathscr{T} := \bigoplus_0^{N-1} \mathscr{A}^{-i}\left(\sum\left[\frac{v_x \cdot i}{N}\right] \cdot x\right)$$

is an $\mathcal{O}_X$-algebra containing $\mathcal{R}'$ and contained in

$$K_Z = \bigoplus_0^{N-1} \mathcal{A}^{-i} \otimes_{\mathcal{O}_X} K.$$

(iii) says that this algebra is normal, which we can prove locally. So we may assume that

$$\mathcal{R}' := A[t]/\langle t^N - \tau^\nu \cdot u\rangle$$

where $A := \mathcal{O}_{X,x}$ and $\nu(= \nu_x) \geqq 2$.

If one defines $d = \gcd(N, \nu)$, $Md = N$ and $\mu d = \nu$ one has $\gcd(M, \mu) = 1$. Consider

$$A_1 := A[v]/\langle v^d - u\rangle.$$

As before, A_1 is normal and unramified over A and, of course, $v \in A_1^*$. One considers the map

$$A_1[t]/\langle t^M - \tau^\mu \cdot v\rangle \to \mathcal{T} = \bigoplus_0^{N-1} A t^i \tau^{-\left[\frac{\nu \cdot i}{N}\right]}$$

defined by

$$t \mapsto t \quad \text{and} \quad v \mapsto t^M \cdot \tau^{-\mu} = t^M \tau^{-\left[\frac{\nu \cdot M}{N}\right]}.$$

Then one has

$$\mathcal{T} = \bigoplus_0^{M-1} A_1 t^j \tau^{-\left[\frac{\mu \cdot j}{M}\right]}.$$

Localize A_1 at a point x_1 above x, and call $\bar{A}_1$ the local ring, with local parameter τ. We now want to show that

$$\bar{\mathcal{T}} := \bigoplus_0^{M-1} \bar{A}_1 t^j \tau^{-\left[\frac{\mu \cdot j}{M}\right]}$$

is a discrete valuation ring.

Let a be the unique integer such that $1 \leqq a \leqq M-1$ and $\mu a = 1 + lM$ for some l with $1 \leqq l$. Set

$$s := t^a \tau^{-\left[\frac{\mu \cdot a}{M}\right]} = t^a \tau^{-l}.$$

Then for $b := a + j$

$$t^j \tau^{-\left[\frac{\mu \cdot j}{M}\right]} s = t^{a+j} \tau^{-\left[\frac{\mu \cdot j}{M}\right] - l} = t^b \tau^{-\left[\frac{\mu \cdot b - 1}{M}\right]}.$$

Regarding this formula for different values of j one finds s to be a local parameter of T:

If $0 < b < M$, then $\frac{\mu b}{M} \notin \mathbb{N}$ and

$$\left[\frac{\mu \cdot b - 1}{M}\right] = \left[\frac{\mu \cdot b}{M}\right].$$

That is

$$t^j \tau^{-[\frac{\mu \cdot j}{M}]} s = t^b \tau^{-[\frac{\mu \cdot b}{M}]}.$$

If $b = M + \beta$ and $0 < \beta < M$, then similarly

$$t^j \tau^{-[\frac{\mu \cdot j}{M}]} s = t^{M+\beta} \tau^{-[\frac{\mu \cdot (M+\beta)}{M}]} = v t^\beta \tau^{-[\frac{\mu \cdot \beta}{M}]}.$$

If $b = M$,

$$t^j \tau^{-[\frac{\mu \cdot j}{M}]} s = t^M \tau^{-[\frac{\mu \cdot M - 1}{M}]} = v \tau .$$

Therefore $\bar{\mathcal{T}} / \langle s \rangle \cong \bar{A}_1 / \langle \tau \rangle$ is a field. This proves that $\bar{\mathcal{T}}$ is a discrete valuation ring, and therefore (ii).

Also one has $s^M = \tau v^a$. This proves that the point defined by s in $\bar{T}$ is totally ramified over x_1, and therefore its ramification index over x is

$$M = \frac{N}{\gcd(N, v_x)}. \quad \square$$

4.4. Remark. In fact, if we remove from our assumptions 4.1 the condition that X has dimension 1 and we just assume that X is a regular scheme over some localization of $\mathbb{Z}$, then the computation above remains word by word the same on $X - \operatorname{Sing}(\Sigma_X)$, where Σ_X is the reduced ramification locus in X. Hence outside of codimension 2 one has:

$$\mathcal{R}|_{X - \operatorname{Sing} \Sigma_X} = \mathcal{T}|_{X - \operatorname{Sing} \Sigma_X}.$$

As $\mathcal{R} = j_* \mathcal{R}|_{X - \operatorname{Sing} \Sigma_X}$, where $j : X - \operatorname{Sing} \Sigma_X \to X$, one still obtains:

$$\mathcal{R} = \bigoplus_0^{N-1} \mathcal{A}^{-i} \left(\sum \left[\frac{v_x \cdot i}{N} \right] \cdot x \right)$$

in this case.

4.5. Assumptions. In the rest of this section, we will assume that $\pi : Y \to X$ is a *tame covering* of Dedekind schemes of degree n, whose *ramification indices* e_y are all *odd*.

We define some natural numbers

$$m_x := \sum_{y \in \pi^{-1}(x)} \frac{e_y^2 - 1}{8} [k(y) : k(x)],$$

$$n_x := \operatorname{lcm} \{e_y, y \in \pi^{-1}(x)\},$$

$$N := \operatorname{lcm} \{n_x, x \in X\} = \operatorname{lcm} \{e_y, y \in Y\}$$

where x is a closed point in X. For any x, any y, one has

$$\gcd(e_y, \operatorname{char} k(\pi(x))) = \gcd(n_x, \operatorname{char} k(x)) = \gcd(N, \operatorname{char} k(x)) = 1 .$$

We define divisors

$$\omega(Y/X) := \sum m_x \cdot x \in \operatorname{Div} X,$$

$$D_{Y/X} := \sum \frac{e_y - 1}{2} \cdot y \in \operatorname{Div} Y,$$

$$D_1 := \sum_{n_x > 1} \frac{N}{n_x} \cdot x \in \operatorname{Div} X.$$

Often we abuse notations by denoting by the same letter the class of the divisor in Pic or in Pic/2.

The covering π being tame, one has $\mathcal{O}_Y(2D_{Y/X}) = \omega_{Y/X}$. Hence there is a trace $\pi_* \mathcal{O}_Y(2D_{Y/X}) \to \mathcal{O}_X$ which defines on the bundle $E := \pi_* \mathcal{O}_Y(D_{Y/X})$ *a unimodular quadratic bilinear form* q_E

$$\begin{array}{ccc} E \times E & \longrightarrow & \pi_* \mathcal{O}_Y(2D_{Y/X}) \\ & \searrow^{q_E} & \downarrow \mathrm{Tr} \\ & & \mathcal{O}_X \end{array}$$

given by $\mathrm{Tr}_{Y/X}(x \cdot y)$.

4.6. Proposition. *Let π be as in* 4.3. *Then there is a Kummer covering $\varphi : Z \to X$ of degree N such that one has*:

Denote by T the normalization of

$$T' := Y \times_X Z,$$

and denote the morphisms as in the diagram

$$\begin{array}{ccccc} T & \xrightarrow{\delta} & T' & \xrightarrow{p_1} & Y \\ & \searrow^{\pi_Z} & \downarrow p_Z & & \downarrow \pi \\ & & Z & \xrightarrow[\varphi]{} & X. \end{array}$$

Then π_Z is an étale covering of irreducible Dedekind schemes.

Proof. Of course $N = 1$ if and only if π is already étale, in which case we take $\varphi = \mathrm{id}$.

Assume $N \geqq 2$. As noted in 3.9 a Dedekind scheme is either a curve over some field and therefore quasi-projective or it is quasi-affine. In both cases we have ample sheaves on X. If X is quasi-affine, then by definition of ampleness one can take $\mathcal{O}_X$.

Assume that X is projective. Then we choose a closed point $x \in X - D_1$. Let $\mathcal{A}$ be an ample sheaf. Replacing $\mathcal{A}$ by some power one can assume that $\mathcal{A}^N(-D_1 - x)$ is generated by global sections. Then $\mathcal{A}^N = \mathcal{O}_X(D)$ where $D = D_1 + x + D_2$ and where D_2 is a divisor with $|D_2| \cap |x + D_1| = \emptyset$.

If X is not projective it is quasi-affine and $\mathcal{O}_X(-D_1)$ is generated by its global sections. We find a divisor D_2 with $|D_2| \cap |D_1| = \emptyset$ and a section s of $\mathcal{O}_X$ with zero set $D = D_1 + D_2$. Multiplying s by a unit, we can assume that s is not a power of some other section.

In fact, if $X = \operatorname{Spec} A \not\supset |D_1|$ we can as well use the Chinese remainder theorem to find a function $f \in A$ whose divisor (f) is of the shape $(f) = D$ where

$$D = D_1 + x + D_2$$

for an effective divisor D_2 and some point $x \notin |D_1|$ with $|D_2| \cap |x + D_1| = \emptyset$.

In any case the conditions of 4.2 are fulfilled. Set $\mathcal{B} := \pi^* \mathcal{A}$. Then $Z = \mathbf{Spec}_{\mathcal{O}_X} \mathcal{R}$, $T = \mathbf{Spec}_{\mathcal{O}_Y} \mathcal{S}$ with

$$\mathcal{R} = \bigoplus_{0}^{N-1} \mathcal{A}^{-i}\left(\sum_{n_x > 1} \left[\frac{i}{n_x}\right] x + \left[\frac{D_2 \cdot i}{N}\right]\right),$$

$$\mathcal{S} = \bigoplus_{0}^{N-1} \mathcal{B}^{-i}\left(\sum_{\substack{n_x > 1 \\ y \in \pi^{-1}(x)}} \left[\frac{i \cdot e_y}{n_x}\right] y + \left[\frac{(\pi^* D_2) \cdot i}{N}\right]\right).$$

As $\pi|_{X - D_1}$ is étale, $\delta|_{X - D_1}$ is the identity, and therefore $\pi_Z|_{X - D_1} = p_2|_{X - D_1}$ is étale.

It $t \in T$, such that $p_1 \delta(t) = y \in \pi^{-1}(x)$, where $x \in D_1$, then the ramification index of t over y is

$$\frac{N}{\gcd\left(N, \dfrac{N e_y}{n_x}\right)} = \frac{n_x}{e_y},$$

and therefore the one of t over x is n_x. The ramification index of $z = \pi_Z(t)$ over x being

$$\frac{N}{\gcd\left(N, \dfrac{N}{n_x}\right)} = n_x,$$

t is unramified over z. Further, separability of the residue field extensions is computed locally, at points above D_1.

We denote again by A and B the local rings of X at x and Y at $y \in \pi^{-1}(x)$, and we set $\nu = \nu_x = \dfrac{N}{n_x}$. Then locally, we obtained the normalizations R and T as follows:

$$\begin{array}{ccccccc}
\dfrac{B[v][\omega][t]}{\langle v^\nu - u, \omega^e - v, t^{n/e} - \tau\omega\rangle} & \xrightarrow{\alpha_4} & \dfrac{B[v](\omega)}{\langle v^\nu - u, \omega^e - v\rangle} & \xrightarrow{\alpha_3} & \dfrac{B[v]}{\langle v^\nu - u\rangle} & \longrightarrow & B \\
 & \searrow^{\pi_Z} & & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \pi} \\
 & & \dfrac{A[v][t]}{\langle v^\nu - u, t^n - \tau v\rangle} & \xrightarrow[\alpha_5]{} & \dfrac{A[v]}{(v^\nu - u)} & \longrightarrow & A\,.
\end{array}$$

The right square is cartesian. Therefore α_2 is separable as π is separable. As

$$(e, \operatorname{char} k(x)) = 1\,,$$

and v is a unit in $B[v]/\langle v^\nu - u\rangle$, α_3 is separable; α_4 and α_5 are totally ramified, hence separable. Therefore $\alpha_5 \circ \pi_Z = \alpha_2 \circ \alpha_3 \circ \alpha_4$ is separable, as well as π_Z.

This shows that π_Z is étale. □

4.7. Notations. Considering φ as in 4.6, we define as in 4.5: $F := \pi_{Z_*}\mathcal{O}_T$, together with its unimodular quadratic form

$$q_F = \operatorname{Tr}_{T/Z}(x \cdot y)\,.$$

As $\delta|_{X-D_1}$ is an isomorphism, one has by base change:

$$\varphi^*(E, q_E)|_{X-D_1} = (F, q_F)|_{X-D_1}\,.$$

Let $j: Z - \varphi^{-1}(D_1) \to Z$ be the open embedding. We define $\mathcal{G}$ to be the subsheaf of $j_*\varphi^* E|_{X-D_1} = j_* F|_{X-D_1}$ generated by $\varphi^* E$ and F. That is, one has a diagram:

$$\varphi^* E \oplus F \xrightarrow{\ =\ } \mathcal{G} \hookrightarrow j_* F|_{X-D_1}\,,$$
$$(\varphi^* e \oplus f) \mapsto \varphi^* e - f\,.$$

As Z is a Dedekind scheme, $\mathcal{G}$ is a locally free $\mathcal{O}_Z$ sheaf (of course, not unimodular quadratic in general).

We denote by $\alpha: F \to \mathcal{G}$ the inclusion. One has an exact sequence

$$0 \to \det F \xrightarrow{\det \alpha} \det \mathcal{G} \to \sum k(z)^{l(z)} \to 0\,.$$

It defines a divisor $\Gamma := \sum l(z) \cdot z \in \operatorname{Div} Z$ which we want to compute in the sequel. We first observe that our definition is symmetric as one has the little

4.8. Lemma.

(i) $w_1(\varphi^* E) = (\det \varphi^* E, \det \varphi^* q_E) = (\det F, \det q_F) = w_1(F)$ *in* $H^1_{\text{ét}}(Z, \mathbb{Z}/2\mathbb{Z})$.

(ii) $c_1(\varphi^* E) = c_1(F)$ *in* $\operatorname{Pic} Z$.

(iii) $\Gamma = c_1(\mathcal{G}) - c_1(F) = c_1(\mathcal{G}) - c_1(\varphi^* E)$ *in* $\operatorname{Pic} Z$.

Proof. As $H^1_{\text{ét}}(Z, \mathbb{Z}/2\mathbb{Z})$ injects into $H^1_{\text{ét}}(K_Z, \mathbb{Z}/2\mathbb{Z})$ and

$$\varphi^*(E, q_E) \otimes_{\mathcal{O}_X} K_Z = (F, q_F) \otimes_{\mathcal{O}_X} K_Z\,,$$

one has (i). Via the map

$$H^1_{ét}(X, \mathbb{Z}/2\mathbb{Z}) \to H^1_{ét}(X, \mathbb{G}_m) = \operatorname{Pic} Z$$

one obtains (ii), which implies (iii). □

4.9. Proposition. *Let Γ and $\omega(Y/X)$ be as in* 4.7 *and* 4.5. *Then one has*

$$l(z) = \sum_{y \in \pi^{-1}(\varphi(z))} \frac{n_x}{e_y} \cdot \frac{(e_y^2 - 1)}{8} [k(y) : k(\varphi(z))]$$

and

$$\Gamma = \varphi^* \omega(Y/X) \quad \text{in} \quad \operatorname{Div} Z/2 .$$

Proof. Base changing by φ one has $\varphi^* E = p_{2_*} p_1^* \mathcal{O}_Y(D_{Y/X})$ (notations are as in 4.6), and one also has $F = p_{2_*} \delta_* \mathcal{O}_T$, where

$$\delta_* \mathcal{O}_T|_{X - D_1} = p_1^* \mathcal{O}_Y(D_{Y/X})|_{X - D_1} = p_1^* \mathcal{O}_Y|_{X - D_1} .$$

Denoting by

$$j' : T' - p_2^{-1} \varphi^{-1}(D_1) \to T'$$

the inclusion, we define $\mathcal{G}'$ to be the subsheaf of

$$j'_* p_1^* \mathcal{O}_Y(D_{Y/X})|_{X - D_1} = j'_* \delta_* \mathcal{O}_T|_{X - D_1}$$

generated by $p_1^* \mathcal{O}_Y(D_{Y/X})$ and $\delta_* \mathcal{O}_T$. That is, one has morphisms

$$p_1^* \mathcal{O}_Y(D_{Y/X}) \oplus \delta_* \mathcal{O}_T \xrightarrow{\ -\ } \mathcal{G}' \hookrightarrow j'_* \delta_* \mathcal{O}_T|_{X - D_1} ,$$

where "$-$" stands for the surjection $(e \oplus f) \mapsto e - f$. As p_2 is finite, one has

$$p_{2_*} \mathcal{G}' = \mathcal{G}, \ \alpha = p_{2_*} \alpha ,$$

where α' is the inclusion $\delta_* \mathcal{O}_T \to \mathcal{G}'$. As $l(z)$ is also the dimension over $k(z)$ of cokernel $\alpha \otimes k(z)$, one has

$$l(z) = \sum_{t' \in p_2^{-1}(z)} a_{t'} [k(t') : k(z)] ,$$

where $a_{t'}$ is the dimension over $k(t')$ of the cokernel of $\alpha' \otimes k(t')$, supported in

$$p_2^{-1} \varphi^{-1}(D_1) \subset T' .$$

Therefore we may compute locally at $x \in D_1 \subset X$. Set

$$V = \operatorname{Spec} A_1 \quad \text{and} \quad W = \operatorname{Spec} B_1 ,$$

with

$$A_1 = A[v]/\langle v^\nu - u \rangle \quad \text{and} \quad B_1 = B[v]/\langle v^\nu - u \rangle ,$$

for

$$A = \mathcal{O}_{X,x} \quad \text{and} \quad B = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_X} \mathcal{O}_Y = \pi^* \mathcal{O}_{X,x} .$$

One has

$$[k(y):k(x)] = \sum_{\substack{w \to y \\ w \to v}} [k(w):k(v)] .$$

Recall that V (resp. W) has ramification index 1 over X (resp. Y). Denote the maps as in the Cartesian squares:

$$\begin{array}{ccccccc} T & \xrightarrow{\delta} & T' & \xrightarrow{p_1} & W & \xrightarrow{q_1} & Y \\ & & {\scriptstyle p_2}\downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} \\ & & Z & \xrightarrow[\varrho]{} & V & \xrightarrow[\sigma]{} & X . \end{array}$$

One has

$$\varrho_* \mathcal{O}_Z = \bigoplus_0^{n_x - 1} A_1 t^i ,$$

$$p_{1_*} \mathcal{O}_{T'} = \bigoplus_0^{n_x - 1} B_1 t^i ,$$

$$p_{1_*} p_1^* q_1^* \mathcal{O}_Y(D_{Y/X}) = p_{1_*} p_1^* \mathcal{O}_W(D_{W/V}) = \bigoplus_0^{n_x - 1} \mathcal{B}_1^{-i} \Bigg(\sum_{\substack{w \to v \\ v \to x}} \frac{e_w - 1}{2} \cdot w \Bigg)$$

as $e_w = e_{q_1(w)}$, where $\mathcal{B}_1^{-1}$ is the invertible sheaf associated to $B_1 t$. One also has

$$p_{1_*} \delta_* \mathcal{O}_T = \bigoplus_0^{n_x - 1} \mathcal{B}_1^{-i} \Bigg(\sum_{\substack{w \to v \\ v \to x}} \left[\frac{e_w \cdot i}{n_x} \right] \cdot w \Bigg) .$$

This says:

$$\mathcal{G}' = \bigoplus_0^{n_x - 1} \mathcal{B}_1^{-i} \Bigg(\sum_{\substack{w \to v \\ v \to x}} \sup \left\{ \left[\frac{e_w \cdot i}{n_x} \right], \frac{e_w - 1}{2} \right\} \cdot w \Bigg) .$$

Therefore

$$\sum_{t' \to w} a_{t'} [k(t'):k(w)] = \sum_{i=0}^{n_x - 1} \sup \left\{ 0, \frac{e_w - 1}{2} - \left[\frac{e_w \cdot i}{n_x} \right] \right\}$$

$$= \sum_{i=0}^{\frac{n_x}{e_w} \cdot \left(\frac{e_w - 1}{2}\right)} \left(\frac{e_w - 1}{2} - \left[\frac{e_w \cdot i}{n_x} \right] \right) = \frac{n_x}{e_w} \cdot \frac{e_w^2 - 1}{8} .$$

This implies, as Z/V is totally ramified, that:

$$l(z) = \sum_{t' \to v} a_{t'} \cdot [k(t'):k(v)] = \sum_{w \to v} [k(w):k(v)] \cdot \sum_{t' \to w} a_{t'} \cdot [k(t'):k(w)]$$

$$= \Bigg(\sum_{w \to v} [k(w):k(v)] \Bigg) \cdot \frac{n_x}{e_w} \cdot \frac{e_w^2 - 1}{8} = \sum_{y \to x} \frac{n_x}{e_y} \cdot \frac{e_y^2 - 1}{8} \sum_{\substack{w \to y \\ w \to v}} [k(w):k(v)]$$

$$= \sum_{y \to x} \frac{n_x}{e_y} \cdot \frac{e_y^2 - 1}{8} [k(y):k(x)] ,$$

as claimed. On the other hand

$$\varphi^* \omega(Y/X) = \sum_{z \to x} \sum_{y \to x} \left(n_x \cdot \frac{e_y^2 - 1}{8} [k(y) : k(x)] \right) \cdot z$$

and as $\left(n_x - \frac{n_x}{e_y} \right) \in 2\mathbb{Z}$, this proves as well that $\Gamma = \varphi^* \omega(Y/X)$. □

§ 5. Stiefel-Whitney classes of quadratic bundles; the splitting principle

5.1. Let X be a scheme over $\mathbb{Z}[\frac{1}{2}]$ and (E, q_E) be a *unimodular quadratic bundle* of rank n. Equivalently E is a vector bundle of rank n together with an isomorphism

$$\tilde{q}_E : E \to E^\vee := \mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X),$$

for which $q_E(x, y) := \tilde{q}_E(y)(x)$ is symmetric. Then (E, q_E) has Stiefel-Whitney classes

$$w_i(E, q_E) \in H^i_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}).$$

The first one is just the isomorphism class of $(\det E, \det q_E)$ in $H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$. There are different ways to understand the higher ones.

Either by computing the cohomology of the classifying space

$$H^*(\mathbf{BO}(n)/X, \mathbb{Z}/2\mathbb{Z}) = H^*(X, \mathbb{Z}/2\mathbb{Z})[w_2, \ldots, w_n]$$

(see section 1) or – as far as w_2 is concerned – by the universal central extension

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \tilde{\mathbf{O}}(n) \to \mathbf{O}(n) \to 0$$

(see 1.17), or via *Grothendieck's splitting principle for quadratic bundles* [18], [4], which we now "recall".

5.2. On the projective bundle $p : \mathbb{P} := \mathbb{P}(E) \to X$ we consider the composite map

$$s : \mathcal{O}_{\mathbb{P}}(-1) \xrightarrow{q^\vee} p^* E^\vee \xrightarrow{\tilde{q}_E^{-1}} p^* E \xrightarrow{q} \mathcal{O}_{\mathbb{P}}(1)$$

which is nothing but the following section of $S^2 E$:

$$\mathcal{O}_X \to E \otimes E^\vee = \mathcal{H}om_{\mathcal{O}_X}(E, E) \xrightarrow{\text{id} \otimes \tilde{q}_E^{-1}} E \otimes E \to S^2 E.$$

The section s is not trivial as q_E is symmetric. We denote by Q its zeroset. In fact Q is the relative quadric of isotropic vectors, which is smooth over X as q_E is unimodular. On $U := \mathbb{P} - Q$, $s|_U$ is an isomorphism and gives $\mathcal{O}(1)|_U$ a unimodular quadratic structure of rank 1. We denote by

$$(\mathcal{O}(1), s) \in H^1_{\text{ét}}(U, \mathbb{Z}/2\mathbb{Z})$$

its first Stiefel-Whitney class. Set $p' := p|_U$.

Claim. *For all m one has a splitting*

$$H^m_{\text{ét}}(U, \mathbb{Z}/2\mathbb{Z}) = \bigoplus_{j=0}^{\mathrm{Min}\{m,n-1\}} p^{*\prime} H^{m-j}_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}) \cup (\mathcal{O}(1), s)^{\cup j}.$$

Proof. The composition of the maps

$$R^i p_* \mathbb{Z}/2\mathbb{Z} \xrightarrow{\alpha_i} R^i p|_{Q*} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\beta_i} R^{i+2} p_* \mathbb{Z}/2\mathbb{Z}$$

vanishes as it is the cup product with the (relative) first Chern class of $\mathcal{O}_{\mathbb{P}}(Q) = \mathcal{O}_{\mathbb{P}}(2)$ in $R^2 p_* \mathbb{Z}/2\mathbb{Z}$. By [4], (3.3), the sheaves

$$R^{2j-1} p|_{Q*} \mathbb{Z}_2$$

are zero and the sheaves

$$R^{2j} p|_{Q*} \mathbb{Z}_2$$

are locally constant $\mathbb{Z}_2$ modules of rank one, for $2j \neq n-2$, and of rank two, for $2j = n-2$. Hence the same holds true if one replaces $\mathbb{Z}_2$ by $\mathbb{Z}/2\mathbb{Z}$. By the "Weak Lefschetz Theorem" and by base change for the proper morphisms p and $p|_Q$ the map α_i is bijective, and hence β_i the zero map, for $i < n-2$. Moreover α_{n-2} is injective. By duality one obtains as well that β_{n-2} is surjective and that, for $i > n-2$, the map β_i is an isomorphism. Therefore one has $R^i p'_* \mathbb{Z}/2\mathbb{Z} = 0$, for $i \geqq n$. As

$$R^{2j+1} p_* \mathbb{Z}/2\mathbb{Z} = R^{2j+1} p|_{Q*} \mathbb{Z}/2\mathbb{Z} = 0,$$

for all $j \geqq 0$, the restriction map

$$R^{2j} p_* \mathbb{Z}/2\mathbb{Z} \to R^{2j} p'_* \mathbb{Z}/2\mathbb{Z}$$

is an isomorphism, for $2j < n$. The residue map

$$R^{2j+1} p'_* \mathbb{Z}/2\mathbb{Z} \to R^{2j} p|_{Q*} \mathbb{Z}/2\mathbb{Z}$$

is an isomorphism for $2j+1 < n-1$. Since the right hand side is the free $\mathbb{Z}/2\mathbb{Z}$ module generated by $\alpha_{2j}(c_1(\mathcal{O}(1))^{\cup j})$, the left hand side is generated by

$$(\mathcal{O}(1), s) \cup c_1(\mathcal{O}(1)|_U)^{\cup j}.$$

In fact, its residue is the cup product of the residue of $(\mathcal{O}(1), s)$ in $p|_{Q*} \mathbb{Z}/2\mathbb{Z}$ with

$$c_1(\mathcal{O}(1)|_Q)^{\cup j} = \alpha_{2j}(c_1(\mathcal{O}(1))^{\cup j}) \in R^{2j} p|_{Q*} \mathbb{Z}/2\mathbb{Z}.$$

If n is even, then $\mathrm{Ker}(\beta_{n-2})$ is a $\mathbb{Z}/2\mathbb{Z}$ module of rank one. Hence

$$\mathrm{Ker}(\beta_{n-2}) = \mathrm{Im}(\alpha_{n-2})$$

and, as above,

$$(\mathcal{O}(1), s) \cup c_1(\mathcal{O}(1)|_U)^{\cup \frac{n-2}{2}}$$

is a generator for $R^{n-1} p'_* \mathbb{Z}/2\mathbb{Z}$.

As $R^{2j} p'_* \mathbb{Z}/2\mathbb{Z}$ is generated by the relative class of

$$c_1(\mathcal{O}(1)|_U)^{\cup j} \in H^{2j}_{\text{ét}}(U, \mathbb{Z}/2\mathbb{Z}) \quad \text{for } 2j < n,$$

and as $R^{2j+1} p'_* \mathbb{Z}/2\mathbb{Z}$ is generated by the relative class of

$$(\mathcal{O}(1), s) \cup c_1(\mathcal{O}(1)|_U)^{\cup j} \in H^{2j+1}_{\text{ét}}(U, \mathbb{Z}/2\mathbb{Z}) \quad \text{for } 2j+1 < n,$$

one obtains the splitting

$$H^m_{\text{ét}}(U, \mathbb{Z}/2\mathbb{Z}) = \bigoplus_{2j \leq \text{Min}\{m, n-1\}} p'^* H^{m-2j}_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}) \cup c_1(\mathcal{O}(1)|_U)^{\cup j}$$
$$\oplus \bigoplus_{2j+1 \leq \text{Min}\{m, n-1\}} p'^* H^{m-2j-1}_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}) \cup (\mathcal{O}(1), s) \cup c_1(\mathcal{O}(1)|_U)^{\cup j}.$$

Finally, one has

$$c_1(\mathcal{O}(1)|_U) = (\mathcal{O}^{(1)}, s)^{\cup 2} + (-1) \cup (\mathcal{O}(1), s)$$

where $(-1) \in p'^* H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$ (see 5.3 below). This proves the claim. □

In particular there are classes $w_{n-j}(E, q_E) \in H^{n-j}_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$ which are defined by the expression

$$(\mathcal{O}(1), s)^{\cup n} = \bigoplus_{j<n} p'^* w_{n-j}(E, q_E) \cup (\mathcal{O}(1), s)^{\cup j}.$$

One defines

$$w_0(E, q_E) = 1 \in H^j_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}),$$

and $w_m(E, q_E) = 0$ for $m > n$. Furthermore, as in Grothendieck's general principle [13], the Stiefel-Whitney classes are functorial and additive.

5.3. For any unit $u \in \Gamma(X, \mathbb{G}_m)$ we denote by (u) its class in $H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$ via the connecting morphism of the Kummer exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{G}_m \xrightarrow{2} \mathbb{G}_m \to 0.$$

As $H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$ injects into $H^1_{\text{ét}}(\operatorname{Spec} K, \mathbb{Z}/2\mathbb{Z}) = K^*/K^{*2}$, (u) is just the quadratic residue of u in K^*/K^{*2}.

For any rank 1 unimodular quadratic bundle $(L, s) \in H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$, one considers its class $c_1(L)$ in $H^2_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$ via

$$H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}) \to H^1_{\text{ét}}(X, \mathbb{G}_m) \to H^2_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}),$$

that is, via the connecting morphism of the exact sequence

$$(*) \qquad 1 \to \mu_2 \to \mu_4 \to \mu_2 \to 1 .$$

Lemma. *One has*

$$c_1(L) = (L,s)^{\cup 2} + (-1) \cup (L,s) .$$

Proof. One can make a direct computation:

as $(*)$ is the twist of the sequence

$$(**) \qquad 0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

by $(-1) \in H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$, one has $\delta_*((L,s)) = \delta_{**}((L,s)) + (-1) \cup (L,s)$ where δ_* (resp. δ_{**}) is the connecting morphism of $(*)$ (resp. $(**)$) (see for example [17], A 3.1).

It remains to show that $\delta_{**}(L,s) = (L,s)^{\cup 2}$. Let $\bar{\zeta}_{\alpha\beta}$ be an étale cocycle for (L,s) in $\mathbb{Z}/2\mathbb{Z}$, of representative $\zeta_{\alpha\beta} \in \{0,1\} \subset \mathbb{Z}$. One has $\delta\zeta_{\alpha\beta} \in 2\mathbb{Z}$, and $\frac{1}{2}\delta\zeta_{\alpha\beta} \bmod 2$ is a cocycle representing $\delta_{**}(L,s)$.

But $(L,s)^{\cup 2}$ is represented by the cocycle $\zeta_{\alpha\beta} \cdot \zeta_{\beta\gamma} \bmod 2$. One has:

$$\begin{aligned} 2\zeta_{\alpha\beta} \cdot \zeta_{\beta\gamma} &= (\zeta_{\alpha\beta} + \zeta_{\beta\gamma})^2 - \zeta_{\alpha\beta}^2 - \zeta_{\beta\gamma}^2 = (\zeta_{\alpha\gamma} + \delta\zeta_{\alpha\beta})^2 - \zeta_{\alpha\beta}^2 - \zeta_{\beta\gamma}^2 \\ &\equiv \zeta_{\alpha\gamma}^2 - \zeta_{\alpha\beta}^2 - \zeta_{\beta\gamma}^2 \pmod 4 \equiv \delta\zeta \pmod 4 , \end{aligned}$$

or $\zeta_{\alpha\beta} \cdot \zeta_{\beta\gamma} \equiv \dfrac{\delta\zeta}{2} \bmod 2.$ □

Remark. If X is smooth proper over an algebraically closed field, then

$$H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}) = {}_2\mathrm{Pic}\, X$$

and $c_1(L) = 0$. But $(-1) = 0$ and $\cup$ is antisymmetric. So the formula does not say anything.

5.4. Notation. On X which we now assume to be regular, one considers the exact sequence

$$0 \to \mathrm{Pic}\, X/2 \to H^2_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}) \to H^2_{\text{ét}}(K, \mathbb{Z}/2\mathbb{Z})$$

coming from the localization

$$\mathbb{Z}/2\mathbb{Z} = j_* \mathbb{Z}/2\mathbb{Z} \to Rj_* \mathbb{Z}/2\mathbb{Z}$$

where $j: \mathrm{Spec}\, K \to X$ is the inclusion of the generic point.

So if $w \in H^2_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$, we denote by $w|_K$ its reduction to the generic point $\mathrm{Spec}\, K$. It is the arithmetical part of w. If $w|_K = 0$, then $w \in \mathrm{Pic}\, X/2$. We say that w is algebraic, that is, supported by algebraic cycles.

5.5. One says that the unimodular quadratic bundle (E, q_E) of rank $2n$ is *split* if there is a subbundle $V^\vee \subset E$ of rank n which is totally isotropic, that is $q_E|_{V^\vee} = 0$. As E is isomorphic to $E^\vee$ via $\tilde{q}_E$ (see 5.1), the quotient $E/V^\vee$ is isomorphic to V, and furthermore for any local sections $v^\vee$ of $V^\vee$, e of E, of image $v \in E/V^\vee \simeq V$, one has

$$q_E(v^\vee, e) = v^\vee(v).$$

Equivalently: (E, q_E) is split if one has a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V^\vee & \longrightarrow & E & \longrightarrow & V & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \tilde{q}_E} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & V^\vee & \longrightarrow & E^\vee & \longrightarrow & V & \longrightarrow & 0. \end{array}$$

Proposition. *One has for a split bundle E*

$$w(E) := \sum_{j \geq 0} w_j(E) = \sum_0^n (1 + (-1))^{n-i} \cup c_i(V)$$

where $c_i(V) = c_i(V^\vee) \in H^{2i}_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$ are the Chern classes of V.

Proof. Assume $n = 1$. By 5.2 one has:

$$(\mathcal{O}(1), s)^{\cup 2} = p'^* w_2(E, q_E) + p'^* w_1(E, q_E) \cup (\mathcal{O}(1), s),$$

and by 5.3 one has

$$(\mathcal{O}(1), s)^{\cup 2} = c_1(\mathcal{O}(1)|_U) + (-1) \cup (\mathcal{O}(1), s).$$

This implies that

$$w_1(E, q_E) = (-1)$$

(which we could have computed directly on Spec$K$!), and that

$$p'^* w_2(E, q_E) = c_1(\mathcal{O}(1)|_U) \quad \text{in} \quad \operatorname{Pic} X/2 \hookrightarrow H^2_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}),$$

as

$$p'^* : \operatorname{Pic} X \to \operatorname{Pic} U$$

is an isomorphism. But one has $p'^* V^\vee|_U \simeq \mathcal{O}(1)|_U$. This is the proposition in this case.

If $n > 1$, we will develop a "*splitting principle for split bundles*":

On $\mathbb{P}$ one has the morphism $\alpha : \mathcal{O}_P(-1) \to p^* V$, which is the composite of the morphism $\tilde{q}_E^{-1} \circ q^\vee$ considered in 5.2 and of $E \to V$. The image of $\alpha|_U$ is a subbundle.

In fact, assume that $x \in U$ such that $\alpha \otimes k(x) \equiv 0$. Then $\tilde{q}_E \circ q^\vee \otimes k(x)$ factorizes through $p^* V^\vee \otimes k(x)$. Hence $q \otimes k(x)$ factorizes through $p^* V \otimes k(x)$ and for $s = q \circ \tilde{q}_E \circ q^\vee$ one has $s \otimes k(x) = 0$, which is excluded.

Let $\varrho : \mathbb{P}_U := \mathbb{P}(p'^* V) \to U$ be the projective bundle and let $\varrho^* p'^* V \to \zeta$ be the map to the tautological bundle ζ of $\mathbb{P}_U$. The morphism

$$\sigma : \varrho^* \mathcal{O}_P(-1)|_U \xrightarrow{\varrho^* \alpha} \varrho^* p'^* V \to \zeta$$

defines a section of $\zeta \otimes \varrho^* \mathcal{O}_P(1)|_U$ whose zero set is denoted by L. Then L is fiberwise for ϱ of degree 1. By homotopy invariance one has

$$\varrho'^* H^*_{\text{ét}}(U, \mathbb{Z}/2\mathbb{Z}) = H^*_{\text{ét}}(\mathbb{A}_U, \mathbb{Z}/2\mathbb{Z}),$$

where

$$\varrho' : \mathbb{A}_U := \mathbb{P}_U - L \to U$$

is the affine bundle.

Since $\zeta^\vee$ is a subbundle of $\varrho'^* p'^* V^\vee$ and since $\sigma|_{\mathbb{A}_U}$ is an isomorphism $\varrho'^* p'^* E$ contains

$$\zeta^\vee \oplus \varrho'^* (\tilde{q}_E^{-1} \circ q^\vee)(\mathcal{O}_U(-1)) =: F$$

as a subbundle. The form $q_{E|F}$ is unimodular and split. In fact, since $\zeta^\vee$ is contained in $V^\vee$ it is totally isotropic. Moreover, $\sigma|_{\mathbb{A}_U}$ gives an isomorphism of $\varrho'^*(\tilde{q}_E^{-1} \circ q^\vee)(\mathcal{O}_U(-1))$ with ζ, and one has $q_E|_F(x^\vee, y) = x^\vee(\sigma(y))$ for all

$$x^\vee \in \zeta^\vee \quad \text{and} \quad y \in \varrho'^*(\tilde{q}_E^{-1} \circ q^\vee)(\mathcal{O}_U(-1)).$$

Let $F^\perp$ be the orthogonal to F in $\varrho'^* p'^* E$. Then $F^\perp$ is isomorphic to $W^\vee \oplus W$, where W is the subbundle which is the kernel of $\varrho'^* p'^* V \to \zeta$, and $q_{E|F^\perp}$ again is split, as $W^\vee$ is totally isotropic.

Applying the case $n = 1$ to $w(F)$ and the induction to $w(F^\perp)$, one obtains:

$$\begin{aligned}
\varrho'^* p'^* w(E, q_E) &= (1 + (-1) + \varrho'^* c_1(\mathcal{O}_U(1))) \cup \sum_0^{(n-1)} (1 + (-1))^{n-1-i} \cup c_i(W) \\
&= \sum_0^n (1 + (-1))^{n-i} \cup (c_i(W) + \varrho'^* c_1(\mathcal{O}_U(1)) \cup c_{i-1}(W)) \\
&= \sum_0^n (1 + (-1))^{n-i} \cup \varrho'^* p'^* c_i(V).
\end{aligned}$$

As $\varrho'^* p'^*$ is injective, one obtains 5.5. □

§ 6. Local contribution to Stiefel-Whitney classes of quadratic bundles

6.1. Definitions and Notations. In this section X is a Dedekind scheme over $\mathbb{Z}[\frac{1}{2}]$ (see 3.8), K is the field of rational functions of X and (E, q_E) and (F, q_F) are two quadratic bundles of rank n such that $(E, q_E)|_K = (F, q_F)|_K$. One example was considered in 4.7 where (E, q_E) is replaced by $\varphi^*(E, q_E)$. We shall use the locally free $\mathcal{O}_X$-sheaf $\mathcal{G}$ defined by

$$E \oplus F \twoheadrightarrow \mathcal{G} \hookrightarrow E|_K.$$

The purpose of this section is to prove the

6.2. Theorem. *The Stiefel-Whitney classes of E, F are related to the Chern classes of E, $\mathscr{G}$ as follows*:

$$w(F \oplus E, q_F \oplus -q_E) = \sum_0^n (1 + (-1))^{n-i} \cup c_i(\mathscr{G}),$$

$$w(E \oplus F, q_E \oplus q_F)\, w(E, -q_E)^2 = \sum_0^{2n} (1 + (-1))^{2n-i} \cup c_i(\mathscr{G} \oplus E).$$

We first show the

6.3. Corollary. $w_2(E, q_E) + w_2(F, q_F) = c_1(\mathscr{G}) + c_1(E)$ *in*

$$\operatorname{Pic} X/2 \hookrightarrow H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}).$$

In particular $w_2(E, q_E) + w_2(F, q_F)$ *is algebraic.*

Proof of corollary. We know by 4.8 (i) that $w_1(E, q_E) = w_1(F, q_F)$, and of course one has $w_1(E, -q_E) = n \cdot (-1) + w_1(E, q_E)$. So the degree 2 part of the left hand side of the second equation of 6.2 becomes

$$\begin{aligned} &w_2(E, q_E) + w_2(F, q_F) + w_1(E, q_E) \cup w_1(E, q_E) \\ &+ (n \cdot (-1) + w_1(E, q_E)) \cup (n \cdot (-1) + w_1(E, q_E)) \\ &= w_2(E, q_E) + w_2(F, q_F) + n^2 \cdot (-1) \cup (-1) \end{aligned}$$

whereas the degree 2 part of the right hand side becomes

$$c_1(\mathscr{G}) + c_1(E) + \binom{2n}{2} \cdot (-1) \cup (-1) = c_1(\mathscr{G}) + c_1(E) + (2n^2 - n) \cdot (-1) \cup (-1). \quad \square$$

Proof of 6.2. For the second equality one considers the quadratic bundle

$$(E \oplus E, q_E \oplus -q_E).$$

This is a split bundle. To prove this one considers the isomorphism

$$E \oplus E \xrightarrow{\ \sigma\ } E \oplus E,$$

$$(x \oplus y) \mapsto \left(\frac{x+y}{2} \oplus \frac{x-y}{2}\right).$$

Then

$$(q_E \oplus -q_E)(\sigma(x \oplus y), \sigma(x \oplus y)) = q_E(x, y).$$

Considering as in 5.1 the isomorphism $\tilde{q}_E : E \to E^\vee$, one has that $(E \oplus E, q_E \oplus -q_E)$ is isomorphic to $(E \oplus E^\vee, c)$, where c is the bilinear form whose quadratic form is the contraction

$$c(x \oplus y^\vee, x \oplus y^\vee) := y^\vee(x).$$

Let $\delta : E \to E \oplus E$ be the diagonal map $x \mapsto (x, x)$, and let $d : E \oplus E \to E^\vee (\simeq E)$ be the map $(x \oplus y) \mapsto \frac{x-y}{2}$. Restricting

$$0 \to E|_K \xrightarrow{\delta} E \oplus E|_K = F \oplus E|_K \xrightarrow{d} E|_K \to 0$$

to $F \oplus E \subset F \oplus E|_K$, one obtains the exact sequence

$$0 \to E \cap F \to F \oplus E \to \mathscr{G} \to 0$$

where $\mathscr{G}$ is the subbundle of $E|_K = F|_K$ generated by

$$(e - f) \quad \text{for} \quad e \in E \quad \text{and} \quad f \in F .$$

Therefore $(F \oplus E, q_F \oplus -q_E)$ is a split bundle with maximal isotropic subbundle $E \cap F \cong \mathscr{G}^\vee$, and

$$(E \oplus E \oplus F \oplus E, q_E \oplus -q_E \oplus q_F \oplus -q_E)$$

is split with maximal isotropic subbundle $\mathscr{G}^\vee \oplus E$. One obtains the second equality using 5.5:

$$w(E \oplus F, q_E \oplus q_F) \cup w(E, -q_E)^2 = w(E \oplus E \oplus F \oplus E, q_E \oplus -q_E \oplus q_F \oplus -q_E)$$
$$= \sum_0^{2n} (1 + (-1))^{2n-i} \cup c_i(E \oplus \mathscr{G}^\vee) = \sum_0^{2n} (1 + (-1))^{2n-i} \cup c_i(E \oplus \mathscr{G}) .$$

For the first equality, one writes

$$w(E \oplus E \oplus F \oplus E, q_E \oplus -q_E \oplus q_F \oplus -q_E)$$
$$= w(E \oplus E, q_E \oplus -q_E) \cup w(F \oplus E, q_F \oplus -q_E) .$$

Again by 5.5, one has

$$w(E \oplus E, q_E \oplus -q_E) = \prod_1^n (1 + (-1) + \alpha_i)$$

where the α_i are the Chern roots of E. The right hand side of the second equality of 6.2 reads:

$$\prod_1^n (1 + (-1) + \alpha_i) \prod_1^n (1 + (-1) + \beta_j)$$

where the β_j are the Chern roots of $\mathscr{G}$. Therefore the first equality is a consequence of the second one. □

§ 7. Serre's formula

7.1. Theorem. *Let X be a Dedekind scheme, and let $\pi : Y \to X$ be a tame covering with odd ramification indices. Consider*

$$(E, q_E) = \left(\pi_* \mathcal{O}_Y\left(\sum_{y \in Y} \frac{e_y - 1}{2}\right) \cdot y, \operatorname{Tr}_{Y/X}(a \cdot b)\right),$$

and as in 4.5 *the divisor*

$$\omega(Y/X) = \sum_{x\in X}\left(\sum_{y\in\pi^{-1}(x)} \frac{e_y^2-1}{8}\cdot[k(y):k(x)]\right)\cdot x$$

supported in the ramification locus of π. *Then one has*

$$\text{(S)} \qquad w_2(E, q_E) + \omega(Y/X) = w_2(\pi) + (2) \cup w_1(E, q_E),$$

where the $w_i(E, q_E)$ *are the Stiefel-Whitney classes of* (E, q_E) *and where* $w_2(\pi)$ *is the class defined in* 3.15.

Proof. Let $\varphi: Z \to X$ be a covering of Dedekind schemes of odd degree as constructed in 4.6. Then

$$\varphi^*: H^i_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}) \to H^i_{\text{ét}}(Z, \mathbb{Z}/2\mathbb{Z})$$

is split (and therefore injective) as for any $w \in H^i_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$ one has $\varphi_*\varphi^* w = (\deg\varphi)\cdot w$ and $\deg\varphi$ acts as a unit on $H^i_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$. Therefore (S) is equivalent to

$$\varphi^*\text{(S)} \qquad \varphi^* w_2(E, q_E) + \varphi^*\omega(Y/X) = \varphi^* w_2(\pi) + (2)\cup\varphi^* w_1(E, q_E).$$

Take the notations of 4.7. By 4.8 (i) one has $\varphi^* w_1(E, q_E) = w_1(F, q_F)$, by the second equality in 4.9 and by 4.8 (iii) one has

$$\varphi^*\omega(Y/X) = c_1(\mathcal{G}) - c_1(F) = c_1(\mathcal{G}) - c_1(\varphi^* E),$$

by 6.3,

$$c_1(\mathcal{G}) - c_1(\varphi^* E) = \varphi^* w_2(E, q_E) + w_2(F, q_F),$$

and by 3.16 one finally has $\varphi^* w_2(\pi) = w_2(\pi_Z)$. Therefore φ^*(S) is equivalent to

$$(\text{S}_{\text{ét}}) \qquad w_2(F, q_F) = w_2(\pi_Z) + (2)\cup w_1(F, q_F)$$

which has been obtained in 2.3. □

7.2. Corollary. *Let* $\varphi: Z \to X$ *be a covering of Dedekind schemes of odd degree and let* T *be the normalization of* $Z \times_X Y$. *Assume that* $\pi_Z: T \to Z$ *is tame with odd ramification indices. Then* $w_2(E, q_E) + \omega(Y/X)$ *is functorial, that is*:

$$\varphi^*\big(w_2(E, q_E) + \omega(Y/X)\big) = w_2(F, q_F) + \omega(T/Z)$$

where

$$(F, q_F) = \left(\pi_{Z*}\mathcal{O}_T\left(\sum_{t\in T}\frac{e_t-1}{2}\cdot t\right), \operatorname{Tr}_{T/Z}(a\cdot b)\right).$$

7.3. Remark. Actually it is this fact, which is transparent in [23] once one believes that there is a functorial definition of $w_2(\pi)$ which inspired the formulation and the proof of 7.1.

Added in proof. Conjecture 2.4 is false. The correct answer is given in *B. Kahn*, Equivariant Stiefel-Whitney classes, preprint 1992.

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